Analysis of Magnetohydrodynamics (MHD) Boundary Layer flow of Nano fluid flow over a Nonlinear Stretching Sheet in the Presence of Thermal Radiation, Chemical Reaction and Partial Slip Boundary Condition

A Thesis Submitted to the Department of Mathematics, Jimma University in Partial Fulfillment for the Requirements of the Degree of Masters of Science in Mathematics.

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Declaration

I, the undersigned declare that, this research paper entitled **” Analysis of Magneto Hydrodynamics (MHD) Boundary Layer Nano fluid flow of Heat Transfer over a Nonlinear Stretching Sheet Presence of Thermal Radiation**, **Chemical Reaction and Partial Slip Boundary condition”** is my own original work and it has not been submitted for the award of any academic degree or the like in any other institution or university, and that all the sources I have used or quoted have been indicated and acknowledged.

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Abstract

This paper presents Analysis of Magneto Hydrodynamics (MHD) Boundary Layer Nano fluid flow of Heat Transfer over a Nonlinear Stretching Sheet Presence of Thermal Radiation, Chemical Reaction and Partial Slip with Boundary condition. The basic partial differential equations are reduced to ordinary differential equations which are solved numerically using Keller-Box Method. This study reveals that the governing parameters, namely, the Chemical Reaction, the radiation parameters and Velocity Slip have major effects on the flow field, Velocity, Temperature, Concentration, skin friction coefficient, the heat transfer rate and mass transfer rate. Comparison between the obtained results and previous works are well in agreement.

Keywords: Heat transfer, Nano fluid, Chemical Reaction, Thermal Radiation,

Velocity Slip boundary condition

Nomenclature

T Temperature of fluid

B_0 Magnetic field strength

c_p Specific heat

T_w Wall temperature

f' Non-dimensional velocity

T_∞ Ambient temperature

C_f Skin friction coefficient

u_w Stretching velocity along x-axis

M Magnetic parameter

N_u Nusselt number

Ec Eckert number

Pr Prandtl number

q_w Wall heat flux

a Stretching rate

k Thermal conductivity

u, v Velocity components in x and y directions respectively

Subscripts

f Fluid

nf Nano fluid

s Solid phase

Greek symbol

λ_1 Velocities slip parameter

λ_2 Thermal slip parameter

λ_3 Concentration slip parameter

τ_w Surface temperature

ϕ Dimensionless volume friction of nanoparticles

ρ Density of fluid

η Similarity variable

θ Dimensionless temperature

ψ Stream function

μ Dynamic viscosity

σ Electric conductivity

τ The ratio of the nanoparticle heat capacity and the base fluid heat capacity

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Chapter one

Introduction

1.1. Background of the study

The flow over a stretching surface is an important role in many engineering processes with applications in industries such as extrusion, melt-spinning, the hot rolling, wire drawing, glass fiber production, manufacture of plastic and rubber sheets, cooling of a large metallic plate in a bath, which may be an electrolyte, etc. (Sakiadis, 1961) analyzed boundary layer behavior on continuous solid surface. (Hayat, 2008)conducted convection flow over a non-linearly stretching sheet of a micro polar fluid using Homotopy analysis method. (Cortell, 2008) was investigated similarity solutions for flow and heat transfer of a quiescent fluid over a non-linearly stretching surface. (Vajravelu, 2001) studied flow and heat transfer in a viscous fluid over a nonlinear stretching sheet without viscous dissipation. The study of magnetic field effects has important applications in physics, chemistry and engineering. Industrial equipment, such as magneto hydrodynamic (MHD) generators, pumps, bearings and boundary layer control are affected by the interaction between the electrically conducting fluid and a magnetic field. The work of many investigators has been studied in relation to these applications. (Prasadet al. 2010) analyzed the fluid properties on the MHD flow and heat transfer over a stretching surface by using Keller-box method.

The Nano particles can be found in metals such as (Cu, Ag), oxides (Al2O3), carbides (SiC), nitrides (AlN, SiN) or nonmetals (graphite, carbon nanotubes). Nanofluids have novel properties that make them potentially useful in many applications in heat transfer including microelectronics, fuel cells, pharmaceutical processes and hybrid powered engines. Nanoparticles provide a bridge between bulk materials, and molecular structure. The Nano fluid term was first used by (Choi, 1995). Boundary layer flow of a Nano fluid over a non-linearly stretching sheet with boundary conditions of convective was investigated by (Makinde and Aziz, 2011). (Sheikholeslami et.al, 2015) investigated Effect of electric field on hydrothermal behavior of Nano fluid in a complex geometry. (Hamad, 2012) investigated the magnetic field effects on free convection flow of past a vertical semi-infinite flat plate of a Nano fluid. (Sheikholeslami and Ganji, 2014) investigates on Nano fluids and heat transfer effects of magnetic fields

Chemical reactions are classified as either homogeneous or heterogeneous. Heat and mass transfer problems with chemical reactions are important in many processes of interest in Engineering and have received significant attention in recent years. These processes include drying, evaporation at the surface of a water body, energy transfer in a wet cooling tower and the flow in a desert cooler. The diffusion of species with chemical reaction in the boundary layer flow has enormous applications in pollution studies, fibrous insulation, oxidation and synthesis materials for instance. (Awang, 2008) investigated the series solution of flow over a nonlinearly stretching sheet with the chemical reaction and magnetic field. Chemical reaction and uniform heat generation effects on stagnation point flow of a Nano fluid on MHD over a porous sheet was investigated by (Anwar I, et al, 2013). The radiation heat transfer is very important for its uses in different engineering areas such as gas cooled nuclear reactors, nuclear power plants, hypersonic fights, gas turbines and space vehicles etc. (Zhang et al., 2015) studied the effects chemical reaction and thermal radiation on Nano fluids of heat transfer through the porous medium. Several authors are examined the combination of chemical reaction and thermal radiation on Nano fluids of heat transfer over a stretching sheet. (Ramya et.al, 2016) investigated on influence of Chemical Reaction on MHD boundary Layer Flow of Nano fluids over a nonlinear stretching Sheet with Thermal Radiation. (M.H. Yazdi et.al, 2011) were studied by Slip MHD liquid flow and heat transfer over non-linear permeable stretching surface with chemical reaction using the Dormand – Prince pair and shooting method. (Ramya et.al, 2016)Boundary layer Viscous Flow of Nano fluids and Heat Transfer over a Nonlinearly Isothermal Stretching Sheet in the Presence of Heat Generation/Absorption and Slip Boundary Conditions.

Motivated by the above investigations the objective of the present study is to find a numerical investigation of Analysis of MHD Boundary Layer Nano fluid flow of Heat Transfer over a Nonlinear Stretching Sheet Presence of Thermal Radiation, chemical reaction and Partial Slip with boundary condition using the implicit finite difference method called Keller box method.

1.2. Statement of the problem

This study was attempted to answers the following basic questions:

- By using similarity transformations to convert a system of partial differential equations into nonlinear ordinary differential equations
- To apply appropriate method to solve the coupled nonlinear ordinary differential equations obtained from partial differential equations using similarity transformations
- Determining the roles of various parameters on the boundary layer Nano fluids flow over a nonlinear stretching sheet
- Analysis of different parameters those affect velocity, temperature, concentration, skin friction coefficient and heat transfer rate

1.3. Objective of the Study

1.3.1. General objective of the Study

The general objective of this study is to find numerical Analysis of Magneto Hydrodynamics (MHD) Boundary Layer Nano fluid flow of Heat Transfer over a Nonlinear Stretching Sheet Presence of Thermal Radiation, Chemical Reaction and Partial Slip with Boundary condition using Keller box method.

1.3.2. Specific Objectives of the Study

The study will have the following specific objectives:

- ✓ Transforming the governing partial differential equations into nonlinear ordinary differential equations using similarity transformations.
- ✓ To solve the nonlinear ordinary differential equations obtained from the partial differential equations by Keller box method.
- ✓ Identify the role of various parameters on boundary layer flow behavior of Nano fluids over a nonlinearly stretching sheet.
- ✓ Determining the effects of various parameters on velocity, temperature, concentration, skin friction coefficient, heat transfer rate and mass transfer rate.

- ✓ Sketching the graphs of various parameters effects on the velocity, temperature, concentration, skin friction coefficient surface, heat transfer rate and mass transfer rate using Mat lab.

1.4. Significance of the Study

The outcomes of this study have the following importance:

- ➢ It will develop the researcher knowledge on applied mathematics research.
- ➢ It may familiarize a researcher with scientific communication in applied mathematics.
- ➢ It will serve for other researchers as a useful reference for future research on this area.

1.5. Delimitation of the Study

This study is delimited to the governing partial differential equations of Boundary layer flow of Nano fluid over a Stretching Sheet focus only on the constructing Keller box method to investigate numerical Analysis of Magneto Hydrodynamics (MHD) Boundary Layer Nano fluid flow of Heat Transfer over a Nonlinear Stretching Sheet Presence of Thermal Radiation, Chemical Reaction and Partial Slip with Boundary condition

1.6. Definition of Important Terms

Magneto hydrodynamics: The study of the interaction between magnetic fields and electrically conducting fluids.

Boundary layer: Is a fluid character that forms in the flow of fluid through a body of surface.

Laminar Flow: Occurs when a fluid flows in the parallel layers, with no disruption between the layers and no cross currents or edges perpendicular to direction of flow.

Similarity transformations: The transformations which reduce the number of independent variables of a system of partial differential equations at least one less than that of the original equation are designated similarity transformations.

A steady Flow: Is a flow in which the various physical phenomena like velocity, pressure and density at any point do not change with time.

Stream Function: Is a function ψ which satisfies continuity equation and defined as**:**

$$\boldsymbol{u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}}$$

Where, u = velocity component in *x*– direction

V = velocity component in *y*– direction.

Chapter two

Literature review

2.1.Magnetohydrodynamics (MHD)

Magneto hydrodynamics is the branch of quantum mechanics which deals with the motion of an electrically conducting fluid in the presence of a magnetic field. The word magneto hydrodynamic (MHD) is derived from: Magneto-meaning magnetic field, Hydro meaning Liquid and Dynamics which means movement. Other variants of nomenclatures are Hydro magnetics, magneto-fluid dynamics, magneto-gas dynamics and so on. The concept of MHD is largely perceived to have been initiated by (Faraday 1812) when he did the first quantitative observation of Magnetohydrodynamics. He did experiments with mercury as a conducting fluid flowing in a glass tube placed in magnetic field and observed that voltage was induced in direction perpendicular to both the direction of flow and magnetic field. He further showed that when an electric field is applied to a conducting fluid in the direction which is perpendicular to magnetic field, a force is exerted on the fluid in the direction perpendicular to both electric field and magnetic field. Since then a lot has been done on MHD and its related fields and (Rao. et al., 1990) studied the heat transfer in porous medium in the presence of transverse magnetic field.

The effects of the heat source parameter and Nusselt number were analyzed. They discovered that the effect of increasing porous parameter is to increase the Nusselt Number. (Kinyanjui et al., 2003) investigated MHD Stokes problem for a vertical infinite plate in dissipative rotating fluid with Hall current as (Sigey et al., 2004) presented an investigation on the numerical study on natural convection turbulent heat transfer in an enclosure. As it is known that, MHD is important branch of fluid dynamics. Many technological problems and natural phenomena are susceptible to MHD analysis. Engineers apply MHD principle, in the design of heat exchangers, in creating novel power generating systems, pumps and flow meters, thermal protection, braking, control and re-entry, in space vehicle propulsion. MHD convection flow problems are also very important in the fields of stellar and planetary magnetosphere’s, aeronautics, electronics and chemical engineering. Hydromagnetic flow of Newtonian fluid and heat transfer over continuous moving flat surface with uniform suction has been studied by (Prasad et al., 2010) and (Kumari et al., 1990) studied the effects of induced magnetic field and heat source/sink on flow and heat transfer characteristic over a stretching surface. (Nazar et al., 2004) investigated the boundary layer over a moving continuous flat plate in an electrically conducting ambient fluid with a step change in applied magnetic field. The Magnetohydrodynamics(MHD) equations play an important role in many areas of astrophysics, space physics and engineering. Typical applications in those areas require one to capture flow on a range of scales in a way that is as dissipation free as possible. As a result, there has been considerable interest in bringing accurate and reliable numerical methods to bear on this problem.

2.2. Thermal Radiation

Radiative heat transfer has important applications in physics and engineering including in space technology and other high-temperature processes. Thermal radiation effects may also play an important role in controlling heat transfer in manufacturing processes where the quality of the final product may depend on heat control factors. High temperature plasmas, the cooling of nuclear reactors, liquid metal fluids, and power generation systems are some important applications of radiative heat transfer from a vertical wall to conductive gray fluids. In high-temperature chemical operations such as in combustion and fire science, it may be necessary to simulate thermal radiation heat transfer effects in combination with conduction, convection, and mass transfer. The effect of radiation on heat transfer problems was studied by (Hossain and Takhar 1996) and (Hamad et al., 2012) studied radiation effects on heat and mass transfer in MHD stagnation point flow over a permeable flat plate with convective surface boundary conditions and temperature dependent viscosity. They observed that the convective heat transfer parameter lowers the fluid velocity and the wall heat transfer rate. Radiation, however, increases the fluid velocity, temperature, and the heat transfer rate. (Singh et al. 2011) studied the effects of thermophoresis on hydro magnetic mixed convection and mass transfer flow past a vertical permeable plate with variable suction and thermal radiation. They observed that thermophoresis has a dominant effect on mass transfer in particle deposition processes. (Salleh et al., 2012) studied free convection over a permeable horizontal flat plate embedded in a porous medium with radiation effects and mixed thermal boundary conditions. They found that the velocity and temperature decreased with an increase in the radiation parameter. A similar solution for laminar thermal boundary layer over a flat plate with a convective surface boundary condition was studied by (Magyari 2009). He found that the wall temperature decreased monotonically with increasing values of Pr for all values of B_i. (Merkin and Pop 2011) studied the forced convection flow of a uniform stream over a flat surface with a convective surface boundary condition. They found that the heat transfer near the leading edge is to be dominated by the surface heat flux. The effect of transpiration on self-similar boundary layer flow over moving surfaces was studied by (Weidman et al., 2006).

2.3. Boundary Layer Flow

Prandtl introduced boundary layer flow layer theory in 1904 to understand the flow behavior of a viscous fluid near a solid boundary. Prandtl gave the concept of a boundary layer in large Reynolds number flows and derived the boundary layer equations by simplifying the Navier-Stokes equations to yield approximate solutions. Prandtls boundary layer equations arise in various physical models of fluid mechanics. The equations of the boundary layer theory have been the subject of considerable interest, since they represent an important simplification of the original Navier-Stokes equations. These equations arise in the study of steady flows produced by wall jets, free jets, and liquid jets, the flow past a stretching plate/surface, flow induced due to a shrinking sheet, and so on. These boundary layer equations are usually solved subject to certain boundary conditions depending upon the specific physical model considered. There are three types of boundary layer flows: velocity boundary layer flow, thermal boundary layer flow and concentration boundary layer flow.

2.3.1. Velocity Boundary Layer Flow

The velocity boundary layer develops whenever there is flow over a surface. It is associated with shear stresses parallel to the surface and results in an increase in velocity through the boundary layer from nearly zero right at the surface to the free.

2.3.2. Thermal Boundary Layer Flow

The thermal boundary layer is associated with temperature gradients near the surface, and develops when there is temperature difference between the fluid free stream and the surface. Right at the fluid-surface interface, heat transfer occurs only through conduction. The thickness of the thermal boundary layer is defined as that point at which the temperature difference between the fluid and surface is 99 percent of the temperature difference between the free stream fluid and the surface.

2.3.3. Concentration Boundary Layer Flow

The concentration boundary layer develops when there is a difference in concentration of a component between the free stream and the surface. A concentration profile develops, and the thickness of the concentration boundary layer is defined as that point at which the difference in concentration between the fluid and the surface is 99 percent of the difference in concentration between the free stream fluid and the surface. (Blasius, 1908) solved the Prandtls boundary layer equations for a flat moving plate problem and gave a power series solution of the problem. (Sakiadis, 1961) initiated the study of the boundary layer flow over a continuously moving rigid surface with a uniform speed. (Crane, 1970) was the first one who studied the boundary layer flow due to a stretching surface and developed the exact solutions of boundary layer equations with parameter. (Gupta and Gupta, 1977) extended the Cranes work and for the first time introduced the concept of heat transfer with the stretching sheet boundary layer flow. The boundary layer thickness, signified by, is simply the thickness of the viscous boundary layer region. Because the main effect of viscosity is to slow the fluid near a wall, the edge of the viscous region is found at the point where the fluid velocity is essentially equal to the free-stream velocity. In a boundary layer, the fluid asymptotically approaches the free-stream velocity as one moves away from the wall, so it never actually equals the free-stream velocity.

Chapter Three

Methodology

3.1 Study Design

This study will employ mixed-design (documentary review design and experimental design) on numerical Analysis of Magneto Hydrodynamics (MHD) Boundary Layer Nano fluid flow of Heat Transfer over a Nonlinear Stretching Sheet Presence of Thermal Radiation, Chemical Reaction and Partial Slip Boundary condition

3.2 Study Site and Period

The study will be conducted in Jimma University under the College of Natural sciences in Mathematics department from September, 2018 to July, 2020. Conceptually, the study focuses on Analysis of Magneto Hydrodynamics (MHD) Boundary Layer Nano fluid flow of Heat Transfer over a Nonlinear Stretching Sheet Presence of Thermal Radiation, Chemical Reaction and Partial Slip Boundary condition.

3.3 Sources of Information

The relevant sources of information for this study will be books, published articles & related studies from internet

3.4. Procedure of the Study

To achieve the stated objectives, the following mathematical procedures will be followed:

- Transforming the governed partial differential equations to nonlinear ordinary differential equations by introducing similarity transformations.
- Reducing the nonlinear ordinary differential equations to a system of first order equations.
- Writing the reduced ordinary differential equations to finite differences.

- Linearizing the algebraic equations by using Newton's method and write those in vector matrix form.
- Solving the linear system by the block tridiagonal elimination technique.
- Finally a sketch will be produced using MATLAB.

CHAPTER FOUR

MATHEMATICAL FORMULATION, RESULT AND DISCUSSION

4.1. **Mathematical Formulation**

Consider a two dimensional, incompressible viscous and steady fluid flow of a water based nanofluid flow past a nonlinear stretching surface. The sheet is extended with velocity $u_w = ax^n$ with fixed origin location, where n is a nonlinear stretching parameter, a is a constant, and x is the coordinate measured along with the stretching surface. The nanofluid flows at $y \geq 0$ where y is the coordinate normal to the surface. The fluid is electrical conducting due to an applied magnetic field $B(x)$normal to the stretching sheet. The magnetic Reynolds number is assumed small and so, the induced magnetic field can be considered negligible. The wall temperature $T_w$and the nanoparticle fraction are assumed constant at the stretching surface. When y tends to infinity, the ambient values of temperature and nanoparticle fraction are denoted by T_∞ and C_∞, respectively. The governing equations of momentum, thermal energy and nanoparticles equations (Mabood et al.) can be written as

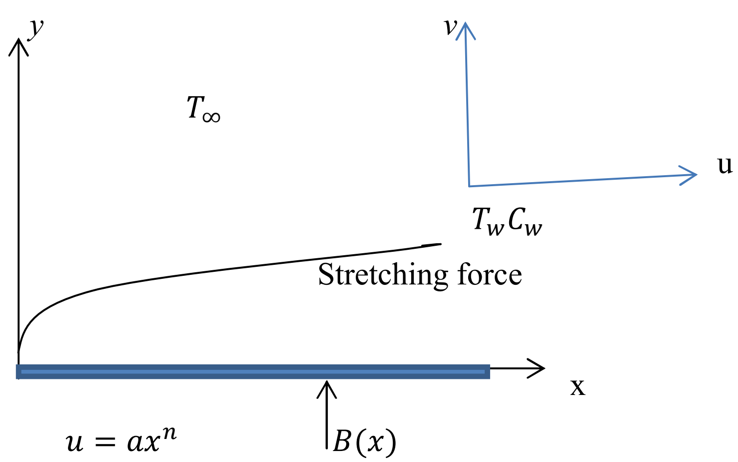

Fig.1 Geometry of the problem

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{1}$$

$$\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=\nu\frac{\partial^2 u}{\partial y^2}-\frac{\sigma B^2(x)}{\rho_f}u-\frac{\mu}{K_p}u \tag{2}$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\frac{\partial^2 T}{\partial y^2}+\tau\left\{D_B\frac{\partial C}{\partial y}\frac{\partial T}{\partial y}+\frac{D_T}{T_\infty}\times\left(\frac{\partial u}{\partial y}\right)^2\right\}-\frac{1}{(\rho cp)_f}\frac{\partial q_r}{\partial y}+\frac{\nu}{c_p}\left(\frac{\partial u}{\partial y}\right)^2+\frac{Q_0}{\rho c_p}(T-T_\infty), \tag{3}$$

$$u\frac{\partial C}{\partial x}+v\frac{\partial C}{\partial y}=D_B\frac{\partial^2 C}{\partial y^2}+\frac{D_T}{T_\infty}\times\left(\frac{\partial u}{\partial y}\right)^2-K_0(C-C_\infty) \tag{4}$$

The boundary conditions (Mabood et al.) are given by

$$u=u_w+K_1\frac{\partial u}{\partial y},v=0,T=T_w+K_2\frac{\partial T}{\partial y},\ C=C_w+K_3\frac{\partial C}{\partial y}\ \text{at } y=0 \text{ and}$$

$$u\to U_\infty, T\to T_\infty, C\to C_\infty \text{ at } y\to\infty \tag{5}$$

$\alpha=\frac{K}{(\rho c)_f}$ is the thermal diffusivity. $\tau=\frac{(\rho c)_p}{(\rho c)_f}$ is the ratio between the effective heat capacity of the nanoparticle material and the heat capacity of the ordinary fluid. Where $u_w=ax^n, K_1$ is the velocity Slipfactor,K_2 is the thermal slip factor, K_3 is the concentration slip factor and K_0 is the chemical rate constant. We assume that the variable magnetic field $B(x)=B_0x^{\frac{n-1}{2}}$ where B_0 is constant.

Using Rosseland approximation for radiation, the radiative heat flux q_r is given by

$$q_r=-\frac{4\sigma^*\partial T^4}{3k^*\partial y} \tag{6}$$

Where σ^* is the Stefan-Boltzmann constant, k^* is the mean absorption coefficient, we assume that the temperature difference with the flow sufficiently small such that the term T^4 may be expanded as a linear function of temperature. This is done by expanding T^4 in a Taylor series about a free stream temperature T^4 and neglecting higher order terms we ge$T^4=4T_\infty^3T-3T_\infty^4$

Hence q_r can be written as:

$$\frac{\partial q_r}{\partial y}=-\frac{16\sigma^*T_\infty^3}{3k^*}\frac{\partial T}{\partial y} \tag{7}$$

Similarity Transformation

In order to Analysis of Magneto Hydrodynamics (MHD) Boundary Layer Nano fluid flow of Heat Transfer over a Nonlinear Stretching Sheet Presence of Thermal Radiation, Chemical Reaction and Partial Slip with Boundary condition; the following dimensionless similarity variables are introduced:

$$\eta = y\sqrt{\frac{a(n+1)}{2v}}x^{\frac{n-1}{2}}, \theta = \frac{T-T_\infty}{T_W-T_\infty} \quad \phi = \frac{C-C_\infty}{C_W-C_\infty} \tag{8}$$

With the velocity components

$$u = \frac{\partial\psi}{\partial y} = ax^n f'(\eta) \tag{9}$$

$$v = -\frac{\partial\psi}{\partial x} = -\sqrt{\frac{av(n+1)}{2}}x^{\frac{n-1}{2}}\left[f(\eta) + \frac{n-1}{n+1}\eta f'(\eta)\right]$$

Whereψ is the stream function, $f(\eta)$ is the non-dimensional stream function, $f'(\eta)$ is the velocity profile, θ is non-dimensional temperature profile and η is similarity variable.

Using Equation (7) and (8) the continuity equation is satisfied as:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = a\left[\frac{\partial x^n}{\partial x}f'(\eta) + x^n f''(\eta)\frac{\partial\eta}{\partial x}\right] + \left[-\sqrt{av_f}f'(\eta)\frac{\partial\eta}{\partial y}\right]$$

$$-\sqrt{\frac{av(n+1)}{2}}x^{\frac{n-1}{2}}\left[f'^{(\eta)}\frac{\partial\eta}{\partial y} + \frac{n-1}{n+1}\frac{\partial\eta}{\partial x}f'^{(\eta)} + \frac{n-1}{n+1}\eta f''^{(\eta)}\frac{\partial\eta}{\partial y}\right]$$

$$= a\left[nx^{n-1}f'(\eta) + \frac{n-1}{2}x^n f''(\eta)y\sqrt{\frac{a(n+1)}{2v}}x^{\frac{n-3}{2}}\right]$$

$$-\sqrt{\frac{av(n+1)}{2}}x^{\frac{n-1}{2}}\frac{\partial\eta}{\partial y}\left[f'(\eta) + \frac{n-1}{n+1}\frac{\partial\eta}{\partial x}f'^{(\eta)} + \frac{n-1}{n+1}\eta f''(\eta)\right]$$

$$= a\left[nx^{n-1}f'(\eta) + \frac{n-1}{2}x^{n-1}f''(\eta)y\sqrt{\frac{a(n+1)}{2v}}x^{\frac{n-1}{2}}\right]$$

$$-\sqrt{\frac{av(n+1)}{2}}x^{\frac{n-1}{2}}\left(\sqrt{\frac{a(n+1)}{2v}}x^{\frac{n-1}{2}}\right)\left[\frac{2n}{n-1}f'(\eta) + \frac{n-1}{n+1}\eta f''(\eta)\right]$$

$$= a\left[nx^{n-1}f'(\eta) + \frac{n-1}{2}x^{n-1}f''(\eta)y\sqrt{\frac{a(n+1)}{2v}}x^{\frac{n-1}{2}}\right]$$

$$-\frac{a(n+1)}{2}x^{n-1}\left[\frac{2n}{n+1}f'(\eta)+\frac{n-1}{n+1}\eta f''(\eta)\right]$$

$$= ax^{n-1}\left[nf'(\eta)+\frac{n-1}{2}f''(\eta)\eta\right]- \ ax^{n-1}\left[nf'(\eta)+\frac{n-1}{2}f''(\eta)\eta\right]$$

$$= 0.$$

And also by using Equations (7) and (8)the governing partial differential equations of Equation (2), (3)and (4) are transformed into nonlinear higher order ordinary differential equations as follows:

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}-\frac{\sigma}{\rho_f}B^2(x)u-\frac{\mu}{K_0}u$$

Where;$u\frac{\partial u}{\partial x}=na^2x^{2n-1}f'^2+\frac{n-1}{2}a^2x^{2n-1}\eta f'f''$,

$$v\frac{\partial u}{\partial y}=-a^2x^{2n-1}\frac{n+1}{2}f''\left(f+\frac{n-1}{n+1}\eta f'\right)$$

$$\nu\frac{\partial^2 u}{\partial y^2}=\frac{n+1}{2}a^2x^{2n-1}f'''$$

$$\frac{\sigma}{\rho_f}B^2(x)u=\frac{\sigma}{\rho_f}{B_0}^2ax^{2n-1}f'$$

$$\frac{\mu}{K_0}u=\frac{\mu}{K_0}ax^nf'$$

Then substitute these from the above equation (2);

$$\Rightarrow na^2x^{2n-1}f'^2+\frac{n-1}{2}a^2x^{2n-1}\eta f'f''+\left[-a^2x^{2n-1}\frac{n+1}{2}f''\left(f+\frac{n-1}{n+1}\eta f'\right)\right]$$

$$=\frac{n+1}{2}a^2x^{2n-1}f'''-\frac{\sigma}{\rho_f}B_0ax^{2n-1}f'-\frac{\mu}{K_0}ax^nf'$$

$$\Rightarrow na^2x^{2n-1}f'^2+\frac{n-1}{2}a^2x^{2n-1}\eta f'f''-\frac{n+1}{2}a^2x^{2n-1}\eta ff''-\frac{n-1}{2}a^2x^{2n-1f'}f''$$

$$=\frac{n+1}{2}a^2x^{2n-1}f'''-\frac{\sigma}{\rho_f}B_0ax^{2n-1}f'-\frac{\mu}{K_0}ax^nf'$$

$$\Rightarrow na^2x^{2n-1}f'^2-\frac{n+1}{2}a^2x^{2n-1}ff''=\frac{n+1}{2}a^2x^{2n-1}f'''-\frac{\sigma}{\rho_f}B_0ax^{2n-1}f'-\frac{\mu}{K_0}ax^nf'$$

$$\Rightarrow\frac{n+1}{2}a^2x^{2n-1}f'''-\frac{n+1}{2}a^2x^{2n-1}ff''+na^2x^{2n-1}f'^2-f'\left(\frac{\sigma}{\rho_f}B_0ax^{2n-1}+\frac{\mu}{K_0}ax^n\right)=0$$

Dividing both sides by $\frac{n+1}{2}a^2x^{2n-1}$ and rearranging we have

$$f''' + ff'' - \frac{2n}{n+1}(f')^2 - f'\left(M + K_p\right) = 0$$

Equation (3) also;

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2} + \tau\left\{D_B\frac{\partial C}{\partial y}\frac{\partial T}{\partial y} + \frac{D_T}{T_\infty}\times\left(\frac{\partial u}{\partial y}\right)^2\right\} - \frac{1}{(\rho cp)_f}\frac{\partial q_r}{\partial y} + \frac{v}{c_p}\left(\frac{\partial u}{\partial y}\right)^2 + \frac{Q_0}{\rho c_p}(T - T_\infty)$$

Where;

$u\frac{\partial T}{\partial x} = \frac{n-1}{2}ax^{n-1}\eta f'\theta'(T_w - T_\infty)$,

$v\frac{\partial T}{\partial y} = -\frac{n+1}{2}ax^{n-1}\eta f\theta'(T_w - T_\infty) - \frac{n-1}{2}ax^{n-1}\eta f'\theta'(T_w - T_\infty)$,

$\alpha\frac{\partial^2 T}{\partial y^2} = \alpha\frac{n+1}{2v}ax^{n-1}\theta''(T_w - T_\infty)$,

$$\tau\left\{D_B\frac{\partial C}{\partial y}\frac{\partial T}{\partial y} + \frac{D_T}{T_\infty}\times\left(\frac{\partial u}{\partial y}\right)^2\right\}$$

$$= \tau D_B\frac{n+1}{2v}ax^{n-1}\theta'(T_w - T_\infty)\phi'(C_w - C_\infty) + \tau\frac{D_T}{T_\infty}\frac{n+1}{2v}ax^{n-1}(\theta'(T_w - T_\infty))^2$$

$$\frac{1}{(\rho cp)_f}\frac{\partial q_r}{\partial y} = \frac{1}{\left(\rho c_p\right)_f}\frac{16\sigma^* T_\infty^3}{3k^*}\frac{n+1}{2v}ax^{n-1}\theta''(T_w - T_\infty)$$

$\frac{v}{c_p}\left(\frac{\partial u}{\partial y}\right)^2 = \frac{v}{c_p}\frac{n+1}{2v}a^3x^{3n-1}(f'')^2$, then substitute these from the above equation (3);

$$\Rightarrow \frac{n-1}{2}ax^{n-1}\eta f'\theta'(T_w - T_\infty) - \frac{n+1}{2}ax^{n-1}\eta f\theta'(T_w - T_\infty) - \frac{n-1}{2}ax^{n-1}\eta f'\theta'(T_w - T_\infty)$$

$$= \alpha\frac{n+1}{2v}ax^{n-1}\theta''(T_w - T_\infty) + \tau D_B\frac{n+1}{2v}ax^{n-1}\theta'(T_w - T_\infty)\phi'(C_w - C_\infty)$$

$$+ \tau\frac{D_T}{T_\infty}\frac{n+1}{2v}ax^{n-1}(\theta'(T_w - T_\infty))^2 - \frac{1}{\left(\rho c_p\right)_f}\frac{16\sigma^* T_\infty^3}{3k^*}\frac{n+1}{2v}ax^{n-1}\theta''(T_w - T_\infty)$$

$$+ \frac{v}{c_p}\frac{n+1}{2v}a^3x^{3n-1}(f'')^2 + \frac{Q_0}{\rho c_p}(T - T_\infty)$$

$$\Rightarrow \alpha \frac{n+1}{2\nu} ax^{n-1}\theta''(T_w - T_\infty) + \frac{n+1}{2} ax^{n-1} f\theta'(T_w - T_\infty)$$

$$+ \tau D_B \frac{n+1}{2\nu} ax^{n-1}\theta'(T_w - T_\infty)\phi'(C_w - C_\infty)$$

$$+ \tau \frac{D_T}{T_\infty}\frac{n+1}{2\nu} ax^{n-1}(\theta'(T_w - T_\infty))^2 + \frac{1}{(\rho c_p)_f}\frac{16\sigma^* T_\infty^3}{3k^*}\frac{n+1}{2\nu} ax^{n-1}\theta''(T_w - T_\infty)$$

$$- \frac{\nu}{c_p}\frac{n+1}{2\nu} a^3 x^{3n-1}(f'')^2 + \frac{Q_0}{\rho c_p}(T - T_\infty) = 0$$

$$\Rightarrow \left[\alpha \frac{n+1}{2\nu} ax^{n-1} + \frac{1}{(\rho c_p)_f}\frac{16\sigma^* T_\infty^3}{3k^*}\frac{n+1}{2\nu} ax^{n-1}\right]\theta''(T_w - T_\infty) + \frac{n+1}{2} ax^{n-1} f\theta'(T_w - T_\infty)$$

$$+ \tau D_B \frac{n+1}{2\nu} ax^{n-1}\theta'(T_w - T_\infty)\phi'(C_w - C_\infty)$$

$$+ \tau \frac{D_T}{T_\infty}\frac{n+1}{2\nu} ax^{n-1}(\theta'(T_w - T_\infty))^2 + \frac{\nu}{c_p}\frac{n+1}{2\nu} a^3 x^{3n-1}(f'')^2 + \frac{Q_0}{\rho c_p}(T - T_\infty)$$

$$= 0$$

Dividing both sides by $\frac{n+1}{2} ax^{n-1}(T_w - T_\infty)$ and rearranging we have

$$\Rightarrow \left[\frac{\alpha}{\upsilon} + \frac{1}{(\rho c_p)_f}\frac{16\sigma^* T_\infty^3}{3k^*}\right]\theta'' + \frac{n+1}{2} ax^{n-1} f\theta' + \tau D_B \frac{n+1}{2\nu} ax^{n-1}\theta'\phi'(C_w - C_\infty)$$

$$+ \tau \frac{D_T}{T_\infty \upsilon}\theta'^2(T_w - T_\infty) + \frac{1}{c_p(T_w - T_\infty)} a^2 x^{2n-1}(f'')^2$$

$$+ \frac{2Q_0}{a(n+1)(\rho c)_f} x^{1-n}\frac{(T - T_\infty)}{(T_w - T_\infty)} = 0$$

$$\Rightarrow \theta'' + \frac{R}{1+R} Pr[f\theta' + \tau N_b \phi'\theta' + \tau N_t(\theta')^2 + Ec(f'')^2 + Q\theta] = 0$$

And equation (4);

$$u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} = D_B \frac{\partial^2 C}{\partial y^2} + \frac{D_T}{T_\infty} \times \left(\frac{\partial u}{\partial y}\right)^2 - k_0(C - C_\infty)$$

Where;

$u\frac{\partial C}{\partial x} = \frac{n-1}{2} ax^{n-1}\eta f'\phi'(C_w - C_\infty)$,

$$v\frac{\partial C}{\partial y} = -\frac{n+1}{2} ax^{n-1}\phi'(C_w - C_\infty)\left(f + \frac{n-1}{n+1}\eta f'\right)$$

$$D_B \frac{\partial^2 C}{\partial y^2} = D_B \frac{n+1}{2\nu} a x^{n-1} \phi''^{(C_w - C_\infty)}$$

$\frac{D_T}{T_\infty} \times \left(\frac{\partial u}{\partial y}\right)^2 = \frac{D_T}{T_\infty} \frac{n+1}{2\nu} a x^{n-1} \theta''(T_w - T_\infty)$, then substitute these from equation (4);

$$\Rightarrow \frac{n-1}{2} a x^{n-1} \eta f' \phi' (C_w - C_\infty) - \frac{n+1}{2} a x^{n-1} \phi' (C_w - C_\infty) \left(f + \frac{n-1}{n+1} \eta f'\right)$$

$$= D_B \frac{n+1}{2\nu} a x^{n-1} \phi''^{(C_w - C_\infty)} + \frac{D_T}{T_\infty} \frac{n+1}{2\nu} a x^{n-1} \theta''(T_w - T_\infty) - k_0 \phi (C_w - C_\infty)$$

$$\Rightarrow \frac{n-1}{2} a x^{n-1} \eta f' \phi' (C_w - C_\infty) - \frac{n+1}{2} a x^{n-1} \phi' (C_w - C_\infty) f$$

$$- \frac{n+1}{2} a x^{n-1} \phi' (C_w - C_\infty) \left(\frac{n-1}{n+1} \eta f'\right)$$

$$= D_B \frac{n+1}{2\nu} a x^{n-1} \phi'' (C_w - C_\infty) + \frac{D_T}{T_\infty} \frac{n+1}{2\nu} a x^{n-1} \theta''(T_w - T_\infty)$$

$$- k_0 \phi (C_w - C_\infty)$$

$$\Rightarrow - \frac{n+1}{2} a x^{n-1} \phi' (C_w - C_\infty) f$$

$$= D_B \frac{n+1}{2\nu} a x^{n-1} \phi'' (C_w - C_\infty) + \frac{n+1}{2} a x^{n-1} \phi' (C_w - C_\infty) f$$

$$+ \frac{D_T}{T_\infty} \frac{n+1}{2\nu} a x^{n-1} \theta''(T_w - T_\infty) - k_0 \phi (C_w - C_\infty)$$

$$\Rightarrow D_B \frac{n+1}{2\nu} a x^{n-1} \phi'' (C_w - C_\infty) + \frac{n+1}{2} a x^{n-1} \phi' (C_w - C_\infty) f + \frac{D_T}{T_\infty} \frac{n+1}{2\nu} a x^{n-1} \theta''(T_w - T_\infty)$$

$$- k_0 \phi (C_w - C_\infty) = 0$$

Dividing both sides by $D_B \frac{n+1}{2\nu} a x^{n-1} (C_w - C_\infty)$and rearranging we have

$$\phi'' + Lef\phi' + \frac{Nt}{Nb} \theta'' - LeR_c \phi$$

Then substitute

$\theta'' = -\frac{R}{1+R} Pr[f\theta' + Nb\phi'\theta' + Nt(\theta')^2 + Ec(f'')^2 + Q\theta]in\phi'' + Lef\phi' + \frac{Nt}{Nb}\theta'' - LeR_c\phi$,

it becomes $\phi'' + Lef\phi' - \frac{R}{1+R} Pr\left[\frac{N_t}{N_b} f\theta' + Nt\phi'\theta' + \frac{(N_t)^2}{N_b}(\theta')^2 + \frac{N_t}{N_b} Ec(f'')^2 + \frac{N_t}{N_b} Q\theta\right] - LeR_c\phi$

So equation (2), (3) and (4) are reduced to

$$f''' + ff'' - \frac{2n}{n+1}(f')^2 - f'\left(M + K_p\right) = 0 \tag{10}$$

$$\theta'' + \frac{R}{1+R} Pr[f\theta' + N_b\phi'\theta' + N_t(\theta')^2 + Ec(f'')^2 + Q\theta] = 0 \quad (11)$$

$$\phi'' + Lef\phi' - \frac{R}{1+R} Pr\left[\frac{N_t}{N_b} f\theta' + Nt\phi'\theta' + \frac{(Nt)^2}{Nb}(\theta')^2 + \frac{N_t}{N_b} Ec(f'')^2 + \frac{N_t}{N_b} Q\theta\right] - LeR_c\phi \quad (12)$$

Where ;

$M = \frac{2\sigma {B_0}^2}{\rho_f a(n+1)}$

$K_p = \frac{\mu}{K_0} R_c$

$R_c = \frac{2K_0}{a(n+1)} x^{1-n}$

$Ec = \frac{{u_w}^2}{c_p(T-T_\infty)}$,

$u_w = ax^n$,

$Pr = \frac{\nu}{\alpha}$,

$\alpha = \frac{\alpha}{(c_p)_f}$,

$N_r = \frac{3k^* k}{16\sigma^* {T^3}_\infty}$

$Q = \frac{2Q_0}{a(n+1)(\rho c)_f} x^{1-n}$

$N_t = \frac{(\rho c)_p D_T(T_w - T_\infty)}{(\rho c)_f T_\infty \nu}$

$N_b = \frac{(\rho c)_p D_B(C_w - C_\infty)}{(\rho c)_f \nu}$

$\lambda_1 = K_1 \sqrt{\frac{a\nu(n+1)}{2}} x^{\frac{n-1}{2}}$

$\lambda_2 = K_2 \sqrt{\frac{a\nu(n+1)}{2}} x^{\frac{n-1}{2}}$,

$\lambda_3 = K_3 \sqrt{\frac{a\nu(n+1)}{2}} x^{\frac{n-1}{2}}$

$Le = \frac{\nu}{D_B}$, (13)

With the following dimensionless boundary conditions;

$$f(0) = 0, f'(0) = 1 + \lambda_1 f''(0), \theta(0) = 1 + \lambda_2\theta'(0), \phi(0) = 1 + \lambda_3\phi'(0) \ \ at\ \eta = 0 \quad (14)$$

$$f'(\infty) = 0,\ \theta(\infty) = 0, \phi(\infty) = 0\ as\ \eta \to \infty.$$

More over;

By using similarity transformations, the boundary conditions (5) were transformed as follow:

$$u = u_w + k_1 \frac{\partial u}{\partial y}$$

$$ax^n f'(\eta) = ax^n + k_1 \frac{\partial}{\partial y}\left(ax^n f'(\eta)\right)$$

$$ax^n f'(\eta) = ax^n + ax^n k_1 \left(\sqrt{\frac{a(n+1)}{2\nu}} x^{\frac{n-1}{2}}\right)$$

$$f'(\eta) = 1 + k_1 \left(\sqrt{\frac{a(n+1)}{2\nu}} x^{\frac{n-1}{2}}\right) f''(\eta)$$

$$f'(\eta) = 1 + \lambda_1 f''(\eta)$$

$f'(0) = 1 + \lambda_1 f''(0)$, at $\eta = 0$

And

$$T = T_w + K_2 \frac{\partial T}{\partial y}$$

$$T_\infty + (T_w - T_\infty)\theta(\eta) = T_w + K_2 \frac{\partial}{\partial y}(T_w - T_\infty)\theta(\eta)$$

$$(T_w - T_\infty)\theta(\eta) = T_w - T_\infty + K_2(T_w - T_\infty)\left(\sqrt{\frac{a(n+1)}{2\nu}} x^{\frac{n-1}{2}}\right)\theta'(\eta)$$

$$(T_w - T_\infty)\theta(\eta) = (T_w - T_\infty)\left[1 + K_2\left(\sqrt{\frac{a(n+1)}{2\nu}} x^{\frac{n-1}{2}}\right)\theta'(\eta)\right]$$

$\theta(\eta) = \lambda_2\, \theta'(\eta)$

$\theta(0) = \lambda_2\, \theta'(0)$ at $y = 0, \eta = 0$

And

$$C = C_w + K_3 \frac{\partial C}{\partial y}$$

$$C_\infty + (C_w - C_\infty)\phi(\eta) = C_w + K_3 \frac{\partial}{\partial y}[C_\infty + (C_w - C_\infty)\phi(\eta)]$$

$$C_\infty + (C_w - C_\infty)\phi(\eta) = C_w + K_3 \frac{\partial}{\partial y}(C_w - C_\infty)\phi'(\eta)\left(\sqrt{\frac{a(n+1)}{2\nu}} x^{\frac{n-1}{2}}\right)$$

$$(C_w - C_\infty)\phi(\eta) = C_w - C_\infty + K_3 \frac{\partial}{\partial y}(C_w - C_\infty)\phi'(\eta)\left(\sqrt{\frac{a(n+1)}{2\nu}} x^{\frac{n-1}{2}}\right)$$

$$(C_w - C_\infty)\phi(\eta) = (C_w - C_\infty)\left(1 + K_3 \sqrt{\frac{a(n+1)}{2\nu}} x^{\frac{n-1}{2}} \phi'(\eta)\right)$$

$$\phi(\eta) = 1 + \lambda_3 \phi'(\eta)$$

$\phi(0) = 1 + \lambda_3 \phi'(0)aty = 0, \eta = 0$

Where;

$K_1 =$ is the velocity slip factor,

$K_2 =$ is the thermal slip factor,

$K_3 =$ is the concentration slip factor,

And also

$$\phi(\eta) = \phi(\infty) = \frac{C - C_\infty}{C_w - C_\infty} = \frac{C_\infty - C_\infty}{C_w - C_\infty} = 0$$

$$\phi(\eta) = \phi(\infty) \to 0 \; as \; \eta \to \infty$$

Similarly

$$T(\eta) = T(\infty) = \frac{T - T_\infty}{T_w - T_\infty} = \frac{T_\infty - T_\infty}{T_w - T_\infty} = 0$$

$$T(\eta) = T(\infty) \to 0 \; as \; \eta \to \infty$$

$M, Pr, Ec,, Nb, Le, Nt, Q, R_c, K_p, \lambda_1, \lambda_2, \lambda_3 and\ Nr$ are denote the Magnetic parameter, Prandtl number, Eckert number , the Brownian motion parameter, the Lewis number, the thermophoresis parameter, heat generation/absorption parameter, Chemical reaction parameter Permeability ,Velocity slip parameter, thermal slip parameter, concentration slip parameter and Thermal radiation parameters respectively.

The special interests and importance physical parameters of the present problem of Nanofluid, in this study skin-friction coefficient C_f, local Nusselt number Nu_x and Sherwood number are defined as;

$C_{fx} = \frac{\tau_w}{\rho u_w^2}$, $Nu_x = \frac{x q_w}{k(T_w - T_\infty)}$, $Sh_x = \frac{x q_m}{D_B(C_w - C_\infty)}$ (15)

Where

$\tau_w = \mu_f \left(\frac{\partial u}{\partial y}\right)$is the wall shear-stress

$q_w = -k \left(\frac{\partial T}{\partial y}\right)_{y=0}$ is the heat flux at wall and (16)

$q_m = -D_B \left(\frac{\partial C}{\partial y}\right)_{y=0}$ is mass flux

Using similarity transformation presented above, (15) can be reduced as

$$C_{fx} = \frac{\tau_w}{\rho u_w^2}$$

$$= \mu_f \frac{\left(\frac{\partial u}{\partial y}\right)}{\rho (a x^n)^2}$$

$$= \frac{\mu_f}{\rho} \frac{ax^n}{a^2x^{2n}} \left(\sqrt{\frac{a(n+1)}{2\nu}} x^{\frac{n-1}{2}} f''(\eta) \right) \text{since} \nu = \frac{\mu_f}{\rho}$$

$$= \sqrt{\frac{\nu}{ax^{n+1}}} \sqrt{\frac{n+1}{2}} f''(\eta)$$

$$= \frac{\sqrt{\frac{n+1}{2}} f''(\eta)}{\sqrt{Re_x}}, \text{ where } Re_x = \frac{U_w x}{\nu} = \frac{ax^{n+1}}{\nu}$$

$$C_{fx}\sqrt{Re_x} = \sqrt{\frac{n+1}{2}} f''(0), \eta = 0$$

And;

$$Nu_x = \frac{xq_w}{k(T_w - T_\infty)}$$

$$= x \left[\frac{-\mathrm{k}\left(\frac{\partial T}{\partial y}\right)}{k(T_w - T_\infty)} \right]$$

$$= \frac{-x(T_w - T_\infty)\left[\left(\sqrt{\frac{a(n+1)}{2\nu}} x^{\frac{n-1}{2}}\right) \theta'(\eta)\right]}{(T_w - T_\infty)}$$

$$= -\left(\sqrt{\frac{a(n+1)x^{n+1}}{2\nu}}\right) \theta'(\eta)$$

$$= \sqrt{\frac{ax^{n+1}}{\nu}} \sqrt{\frac{n+1}{2}} \theta'(\eta)$$

$$N_{ux} = \sqrt{Re_x} \sqrt{\frac{n+1}{2}} \theta'(\eta)$$

$$N_{ux} {R_{ex}}^{-1/2} = \sqrt{\frac{n+1}{2}} \theta'(0), \eta = 0$$

Finally;

$$Sh_x = \frac{xq_m}{D_B(C_w - C_\infty)}$$

$$= x\left[-D_B\left(\frac{\partial C}{\partial y}\right)\right]$$

$$= \frac{-x(C_w - C_\infty)}{C_w - C_\infty} \sqrt{\frac{a(n+1)}{2\nu}} x^{\frac{n-1}{2}} \phi'^{(\eta)}$$

$$= \sqrt{Re_x} \sqrt{\frac{n+1}{2}} \phi'^{(\eta)}$$

$Sh_x {R_{ex}}^{-1/2} = \sqrt{\frac{n+1}{2}} \phi'(0), \eta = 0$

Where the local Reynolds number is given by $Re_x = \frac{U_w x}{\nu}$.

4.2 Method of Solution

Equations (10) – (12) subject to the boundary conditions (13) were solved numerically by Keller box method which is implemented in Matlab. The Keller box method is an implicit finite difference method that can be used to solve differential equations. This method has four basic steps:

1. Converting Equations (2) – (4) into a system of first order nonlinear ordinary differential equations.
2. By approximating the derivatives in system of first order equations with central difference approximations.
3. Linearizing the nonlinear algebraic equations with Newton′s method and then casting as the matrix vector form.
4. Finally solving the system of linear equations using block tridiagonal elimination scheme with the suitable initial solution.

In this method the transformed differential equations (9) and (10) are written in terms of first order system (Mitiku and Devaraj, 2017) for that introduce new dependent variable u, v, g *and* s such that

$$\begin{aligned} f' &= u, \\ u' &= v, \\ \theta' &= g, \\ \phi' &= s, \end{aligned} \tag{17}$$

So equation (9) and (10) can be written as

$$v' + fv - \frac{2n}{n+1} u^2 - (M + K_p)u = 0 \tag{18}$$

$$g' + \frac{1}{1+\frac{4}{3}R} Pr[fg + N_b kg + N_t tg^2 + Ecv^2 + Q\theta] = 0 \tag{19}$$

$$s' + Lefs - \frac{1}{1+\frac{4}{3}R}\frac{N_t}{N_b} Pr[fg + N_b sg + N_t g^2 + Ecv^2 + Q\theta] - LeR_c\phi = 0 \tag{20}$$

And the transformed boundary conditions for the problem are

$$f(0) = 0, u(0) = 1 + \lambda_1 v(0), \theta(0) = 1 + \lambda_2 g(0), \phi(0) = 1 + \lambda_3 s(0) \;\; at\; \eta = 0 \tag{21}$$

$$u(\infty) = 0,\; \theta(\infty) = 0, \phi(\infty) = 0 \; as\; \eta \to \infty.$$

Now, consider the net rectangle in $x - \eta$ plane as show in figure 4.2 and the net points are defined as follow:

$$x_0 = 0, x_{n-1} + k_n, n = 1,2, \dots N$$

$$\eta_0 = 0, \eta_{j-1} = 0, j = 1,2, \dots J \tag{22}$$

Where;k_n is the $\Delta x-$ spacing and h_j is the $\Delta\eta$ –spacing. Here h and j are the sequence of number that indicate the coordinate location.

Now write the finite difference approximation of the ODE, Equation (17) for the midpoint(x_{n-1}, η_{j-1}) of the segment P_1P_2 using centered difference derivatives, this is called centering about (x_{n-1}, η_{j-1})

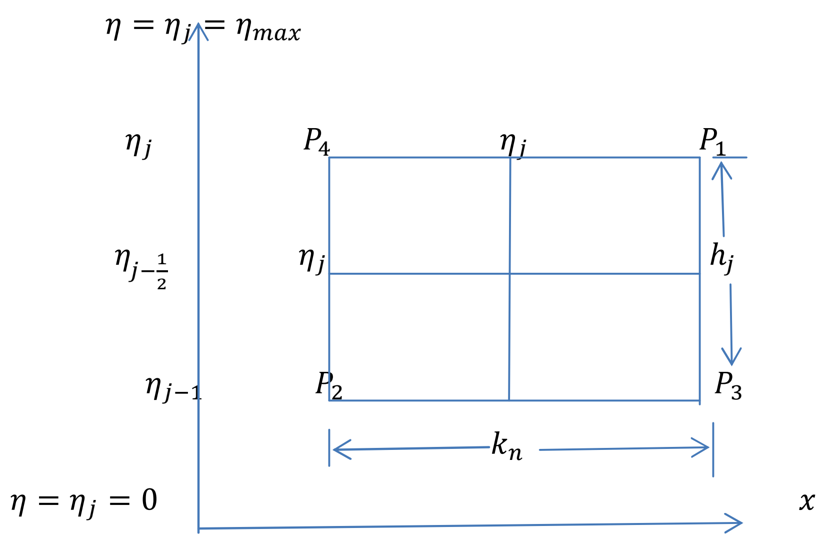

Figure 4.2: typical grid structure for difference approximations.

$$\frac{f_j - f_{j-1}}{h_j} = u_{j-\frac{1}{2}} = \frac{u_j + u_{j-1}}{2}$$

$$\frac{u_j-u_{j-1}}{h_j}=v_{j-\frac{1}{2}}=\frac{v_j+v_{j-1}}{2} \tag{23}$$

$$\frac{\theta_j-\theta_{j-1}}{h_j}=g_{j-\frac{1}{2}}=\frac{g_j+g_{j-1}}{2}$$

$$\frac{\phi_j-\phi_{j-1}}{h_j}=s_{j-\frac{1}{2}}=\frac{s_j+s_{j-1}}{2}$$

An ODE of Equations (18), (19) and (20) are approximated by the centering about the midpoint (x_{n-1},η_{j-1}) of the rectangle$P_1P_2P_3P_4$.

$$v'+fv-\frac{2n}{n+1}u^2-(M+K_p)u=0$$

$$\frac{v_j-v_{j-1}}{h_j}+f_{j-\frac{1}{2}}v_{j-\frac{1}{2}}-\frac{2n}{n+1}\left(u_{j-\frac{1}{2}}\right)^2-(M+K_p)u_{j-\frac{1}{2}}=0$$

$$\Rightarrow\ \frac{v_j-v_{j-1}}{h_j}+\left(\frac{f_j+f_{j-1}}{2}\right)\left(\frac{v_j+v_{j-1}}{2}\right)-\frac{2n}{n+1}\left(\frac{u_j+u_{j-1}}{2}\right)^2-(M+K_p)\left(\frac{u_j+u_{j-1}}{2}\right)=0$$

$$\Rightarrow v_j-v_{j-1}+h_j\left(\frac{f_j+f_{j-1}}{2}\right)\left(\frac{v_j+v_{j-1}}{2}\right)-h_j\frac{2n}{n+1}\left(\frac{u_j+u_{j-1}}{2}\right)^2$$

$$-h_j(M+K_p)M\left(\frac{u_j+u_{j-1}}{2}\right)=0$$

$$\Rightarrow\ \delta v_j-\delta v_{j-1}+v_j-v_{j-1}+\frac{h_j}{4}\left[(\delta f_j+\delta f_{j-1}+f_j+f_{j-1})(\delta v_j+\delta v_{j-1}+v_j+v_{j-1})\right]-$$

$$\frac{2n}{n+1}\frac{h_j}{2}(\delta u_j+\delta u_{j-1}+u_j+u_{j-1})^2-(M+K_p)\frac{h_j}{2}(\delta u_j+\delta u_{j-1}+u_j+u_{j-1})=0 \tag{24}$$

And

$$g'+\frac{R}{1+R}Pr[fg+N_b sg+N_t g^2+Ecv^2+Q\theta]=0$$

$$\Rightarrow\frac{g_j-g_{j-1}}{h_j}+\frac{R}{1+R}pr\left[\left(\frac{f_j+f_{j-1}}{2}\right)\left(\frac{g_j+g_{j-1}}{2}\right)+N_b\left(\frac{s_j+s_{j-1}}{2}\right)\left(\frac{g_j+g_{j-1}}{2}\right)\right.$$

$$\left.+N_t\left(\frac{g_j+g_{j-1}}{2}\right)^2+Ec\left(\frac{v_j+v_{j-1}}{2}\right)^2+Q\left(\frac{\theta_j+\theta_{j-1}}{2}\right)\right]=0$$

$$\Rightarrow g_j-g_{j-1}+\frac{R}{1+R}h_jPr\left[\left(\frac{f_j+f_{j-1}}{2}\right)\left(\frac{g_j+g_{j-1}}{2}\right)+N_b\left(\frac{s_j+s_{j-1}}{2}\right)\left(\frac{g_j+g_{j-1}}{2}\right)\right.$$

$$\left.+N_t\left(\frac{g_j+g_{j-1}}{2}\right)^2+Ec\left(\frac{v_j+v_{j-1}}{2}\right)^2+\frac{h_j}{2}Q\left(\frac{\theta_j+\theta_{j-1}}{2}\right)\right]=0$$

$$\Rightarrow \delta g_j - \delta g_{j-1} + g_{j-1} - g_j + \frac{R}{1+R} Pr\left[\frac{h_j}{4}(\delta f_j + \delta f_{j-1} + f_j + f_{j-1})(\delta g_j + \delta g_{j-1} + g_j + g_{j-1}) + N_b \frac{h_j}{4}(\delta s_j + \delta s_{j-1} + s_j + s_{j-1})(\delta g_j + \delta g_{j-1} + g_j + g_{j-1}) + N_t \frac{h_j}{4}(\delta g_j + \delta g_{j-1} + g_j + g_{j-1})^2 + Ec \frac{h_j}{4}(\delta v_j + \delta v_{j-1} + v_j + v_{j-1})^2 + \frac{h_j}{2} Q(\delta\theta_j + \delta\theta_{j-1} + \theta_j + \theta_{j-1})\right] = 0 \quad (25)$$

And

$$s' - \frac{R}{1+R}\frac{N_t}{N_b} Pr[fg + N_t sg + N_t g^2 + v^2 + Q\theta] - LeR_c\phi = 0$$

$$\frac{s_j - s_{j-1}}{h_j} - \frac{R}{1+R}\frac{N_t}{N_b} Pr\left[\left(\frac{f_j + f_{j-1}}{2}\right)\left(\frac{g_j + g_{j-1}}{2}\right) + N_b\left(\frac{s_j + s_{j-1}}{2}\right)\left(\frac{g_j + g_{j-1}}{2}\right) + Nt\left(\frac{g_j + g_{j-1}}{2}\right)^2 + \left(\frac{v_j + v_{j-1}}{2}\right)^2 + \frac{h_j}{2} Q\left(\frac{\theta_j + \theta_{j-1}}{2}\right)\right] - LeR_c\left(\frac{\phi_j + \phi_{j-1}}{2}\right)$$

$$s_j - s_{j-1} - \frac{R}{1+R} h_j \frac{N_t}{N_b} Pr\left[\left(\frac{f_j + f_{j-1}}{2}\right)\left(\frac{g_j + g_{j-1}}{2}\right) + N_b\left(\frac{s_j + s_{j-1}}{2}\right)\left(\frac{g_j + g_{j-1}}{2}\right) + N_t\left(\frac{g_j + g_{j-1}}{2}\right)^2 + \left(\frac{v_j + v_{j-1}}{2}\right)^2 + \frac{h_j}{2} Q\left(\frac{\theta_j + \theta_{j-1}}{2}\right)\right] - LeR_c h_j\left(\frac{\phi_j + \phi_{j-1}}{2}\right)$$

$$\delta s_j - \delta s_{j-1} + s_j - s_{j-1} - \frac{R}{1+R}\frac{N_t}{N_b} Pr\left[\frac{h_j}{4}(\delta f_j + \delta f_{j-1} + f_j + f_{j-1})(\delta g_j + \delta g_{j-1} + g_j + g_{j-1}) + N_b \frac{h_j}{4}(\delta s_j + \delta s_{j-1} + s_j + s_{j-1})(\delta g_j + \delta g_{j-1} + g_j + g_{j-1}) + N_t \frac{h_j}{4}(\delta g_j + \delta g_{j-1} + g_j + g_{j-1})^2 + Ec \frac{h_j}{4}(\delta v_j + \delta v_{j-1} + v_j + v_{j-1})^2 + \frac{h_j}{2} Q(\delta\theta_j + \delta\theta_{j-1} + \theta_j + \theta_{j-1})\right] - LeR_c \frac{h_j}{2}(\delta\phi_j + \delta\phi_{j-1} + \phi_j + \phi_{j-1}) = 0 \quad (26)$$

Now linearize the nonlinear system of equation (23) - (26) using the Newton's linearization scheme. That is, we assume for $(i+1)^{th}$ iterations.

$f_j{}^{i+1}, v_j{}^{i+1}$, etc. (27)

Substituting equation (27) into the above equations and dropping the quadratic terms in $\delta f^i{}_j, \delta u^i{}_j, \delta v^i{}_j, \delta\theta^i{}_j, \delta g^i{}_j, \delta\phi^i{}_j, \delta s^i{}_j$ we obtain tridiagonal system of algebraic equations.

$$\delta f_j - \delta f_{j-1} - \frac{h_j}{2}(\delta u_j + \delta u_{j-1}) = (r_1)_j \quad (28)$$

$$\delta u_j - \delta u_{j-1} - \frac{h_j}{2}(\delta v_j + \delta v_{j-1}) = (r_2)_j \quad (29)$$

$$\delta\theta_j - \delta\theta_{j-1} - \frac{h_j}{2}(\delta g_j + \delta g_{j-1}) = (r_3)_j \quad (30)$$

$$\delta\phi_j - \delta\phi_{j-1} - \frac{h_j}{2}\left(\delta s_j + \delta s_{j-1}\right) = (r_4)_j \tag{31}$$

$$(a_1)_j \delta v_j + +(a_2)_j \delta v_{j-1} + (a_3)_j \delta f_j + (a_4)_j \delta f_{j-1} + (a_5)_j \delta u_j + (a_6)_j \delta u_{j-1} = (r_5)_j \tag{32}$$

$$(b_1)_j \delta g_j + (b_2)_j \delta g_{j-1} + (b_3)_j \delta f_j + (b_4)_j \delta f_j + (b_5)_j \delta s_j + (b_6)_j \delta s_{j-1} + (b_7)_j \delta v_j + (b_8)_j \delta v_{j-1} + (b_9)_j \delta \theta_j + (b_{10})_j \delta \theta_{j-1} = (r_6)_j \tag{33}$$

$$(c_1)_j \delta s_j + (c_2)_j \delta s_{j-1} + (c_3)_j \delta f_j + (c)_j \delta f_j + (c_5)_j \delta g_j + (c_6)_j \delta g_{j-1} + (c_7)_j \delta v_j + (c_8)_j \delta v_{j-1} + (c_9)_j \delta \theta_j + (c_{10})_j \delta \theta_{j-1} + (c_{11})_j \delta \phi_j + (c_{12})_j \delta \phi_{j-1} = (r_7)_j \tag{34}$$

Where

$$(a_1)_j = 1 + \frac{h_j}{4}\left(f_j + f_{j-1}\right)$$

$$(a_2)_j = -1 + \frac{h_j}{4}\left(f_j + f_{j-1}\right)$$

$$(a_3)_j = (a_4)_j = \frac{h_j}{4}\left(v_j + v_{j-1}\right)$$

$$(a_5)_j = (a_6)_j = -\frac{h_j}{2}\left[\frac{2n}{n+1}\left(u_j + u_{j-1}\right) + \left(M + K_p\right)\right] \tag{35}$$

$$(b_1)_j = 1 + \frac{R}{1+R}\frac{h_j}{4}Pr\left[\left(f_j + f_{j-1}\right) + N_b\frac{h_j}{4}\left(s_j + s_{j-1}\right) + 2N_t\left(g_j + g_{j-1}\right)\right]$$

$$(b_2)_j = -1 + \frac{R}{1+R}\frac{h_j}{4}Pr\left[\left(f_j + f_{j-1}\right) + N_b\left(s_j + s_{j-1}\right) + 2N_t\left(g_j + g_{j-1}\right)\right]$$

$$(b_3)_j = (b_4)_j = \frac{R}{1+R}\frac{h_j}{4}Pr\left(g_j + g_{j-1}\right)$$

$$(b_5)_j = (b_6)_j = \frac{R}{1+R}N_b\frac{h_j}{4}Pr\left(g_j + g_{j-1}\right)$$

$$(b_7)_j = (b_8)_j = \frac{R}{1+R}Ec\frac{h_j}{2}Pr\left(v_j + v_{j-1}\right)$$

$$(b_9)_j = (b_{10})_j = \frac{R}{1+R}\frac{h_j}{2}Pr\mathrm{Q} \tag{36}$$

$$(c_1)_j = 1 + Le\frac{h_j}{4}\left(f_j + f_{j-1}\right) - N_t\frac{R}{1+R}\frac{h_j}{4}Pr\left(g_j + g_{j-1}\right)$$

$$(c_2)_j = -1 + Le\frac{h_j}{4}\left(f_j + f_{j-1}\right) - N_t\frac{R}{1+R}\frac{h_j}{4}Pr\left(g_j + g_{j-1}\right)$$

$$(c_3)_j = (c_4)_j = Le\frac{h_j}{4}\left(s_j + s_{j-1}\right) - \frac{N_t}{N_b}\frac{R}{1+R}\frac{h_j}{4}Pr\left(g_j + g_{j-1}\right)$$

$$(c_5)_j = (c_6)_j = -\frac{R}{1+R}\frac{N_t}{N_b}\frac{h_j}{4}Pr\left[\left(f_j + f_{j-1}\right) + N_b\left(s_j + s_{j-1}\right) + 2N_t\left(g_j + g_{j-1}\right)\right]$$

$$(c_7)_j = (c_8)_j = -\frac{N_t}{N_b}\frac{R}{1+R}\frac{h_j}{2}Pr\left(v_j + v_{j-1}\right)$$

$$(c_9)_j = (c_{10})_j = -\frac{N_t}{N_b}\frac{R}{1+R}\frac{h_j}{2}PrQ$$

$(c_{11})_j = (c_{12})_j = -LeR_c\frac{h_j}{2}$(37)

$$(r_1)_j = f_{j-1} - f_j + \frac{h_j}{2}\left(u_j + u_{j-1}\right)$$

$$(r_2)_j = u_{j-1} - u_j + \frac{h_j}{2}\left(v_j + v_{j-1}\right)$$

$(r_3)_j = \theta_{j-1} - \theta_j + \frac{h_j}{2}\left(g_j + g_{j-1}\right)$

$$(r_4)_j = \phi_{j-1} - \phi_j + \frac{h_j}{2}\left(s_j + s_{j-1}\right)$$

$$(r_5)_j = v_{j-1} - v_j - \frac{h_j}{4}\left(f_j + f_{j-1}\right)\left(v_j + v_{j-1}\right) + \frac{2n}{n+1}\frac{h_j}{4}\left(u_j + u_{j-1}\right)^2 + \left(M + K_p\right)\frac{h_j}{2}\left(u_j + u_{j-1}\right)$$

$$(r_6)_j = g_{j-1} - g_j - \frac{R}{1+R}\frac{h_j}{4}Pr\left[\left(f_j + f_{j-1}\right)\left(g_j + g_{j-1}\right) + Nb\left(s_j + s_{j-1}\right)\left(g_j + g_{j-1}\right) + Nt\left(g_j + g_{j-1}\right)^2 + Ec\left(v_j + v_{j-1}\right)^2 + 2Q\left(\theta_j + \theta_{j-1}\right)\right]$$

$(r_7)_j = s_{j-1} - s_j - Le\frac{h_j}{4}\left(f_j + f_{j-1}\right)\left(s_j + s_{j-1}\right) + \frac{R}{1+R}\frac{N_t}{N_b}\frac{h_j}{4}Pr\left[\left(f_j + f_{j-1}\right)\left(g_j + g_{j-1}\right) + N_b\left(s_j + s_{j-1}\right)\left(g_j + g_{j-1}\right) + N_t\left(g_j + g_{j-1}\right)^2 + Ec\left(v_j + v_{j-1}\right)^2 + 2Q\left(\theta_j + \theta_{j-1}\right)\right] + LeR_c\frac{h_j}{2}\left(\phi_j + \phi_{j-1}\right)$ (38)

With the boundary conditions;

$f(0) = 0,\ \delta u(0) = 1 + \lambda_1\delta v(0),\ \delta\theta(0) = 1 + \lambda_2\delta g(0),\ \delta\phi(0) = 1 + \lambda_3\delta s(0)\ \ at\ \eta = 0$ (39)

$$\delta u(\infty) \to 0, \quad \delta\theta(\infty) \to 0, \quad \delta\phi(\infty) \to 0\ as\ \eta \to \infty$$

Hence the linearized system of equations (28) – (36) can be written in the matrix

Form as: $A = \delta r$ (40)

Where

$$
\begin{bmatrix}
[A_1] & [C_1] & & & & & \\
[B_2] & [A_2] & [C_2] & & & & \\
 & & \cdot & \cdot & \cdot & & \\
 & & & \cdot & \cdot & \cdot & \\
 & & & & \cdot & \cdot & \cdot \\
 & & & & [B_{J-1}] & [A_{J-1}] & [C_{J-1}] \\
 & & & & & [B_J] & [A_J]
\end{bmatrix}
\delta = \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \\ \delta_5 \\ \delta_6 \\ \delta_7 \end{bmatrix}
r = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \\ r_6 \\ r_7 \end{bmatrix}
$$

The elements of matrix A are block matrix of order 7×7

$$
A_1 = \begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
d_1 & 0 & 0 & 0 & d_1 & 0 & 0 \\
0 & d_1 & 0 & 0 & 0 & d_1 & 0 \\
0 & 0 & d_1 & 0 & 0 & 0 & d_1 \\
(a_2)_1 & 0 & 0 & (a_3)_1 & (a_1)_1 & 0 & 0 \\
(b_8)_1 & (b_2)_1 & (b_6)_1 & (b_3)_1 & (b_7)_1 & (b_1)_1 & (b_5)_1 \\
(c_8)_1 & (c_6)_1 & (c_2)_1 & (c_3)_1 & (c_7)_1 & (c_5)_1 & (c_1)_1
\end{bmatrix},
$$

$$
A_j = \begin{bmatrix}
d_j & 0 & 0 & 1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & d_j & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & d_j & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & d_j \\
(a_6)_j & 0 & 0 & (a_3)_j & (a_1)_j & 0 & 0 \\
0 & (b_{10})_j & 0 & (b_3)_j & (b_7)_1 & (b_1)_j & (b_5)_j \\
0 & (c_{10})_j & (c_{12})_j & (c_3)_j & (c_7)_j & (c_5)_j & (c_1)_j
\end{bmatrix}, 2 \le j \le J
$$

$$
B_j = \begin{bmatrix}
0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & d_j & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & d_j & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & d_j \\
0 & 0 & 0 & (a_4)_j & (a_2)_j & 0 & 0 \\
0 & 0 & 0 & (b_4)_j & (b_8)_j & (b_2)_j & (b_6)_j \\
0 & 0 & 0 & (c_4)_j & (c_8)_j & (c_6)_j & (c_2)_j
\end{bmatrix}, \text{for } 1 \le j \le J
$$

$$C_j = \begin{bmatrix} d_j & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ (a_5)_j & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & (b_9)_j & 0 & 0 & 0 & 0 & 0 \\ 0 & (c_9)_j & (c_{11})_j & 0 & 0 & 0 & 0 \end{bmatrix}, \text{ for } 1 \le j \le J$$

Here $d_j = -\frac{h_j}{2}$ and $\delta_1 = \begin{bmatrix} \delta v_0 \\ \delta g_0 \\ \delta s_0 \\ \delta f_1 \\ \delta v_1 \\ \delta g_1 \\ \delta s_1 \end{bmatrix}$ $for\ j = 1$, $\delta_j = \begin{bmatrix} \delta u_{j-1} \\ \delta \phi_{j-1} \\ \delta \theta_{j-1} \\ \delta f_j \\ \delta v_j \\ \delta g_j \\ \delta s_j \end{bmatrix}$, $for\ 2 \le j \le J$, $r_j = \begin{bmatrix} (r_1)_j \\ (r_2)_j \\ (r_3)_j \\ (r_4)_j \\ (r_5)_j \\ (r_6)_j \\ (r_7)_j \end{bmatrix}$,

$for\ 1 \le j \le J$

The solution of equation (41) can obtained using block elimination method which consists of forward and backward sweeps

Forward sweep

To solve equation (38), we use LU factorization for decomposing matrix A into a product of a lower triangular matrix L and an upper triangular matrix U as follow (Chaltu etal.,2017, adhikarri and sanyal,2013),

A = LU (42)

Where

$$L = \begin{bmatrix} [\alpha_1] & & & & & \\ [\beta_2] & [\alpha_2] & & & & \\ & [\beta_3] & [\alpha_3] & & & \\ & \dots & \dots & & & \\ & & \dots & \dots & & \\ & & & [\beta_{J-1}] & [\alpha_{J-1}] & \\ & & & & [\beta_J] & [\alpha_J] \end{bmatrix}, \quad U = \begin{bmatrix} [I] & [\Gamma_1] & & & & \\ & [I] & [\Gamma_2] & & & \\ & & [I] & [\Gamma_3] & & \\ & & \dots & \dots & & \\ & & & \dots & \dots & \\ & & & & [I] & [\Gamma_{J-1}] \\ & & & & & [I] \end{bmatrix}$$

[I] is the identity matrix of order 7 by 7 and $[\alpha_j]$ and $[\Gamma_j]$ are 7 by 7 matrices which elements are determined by the following equations:

$$[\alpha_1] = [A_1][A_1][\Gamma_1] - [C_1]$$

$$[\alpha_j] = [A_j] - [B_1][\Gamma_{j-1}] \qquad j = 2, 3, \ldots, J \tag{43}$$

$$[\alpha_j][\Gamma_j] = [C_j] \qquad j = 2, 3, \ldots, J-1$$

Backward sweep

Equation (41) can be substituted into equation (42), and so we get

$$LU\delta = r \tag{44}$$

If we define, $U\delta = W$ (45)

Then equation (44) becomes

$$LW = r \tag{46}$$

Where

$$W = [\mathrm{w}_1 \mathrm{w}_2 \ . \ . \ . \ \mathrm{w}_{j-1} \mathrm{w}_j]^{Tra} \tag{47}$$

w_j *are* 6×1 Column matrices. The elements W can be found by the solving equation (46)

$$[\alpha_1][\mathrm{w}_1] = [r_1] \tag{48}$$

$$[\alpha_j][\mathrm{w}_j] = [r_j] - [\beta_j][\mathrm{w}_{j-1}], \ j \le 2 \le J \tag{49}$$

Once the element of W are found, we can find the solution of eqn.(45) using the recurrent relations

$$[\delta_J] = [\mathrm{w}_j], \tag{50}$$

$$[\delta_J] = [\mathrm{w}_j] - [\Gamma_j][\delta_{j+1}], \qquad 1 \le j \le J-1 \tag{51}$$

These calculations are repeated until convergence criterion is satisfied and calculations are stopped when $|\delta v_0{}^{(i)}| < \varepsilon$, where ε is the desired level of accuracy. In this study, the value of $\varepsilon = 10^{-6}$.

4.3 Numerical Results and Discussion

Table1. Comparison of $-\theta'(0)$ for different values of Pr and n when $\mathrm{Nb} = \mathrm{Nt} = \mathrm{Le} = \mathrm{M} = \mathrm{Kp} = \mathrm{Nr} = \mathrm{Q} = \mathrm{Ec} = \lambda_1 = \lambda_2 = \lambda_3 = 0$.

Pr	n	(Mabood et al 2015)	(Dodda et al., 2016)			(Chaltu et al, 2017)	**Present results for different grid size (h)**		
			h=0.1	h=0.01	h=0.006		h = 0.1	h = 0.01	h=0.006
1	0	_	_	_	_	_ __ _ _	0.6276567	0.6275566	0.6275560
	0.1	_	_	_	_	_ __ _ _	0.6178463	0.6177554	0.6177548
	0.2	0.61131	0.6102	0.6102	0.6112	0.611310	0.6102874	0.6102041	0.6102036
	0.3	_	_	_	_	_ __ _ _	0.6042700	0.6041934	0.6041929
	0.4	_	_	_	_	_ __ _ _	0.5993587	0.5992879	0.5992874
	0.5	0.59668	0.5952	0.5952	0.5966	0.596688	0.5952701	0.5952043	0.5952038
	0.6					_ __ _ _	0.5918109	0.5917496	0.5917492
	1	_	_	_	_	0.583866	0.5820296	0.5819817	0.5819814
	1.5	0.57686	0.5747	0.5747	0.5768	0.576870	0.5747726	0.5747357	0.5747355
	2	_	_	_	_	0.572456	0.5701808	0.5701513	0.5701512
	3	0.56719	0.5647	0.5646	0.5671	0.567191	0.5646906	0.5646704	0.5646703
	4					0.564156	0.5615193	0.5615048	0.5615047
	5	_	_	_	_	_ __ _ _	0.5594532	0.5594424	0.5594423
	6						0.5579999	0.5579919	0.5579918
	8					0.562182	0.5560907	0.5560862	0.5560862

	10	0.55783	0.5548	0.5549	0.5578	0.558974	0.5548921	0.5548900	0.5548900
[illegible]	0	_	_	_	_	_ __ _ _	1.6344894	1.6318473	1.6318303
	0.1	_	_	_	_	_ __ _ _	1.6208116	1.6183021	1.6182859
	0.2	1.60757	1.6102	1.6077	1.6074	_ __ _ _	1.6102210	1.6078109	1.6077954
	0.3	_	_	_	_	_ __ _ _	1.6017539	1.5994214	1.5994063
	0.4	_	_	_	_	_ __ _ _	1.5948172	1.5925470	1.5925323
	0.5	1.58658	1.5890	1.5867	1.5864	_ __ _ _	1.5890234	1.5868044	1.5867901
	0.6							1.5819313	1.5819172
	1	_	_	_	_	1.567891	1.5701311	1.5680745	1.5680612
	1.5	1.55751	1.5596	1.5576	1.5574	1.557539	1.5596844	1.5577146	1.5577019
	2	_	_	_	_	1.550951	1.5530377	1.5511221	1.5511098
	3	1.54271	1.5450	1.5431	1.5429	1.543034	1.5450510	1.5431998	1.5431878
	4					_ __ _ _	1.5404174	1.5386030	1.5385913
	5	_	_	_	_	_ __ _ _	1.5373902	1.5355998	1.5355882
	6					_ __ _ _	1.5352570	1.5334834	1.5334720
	8					_ __ _ _	1.5324494	1.5306978	1.5306865
	10	1.52877	1.5306	1.5288	1.5286	1.528791	1.5306839	1.5289461	1.5289349

In order to validate the numerical results, we first compare the present result with the results obtained by (Mabood et al., 2015), (Dodda et al., 2016) and (Chaltu et al., 2017) in the above Table without the presence of magnetic field parameter, Slip parameter, permeability, Brownian motion, Thermophoresis number, chemical reaction parameter, Eckert number, thermal radiation parameter and Heat source/sink parameter. The values of $-\theta'(0)$from the table are found to be in a good agreement.

In this study, the system of nonlinear higher order ordinary differential equations (10) – equation (12) subject to boundary condition equation (13) were solved numerically employing Keller box method. By taking the step size $\Delta\eta = 0{:}\,01$ in η and within the interval [0; η_∞]. The effect of different parameters like power index (n), Slip parameter, Magnetic field parameter (M), Permeability (Kp), Eckert number (Ec), Heat source/sink parameter (Q), Thermophoresis (Nt), Lewis number (Le), Chemical reaction parameter (R_c), Prandtl number(Pr), Thermal radiation parameter (Nr) on velocity, temperature and concentration profiles, skin friction coefficient, surface heat transfer rate and mass transfer rate have been analyzed graphically.

For instance, Figure 4.3 – 4.6 shows the effect of power index, magnetic field parameter, Slip parameter and Permeability on velocity profile respectively, while the other parameters are constant.

Figure 4.3 indicates that the effect of power index on velocity profile. As we have seen from the figure, decreasing velocity profile due to increment of power index. From Figure 4.4, it is observed that effect of magnetic parameter on the velocity of the fluid. It is found that increasing of magnetic parameter decreases the velocity profile. The momentum boundary layer thickness is decreases with increasing the values of magnetic parameter. As the result of this aspect is that application of magnetic field to an electrically conducting fluid gives to a resistive type force called the Lorentz force which opposes the fluid motion. Figure 4.5illustratesthe effect of Permeability parameter on velocity profile. This shows that velocity profile decrease for higher values of the Permeability parameter. That is the boundary layer thickness decreases for large values of porosity parameter. Figure 4.6describes that the effect of velocity slip parameter (γ)on the velocity. As velocity slip parameter (γ) increases then there is a decrement in velocity profile. It is observed that slip velocity is increases and consequently fluid velocity diminished because of the slip condition at the boundary. The pulling of the stretching sheet can only partly be transmitted to the fluid. It is found that velocity slip (γ) has a substantial effect on the solutions.Figure 7 - 11 shows the effect of Prandtl number, Eckert number, Heat source/sink parameter, Thermophoresis parameter, and thermal radiation on temperature profile respectively, and the other parameters are constant. Figure 4.7 indicates that increase Prandntl number decrease temperature. Figure 4.8reveals that enhancement of Eckert number rises of fluids temperature. From Figure.4.9 indicates the presence of a heat source or a heat generation effect, which shows the thermal state of the fluid increase causing the thermal boundary layer to increase. So, rise temperature of the nanofluid due to increase a heat source or a heat generation. In Figure4.10 and Figure 4.11, it is observed that increasing thermophoresis parameter and thermal radiation rises temperature of the fluid. Effects of Lewis number and Chemical reaction parameter on concentration of thenanoparticles are displayed on Figure 4.12 and 4.14 respectively. From both Figures it is indicated that increasing each of the parameters reduces concentration of the nanoparticles. Figure 13 shows Brownian motion parameter on concentration of the nanoparticles. As the Brownian motion parameter of the fluid increases, it leads to a decreases in the concentration inside the boundary layer.

Fig. 4.15 shows that the effect of slip parameter and power index on skin coefficient friction. As we observed from the figure increasing in the Slip parameter and power index, it will be increasing the skin friction coefficient. Figure 4.16 describe the effect of magnetic parameter and thermal radiation on Nusselt number. The Figure tells us that the surface heat transfer rate decreasing with increasing both magnetic parameter and thermal radiation. The rate of heat transfer reduces with higher values of M and Nr. Figure 17 describe that the effect of Permeability and Lewis number on Sherwood number, surface mass transfer rate is higher, in the case of increasing Permeability and Lewis number on the concentration nanoparticles.

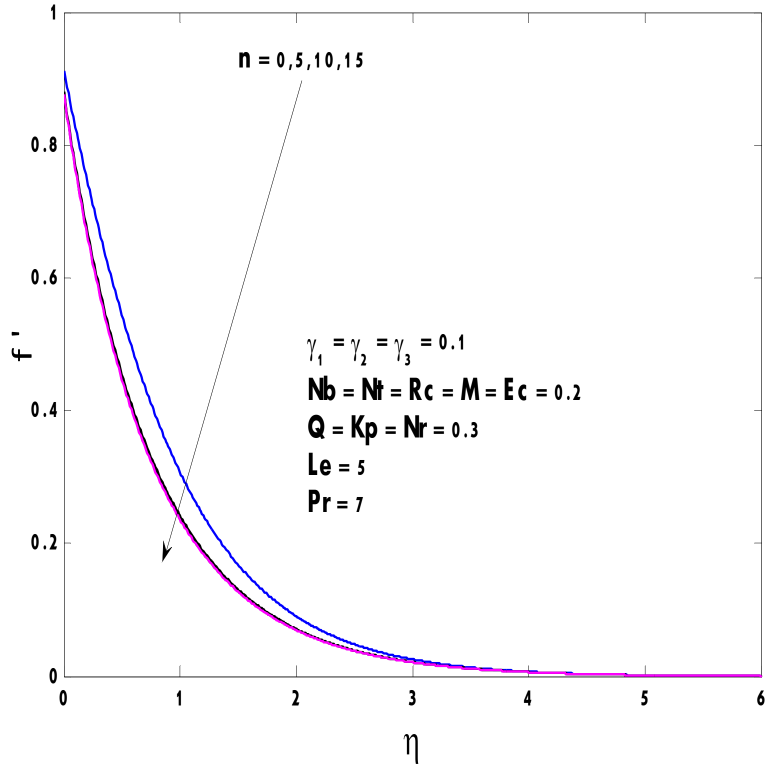

Figure 4.3: Effect of power index (n) on velocity profile

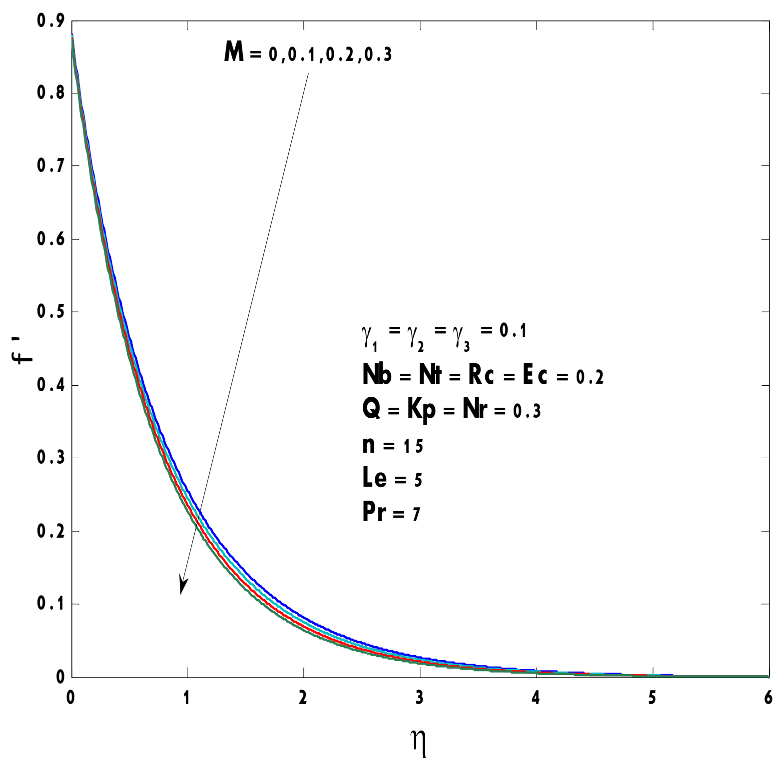

Figure 4.4: Effect of Magnetic parameter (M) on velocity profile

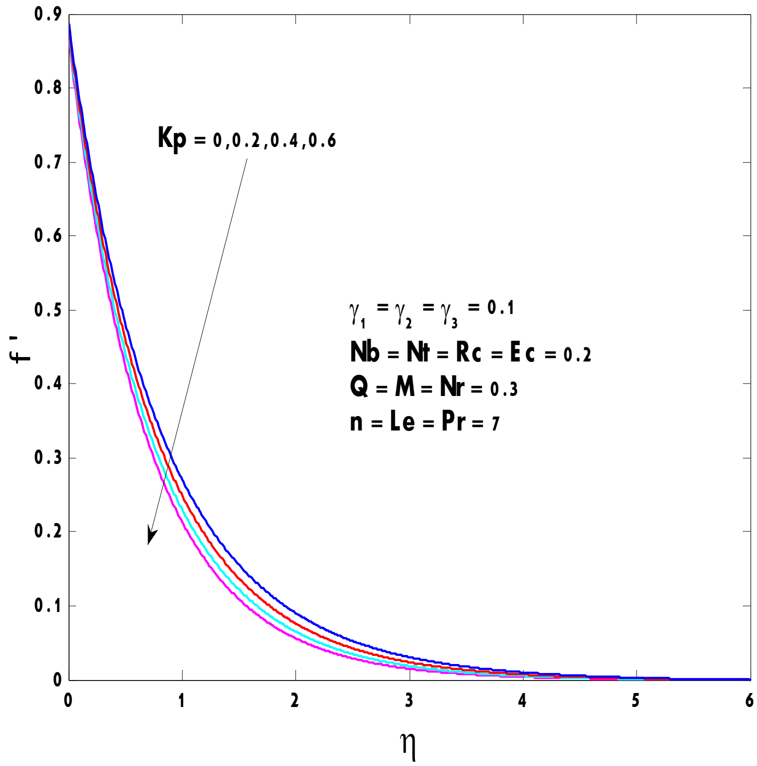

Figure 4.5: Effect of permeability parameter (Kp) on velocity profile

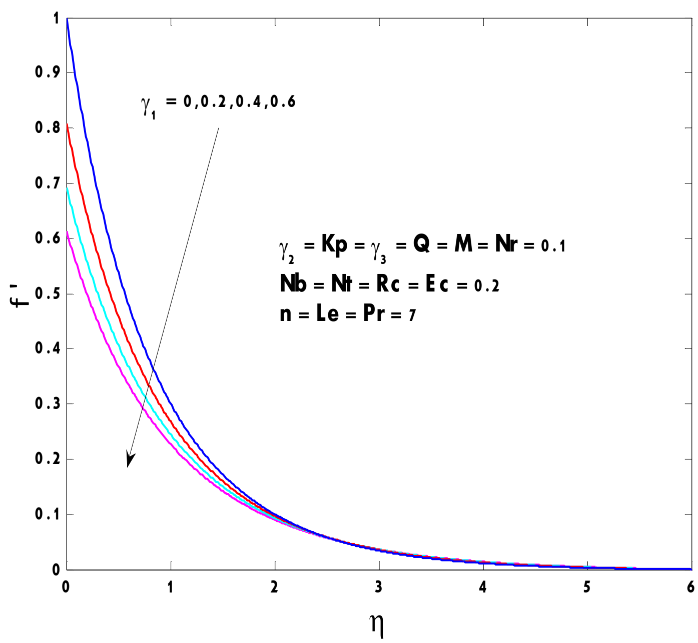

Figure 4.6: Effect of Slip parameter (γ) on velocity profile

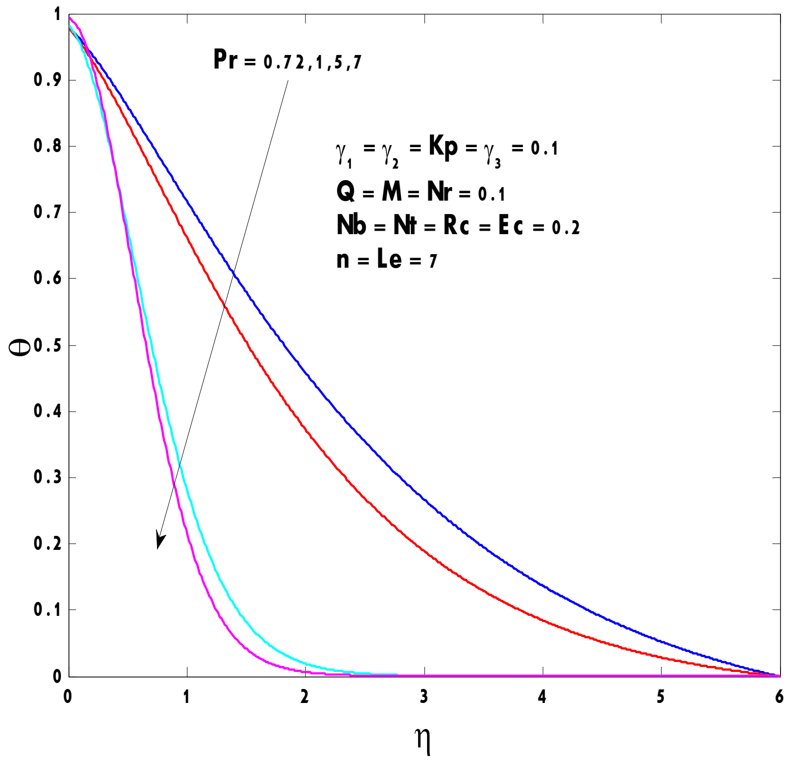

Figure 4.7: Effect of Prandtl number (Pr) on temperature profile

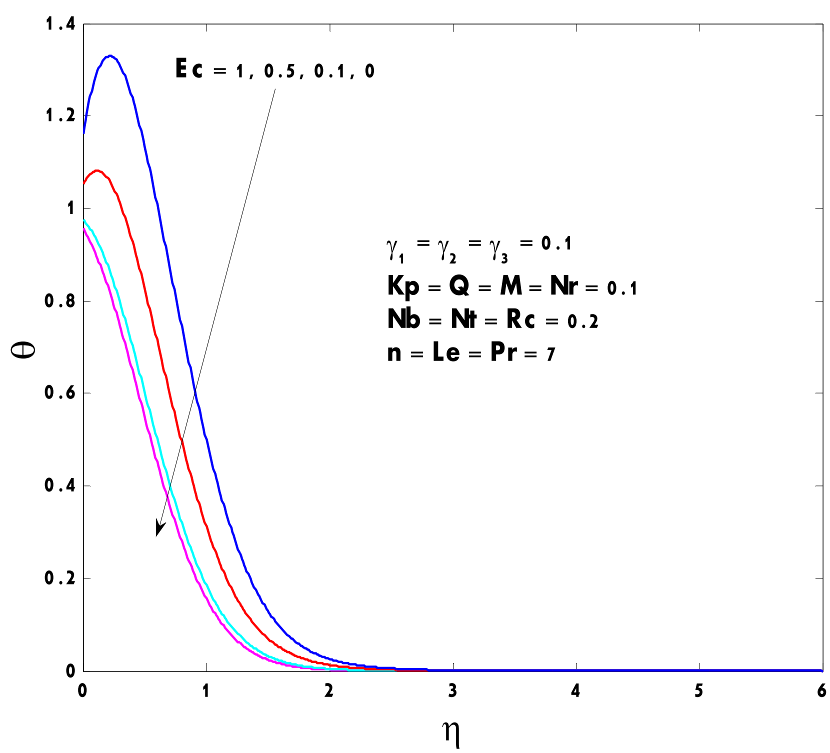

Figure 4.8: Effect of Eckert number (Ec) on temperature profile

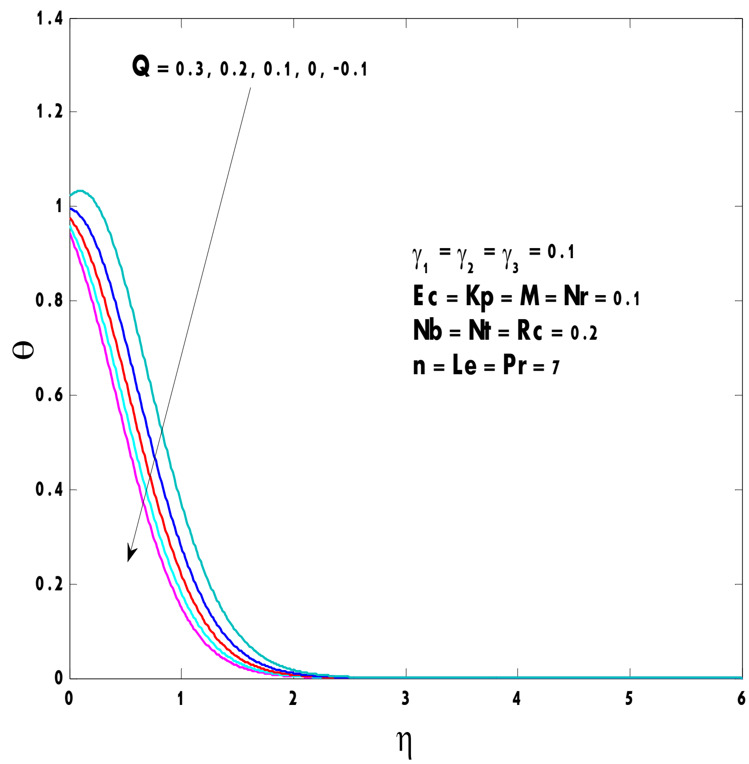

Figure 4.9: Effect of Heat source-sink parameter (Q) on temperature profile

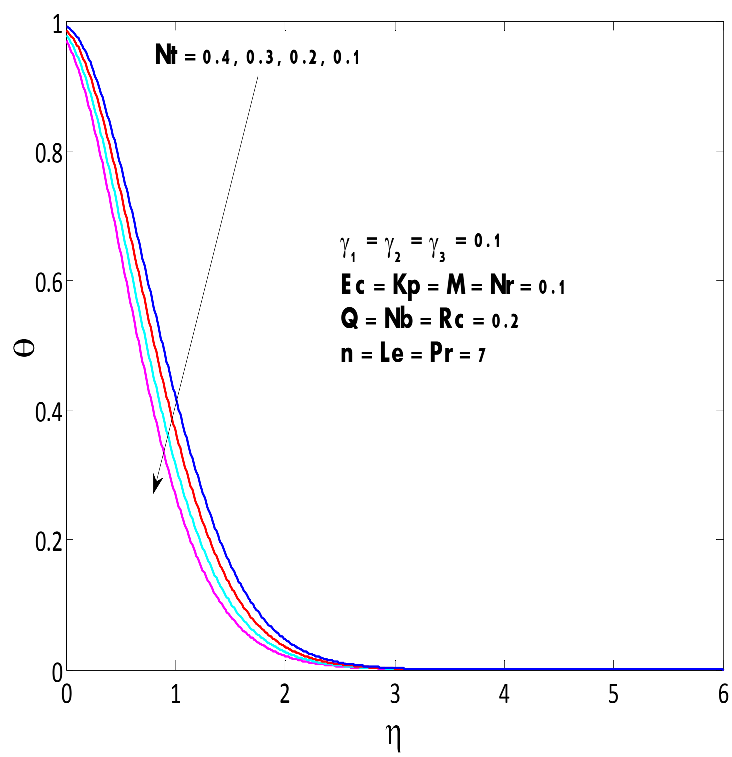

Figure 4.10: Effect of thermophoresis parameter (Nt) on temperature profile

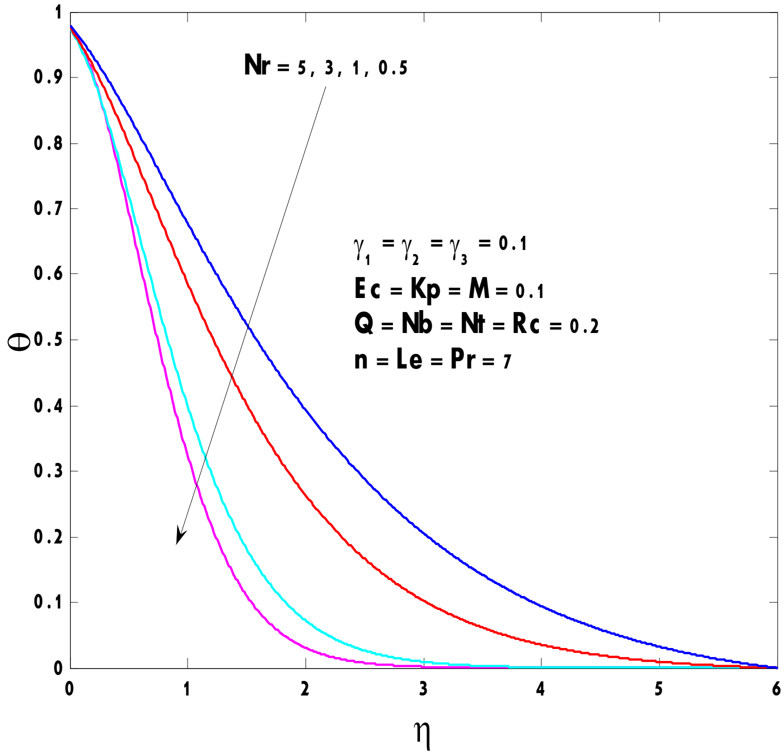

Figure 4.11: Effect of thermal radiation parameter (Nr) on temperature profile

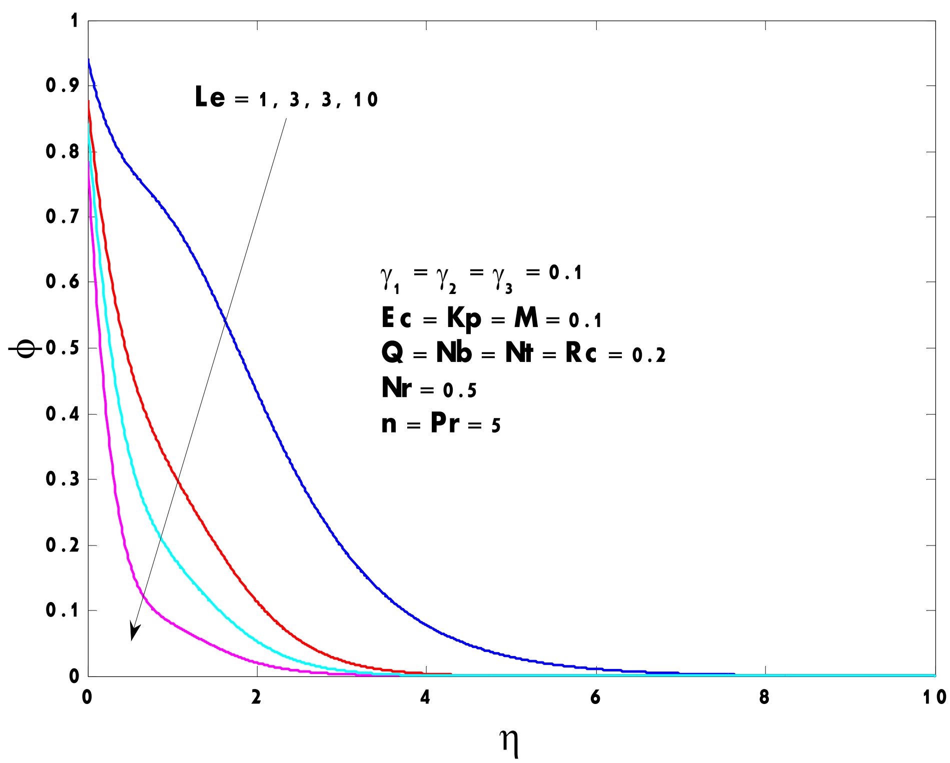

Figure 4.12: Effect of Lewis number (Le) on concentration profile

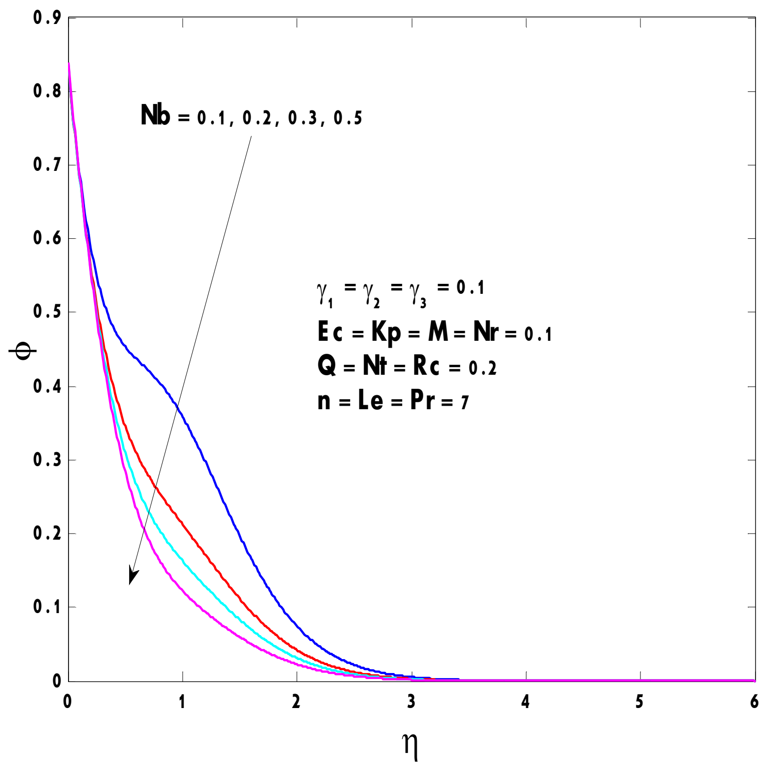

Figure 4.13: Effect of Brownian motion parameter (Nb) on concentration profile.

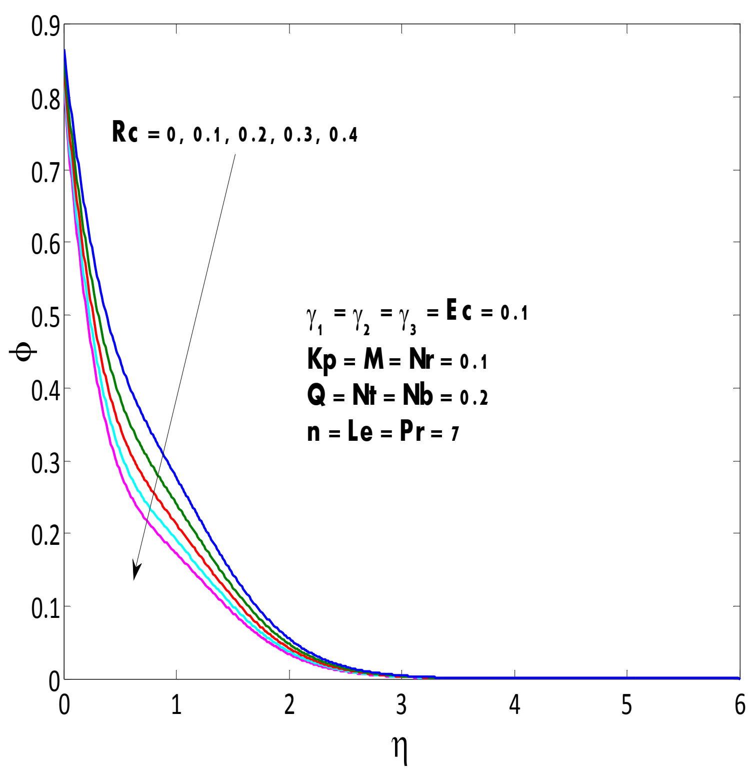

Figure 4.14: Effect of Chemical reaction (R_c) on concentration profile.

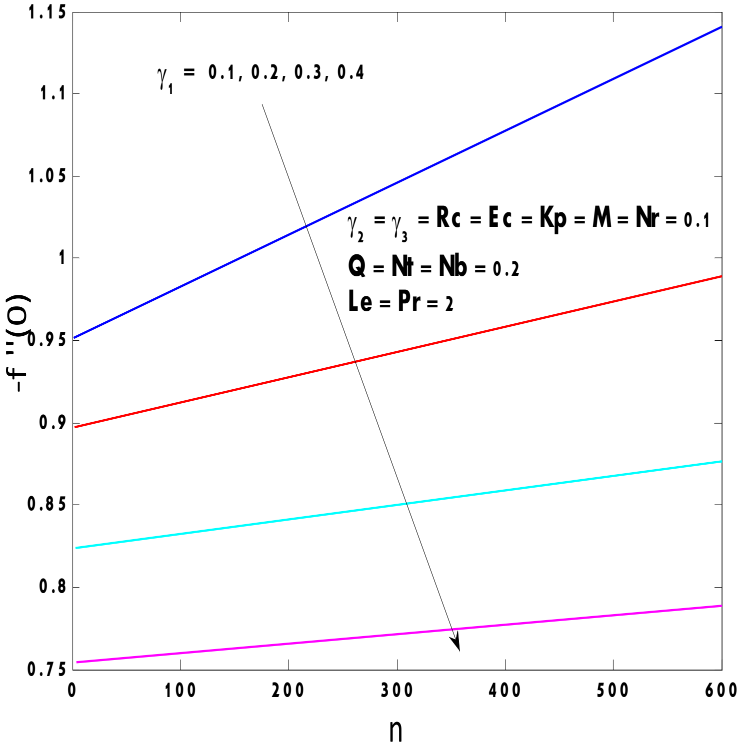

Figure 4.15: Effect of Slip parameter and index power on skin friction coefficient.

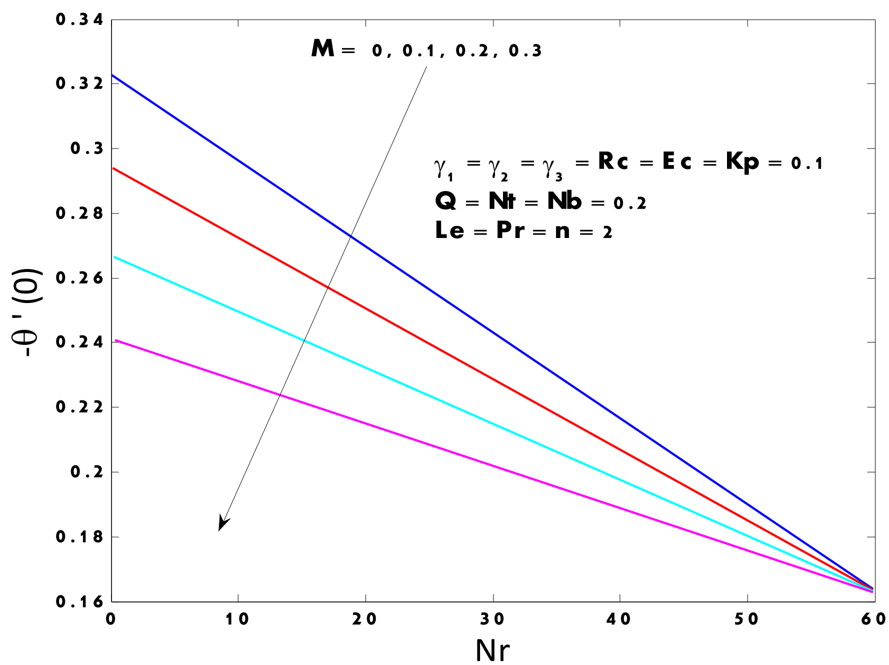

Figure 4.16: Effect of magnetic parameter (M) and thermal radiation parameter (Nr)on heat transfer rate

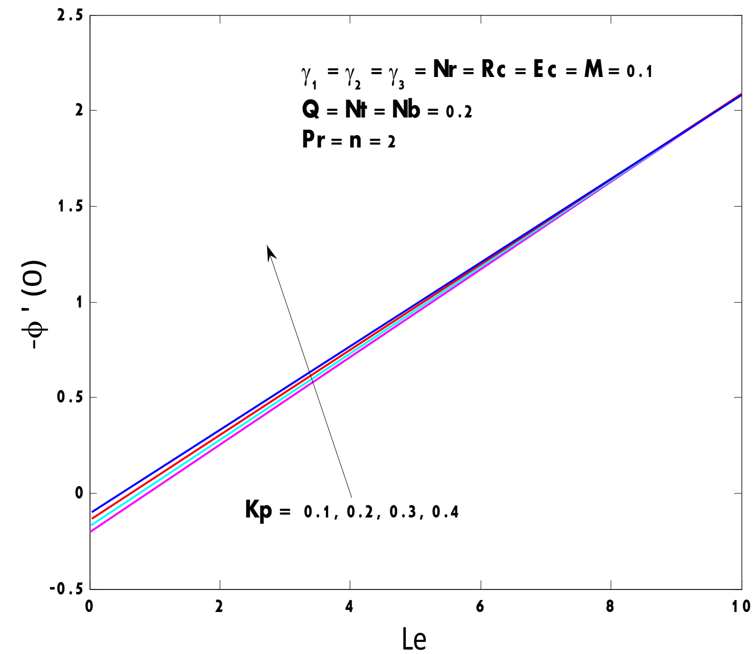

Figure 4.17: Effect of permeability parameter (K_p) and Lewis number (Le) on mass transfer rate

Chapter 5

Conclusion and Scope for the future work

5.1 Conclusion

In this study, Analysis of Magneto Hydrodynamics (MHD) Boundary Layer Nano fluid flow of Heat Transfer over a Nonlinear Stretching Sheet Presence of Thermal Radiation, Chemical Reaction and Partial Slip with Boundary condition was considered and the governing nonlinear partial differential equation was transformed into higher order nonlinear ordinary differential equation and solved numerically by Keller box method.

The velocity, temperature, and concentration profiles along a nonlinear stretching sheet presence of thermal radiation, chemical reaction and partial slip were studied and the results were shown graphically. The skin-friction coefficient, the rate of heat transfer and the rate of mass transfer were analyzed.

From this study, we found that:

1. Increasing the power index (n), magnetic parameter (M), permeability parameter (Kp) and Slip parameter (γ) reduces velocity of theNano fluid.
2. Increase Pradntl number (Pr) reduce temperature of the Nano fluid.
3. Enhancement of Eckert number (Ec), heat source/sink parameter (Q), Brownian motion parameter (Nb) thermophoresis parameter (Nt) and thermal radiation (Nr) rise temperature of the Nano fluid.
4. Increasing Lewis number (Le), Brownian motion parameter (Nb) and chemical reaction parameter (Rc) reduces concentration of the nanoparticles.
5. Increasing slip parameter (γ) and index power (n) increase the skin friction coefficient.
6. Rising of thermal radiation (Nr) and magnetic parameter (M) reduce surface heat transfer rate.
7. Increasing Lewis number (Le) and permeability parameter (Kp) rises themass transfer rate.

5.2 Scope for the future work

In this thesis, Analysis of Magneto Hydrodynamics (MHD) Boundary Layer Nano fluid flow of Heat Transfer over a Nonlinear Stretching Sheet Presence of Thermal Radiation, Chemical Reaction and Partial Slip with Boundary condition by employing the implicit finite difference scheme called Keller Box method. So, one can find the solution for the problem discussed above by consider unsteady, turbulent flow, two dimensional MHD boundary layer flow of a viscous, incompressible, electrically conducting fluid over a nonlinear stretching sheet presence of thermal radiation and partial slip with permeability Kpin the presence of non-uniform transverse magnetic field.

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