Fixed Point Theorems for α-F-Convex Contraction Mappings in b-Metric Spaces

A RESEARCH SUBMITTED TO THE DEPARTMENT OF MATHEMATICS IN PARTIAL FULFILLMENT FOR THE REQUIREMENTS OF THE DEGREE OF MASTERS OF SCIENCE IN MATHEMATICS

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Declaration

I, the undersigned declare that, this research paper entitled " Fixed Point Theorems for α-**F**-Convex Contraction Mappings in b-Metric spaces " is my own original work and it has not been submitted for the award of any academic degree or the like in any other institution or university, and that all the sources I have used or quoted have been indicated and acknowledged.

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Abstract

In this research we established fixed point theorems for α-$\mathbf{F}$-convex contraction mappings in b-metric space and proved the existence and uniqueness of fixed points for such mappings. Our result extend and generalize the work of Eke et al. (2019). In this research undertaking, we followed analytical study design and used secondary sources of data, such as published articles and related books. Finally, We also provided examples in support of our main findings.

Contents

Chapter 1

Introduction

1.1 Background of the study

Let X be a non-empty set and $T : X \to X$ is said to a self-map of X. An element $x \in X$ is called a fixed point of T if $T(x) = x$ and denote by F_T or $Fix(T)$ is the set of all fixed points of T. Fixed point theory is an important tool in the study of nonlinear analysis. It is considered to be the key connection between pure and applied mathematics. It is also widely applied in different fields of study such as Economics, Chemistry, Physics and almost all engineering fields. The contraction mapping principle introduced by Banach (1922) has wide range of applications in a fixed point theory. In 1922, Banach proved the following famous fixed point theorem.

Let (X,d) be a complete metric space, $T : X \to X$ be a contraction, there exists a unique fixed point $x_0 \in X$ of T. This theorem, called the Banach contraction principle is a forceful tool in nonlinear analysis. Another category of contraction which is separate from Banach contraction, and does not imply continuity, was proposed by Kannan(1968) who also established in the same work that such mappings necessarily have unique fixed points in complete metric spaces. Mappings belonging to this category are known as Kannan type.

In 1972, a new concept which is different from that of Banach (1922) and Kannan (1968) for contraction type mapping was introduced by Chatterjea (1972) which gives a new direction to the study of fixed point theory. There are the classes of contractive mappings which are different from Banach contraction and have unique fixed point in complete metric spaces.

Many authors generalized Banach contraction principle and proved the existence of the fixed point without the continuity of the mapping in the whole domain X (see, Choudhury, 2009; Eke, 2016a; Eke, Oghonyon, & Davvaz, 2018; Eke, 2016b, Eke, Imaga, & Odetunmibi, 2017; Muresan & Muresan, 2015). Hardy & Rogers (1973), the existence of the fixed point of Hardy and Rogers contractive mappings does not require the mapping to be continuous in the entire domain X. Rather, a mapping satisfying Hardy and Rogers contraction turns out to be continuous at the fixed point. The family of contractive mappings in metric spaces is a great interest and has already been studied in the literature since long time.

Istratescu (1981) introduced the class of convex contraction mappings in metric spaces and generalized the well-known Banach's contraction principle. Recently, some works have appeared on the generalization of such classes of mappings in the setting of various spaces. Czerwik (1993) introduced the concept of b-metric spaces and proved the Banach contraction mapping principle in the setting of b-metric spaces. Afterwards, several research papers were published on the existence of fixed point results for single-valued and multi-valued mappings in the setting of b-metric spaces.

Very recently, Eke et al. (2019) introduced the notion of convex contractive mappings in metric spaces and proved a fixed point theorem for convex contractive mappings defined on complete metric spaces. Inspired and motivated by the results of Eke et al. (2019) the aim of this research is to extend and generalize the main theorem of Eke et al. (2019) in the setting of *b*-metric spaces.

1.2 Statements of the problem

In this study we focused on establishing and proving fixed point theorems for α-**F**-convex contraction mappings in the setting of *b*-metric spaces.

1.3 Objectives of the study

1.3.1 General objective

The main objective of this study was to establish fixed point theorems for α-**F**-convex contraction mappings in the setting of *b*-metric spaces.

1.3.2 Specific objectives

This study has the following specific objectives:

- To prove the existence of fixed points for α-**F**-convex contraction mappings in the setting of b-metric spaces.
- To verify the uniqueness of the fixed points.
- To verify the applicability of the main results obtained using specific examples.

1.4 Significance of the study

The study may have the following importance:

- It may provide basic research skills to the researcher.
- The outcome of this study may contribute to research activities on study area.
- It may help to show existence and uniqueness of solution of some problems involving integral and differential equations.

1.5 Delimitation of the Study

This study was delimited to establishing and proving fixed point theorems for α-**F**-convex contraction mappings in the setting of b-metric spaces.

Chapter 2

Review of Related Literatures

Fixed point theory is very important in diverse disciplines of mathematics since it can be applied for solving various problems and it is one of the most dynamic research subjects in nonlinear analysis. A very interesting useful result in fixed point theory is due to the Banach contraction principle. This theorem has witnessed numerous generalizations and extensions in the literature because of its simplicity and contractive approaches. For this reason generalizations of Banach's contraction principle have been investigated heavily by many researchers. Banach (1922) Banach contraction principle was introduced as follows: Let (X,d) be a metric space and $T : X \to X$. Then T is called a Banach contraction mapping if there exists $k \in [0,1)$ such that $d(Tx,Ty) \leq kd(x,y)$ for all $x,y \in X$. If (X,d) is a complete metric space, then T has a unique fixed point.

Kannan (1968) The concept of Kannan mapping was introduced in 1968 as follows: Let (X,d) be a metric space and $T : X \to X$. Then T is called a Kannan mapping if there exists $k \in [0,1/2)$ such that $d(Tx,Ty) \leq k[d(x,Tx)+d(y,Ty)]$ for all $x,y \in X$. If (X,d) is a complete metric space, then T has a unique fixed point. In 1972, the concept of Chatterjea type mapping was introduced as follow: Let (X,d) be a metric space and $T : X \to X$. Then T is called Chatterjea mapping if there exists $k \in [0,1/2)$ such that $d(Tx,Ty) \leq k[d(x,Ty)+d(y,Tx)]$ for all $x,y \in X$. If (X,d) is a complete metric space, then T has a unique fixed point.

Istratescu (1981) introduced the concept of convex contraction mappings as follows: Let (X,d) be a metric space and a continuous mapping $T : X \to X$ is called a convex contraction mapping of order 2, if there exists a, b $\in$ (0, 1) such that : $d(T^2x,T^2y) \leq ad(Tx,Ty)+bd(x,y)$ for all $x,y \in X$ and $a+b<1$.
Generalizing the Banach contraction principle, In 2012, Wardowski introduce the notion of F-contraction and proved a new fixed point theorem concerning **F**-contractions.

Chapter 3

Methodology

3.1 Study area and period

The study was conducted at Jimma University under the supervision of the department of mathematics from September, 2019 G. C. to August, 2020 G. C.

3.2 Study Design

In this study, we followed analytical design using standard mathematical procedures.

3.3 Source of Information

The relevant sources of information for this study where books and published articles related to the area of the study.

3.4 Mathematical Procedure of the Study

In this study, we followed the procedures stated below:

- Establishing theorems.
- Constructing sequences.
- Showing the convergence of the sequences.
- Showing the constructed sequence are b-Cauchy .
- Proving the existence of fixed point.
- Showing uniqueness of the fixed point.
- Providing examples in support of our main findings.

Chapter 4

Preliminaries and Main Result

4.1 Preliminaries

Notation: Throughout this research denote, $\mathbb{R}$ is the set of all real numbers, $\mathbb{R}^+$ is the set of all non-negative real numbers and $\mathbb{N}$ is the set of all natural numbers.

- $\mathbb{R} = (-\infty, \infty)$.
- $\mathbb{R}^+ = [0, \infty)$.
- $\mathbb{N}$= 1, 2, 3, $\cdots$.
- Φ denotes the set of all functions $F : \mathbb{R}^+ \longrightarrow \mathbb{R}$ be a mapping satisfying the following conditions:

  (F_1) F is strictly increasing, that is, for all $x, y \in \mathbb{R}^+$ if $x < y$ then $F(x) < F(y)$;

  (F_2) For each sequence $\{\alpha_n\}$ of positive numbers, $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$;

  (F_3) There exists $k \in (0,1)$ such that $\lim_{\alpha\to 0^+} (\alpha)^k F(\alpha) = 0$.

Definition 4.1.1 *(Samet et al., 2012) Let $T\colon X \to X$ be a self-mapping on a non-empty set X and $\alpha\colon X \times X \to \mathbb{R}^+$ be a function. We say that T is an α-admissible if $x, y \in X$, $\alpha(x,y) \geq 1 \Rightarrow \alpha(Tx, Ty) \geq 1$.*

Definition 4.1.2 *(Singh et al., 2018) An α-admissible mapping T is said to be an α^*-admissible, if for each $x, y \in Fix(T) \neq \varnothing$ we have $\alpha(x,y) \geq 1$. If $Fix(T) = \varnothing$ we say that T is vacuously α^*-admissible.*

Definition 4.1.3 *(Ciric, 1971) Let (X,d) be a metric space and $T\colon X \to X$ is said to be an orbitally continuous on X if $\lim_{n\to\infty} T^n x = z$ implies that $\lim_{n\to\infty} T(T^n x) = Tz$.*

Definition 4.1.4 *(Bakhtin, 1989; Czerwik, 1993) Let X be a non-empty set and $s \geq 1$ be a given real number. A function $d : X \times X \to \mathbb{R}^+$ is said to be a b-metric if and only if for all $x, y, z \in X$, the following conditions are satisfied:*

(a) $d(x,y) = 0$ *if and only if* $x = y$;

(b) $d(x,y) = d(y,x)$*;*

(c) $d(x,y) \leq s[d(x,z) + d(z,y)]$.

The pair (X,d) is called a b-metric space.

It should be noted that, the class of b-metric spaces is effectively larger than that of metric spaces, since a b-metric is a metric when $s = 1$. But, in general, the converse is not true.
Example 1.1 (Czerwik, 1993) Let $X = \{-1, 0, 1\}$ and $d : X \times X \to \mathbb{R}^+$ be given by $d(x,y) = d(y,x)$ for all $x, y \in X$, $d(x,x) = 0$, $d(-1,0) = 3$,
$d(-1,1) = d(0,1) = d(1,0) = 1$ then (X,d) is a b-metric on X with $s = 3/2$, but it is not a metric space. let $x = -1, y = 0, z = 1$, then
$d(x,y) \leq d(x,z) + d(z,y)$ since $d(-1,0) = 3 \not< 2 = d(-1,1) + d(1,0)$.
Hence the triangle inequality for a metric does not satisfied.
Example 1.2 (Czerwik, 1993) Let $X = \{0, 1, 2\}$. Define $d : X \times X \to \mathbb{R}^+$ be given by $d(x,x) = 0$ for all $x \in X$, $d(0,1) = d(1,0) = 1, d(1,2) = d(2,1) = 2$,
$d(0,2) = d(2,0) = 6$. Then d is a b-metric on X with $s = 2$. But it is not a metric on X. For, let $x = 0, y = 2, z = 1$, then
$d(0,2) = 6 > 3 = d(0,1) + d(1,2)$. Hence (X,d) is not a metric space.

Definition 4.1.5 *(Boriceanu, 2009) Let (X,d) be a b-metric space with the coefficient $s \geq 1$ and let $T : X \to X$ be a given mapping. We say that T is continuous at $x_o \in X$ if and only if for every sequence $x_n \in X$, we have $x_n \to x_o$ as $n \to \infty$ then $Tx_n \to Tx_o$ as $n \to \infty$. If T is continuous at each point of $x_0 \in X$ then we say that T is continuous on X.*
In general, a b-metric is not necessarily continuous.
Example 1.3 *(Hussain et al., 2012) Let $X = \mathbb{N}\ U\ \{\infty\}$.*
Define a mapping $d : X \times X \to \mathbb{R}^+$ as follows:

$$d(m,n)=\begin{cases}0 & \text{if } m=n,\\ |\frac{1}{m}-\frac{1}{n}| & \text{if one of m and n is even and the other even or } \infty,\\ 5 & \text{if one of m and n is odd and the other odd or } \infty,\\ 2 & \text{if others.}\end{cases}$$

$d(m,p) \leq 3[d(m,n)+d(n,p)]$ *for all* $m,n,p \in X$.

Then (X,d) *is a b-metric space with* $s=\frac{3}{2}$.

Choose $x_n = 2n$ *for each* $n \in \mathbb{N}$. *Then*

$$d(x_n,\infty)=d(2n,\infty)=\frac{1}{2n} \to 0 \text{ as } n \to \infty$$

that is, $x_n \to \infty$ *as* $n \to \infty$.

But, $d(x_n,1)=2 \nrightarrow 5 = d(\infty,1)($ *as* $n \to \infty)$.

Hence it is not cotinuous.

Definition 4.1.6 *(Boriceanu, 2009) Let X be a b-metric space and* $\{x_n\}$ *be a sequence in X we say that*

(a) $\{x_n\}$ *b-converges to* $x \in X$ *if* $d(x_n,x) \to 0$ *as* $n \to \infty$;

(b) $\{x_n\}$ *is a b-Cauchy sequence if* $d(x_n,x_m) \to 0$ *as* $n,m \to \infty$;

(c) (X,d) *is b-complete if every b-Cauchy sequence in X is b-convergent.*

Definition 4.1.7 *(Istratescu, 1981) A mapping* $T: X \to X$ *defined on a metric space X is called two-sided convex contraction mapping if there exist positive numbers* $a_1,a_2,b_1,b_2 \in (0,1)$ *such that the following inequality holds:*

$$d(T^2x,T^2y) \leq a_1 d(x,Tx)+a_2 d(Tx,T^2x)+b_1 d(y,Ty)+b_2 d(Ty,T^2y)$$

for all $x,y \in X$ *and* $a_1+a_2+b_1+b_2 < 1$.

Theorem 4.1.1 *(Istratescu, 1981) Let (X,d) be a complete metric space and $T : X \longrightarrow X$ be any two sided convex contraction mapping. Then T has a unique fixed point.*

Definition 4.1.8 *(Wardowski, 2012) Let (X,d) be a metric space and $T : X \to X$ be a self-map, then T is said to be an* **F***- contraction mapping on (X,d) if there exists a $\mathbf{F} \in \Phi$ and $\tau > 0$ such that for all $x,y \in X$.*

$$d(Tx,Ty) > 0 \Longrightarrow \tau + F(d(Tx,Ty)) \leq F(d(x,y)). \tag{4.1}$$

Remark 4.1.1 *From (F_1) and if T satisfies Inequality (4.1), then T is contractive, i.e.,*

$$F(d(Tx,Ty)) \leq F(d(x,y)) - \tau < F(d(x,y)) \Longrightarrow d(Tx,Ty) < d(x,y)$$

for all $x,y \in X$ such that $Tx \neq Ty$. Also T is a continuous mapping.

Definition 4.1.9 *(Singh et al., 2018) Let (X,d) be a metric space. A mapping $T : X \to X$ is said to an α-**F**- convex contraction on (X,d) if there exist two function $\alpha : X \times X \to \mathbb{R}^+$, $\mathbf{F} \in \Phi$ and $\tau > 0$ such that for all $x,y \in X$,*

$$d(T^2x,T^2y) > 0 \Longrightarrow \tau + F(\alpha(x,y)d(T^2x,T^2y)) \leq F(ad(Tx,Ty) + bd(x,y))$$

where $a,b \geq 0$ and $a + b < 1$.

Theorem 4.1.2 *(Singh et al., 2018) Let (X,d) be a complete metric space and $T : X \to X$ be an α-**F**-convex contraction satisfying the following conditions:*

(a) T is α-admissible;

(b) There exists $x_0 \in X$ such that $\alpha(x_0,Tx_0) \geq 1$;

(c) T is orbitally continuous on X. Then, T has a fixed point in X.

Definition 4.1.10 *(Eke et al., 2019) A mapping $T : X \to X$ defined on a metric space X is called Chatterjea two sided convex contraction mappings if there exist positive numbers $a_1, a_2, b_1, b_2 \in (0,1)$ such that the following inequality holds:*

$$d(T^2x, T^2y) \leq a_1 d(x, Ty) + a_2 d(Ty, T^2y) + b_1 d(y, Tx) + b_2 d(Tx, T^2x)$$

for all $x, y \in X$ and $a_1 + a_2 + b_1 + b_2 < 1$.

Theorem 4.1.3 *(Eke et al., 2019) Let (X,d) be a complete metric space and T be a self-mapping satisfying the Chatterjea two sided convex contraction conditions. Suppose T is orbitally continuous. Then T has a unique fixed point in X. For any $x_0 \in X$, the Picard iteration x_n given by $x_{n+1} = Tx_n$, $n \geq 0$ converges to the fixed point of T.*

Definition 4.1.11 *(Eke et al., 2019) A mapping $T : X \to X$ defined on a metric space X is called Hardy and Rogers convex contraction mapping of type 2 if there exist positive numbers $a_1, a_2, b_1, b_2, c_1, c_2, e_1, e_2, f_1, f_2 \in (0,1)$ such that the following inequality holds:*

$$\begin{aligned} d(T^2x, T^2y) \quad \leq \quad & a_1 d(x,y) + a_2 d(Tx, Ty) + b_1 d(x, Tx) + b_2 d(Tx, T^2x) \\ & + c_1 d(y, Ty) + c_2 d(Ty, T^2y) + e_1 d(x, Ty) + e_2 d(Ty, T^2y) \\ & + f_1 d(y, Tx) + f_2 d(Tx, T^2x) \end{aligned}$$

for all $x, y \in X$ and $a_1 + a_2 + b_1 + b_2 + c_1 + c_2 + e_1 + e_2 + f_1 + f_2 < 1$.

Theorem 4.1.4 *(Eke et al., 2019) Let (X,d) be a complete metric space and T be a self-mapping satisfying Hardy and Rogers convex contraction conditions. Suppose T is orbitally continuous. Then T has a unique fixed point in X. For any $x_0 \in X$, the Picard iteration $\{x_n\}$ given by x_{n+1}=Tx_n, $n \geq 0$ converges to the fixed point of T.*

4.2 Main Results

In this section, we proved fixed point results for α-**F**-convex contraction mappings in the setting of b-metric spaces.

Definition 4.2.1 *Let (X,d) be a b-metric space with parameter $s \geq 1$, $T : X \to X$, $\alpha : X \times X \to \mathbb{R}^+$ and $F \in \Phi$. Then T is called Chatterjea two sided α-**F**-convex contraction mapping if there exist $a_i, b_i \in [0,1)$ with $\sum_{i=1,2}(a_i + b_i) < 1/s$ and satisfies the following condition:*

$$\begin{aligned} d(T^2x, T^2y) &> 0 \\ \Longrightarrow \tau + F\Big(\alpha(x,y)d(T^2x,T^2y)\Big) &\leq F\Big[a_1 d(x,Ty) + a_2 d(Ty,T^2y) \\ &\quad + b_1 d(y,Tx) + b_2 d(Tx,T^2x)\Big] \end{aligned} \tag{4.2}$$

for all $x, y \in X$ and $\tau > 0$.

Theorem 4.2.1 *Let (X,d) be a complete b-metric space with $s \geq 1$.*
*$T : X \to X$ be Chatterjea two sided α-**F**-convex contraction mapping satisfying the following conditions:*

(a) T is α-admissible;

(b) There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;

(c) T is orbitally continuous on X.

Then, T has a fixed point in X. Further, if T is α^*-admissible, then T has a unique fixed point $z \in X$. Moreover, for any $x_0 \in X$ if $x_{n+1} = T^{n+1}x_0 \neq T^n x_0$ for all $n \geq 0$, then $\lim\limits_{n\to\infty} T^n x_0 = z$.

Proof: By (b) there exists a point $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and define a sequence $\{x_n\}$ by
$x_1 = Tx_0$, $x_2 = Tx_1$, $x_3 = Tx_2, \cdots$, $x_{n+1} = Tx_n$ for all $n = 0, 1, 2, \cdots$.
If $x_n = x_{n+1}$ for some n, $x_n = x_{n+1} = Tx_n$, x_n is fixed point of T.
Assume $x_n \neq x_{n+1}$ for all $n = 0, 1, 2, \cdots$.
Then $d(x_n, x_{n+1}) > 0$ for all $n = 0, 1, 2, \cdots$.

Since T is α-admissible, $\alpha(x_0,Tx_0)\geq 1 \Longrightarrow \alpha(x_1,x_2)=\alpha(Tx_0,T^2x_0)\geq 1$.
Therefore, one can obtain inductively that $\alpha(x_n,x_{n+1})=\alpha(T^nx_0,T^{n+1}x_0)\geq 1$ for all $n=0,1,2,\cdots$.
From Eq. (4.2) by using $x=x_0$ and $y=Tx_0$, we have

$$\begin{aligned}
d(T^2x_0,T^3x_0) &> 0 \Longrightarrow \tau+F\Big(\alpha(x_0,Tx_0)d(T^2x_0,T^3x_0)\Big) \\
&\leq F\Big[a_1d(x_0,T^2x_0)+a_2d(T^2x_0,T^3x_0)+b_1d(Tx_0,Tx_0)+b_2d(Tx_0,T^2x_0)\Big] \\
&= F\Big[a_1d(x_0,T^2x_0)+a_2d(T^2x_0,T^3x_0)+b_2d(Tx_0,T^2x_0)\Big] \\
&\leq F\Big[a_1(sd(x_0,Tx_0)+sd(Tx_0,T^2x_0))+a_2d(T^2x_0,T^3x_0)+b_2d(Tx_0,T^2x_0)\Big] \\
&= F\Big[a_1sd(x_0,Tx_0)+a_1sd(Tx_0,T^2x_0)+a_2d(T^2x_0,T^3x_0)+b_2d(Tx_0,T^2x_0)\Big] \\
&= F\Big[a_1sd(x_0,Tx_0)+(a_1s+b_2)d(Tx_0,T^2x_0)+a_2d(T^2x_0,T^3x_0)\Big] \\
&\leq F\Big[(2a_1s+b_2)max\{d(x_0,Tx_0),d(Tx_0,T^2x_0)\}+a_2d(T^2x_0,T^3x_0)\Big] \\
&= F\Big[(2a_1s+b_2)v+a_2d(T^2x_0,T^3x_0)\Big],
\end{aligned}$$

where $v=max\{d(x_0,Tx_0),d(Tx_0,T^2x_0)\}$, since F is strictly increasing, and $\tau>0$,

$$\begin{aligned}
d(T^2x_0,T^3x_0) &< (2a_1s+b_2)v+a_2d(T^2x_0,T^3x_0) \\
(1-a_2)d(T^2x_0,T^3x_0) &< (2a_1s+b_2)v \\
d(T^2x_0,T^3x_0) &< \frac{2a_1s+b_2}{1-a_2}v, 1-a_2>0 \\
d(T^2x_0,T^3x_0) &< \lambda v,
\end{aligned}$$

where $\lambda = \frac{2a_1s+b_2}{1-a_2}$.
From Eq. (4.2) by using $x = Tx_0$ and $y = T^2x_0$, we have

$$\begin{aligned}
d(T^3x_0,T^4x_0) &> 0 \Longrightarrow \tau + F\Big(\alpha(Tx_0,T^2x_0)d(T^3x_0,T^4x_0)\Big) \\
&\leq F\Big[a_1d(Tx_0,T^3x_0)+a_2d(T^3x_0,T^4x_0)+b_1d(T^2x_0,T^2x_0)+b_2d(T^2x_0,T^3x_0)\Big] \\
&= F\Big[a_1d(Tx_0,T^3x_0)+a_2d(T^3x_0,T^4x_0)+b_2d(T^2x_0,T^3x_0)\Big] \\
&\leq F\Big[a_1(sd(Tx_0,T^2x_0)+sd(T^2x_0,T^3x_0))+a_2d(T^3x_0,T^4x_0)+b_2d(T^2x_0,T^3x_0)\Big] \\
&= F\Big[a_1sd(Tx_0,T^2x_0)+a_1sd(T^2x_0,T^3x_0)+a_2d(T^3x_0,T^4x_0)+b_2d(T^2x_0,T^3x_0)\Big] \\
&= F\Big[a_1sd(Tx_0,T^2x_0)+(a_1s+b_2)d(T^2x_0,T^3x_0)+a_2d(T^3x_0,T^4x_0)\Big] \\
&\leq F\Big[(a_1s)v+(a_1s+b_2)\lambda v+a_2d(T^3x_0,T^4x_0)\Big],
\end{aligned}$$

since F is strictly increasing, and $\tau > 0$,

$$\begin{aligned}
d(T^3x_0,T^4x_0) &< (2a_1s+b_2)v+a_2d(T^3x_0,T^4x_0) \\
(1-a_2)d(T^3x_0,T^4x_0) &< (2a_1s+b_2)v \\
d(T^3x_0,T^4x_0) &< \frac{2a_1s+b_2}{1-a_2}v, 1-a_2>0 \\
d(T^3x_0,T^4x_0) &< \lambda v.
\end{aligned}$$

From Eq. (4.2) by using $x = T^2x_0$ and $y = T^3x_0$, we have

$$\begin{aligned}
d(T^4x_0,T^5x_0) &> 0 \Longrightarrow \tau + F\Big(\alpha(T^2x_0,T^3x_0)d(T^4x_0,T^5x_0)\Big) \\
&\leq F\Big[a_1d(T^2x_0,T^4x_0)+a_2d(T^4x_0,T^5x_0)+b_1d(T^3x_0,T^3x_0)+b_2d(T^3x_0,T^4x_0)\Big] \\
&= F\Big[a_1d(T^2x_0,T^4x_0)+a_2d(T^4x_0,T^5x_0)+b_2d(T^3x_0,T^4x_0)\Big] \\
&\leq F\Big[a_1(sd(T^2x_0,T^3x_0)+sd(T^3x_0,T^4x_0))+a_2d(T^4x_0,T^5x_0)+b_2d(T^3x_0,T^4x_0)\Big] \\
&= F\Big[a_1sd(T^2x_0,T^3x_0)+a_1sd(T^3x_0,T^4x_0))+a_2d(T^4x_0,T^5x_0)+b_2d(T^3x_0,T^4x_0)\Big] \\
&= F\Big[a_1sd(T^2x_0,T^3x_0)+(a_1s+b_2)d(T^3x_0,T^4x_0)+a_2d(T^4x_0,T^5x_0)\Big] \\
&\leq F\Big[a_1s(\lambda)v+(a_1s+b_2)(\lambda)v+a_2d(T^4x_0,T^5x_0)\Big],
\end{aligned}$$

since F is strictly increasing , and $\tau > 0$,

$$\begin{aligned}
d(T^4x_0, T^5x_0) &< a_1s(\lambda)v + (a_1s + b_2)(\lambda)v + a_2d(T^3x_0, T^4x_0) \\
(1 - a_2)d(T^3x_0, T^4x_0) &< (2a_1s + b_2)\lambda v \\
d(T^4x_0, T^5x_0) &< \frac{2a_1s + b_2}{1 - a_2}\lambda v, 1 - a_2 > 0 \\
d(T^4x_0, T^5x_0) &< \lambda^2 v.
\end{aligned}$$

From Eq. (4.2) by using $x = T^3x_0$ and $y = T^4x_0$, we have

$$\begin{aligned}
d(T^5x_0, T^6x_0) > 0 &\Longrightarrow \tau + F\Big(\alpha(T^3x_0, T^4x_0)d(T^5x_0, T^6x_0)\Big) \\
&\leq F\Big[a_1d(T^3x_0, T^5x_0) + a_2d(T^5x_0, T^6x_0) + b_1d(T^4x_0, T^4x_0) + b_2d(T^4x_0, T^5x_0)\Big] \\
&= F\Big[a_1d(T^3x_0, T^5x_0) + a_2d(T^5x_0, T^6x_0) + b_2d(T^4x_0, T^5x_0)\Big] \\
&\leq F\Big[a_1(sd(T^3x_0, T^4x_0) + sd(T^4x_0, T^5x_0)) + a_2d(T^5x_0, T^6x_0) + b_2d(T^4x_0, T^5x_0)\Big] \\
&= F\Big[a_1sd(T^3x_0, T^4x_0) + a_1sd(T^4x_0, T^5x_0) + a_2d(T^5x_0, T^6x_0) + b_2d(T^4x_0, T^5x_0)\Big] \\
&= F\Big[a_1sd(T^3x_0, T^4x_0) + (a_1s + b_2)d(T^4x_0, T^5x_0) + a_2d(T^5x_0, T^6x_0)\Big] \\
&\leq F\Big[a_1s(\lambda v) + (a_1s + b_2)(\lambda^2 v) + a_2d(T^5x_0, T^6x_0)\Big],
\end{aligned}$$

since F is strictly increasing, and $\tau > 0$,

$$\begin{aligned}
d(T^5x_0, T^6x_0) &< (2a_1s + b_2)\lambda v + a_2d(T^5x_0, T^6x_0) \\
(1 - a_2)d(T^5x_0, T^6x_0) &< (2a_1s + b_2)\lambda v \\
d(T^5x_0, T^6x_0) &< \frac{2a_1s + b_2}{1 - a_2}\lambda v, 1 - a_2 > 0 \\
d(T^5x_0, T^6x_0) &< \lambda^2 v.
\end{aligned}$$

Continuing this process inductively, we get

$$d(T^m x_0, T^{m+1}x_0) < \lambda^l v.$$

When m is even or $m = 2l$ and m is odd or $m = 2l + 1$ for $l \geq 1$.
Now we show that $\{x_n\}$ is a b- Cauchy sequence in X.
Let $m, n > 0$ with $n > m$.

Case 1: For $m = 2l$ or even, $l \geq 1$.

$$\begin{aligned}
d(T^m x_0, T^n x_0) &= d(T^{2l} x_0, T^n x_0) \\
&\leq s\Big(d(T^{2l} x_0, T^{2l+1} x_0) + d(T^{2l+1} x_0, T^n x_0)\Big) \\
&\leq sd(T^{2l} x_0, T^{2l+1} x_0) + s^2 d(T^{2l+1} x_0, T^{2l+2} x_0) + s^3 d(T^{2l+2} x_0, T^{2l+3} x_0) + \\
&\quad s^4 d(T^{2l+3} x_0, T^{2l+4} x_0) + s^5 d(T^{2l+4} x_0, T^{2l+5} x_0) + \cdots + s^{n-1} d(T^{n-1} x_0, T^n x_0) \\
&\leq s\lambda^l v + s^2 \lambda^l v + s^3 \lambda^{l+1} v + s^4 \lambda^{l+1} v + s^5 \lambda^{l+2} v + s^6 \lambda^{l+2} v + \cdots \\
&= s\lambda^l \Big[1 + s^2\lambda + s^4\lambda^2 + s^6\lambda^3 + \cdots\Big] v + s^2\lambda^l \Big[1 + s^2\lambda + s^4\lambda^2 + s^6\lambda^3 + \cdots\Big] v \\
&= (s + s^2)(\lambda^l) \Big[1 + s^2\lambda + s^4\lambda^2 + s^6\lambda^3 + \cdots\Big] v \\
&\leq (s + s^2)(\lambda^l) \frac{1}{1 - s^2\lambda} v \\
d(T^m x_0, T^n x_0) &\leq (s + s^2)(\lambda^l) \frac{1}{1 - s^2\lambda} v \to 0 \text{ as } l \to \infty.
\end{aligned}$$

Case 2: For $m = 2l + 1$ or odd, $l \geq 1$.

$$\begin{aligned}
d(T^m x_0, T^n x_0) &= d(T^{2l+1} x_0, T^n x_0) \\
&\leq s\Big(d(T^{2l+1} x_0, T^{2l+2} x_0) + d(T^{2l+2} x_0, T^n x_0)\Big) \\
&\leq sd(T^{2l+1} x_0, T^{2l+2} x_0) + s^2 d(T^{2l+2} x_0, T^{2l+3} x_0) + s^3 d(T^{2l+3} x_0, T^{2l+4} x_0) + \\
&\quad s^4 d(T^{2l+4} x_0, T^{2l+5} x_0) + s^5 d(T^{2l+5} x_0, T^{2l+6} x_0) + \cdots + s^{n-1} d(T^{n-1} x_0, T^n x_0) \\
&\leq s\lambda^l v + s^2 \lambda^{l+1} v + s^3 \lambda^{l+1} v + s^4 \lambda^{l+2} v + s^5 \lambda^{l+2} v + s^6 \lambda^{l+3} v + \cdots \\
&= s\lambda^l \Big[1 + s^2\lambda + s^4\lambda^2 + s^6\lambda^3 + \cdots\Big] v + s^2\lambda^l \Big[1 + s^2\lambda + s^4\lambda^2 + s^6\lambda^3 + \cdots\Big] v \\
&= (s + s^2)(\lambda^l) \Big[1 + s^2\lambda + s^4\lambda^2 + s^6\lambda^3 + \cdots\Big] v \\
&\leq (s + s^2)(\lambda^l) \frac{1}{1 - s^2\lambda} v \\
d(T^m x_0, T^n x_0) &\leq (s + s^2)(\lambda^l) \frac{1}{1 - s^2\lambda} v \to 0 \text{ as } l \to \infty.
\end{aligned}$$

This shows that $\{x_n\}$ is a b-Cauchy sequence in X.
Since X is b- complete, there exists $z \in X$ such that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} T^n x_0 \longrightarrow z$.
Now we prove that z is a fixed point of T.
By the continuity of T, we obtain $z = \lim_{n\to\infty} T(T^n x_0) = Tz$.
This shows that z is a fixed point of T.

Uniqueness

We suppose that T is α^*-admissible. Since $Fix(T) \neq \emptyset$,
let $z, z^* \in Fix(T)$, by α^*- admissible of T, we have $\alpha(z, z^*) \geq 1$. From Eq. (4.2)

$$\begin{aligned} F(d(z,z^*)) &= F(d(T^2z,T^2z^*) = F(\alpha(z,z^*)d(T^2z,T^2z^*)) \\ &\leq F\Big[a_1d(z,Tz^*)+a_2d(Tz^*,T^2z^*)+b_1d(z^*,Tz)+b_2d(Tz,T^2z)\Big]-\tau \\ &\leq F\Big[a_1d(z,Tz^*)+b_1d(z^*,Tz)\Big]-\tau. \end{aligned}$$

Since $\tau > 0$ and F is strictly increasing, we obtain

$$\begin{aligned} d(z,z^*) &< a_1d(z,Tz^*)+b_1d(z^*,Tz) \\ &< (a_1+b_1)d(z,z^*) \end{aligned}$$

$d(z,z^*) < d(z,z^*)$, a contradiction, which in turn gives $z^* = z$.
Hence T has a unique fixed point in X. □

Now we give an example in support of Theorem 4.2.1

Example 4.2.1 Let $X = [0,1]$ and $d : X \times X \longrightarrow \mathbb{R}^+$ be given by $d(x,y) = |x-y|^2$ for $x, y \in X$. Then (X,d) a complete b-metric space with $s = 2$.
We define a mapping $T : X \longrightarrow X$ by

$$T(x) = \begin{cases} 0, & \text{if } x \in [0, \frac{1}{2}), \\ \frac{x^2}{5} + \frac{1}{10}, & \text{if } x \in [\frac{1}{2}, 1]. \end{cases}$$

and

$$\alpha(x,y) = \begin{cases} 1, & \text{for all } x, y \in X, \\ 0, & \text{otherwise.} \end{cases}$$

Then T is α-admissible. Setting $F \in \Phi$ such that $F : \mathbb{R}^+ \longrightarrow \mathbb{R}$ given by $F(\gamma) = ln\gamma$ and $\gamma > 0$. Then for $x, y \in X$. with $x \neq y$, we obtain

$$\begin{aligned}
|Tx - Ty|^2 &= \frac{1}{5}|x^2 - y^2|^2 \leq |x - y|^2 \\
\alpha(x,y)\left|T^2x - T^2y\right|^2 &= \left|T^2x - T^2y\right|^2 \\
&= \left|(\frac{x^2}{5} + \frac{1}{10})^2 - (\frac{y^2}{5} + \frac{1}{10})^2\right|^2 \\
&= \left|\frac{x^4}{25} + \frac{x^2}{25} + \frac{1}{100} - (\frac{y^4}{25} + \frac{y^2}{25} + \frac{1}{100})\right|^2 \\
&= \left|\frac{x^4}{25} + \frac{x^2}{25} - (\frac{y^4}{25} + \frac{y^2}{25})\right|^2 \\
&= \frac{1}{25}\left|x^4 - y^4 + x^2 - y^2\right|^2 \\
&\leq \frac{1}{5}\left|Tx - Ty\right|^2 + \frac{1}{10}\left|x - y\right|^2 \\
&\leq \frac{1}{25}d(x,Ty) + \frac{1}{30}d(Ty,T^2y) + \frac{1}{25}d(y,Tx) + \frac{1}{30}d(Tx,T^2x) \\
&\leq \frac{55}{375}Max\left\{|x - Ty|^2, |Ty - T^2y|^2, |y - Tx|^2, |Tx - T^2x|^2\right\} \\
&\leq e^{-\tau}\Big(|x - Ty|^2, |Ty - T^2y|^2, |y - Tx|^2, |Tx - T^2x|^2\Big),
\end{aligned}$$

where $-\tau = ln(\frac{55}{375})$. Taking natural logarithm on both sides, we obtain
$\tau + F\Big(\alpha(x,y)d(T^2x, T^2y)\Big) \leq F\Big[a_1d(x,Ty) + a_2d(Ty,T^2y) + b_1d(y,Tx) + b_2d(Tx,T^2y)\Big]$,
where $a_1 = b_1 = \frac{1}{25}$, & $a_2 = b_2 = \frac{1}{30}$.
This shows that T is Chatterjea two sided α-**F**-convex contraction mapping.
Let $T^nx = \frac{x^{2n}}{5} + \frac{1}{10^n} \longrightarrow 0$ as $n \longrightarrow \infty$. Then
$T(\frac{1}{2}) = \frac{3}{20}$, $T^n(\frac{1}{2}) \longrightarrow 0$ as $n \longrightarrow \infty$.
$T(T^nx) \longrightarrow T(0) = 0$ as $n \longrightarrow \infty$. This shows that T is an orbital continuous.
The conditions of Theorem 4.2.1 are satisfied and the unique fixed point of T is 0.

Definition 4.2.2 *Let (X,d) be a b-metric space with parameter $s \geq 1$, $T : X \to X$, $\alpha : X \times X \to \mathbb{R}^+$ and $F \in \Phi$. Then T is called Hardy and Rogers α-$\mathbf{F}$-convex contraction mapping if there exists $a_i, b_i, c_i, e_i, f_i \in [0,1)$ with $\sum_{i=1,2}(a_i + b_i + c_i + e_i + f_i) < 1/s$ and satisfies the following condition:*

$$\begin{aligned} d(T^2x, T^2y) &> 0 \\ \Longrightarrow \tau + F\Big(\alpha(x,y)d(T^2x,T^2x)\Big) &\leq F\Big[a_1d(x,y) + a_2d(Tx,Ty) + b_1d(x,Tx) + \\ & b_2d(Tx,T^2x) + c_1d(y,Ty) + c_2d(Ty,T^2y) \\ & + e_1d(x,Ty) + e_2d(Ty,T^2y) + f_1d(y,Tx) \\ & + f_2d(Tx,T^2x)\Big] \qquad (4.3) \end{aligned}$$

for all $x,y \in X$ and $\tau > 0$.

Theorem 4.2.2 *Let (X,d) be a complete b-metric space with $s \geq 1$. $T : X \to X$ be Hardy and Rogers α-$\mathbf{F}$-convex contraction mapping satisfying the following conditions:*

(a) T is α-admissible;

(b) There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;

(c) T is orbitally continuous on X.

Then, T has a fixed point in X. Further, if T is α^*-admissible, then T has a unique fixed point $z \in X$. Moreover, for any $x_0 \in X$ if $x_{n+1} = T^{n+1}x_0 \neq T^n x_0$ for all $n \geq 0$, then $\lim_{n\to\infty} T^n x_0 = z$.

Proof: By (b) there exists a point $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and define a sequence $\{x_n\}$ by
$x_1 = Tx_0$, $x_2 = Tx_1$, $x_3 = Tx_2, \cdots$, $x_{n+1} = Tx_n$ for all $n = 0,1,2,\cdots$.
If $x_n = x_{n+1}$ for some n, $x_n = x_{n+1} = Tx_n$, x_n is fixed point of T.
Assume $x_n \neq x_{n+1}$ for all $n = 0,1,2,\cdots$.
Then $d(x_n, x_{n+1}) > 0$ for all $n = 0,1,2,\cdots$.
Since T is α-admissible, $\alpha(x_0, Tx_0) \geq 1 \Longrightarrow \alpha(x_1,x_2) = \alpha(Tx_0, T^2x_0) \geq 1$. Therefore, one can obtain inductively that $\alpha(x_n, x_{n+1}) = \alpha(T^n x_0, T^{n+1}x_0) \geq 1$ for all

$n = 0, 1, 2, \cdots$.
From Eq. (4.3) by using $x = x_0$ and $y = Tx_0$, we have

$$\begin{aligned}
d(T^2x_0, T^3x_0) &> 0 \Longrightarrow \tau + F\Big(\alpha(x_0, Tx_0)d(T^2x_0, T^3x_0)\Big) \\
&\leq F\Big[a_1d(x_0, Tx_0) + a_2d(Tx_0, T^2x_0) + b_1d(x_0, Tx_0) + b_2d(Tx_0, T^2x_0) \\
&\quad + c_1d(Tx_0, T^2x_0) + c_2d(T^2x_0, T^3x_0) + e_1d(x_0, T^2x_0) + e_2d(T^2x_0, T^3x_0) \\
&\quad + f_1d(Tx_0, Tx_0) + f_2d(Tx_0, T^2x_0)\Big] \\
&= F\Big[a_1d(x_0, Tx_0) + a_2d(Tx_0, T^2x_0) + b_1d(x_0, Tx_0) + b_2d(Tx_0, T^2x_0) \\
&\quad + c_1d(Tx_0, T^2x_0) + c_2d(T^2x_0, T^3x_0) + e_1d(x_0, T^2x_0) + e_2d(T^2x_0, T^3x_0) \\
&\quad + f_2d(Tx_0, T^2x_0)\Big] \\
&\leq F\Big[a_1d(x_0, Tx_0) + a_2d(Tx_0, T^2x_0) + b_1d(x_0, Tx_0) + b_2d(Tx_0, T^2x_0) \\
&\quad + c_1d(Tx_0, T^2x_0) + c_2d(T^2x_0, T^3x_0) + e_1(sd(x_0, Tx_0) + sd(Tx_0, T^2x_0)) \\
&\quad + e_2d(T^2x_0, T^3x_0) + f_2d(Tx_0, T^2x_0)\Big] \\
&= F\Big[a_1d(x_0, Tx_0) + a_2d(Tx_0, T^2x_0) + b_1d(x_0, Tx_0) + b_2d(Tx_0, T^2x_0) \\
&\quad + c_1d(Tx_0, T^2x_0) + c_2d(T^2x_0, T^3x_0) + e_1sd(x_0, Tx_0) + e_1sd(Tx_0, T^2x_0) \\
&\quad + e_2d(T^2x_0, T^3x_0) + f_2d(Tx_0, T^2x_0)\Big] \\
&= F\Big[a_1 + b_1 + e_1s)d(x_0, Tx_0) + (a_2 + b_2 + c_1 + e_1s + f_2)d(Tx_0, T^2x_0) + \\
&\quad (c_2 + e_2)d(T^2x_0, T^3x_0)\Big] \\
&\leq F\Big[a_1 + b_1 + c_1 + 2se_1 + a_2 + b_2 + f_2)max\{d(x_0, Tx_0), d(Tx_0, T^2x_0)\} + \\
&\quad (c_2 + e_2)d(T^2x_0, T^3x_0)\Big] \\
&= F\Big[a_1 + b_1 + c_1 + 2e_1s + a_2 + b_2 + f_2)v + (c_2 + e_2)d(T^2x_0, T^3x_0)\Big],
\end{aligned}$$

where $v = max\{d(x_0, Tx_0), d(Tx_0, T^2x_0)\}$. Since F is strictly increasing, and $\tau > 0$,

$$\begin{aligned}
d(T^2x_0, T^3x_0) &< (a_1 + b_1 + 2e_1s + a_2 + b_2 + f_2)v + (c_2 + e_2)d(T^2x_0, T^3x_0) \\
(1 - c_2 - e_2)d(T^2x_0, T^3x_0) &< (a_1 + b_1 + c_1 + 2se_1 + a_2 + b_2 + f_2)v \\
d(T^2x_0, T^3x_0) &< \frac{(a_1 + b_1 + c_1 + 2e_1s + a_2 + b_2 + f_2)}{(1 - c_2 - e_2)}v, (1 - c_2 - e_2) > 0 \\
d(T^2x_0, T^3x_0) &< \lambda v,
\end{aligned}$$

where $\lambda = \dfrac{a_1+b_1+c_1+2e_1s+a_2+b_2+f_2}{(1-c_2-e_2)}$.

From Eq. (4.3) by using $x = Tx_0$ and $y = T^2x_0$, we have

$$
\begin{aligned}
(T^3x_0,T^4x_0) &> 0 \Longrightarrow \tau + F\Big(\alpha(Tx_0,T^2x_0)d(T^3x_0,T^4x_0)\Big) \\
&\le F\Big[a_1d(Tx_0,T^2x_0)+a_2d(T^2x_0,T^3x_0)+b_1d(Tx_0,T^2x_0)+b_2d(T^2x_0,T^3x_0)+ \\
&\quad c_1d(T^2x_0,T^3x_0)+c_2d(T^3x_0,T^4x_0)+e_1d(Tx_0,T^3x_0)+e_2d(T^3x_0,T^4x_0)+ \\
&\quad f_1d(T^2x_0,T^2x_0)+f_2d(T^2x_0,T^3x_0)\Big] \\
&= F\Big[a_1d(Tx_0,T^2x_0)+a_2d(T^2x_0,T^3x_0)+b_1d(Tx_0,T^2x_0)+b_2d(T^2x_0,T^3x_0)+ \\
&\quad c_1d(T^2x_0,T^3x_0)+c_2d(T^3x_0,T^4x_0)+e_1d(Tx_0,T^3x_0)+e_2d(T^3x_0,T^4x_0)+ \\
&\quad f_2d(T^2x_0,T^3x_0)\Big] \\
&\le F\Big[a_1d(Tx_0,T^2x_0)+a_2d(T^2x_0,T^3x_0)+b_1d(Tx_0,T^2x_0)+b_2d(T^2x_0,T^3x_0)+ \\
&\quad c_1d(T^2x_0,T^3x_0)+c_2d(T^3x_0,T^4x_0)+e_1(sd(Tx_0,T^2x_0)+sd(T^2x_0,T^3x_0))+ \\
&\quad e_2d(T^3x_0,T^4x_0)+f_2d(T^2x_0,T^3x_0)\Big] \\
&= F\Big[a_1d(Tx_0,T^2x_0)+a_2d(T^2x_0,T^3x_0)+b_1d(Tx_0,T^2x_0)+b_2d(T^2x_0,T^3x_0)+ \\
&\quad c_1d(T^2x_0,T^3x_0)+c_2d(T^3x_0,T^4x_0)+e_1sd(Tx_0,T^2x_0)+e_1sd(T^2x_0,T^3x_0)+ \\
&\quad e_2d(T^3x_0,T^4x_0)+f_2d(T^2x_0,T^3x_0)\Big] \\
&= F\Big[a_1+b_1+e_1s)d(Tx_0,T^2x_0)+(a_2+b_2+c_1+e_1s+f_2)d(T^2x_0,T^3x_0)+ \\
&\quad (c_2+e_2)d(T^3x_0,T^4x_0)\Big] \\
&\le F\Big[a_1+b_1+e_1s)v+(a_2+b_2+c_1+e_1s+f_2)\lambda v+(c_2+e_2)d(T^3x_0,T^4x_0)\Big],
\end{aligned}
$$

since F is strictly increasing , and $\tau > 0$,

$$
\begin{aligned}
d(T^3x_0,T^4x_0) &< (a_1+b_1+2se_1+a_2+b_2+f_2)v+(+c_2+e_2)d(T^3x_0,T^4x_0) \\
(1-c_2-e_2)d(T^3x_0,T^4x_0) &< (a_1+b_1+c_1+2se_1+a_2+b_2+f_2)v \\
d(T^3x_0,T^4x_0) &< \frac{(a_1+b_1+c_1+2e_1s+a_2+b_2+f_2)}{(1-c_2-e_2)}v, (1-c_2-e_2)>0 \\
d(T^3x_0,T^4x_0) &< \lambda v.
\end{aligned}
$$

Continuing this process inductively, we get

$$d(T^m x_0, T^{m+1} x_0) < \lambda^l v.$$

When m is even or $m = 2l$ and m is odd or $m = 2l+1$ for $l \geq 1$.
Now we show that $\{x_n\}$ is a b-Cauchy sequence in X.
Let $m, n > 0$ with $n > m$.
Case 1: For $m = 2l$ or even, $l \geq 1$.

$$\begin{aligned}
d(T^m x_0, T^n x_0) &= d(T^{2l} x_0, T^n x_0) \\
&\leq s\Big(d(T^{2l} x_0, T^{2l+1} x_0) + d(T^{2l+1} x_0, T^n x_0)\Big) \\
&\leq sd(T^{2l} x_0, T^{2l+1} x_0) + s^2 d(T^{2l+1} x_0, T^{2l+2} x_0) + \\
&\quad s^3 d(T^{2l+2} x_0, T^{2l+3} x_0) + s^4 d(T^{2l+3} x_0, T^{2l+4} x_0) + \\
&\quad s^5 d(T^{2l+4} x_0, T^{2l+5} x_0) + \cdots + s^{n-1} d(T^{n-1} x_0, T^n x_0) \\
&\leq s\lambda^l v + s^2 \lambda^l v + s^3 \lambda^{l+1} v + s^4 \lambda^{l+1} v + s^5 \lambda^{l+2} v + s^6 \lambda^{l+2} v + \cdots \\
&= s\lambda^l \Big[1 + s^2\lambda + s^4\lambda^2 + s^6\lambda^3 + \cdots\Big] v + s^2\lambda^l \Big[1 + s^2\lambda + s^4\lambda^2 + s^6\lambda^3 + \cdots\Big] v \\
&= (s+s^2)(\lambda^l)\Big[1 + s^2\lambda + s^4\lambda^2 + s^6\lambda^3 + \cdots\Big] v \\
&\leq (s+s^2)(\lambda^l)\frac{1}{1-s^2\lambda} v \\
d(T^m x_0, T^n x_0) &\leq (s+s^2)(\lambda^l)\frac{1}{1-s^2\lambda} v \to 0 \text{ as } l \to \infty.
\end{aligned}$$

Case 2: For $m = 2l+1$ or odd, $l \geq 1$.

$$\begin{aligned}
d(T^m x_0, T^n x_0) &= d(T^{2l+1}x_0, T^n x_0) \\
&\leq s\Big(d(T^{2l+1}x_0, T^{2l+2}x_0) + d(T^{2l+2}x_0, T^n x_0)\Big) \\
&\leq sd(T^{2l+1}x_0, T^{2l+2}x_0) + s^2 d(T^{2l+2}x_0, T^{2l+3}x_0) + \\
&\quad s^3 d(T^{2l+3}x_0, T^{2l+4}x_0) + s^4 d(T^{2l+4}x_0, T^{2l+5}x_0) + \\
&\quad s^5 d(T^{2l+5}x_0, T^{2l+6}x_0) + \cdots + s^{n-1} d(T^{n-1}x_0, T^n x_0) \\
&\leq s\lambda^l v + s^2\lambda^{l+1} v + s^3\lambda^{l+1} v + s^4\lambda^{l+2} v + s^5\lambda^{l+2} v + s^6\lambda^{l+3} v + \cdots \\
&= s\lambda^l\Big[1 + s^2\lambda + s^4\lambda^2 + s^6\lambda^3 + \cdots\Big]v + s^2\lambda^l\Big[1 + s^2\lambda + s^4\lambda^2 + s^6\lambda^3 + \cdots\Big]v \\
&= s+s^2)(\lambda^l)\Big[1 + s^2\lambda + s^4\lambda^2 + s^6\lambda^3 + \cdots\Big]v \\
&\leq (s+s^2)(\lambda^l)\frac{1}{1-s^2\lambda}v \\
d(T^m x_0, T^n x_0) &\leq (s+s^2)(\lambda^l)\frac{1}{1-s^2\lambda}v \to 0 \text{ as } l \to \infty.
\end{aligned}$$

This shows that $\{x_n\}$ is a b-Cauchy sequence in X.
Since X is b- complete, there exists $z \in X$ such that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} T^n x_0 \longrightarrow z$.
Now we prove that z is a fixed point of T.
By continuity of T , we obtain $z = \lim_{n\to\infty} T(T^n x_0) = Tz$.
This shows that z is a fixed point of T.
Uniqueness
We suppose that T is α^*-admissible. Since $Fix(T) \neq \emptyset$,
let $z, z^* \in Fix(T)$, by α^*-admissible of T, we have $\alpha(z, z^*) \geq 1$. From Eq. (4.3)

$$\begin{aligned}
F(d(z,z^*)) &= F(d(T^2z, T^2z^*) = F(\alpha(z,z^*)d(T^2z, T^2z^*)) \\
&\leq F\Big[a_1 d(z,z^*) + a_2 d(Tz, Tz^*) + b_1 d(z, Tz) + b_2 d(Tz, T^2z) + c_1 d(z^*, Tz^*) + \\
&\quad c_2 d(Tz^*, T^2z^*) + e_1 d(z, Tz^*) + e_2 d(Tz^*, T^2z^*) + f_1 d(z^*, Tz) + f_2 d(Tz, T^2z)\Big] - \tau \\
&\leq F\Big[a_1 d(z,z^*) + a_2 d(z,z^*) + e_1 d(z,z^*) + f_1 d(z^*, z)\Big] - \tau,
\end{aligned}$$

since $\tau > 0$ and F is strictly increasing, we obtain

$$\begin{aligned}
d(z,z^*)) &< a_1 d(z,z^*) + a_2 d(z,z^*) + e_1 d(z,z^*) + f_1 d(z^*, z) \\
&< (a_1 + a_2 + e_1 + f_1) d(z,z^*)
\end{aligned}$$

$d(z,z^*) < d(z,z^*)$, a contradiction, which in turn gives $z^* = z$.
Hence T has a unique fixed point in X. $\square$

Now we give an example in support of Theorem 4.2.2.

Example 4.2.2 Let $X = [0,1]$ and $d : X \times X \longrightarrow \mathbb{R}^+$ be given by $d(x,y) = |x-y|^2$ for $x,y \in X$. Then (X,d) a complete b-metric space with $s = 2$.
Define a mapping $T : X \longrightarrow X$ by

$$T(x) = \begin{cases} 0, & \text{if } x \in [0, \frac{1}{5}), \\ \frac{x^2}{2} + \frac{1}{4}, & \text{if } x \in [\frac{1}{5}, 1]. \end{cases}$$

With $\alpha(x,y) = 1$ for all $x,y \in X$. Then, T is α-admissible. Let $F \in \Phi$ such that $F : \mathbb{R}^+ \longrightarrow \mathbb{R}$ given by $F(\gamma) = ln\gamma$, $\gamma > 0$. Since we have
$|Tx - Ty|$ =$\frac{1}{2}|x^2 - y^2| \leq |x-y|$ for all $x \in X$.

$$\begin{aligned}
\alpha(x,y)\left|T^2x - T^2y\right|^2 &= \left|T^2x - T^2y\right|^2 \\
&= \frac{1}{32}(\left|4x^4 - 4y^4 + 4x^2 - 4y^2\right|^2) \\
&= \frac{1}{8}(\left|x^4 - y^4 + x^2 - y^2\right|^2) \\
&\leq \frac{1}{8}(\left|x^4 - y^4\right|^2 + \left|x^2 - y^2\right|^2) \\
&\leq \frac{1}{2}(|Tx - Ty|^2 + \frac{1}{4}|x-y|^2) \\
&\leq \frac{1}{14}d(x,y) + \frac{1}{16}d(Tx,Ty) + \frac{1}{30}d(x,Tx) + \\
&\quad \frac{1}{25}d(Tx,T^2x) + \frac{1}{14}d(y,Ty) + \frac{1}{16}d(Ty,T^2y) + \\
&\quad \frac{1}{30}d(x,Ty) + \frac{1}{40}d(Ty,T^2y) + \frac{1}{25}d(y,Tx) + \frac{1}{40}d(Tx,T^2x) \\
&\leq \frac{1951}{4200}max\Big\{|x-y|^2, |Tx-Ty|^2, |x-Tx|^2, \\
&\quad |Tx - T^2x|^2, |y-Ty|^2, |Ty - T^2y|^2, \\
&\quad |x-Ty|^2, |Ty-T^2y|^2, |y-Tx|^2, |Tx-T^2x|^2\Big\} \\
&\leq e^{-\tau}\Big(d(x,y), d(Tx,Ty), d(x,Tx), d(Tx,T^2x), (y,Ty), \\
&\quad d(Ty,T^2y), d(x,Ty), d(Ty,T^2y), d(y,Tx), d(Tx,T^2x)\Big),
\end{aligned}$$

where $-\tau = ln\left(\frac{1951}{4200}\right)$. Taking natural logarithm on both sides, we obtain

$$\begin{aligned}\tau + F\Big(\alpha(x,y)d(T^2x,T^2y)\Big) \leq\; & F\Big[a_1d(x,y)+a_2d(Tx,Ty)+b_1d(x,Tx)+b_2d(Tx,T^2x)\\ & +c_1d(y,Ty)+c_2d(Ty,T^2y)+e_1d(x,Ty)+e_2d(Ty,T^2y)\\ & +f_1d(y,Tx)+f_2d(Tx,T^2x)\Big],\end{aligned}$$

where $a_1 = c_1 = \frac{1}{14}, a_2 = c_2 = \frac{1}{16}, b_1 = e_1 = \frac{1}{30}, b_2 = f_1 = \frac{1}{25}, e_2 = f_2 = \frac{1}{40}$.

This shows that T is Hardy and Rogers an α-**F**-convex contraction mapping.
Let $T^nx = \frac{x^{2n}}{2} + \frac{1}{4^n} \longrightarrow 0$ as $n \longrightarrow \infty$. Then
$T(\frac{1}{5}) = \frac{27}{100}$, $T^n(\frac{1}{5}) \longrightarrow 0$ as $n \longrightarrow \infty$.
$T(T^nx) \longrightarrow T(0) = 0$ as $n \longrightarrow \infty$. This shows that T is an orbital continuous.
Thus all conditions of Theorem 4.2.2 are satisfied and the unique fixed point of T is 0.

Remark 4.2.1 *If $e_1 = e_2 = f_1 = f_2 = 0$ and we take $s = 1$ in Theorem 4.2.2, then we get the result of Singh et al. (2018). If $a_1 = a_2 = b_1 = b_2 = c_1 = c_2 = 0$ in Theorem 4.2.2, then we get the result of Theorem 4.2.1.*

Chapter 5

Conclusion and Future scope

5.1 Conclusion

Eke, K. S., Olisama, V. O., & Bishop, S. A. (2019). Established fixed point theorems for convex contractive mappings in complete metric spaces and proved the existence and uniqueness of fixed points. In this research work, we established fixed point theorems for α-$\mathbf{F}$-convex contraction mappings, and proved the existence and uniqueness of fixed points in the setting of complete b-metric spaces. Our results extend and generalize the comparable results in the literature. We have also supported the main results of this research work by applicable examples.

5.2 Future scope

There are some published results related to the existence of fixed point theorems of mappings defined on b-metric space. The researchers believe the search for the existence and uniqueness of fixed points of self-mappings satisfying α-$\mathbf{F}$-convex contraction conditions in b-metric spaces is an active area of study. So, any interested researchers can use this opportunity and conduct their research work in this area.

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