HYDRODYNAMICS INSTABILITY AROUND COMPACT OBJECT IN SDS BACKGROUND

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To my mother who was faithful and praying for my success.

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Abstract

The success of general theory of relativity (GR) over a wide range of astrophysical observations is the manifestation of progress in astronomy and astrophysical studies. Whilst, there are encouraging past success of GR and future hopes there is an outstanding debates on GR field equations dated back to their origin, where Einstein himself was also puzzled in the solutions of the equations. As the consequence, Einstein did introduce a positive cosmological constant to his original field equations. Since then, the cosmological constant has remained with debate where it is being cast out at a time and reintroduced at other time. But a firm considers of is triggered in the 1960's when an excess quasi-stellar objects (QSO's) near the redshift $Z \cong 1.95$ were observed. Moreover, the recent discovery of expanding universe at an accelerated rate favors a flat low density Cold Dark Matter with dark energy in the form of cosmological constant $\Lambda - CDM$ model is more or less consistent with all the current cosmological observations. But the general perception, owing to its tiny value, questions its significance on a local gravitational phenomenon. But, a local effect of cosmological constant is claimed to be observable from relativistic accretion phenomena around massive BHs which involve distancescales. So the effect of on the dynamics of objects including jets around massive objects at kiloparsecs or more astrophysical distances need investigations. So far all the works on the effect of Λ on accreting systems were carried out under some restricted conditions owing to the complex and nonlinear character of the equations in GR; if not most were under Newtonian. Motivated by this short scientific rationale, we studied the effect of cosmological constant on dynamical systems including magnetohydrodynamic (MHD) instabilities around massive objects like BHs in the current standard $\Lambda - CDM$ model where the Schwarzschild de Sitter (SdS) background is being considered then the interior and exterior solutions of Einsteins equations with a non-zero cosmological constant are given for static and spherically symmetric configurations of uniform density. The potential and pressure are determined for both positive and negative values of the cosmological constant. Limits on the outer radius of the interior solutions are established.The potential around a compact object between the horizons considering mean density of universe is discussed.

Key words: GR, cosmological constant, SdS.

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Introduction

The success of theory of general relativity (GR) in the observation of deflection of light [?], radar echo delay [?], precession of planetary motion [?] and gravitational redshift [?] by gravity are the manifestation of progress in astronomy and astrophysical studies. The discovery of the expanding universe at an accelerating phase [?, ?] and the direct confirmation of gravitational wave detection are other outstanding progresses in astronomy and astrophysics. Nowadays, the end products of stellar evolution called compact objects (White Dwarf (WD), Neutron Star (NS) and Black Hole (BH)) act as laboratory for the Theory of General Relativity tests over a wide range including origins and future determinations. These objects provide important information about the age of astrophysical objects; constrain models of galactic and cosmological evolutionary history from small scale to large scale structure. Currently, the development of astronomy has led an expansion of human knowledge reaching out, ever farther from our home where the observational tools were solely dependent on the information carried by electromagnetic waves (EMWs). But, due to EMWs interaction with matter there are limitations where these waves unable to penetrate a great deal of objects including compact objects. However, the transparency of media to GWs is hoped as a laboratory for general relativity and a window to energetic astrophysical phenomena. The information that carried by gravitational waves is very different from that carried by electromagnetic waves. While, EM waves permit to imaging objects, GW observations do not generally allow imaging, instead used to extract information from waveforms proceeds with audiolike methods such as timefrequency analysis [?].

The recent observations of high redshift Type Ia supernovae [?],[?] and temperature fluctuations of the cosmic microwave background [?] that presently the universe is expanding at an accelerating phase with approximately flat geometry. As reported by these observational data analysis group a flat low density Cold Dark Matter with dark energy in the form of cosmological constant ($\Lambda + CDM$) universe with $\Omega_m = 0.3$ and $\Omega_\Lambda = 0.7$, with an approximately flat metric is favored over a wide range of observational data ranging from large and intermediate angle Cosmic Microwave Background Radiation (CMBR) anisotropies to observations of galaxy clustering on large scales. On the theoretical ground the component, that represents approximately 70 *percent* of the total matter energy content of the universe today is characterized by a negative pressure, and should be responsible for the expansion of the universe.

In the presence of a repulsive cosmological constant (positive) $'+'$ the spacetime geometry exterior to a static spherically symmetric metric is Schwarzschild-de Sitter (SdS), which describes an isolated black hole (BH) in a spatially inflated Universe, rather than Schwarzschild metric. Therefore, the cosmological constant may affect any local gravitational phenomenon like perihelion shift of the orbits of gravitationally bound systems [**?**].

The $\Lambda - CDM$ model is more or less consistent with all the current cosmological observations [**?**] though the origin of cosmological constant still remains elusive. Therefore, the cosmological constant may affect any local gravitational phenomenon like perihelion shift of the orbits of gravitationally bound systems [**?**], gravitational bending of light [**?**] , geodesic precession [**?**], but the general perception is that owing to its tiny value, cosmological constant does not lead to any significant observable effects in a local gravitational phenomenon. However, the contribution of repulsive Λ could be significant (larger than the second order term) even in a local gravitational phenomenon when kiloparsecs to megaparsecs-scale distances are involved, such as the gravitational bending of light by cluster of galaxies [**?**]. Moreover, the recently confirmed gravitational wave presence shines on the matter to study high precision astrophysical phenomena at small scale level. Probably, a local effect of cosmological constant is claimed to be observable from relativistic accretion phenomena around massive BHs which involve distance-scale of the order of hundreds of parsecs or even more

[?] and the references therein. However, a few studies have been carried out so far to investigate the effect of Λ in astrophysical jet/accretion flow paradigm [?],[?]. So the effect of Λ on the dynamics of kiloparsecs to megaparsecsscale astrophysical objects including jets need investigations. The work of [?] showed that Λ has considerable strong collimation effect on jets on the dynamical stability of accretion with constant angular momentum around a Sds BH.So due to this facts we are studied hydrodynamic instability around compact object in Sds background.We Provide preliminary boundary conditions to derive the relevant set of dynamical equations from the GR equations in the SdS background also Study and examine the effects of the relevant parameters derived from the equations and Numerically generate some theoretical data from the formalism using computation.

The paper is organized as follows: in the first section the Einstein equations and the notation are introduced.In Section 2, Einsteins equations with a non-zero cosmological constant and the conservation law of energy-momentum tensor are used in the case of spherically symmetric spacetimes to give the equations of structure of spherically symmetric and static configurations representing relativistic stars and the equations of structure are explicitly integrated for the configurations of uniform density, and the pressure and potential inside of these configurations are given. In section 3 using the pressure the exterior and interior solution with non-zero cosmological constant is determined and discussed.At the end in section 4 summery and conclusion given.

Chapter 1

Gravitation and General Theory Of Relativity

1.1 Introduction to Einstein General Relativity

After many years of development Einstein presented his general theory of relativity in 1915, it was then published the following year in [?]. General relativity is an extension of special relativity which includes a modification of Newtons law of gravity. It provides a relativistic description of the gravitational field exerted by a massive object and its effects on the geometric structure of the surrounding spacetime. The theory states that the gravitational interaction due to the presence of matter causes spacetime to curve hence distorting the path of a nearby object. This differs from the original foundations of Newtons laws of gravitation, where gravity is an attractive force between two massive objects which interacts instantaneously. In this description, planetary orbits are a consequence of this gravitational pull emanating from the sun, therefore in this theory the suns gravitational field interacts directly with the planet as opposed to the surrounding spacetime. However given certain circumstances Newtonian theory provides an accurate description of the gravitational interaction, this includes a weaker gravitational field. This is known as the Newtonian limit in which spacetime is asymptotically flat and the field equations can be approximated with Newtons laws of motion. General relativity is required for a more significant gravitational field, when Newtonian gravity no longer agrees with observation. For instance, the observation of the precession of the perihelion of Mercury deviated slightly from the predictions of Newtons equations, whereas solutions in general relativity describe this orbit correctly

1.2 Space-time Geometry Of Gravitation

In general relativity, spacetime has the structure of a four-dimensional pseudo -Riemannian manifold $\mathcal{M}$, this is equipped with a metric $g_{\mu\nu}$ which can be used to determine local geometric quantities such as angles and lengths. The metric associated with a pseudo-Riemannian manifold is not positive definite, therefore it will have signature $(1,3)$ or $(3,1)$, for the purposes of this thesis I will consider a metric with signature $(-,+,+,+)$ for results in general relativity unless otherwise stated. The metric $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$ are symmetric so that $g_{\mu\nu} = g_{\nu\mu}$ and $g^{\mu\nu} = g^{\nu\mu}$, where $g^{\mu\sigma} g_{\nu\sigma} = \delta^{\mu}_{\nu}$. The line element

$$ds^2 = g_{\mu\nu} dx^{\mu} dx^{\nu} \tag{1.2.1}$$

is invariant under arbitrary invertible transformations known as diffeomorphisms.

1.3 Tensor In General Relativity

Principle of General Covariance by virtue of the Einstein Equivalence Principle, a physical equation holds in an arbitrary gravitational field if

1. the equation holds in the absence of gravity, i.e. when $g_{\mu\nu} = \eta_{\mu\nu}$ $,\Gamma^{\mu}_{\nu\lambda} = 0$
2. the equation is generally covariant, i.e. preserves its form under a general coordinate transformation

In order to construct generally covariant equations, we need objects that transform in a simple way under coordinate transformations. The prime examples of such objects are **tensors** .[?]

1.3.1 Metric tensor and Affine Connection

Affine connection is the field that determines the gravitational force and used as to represent the gravitational field. It also called as the Christoffel second symbol which denoted as $\{\mu\nu, \lambda\}$ or $\{{\lambda \atop \mu\nu}\}$ or $\Gamma^{\lambda}_{\mu\nu}$ .the metric tensor is used to determine the proper time interval between two events with a given infinitesimal coordinate separation and also the gravitational potential.Its derivative helps to determine the field $\Gamma^{\lambda}_{\mu\nu}$ as well as denoted as $g_{\mu\nu}$. The mathematical definition of $g_{\mu\nu}$ and $\Gamma^{\lambda}_{\mu\nu}$ as,

$$g_{\mu\nu} \equiv \frac{\partial\xi^{\alpha}}{\partial x^{\mu}}\frac{\partial\xi^{\beta}}{\partial x^{\nu}}\eta^{\alpha\beta}$$

$$\Gamma^{\lambda}_{\mu\nu} \equiv \frac{\partial x^{\lambda}}{\partial\xi^{\alpha}}\frac{\partial^2\xi^{\alpha}}{\partial x^{\mu}\partial x^{\nu}} \tag{1.3.1}$$

where ξ^{α} and ξ^{β} are local coordinates. The infinitesimal line element and the motion of particle in a gravitational field can be written as,

$$d\tau^2 = g_{\mu\nu}dx^{\mu}dx^{\nu},$$

$$\frac{d^2x^{\lambda}}{d\tau^2} + \Gamma^{\lambda}_{\mu\nu}\frac{dx^{\mu}}{d\tau}\frac{dx^{\nu}}{d\tau} = 0 \tag{1.3.2}$$

Now differentiating the metric tensor in a gravitational field with respect to the general coordinate system x^{λ}

$$\frac{\partial g_{\mu\nu}}{\partial x^{\lambda}} = \frac{\partial}{\partial x^{\lambda}}\left(\frac{\partial\xi^{\alpha}}{\partial x^{\mu}}\frac{\partial\xi^{\beta}}{\partial x\nu}\eta^{\alpha\beta}\right)$$

$$\frac{\partial g_{\mu\nu}}{\partial x^{\lambda}} = \frac{\partial^2\xi^{\alpha}}{\partial x^{\lambda}\partial x^{\mu}}\frac{\partial\xi^{\alpha}}{\partial x^{\mu}}\eta_{\alpha\beta} + \frac{\partial\xi^{\alpha}}{\partial x^{\mu}}\frac{\partial^2\xi^{\beta}}{\partial x^{\mu}\partial x^{\nu}}\eta_{\alpha\beta} \tag{1.3.3}$$

Equation(1.3.3) can be written as,

$$\frac{\partial g_{\mu\nu}}{\partial x^{\lambda}} = \Gamma^{\rho}_{\lambda\mu}\frac{\partial\xi^{\alpha}}{\partial x^{\rho}}\frac{\partial\xi^{\beta}}{\partial x^{\nu}}\eta_{\alpha\beta} + \Gamma^{\rho}_{\lambda\nu}\frac{\partial\xi^{\alpha}}{\partial x^{\mu}}\frac{\partial\xi^{\beta}}{\partial x^{\rho}}\eta_{\alpha\beta} \tag{1.3.4}$$

where

$$\Gamma^{\rho}_{\lambda\mu} = \frac{\partial x^{\rho}}{\partial\xi^{\alpha}}\frac{\partial^2\xi^{\alpha}}{\partial x^{\lambda}\partial x^{\mu}}$$

$$\Gamma^{\rho}_{\lambda\nu} = \frac{\partial x^{\rho}}{\partial\xi^{\alpha}}\frac{\partial^2\xi^{\alpha}}{\partial x^{\lambda}\partial x^{\nu}}$$

Equation(1.3.4)can be written as,

$$\frac{\partial g_{\mu\nu}}{\partial x^{\lambda}} = \Gamma^{\rho}_{\lambda\mu} g_{\rho\nu} + \Gamma^{\rho}_{\lambda\nu} g_{\rho\mu} \tag{1.3.5}$$

where,

$$g_{\rho\nu} = \frac{\partial \xi^{\alpha}}{\partial x^{\rho}} \frac{\partial \xi^{\beta}}{\partial x^{\nu}} \eta_{\alpha\beta}$$

$$g_{\rho\mu} = \frac{\partial \xi^{\alpha}}{\partial x^{\mu}} \frac{\partial \xi^{\beta}}{\partial x^{\rho}} \eta_{\alpha\beta}$$

The two $\Gamma^{\rho}_{\lambda\mu}$ and $\Gamma^{\rho}_{\lambda\nu}$ are the affine connections. If we considering freely falling particle affine connection is field that determine the gravitational force. Now using the symmetry property of affine connection with the exchange of lower indices ,i.e $\Gamma^{\rho}_{\lambda\mu} = \Gamma^{\rho}_{\mu\lambda}$. To solve for the affine connection ,it is a matter of adding to equation (**??**) the same equation with $\mu$and λ inter changing and subtract the same equation with ν and λ interchange.It shows,

$$\frac{\partial g_{\mu\nu}}{\partial x^{\lambda}} + \frac{\partial g_{\lambda\nu}}{\partial x^{\mu}} - \frac{\partial g_{\mu\lambda}}{\partial x^{\nu}} = \Gamma^{\rho}_{\lambda\mu} g_{\rho\nu} + \Gamma^{\rho}_{\lambda\nu} g_{\rho\mu} + \Gamma^{\rho}_{\mu\lambda} g_{\rho\nu} + \Gamma^{\rho}_{\mu\nu} g_{\rho\lambda} - \Gamma^{\rho}_{\nu\mu} g_{\rho\lambda} - \Gamma^{\rho}_{\nu\lambda} g_{\rho\mu} \tag{1.3.6}$$

From the symmetry property of affine connection,$\Gamma^{\rho}_{\mu\nu}$ and the metric tensor,$g_{\mu\nu}$,then

$$\frac{\partial g_{\mu\nu}}{\partial x^{\lambda}} + \frac{\partial g_{\lambda\nu}}{\partial x^{\nu}} - \frac{\partial g_{\mu\lambda}}{\partial x^{\nu}} = 2\Gamma^{\rho}_{\lambda\mu} g_{\rho\nu} \tag{1.3.7}$$

Now let us define metric $g^{\nu}\sigma$ as the inverse of $g_{\nu}\sigma$.

$$g^{\nu\sigma} g_{\rho\nu} = \delta^{\sigma}_{\rho}$$

where, δ^{σ}_{ρ} is the kronecker delta define as $\delta^{\sigma}_{\rho} = 1$ for $\sigma = \rho$ and zero for else. Therefore, applying $(\sigma = \rho)$ to kronecker delta,thus

$$\Gamma^{\sigma}_{\lambda\mu} = \frac{1}{2} g^{\nu\sigma} \left(\frac{\partial g_{\mu\nu}}{\partial x^{\lambda}} + \frac{\partial g_{\lambda\nu}}{\partial x^{\mu}} - \frac{\partial g_{\mu\lambda}}{\partial x^{\nu}} \right) \tag{1.3.8}$$

Equation (1.3.8) is the relation developed between the metric tensor and affine connection in a gravitational field. Here both of them represent the presence of gravitational effect. Consider the case where a particle is moving slowly in a weak stationary gravitational field.

For sufficiently slow motion of a particle,the equation of motion of a particle can be written as,

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{00}\left(\frac{dt}{d\tau}\right) = 0 \tag{1.3.9}$$

This is from the equation of motion of the particle in a gravitational field which can be given by,

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\lambda\nu}\frac{dx^\lambda}{d\tau}\frac{dx^\nu}{d\tau} = 0 \tag{1.3.10}$$

For which $\lambda = \nu = 0$ and $dx^0 = dt$.Recall the relation given by, (**??**)

$$\Gamma^\sigma_{\lambda\mu} = \frac{1}{2}g^{\nu\sigma}\left(\frac{\partial g_{\mu\nu}}{\partial x^\lambda} + \frac{\partial g_{\lambda\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\lambda}}{\partial x^\nu}\right)$$

Since the field is stationary,all time derivatives of $g_{\mu\nu}$ vanish,so that

$$\Gamma^\lambda_{00} = -\frac{1}{2}g^{\lambda\nu}\frac{\partial g_{00}}{\partial x^\nu} \tag{1.3.11}$$

For a weak static field produced by non-relativistic mass density ρ,

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$$

where, $\parallel h_{\alpha\beta} \parallel << 1$ and $\eta_{\alpha\beta}$ is is minkowski metric tensor. For $\alpha = \beta = 0$ and applying the relation $\eta_{00} = 1$.

$$g_{00} = 1 + h_{00} \tag{1.3.12}$$

Therefore we have,

$$\Gamma^\alpha_{00} = -\frac{1}{2}\eta^{\alpha\beta}\frac{\partial h_{00}}{\partial x^\beta} \tag{1.3.13}$$

Now the equation of motion has take the form of,

$$\frac{d^2x^\mu}{d\tau} = \frac{1}{2}\eta^{\alpha\beta}\left(\frac{\partial h_{00}}{\partial x^\beta}\right)\left(\frac{dt}{d\tau}\right)^2 \tag{1.3.14}$$

For $\alpha = \beta = 1, 2, 3$ the minkowski metric tensor,$\eta_{\alpha\beta} = \eta^{\alpha\beta}$,then the above equation can be written as,

$$\frac{d^2x}{d\tau^2} = -\frac{1}{2}[\frac{dt}{d\tau}]^2\nabla h_{00} \tag{1.3.15}$$

Where, $\frac{\partial h_{00}}{\partial x} = \nabla h_{00}$. Once rearranging the equation that gives,

$$\frac{d^2x}{dt^2} = -\frac{1}{2}\nabla h_{00} \tag{1.3.16}$$

Now the corresponding Newtonian result is,

$$\frac{d^2x}{d\tau^2} = -\nabla\phi \tag{1.3.17}$$

Where, ϕ is the Newtonian potential. The comparison of equations result,

$$\frac{1}{2}\nabla h_{00} = \nabla\phi$$

$$\nabla h_{00} = 2\nabla\phi$$

$$h_{00} = 2\phi + constant \tag{1.3.18}$$

Furthermore, the coordinates system must become Minkowskian at great distance so h_{00} vanish at infinity. Then if ϕ defined to vanish at infinity $\big($where $\phi = \frac{-GM}{r}$, r is the distance from the center of a spherical body of mass M$\big)$. By recall the relation for a weak static field given by,

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$$

Therefore

$$g_{00} = 1 + h_{00} \tag{1.3.19}$$

$$g_{00} = 1 + 2\phi \tag{1.3.20}$$

1.3.2 Curvature Tensor

If we use only $g_{\mu\nu}$ and its first derivatives , then no new tensor can be contracted , for at any point we can find a coordinate system in which the first derivative of the metric tensor vanish, so in this coordinate system the desired tensor must be equal to one of those that can constructed out of the metric tensor alone, and since this is an equality between tensors alone, and since this is an equality between tensors it must be true in all coordinate systems[**?**].This simplest possibility is to construct a tensor out of the metric tensor and its first and second derivatives.To do this it is possible to write the transformation rule of affine connection as,

$$\Gamma'^{\lambda}_{\mu\nu} = \frac{\partial x'^{\lambda}}{\partial \xi^{\alpha}} \frac{\partial^2 \xi^{\alpha}}{\partial x'^{\mu} \partial x'^{\nu}}$$

or it can be written as,

$$\Gamma'^{\lambda}_{\mu\nu} = \frac{\partial x'^{\lambda}}{\partial x^{\rho}} \frac{\partial x^{\rho}}{\partial \xi^{\alpha}} \frac{\partial}{\partial x'^{\mu}} \left(\frac{\partial \xi^{\alpha}}{\partial x^{\sigma}} \frac{\partial x^{\sigma}}{\partial x'^{\nu}} \right) \tag{1.3.21}$$

but,

$$\frac{\partial}{\partial x'^{\mu}} \left(\frac{\partial \xi^{\alpha}}{\partial x^{\sigma}} \frac{\partial x^{\sigma}}{\partial x'^{\nu}} \right) = \frac{\partial \xi^{\alpha}}{\partial x^{\sigma}} \left(\frac{\partial^2 x^{\sigma}}{\partial x'^{\mu} \partial x'^{\nu}} \right) + \frac{\partial x^{\sigma}}{\partial x'^{\nu}} \left(\frac{\partial^2 \xi^{\alpha}}{\partial x^{\tau} \partial x^{\sigma}} \frac{\partial x^{\tau}}{\partial x'^{\mu}} \right)$$

Therefore the transformation of affine connection becomes,

$$\begin{aligned} \Gamma'^{\lambda}_{\mu\nu} &= \frac{\partial x'^{\lambda}}{\partial x^{\rho}} \frac{\partial x^{\rho}}{\partial \xi^{\alpha}} \left(\frac{\partial \xi^{\alpha}}{\partial x^{\sigma}} \frac{\partial^2 x^{\sigma}}{\partial x'^{\mu} \partial x'^{\nu}} + \frac{\partial x^{\sigma}}{\partial x'^{\nu}} \frac{\partial^2 \xi^{\alpha}}{\partial x^{\tau} \partial x^{\sigma}} \frac{\partial x^{\tau}}{\partial x'^{\mu}} \right) \\ &= \frac{\partial x'^{\lambda}}{\partial x^{\rho}} \frac{\partial x^{\tau}}{\partial x'^{\mu}} \frac{\partial x^{\sigma}}{\partial x'^{\nu}} \left(\frac{\partial x^{\rho}}{\partial \xi^{\alpha}} \frac{\partial^2 \xi^{\alpha}}{\partial x^{\tau} \partial x^{\sigma}} \right) + \frac{\partial x'^{\lambda}}{\partial x^{\rho}} \frac{\partial x^{\rho}}{\partial \xi^{\alpha}} \frac{\partial \xi^{\alpha}}{\partial x^{\sigma}} \left(\frac{\partial^2 x^{\sigma}}{\partial x'^{\mu} \partial x'^{\nu}} \right) \end{aligned} \tag{1.3.22}$$

Using the relation given by affine connection and kronecker delta into equation (1.3.21) which are,

$$\Gamma^{\rho}_{\tau\sigma} = \frac{\partial x^{\rho}}{\partial \xi^{\alpha}} \frac{\partial^2 \xi^{\alpha}}{\partial x^{\tau} \partial x^{\sigma}}$$

$$\frac{\partial x^{\rho}}{\partial \xi^{\alpha}} \frac{\partial \xi^{\alpha}}{\partial x^{\sigma}} = \delta^{\rho}_{\sigma}$$

Where , $\delta^{\rho}_{\sigma} = 1$ for $\rho = \sigma$ else zero.

$$\Gamma'^{\lambda}_{\mu\nu} = \frac{\partial x'^{\lambda}}{\partial x^{\rho}} \frac{\partial x^{\tau}}{\partial x'^{\mu}} \frac{\partial x^{\sigma}}{\partial x'^{\nu}} \Gamma^{\rho}_{\tau\sigma} + \frac{\partial x'^{\lambda}}{\partial x^{\rho}} \left(\frac{\partial^2 x^{\rho}}{\partial x'^{\mu} \partial x'^{\nu}} \right) \tag{1.3.23}$$

According to the statement given by general coordinate transformation ,equation (1.3.22) implies that $\Gamma^{\lambda}_{\mu\nu}$ is not a tensor. If $\Gamma^{\lambda}_{\mu\nu}$ is a tensor the expected term will be ,$\frac{\partial x'^{\lambda}}{\partial x^{\rho}} \frac{\partial x^{\tau}}{\partial x'^{\mu}} \frac{\partial x^{\sigma}}{\partial x'^{\nu}} \Gamma^{\rho}_{\tau\sigma}$.Now invert equation (1.3.22) as,

$$\Gamma^{\lambda}_{\mu\nu} = \frac{\partial x^{\lambda}}{\partial x'^{\tau}} \frac{\partial x'^{\rho}}{\partial x^{\mu}} \frac{\partial x'^{\sigma}}{\partial x^{\nu}} \Gamma'^{\tau}_{\rho\sigma} + \frac{\partial x^{\lambda}}{x'^{\tau}} \frac{\partial^2 x'^{\tau}}{\partial x^{\mu} \partial x^{\nu}}$$

Thus,

$$\frac{\partial^2 x'^\tau}{\partial x^\mu \partial x^\nu} = \frac{\partial x'^\tau}{\partial x^\lambda}\Gamma^\lambda_{\mu\nu} - \frac{\partial x'^\rho}{\partial x^\mu}\frac{\partial x'^\sigma}{\partial x^\nu}\Gamma'^\tau_{\rho\sigma} \tag{1.3.24}$$

Differentiating this equation with respect to x^κ gives,

$$\frac{\partial^3 x'^\tau}{\partial x^\kappa \partial x^\mu \partial x^\nu} = \frac{\partial^2 x'^\tau}{\partial x^\kappa \partial x^\lambda}\Gamma^\lambda_{\mu\nu} + \frac{\partial x'^\tau}{\partial x^\lambda}\frac{\partial}{\partial x^\kappa}\Gamma^\lambda_{\mu\nu} - \frac{\partial^2 x'^\rho}{\partial x^\kappa \partial x^\mu}\frac{\partial x'^\sigma}{\partial x^\nu}\Gamma'^\tau_{\rho\sigma} + \frac{\partial x'^\rho}{\partial x^\mu}\frac{\partial^2 x'^\sigma}{\partial x^\kappa \partial x^\nu}\Gamma'^\tau_{\rho\sigma} - \frac{\partial x'^\rho}{\partial x^\mu}\frac{\partial x'^\sigma}{\partial x^\nu}\frac{\partial}{\partial x^\kappa}\Gamma'^\tau_{\rho\sigma} \tag{1.3.25}$$

According to the relation given by the equation(**??**) it is possible to write the following,

$$\frac{\partial^2 x'^\tau}{\partial x^\kappa \partial x^\lambda} = \frac{\partial x'^\tau}{\partial x^\eta}\Gamma^\eta_{\kappa\lambda} - \frac{\partial x'^\rho}{\partial x^\kappa}\frac{\partial x'^\sigma}{\partial x^\lambda}\Gamma'^\tau_{\rho\sigma}$$

$$\frac{\partial^2 x'^\rho}{\partial x^\kappa \partial x^\mu} = \frac{\partial x'^\rho}{\partial x^\eta}\Gamma^\eta_{\kappa\mu} - \frac{\partial x'^\eta}{\partial x^\kappa}\frac{\partial x'^\xi}{\partial x^\mu}\Gamma'^\rho_{\eta\xi}$$

$$\frac{\partial^2 x'^\sigma}{\partial x^\kappa \partial x^\nu} = \frac{\partial x'^\sigma}{\partial x^\eta}\Gamma^\eta_{\kappa\nu} - \frac{\partial x'^\eta}{\partial x^\kappa}\frac{\partial x'^\xi}{\partial x^\nu}\Gamma'^\sigma_{\eta\xi}$$

Substituting those equation into equation(**??**), we get,

$$\frac{\partial^3 x'^\tau}{\partial x^\kappa \partial x^\mu \partial x^\nu} = \left(\frac{\partial x'^\tau}{\partial x^\eta}\Gamma^\eta_{\kappa\lambda} - \frac{\partial x'^\rho}{\partial x^\kappa}\frac{\partial x'^\sigma}{\partial x^\lambda}\Gamma'^\tau_{\rho\sigma}\right)\Gamma^\lambda_{\mu\nu} + \frac{\partial \Gamma^\lambda_{\mu\nu}}{\partial x^\kappa}\frac{\partial x'^\tau}{\partial x^\lambda} - \Gamma'^\tau_{\rho\sigma}\frac{\partial x'^\sigma}{\partial x^\mu}\left(\frac{\partial x'^\rho}{\partial x^\eta}\Gamma^\eta_{\kappa\mu} - \frac{\partial x'^\eta}{\partial x^\kappa}\frac{\partial x'^\xi}{\partial x^\mu}\Gamma'^\rho_{\eta\xi}\right)$$

$$- \frac{\partial x'^\rho}{\partial x^\mu}\frac{\partial x'^\sigma}{\partial x^\nu}\frac{\partial \Gamma'^\tau_{\rho\sigma}}{\partial x^\kappa} - \Gamma'^\tau_{\rho\sigma}\frac{\partial x'^\rho}{\partial x^\mu}\left(\frac{\partial x'^\sigma}{\partial x^\eta}\Gamma^\eta_{\kappa\nu} - \frac{\partial x'^\eta}{\partial x^\kappa}\frac{\partial x'^\xi}{\partial x^\nu}\Gamma'^\sigma_{\eta\xi}\right) \tag{1.3.26}$$

Now collect similar terms and juggle indices a bit gives,

$$\frac{\partial^3 x'^\tau}{\partial x^\kappa \partial x^\mu \partial x^\nu} = \frac{\partial x'^\tau}{\partial x^\lambda}\left(\frac{\partial \Gamma^\lambda_{\mu\nu}}{\partial x^\kappa} + \Gamma^\eta_{\mu\nu}\Gamma^\lambda_{\kappa\eta}\right) - \frac{\partial x'^\rho}{\partial x^\mu}\frac{\partial x'^\sigma}{\partial x^\nu}\frac{\partial'^\eta}{\partial x^\kappa}\left(\frac{\partial \Gamma'^\tau_{\rho\sigma}}{\partial x'^\eta} - \Gamma'^\tau_{\rho\lambda}\Gamma'^\lambda_{\eta\sigma} - \Gamma'^\tau_{\lambda\sigma}\Gamma'^\lambda_{\eta\rho}\right)$$

$$- \Gamma'^\tau_{\rho\sigma}\frac{\partial x'^\sigma}{\partial x^\lambda}\left(\Gamma^\lambda_{\mu\nu}\frac{\partial x'^\sigma}{\partial x^\kappa} + \Gamma^\lambda_{\kappa\nu}\frac{\partial x'^\rho}{\partial x^\mu} + \Gamma^\lambda_{\kappa\mu}\frac{\partial x'^\rho}{\partial x^\nu}\right) \tag{1.3.27}$$

then after subtracting the same equation with interchanging $\nu \longleftrightarrow \kappa$ at the drop out the product of Γ and Γ' ,so that

$$0 = \frac{\partial x'^\tau}{\partial x^\lambda}\left(\frac{\partial \Gamma^\lambda_{\mu\nu}}{\partial x^\kappa} - \frac{\partial \Gamma^\lambda_{\mu\kappa}}{\nu} + \Gamma^\eta_{\mu\nu}\Gamma^\lambda_{\kappa\eta} - \Gamma^\eta_{\mu\kappa}\Gamma^\lambda_{\nu\eta}\right)$$

$$- \frac{\partial x'^\rho}{\partial x^\mu}\frac{\partial x'^\sigma}{\partial x^\nu}\frac{\partial'^\eta}{\partial x^\kappa}\left(\frac{\partial \Gamma'^\tau_{\rho\sigma}}{\partial x'^\eta} - \frac{\partial \Gamma'^\tau_{\rho\eta}}{\partial x'^\sigma} - \Gamma'^\tau_{\lambda\sigma}\Gamma'^\lambda_{\eta\rho} + \Gamma'^\tau_{\lambda\eta}\Gamma'^\lambda_{\sigma\rho}\right) \tag{1.3.28}$$

This may be written as a transformation rule,

$$R'^{\tau}_{\rho\sigma\eta} = \frac{\partial x'^{\tau}}{\partial x^{\lambda}} \frac{\partial x^{\mu}}{\partial x'^{\rho}} \frac{\partial x^{\nu}}{\partial x'^{\sigma}} \frac{\partial x^{\kappa}}{\partial x'^{\eta}} R^{\lambda}_{\mu\nu\kappa} \tag{1.3.29}$$

From the term in the first bracket of equation(**??**)using the curvature tensor notation as,

$$R^{\lambda}_{\mu\nu\kappa} = \frac{\partial \Gamma^{\lambda}_{\mu\nu}}{\partial x^{\kappa}} - \frac{\partial \Gamma^{\lambda}_{\mu\kappa}}{\nu} + \Gamma^{\eta}_{\mu\nu}\Gamma^{\lambda}_{\kappa\eta} - \Gamma^{\eta}_{\mu\kappa}\Gamma^{\lambda}_{\nu\eta} \tag{1.3.30}$$

$R^{\lambda}_{\mu\nu\kappa}$ is called Riemann-Christoffel curvature tensor plays an important role in specifying the geometrical properties of space-time.The space-time is considered flat,if the Riemann tensor vanishes every where. It is possible to write the Riemann curvature tensor in it fully covariant form as,

$$R_{\lambda\mu\nu\kappa} = g_{\lambda\sigma} R^{\sigma}_{\mu\nu\kappa} \tag{1.3.31}$$

Riemann curvature tensor can also be written directly in terms of the space-time metric, using the definition of affine connection,

$$\Gamma^{\sigma}_{\lambda\mu} = \frac{1}{2} g^{\nu\sigma} \left(\frac{\partial g_{\mu\nu}}{\partial x^{\lambda}} + \frac{\partial g_{\lambda\nu}}{\partial x^{\mu}} - \frac{\partial g_{\mu\lambda}}{\partial x^{\nu}} \right)$$

Thus,

$$R_{\lambda\mu\nu\kappa} = \frac{1}{2} g_{\lambda\sigma} \frac{\partial g^{\sigma\rho}}{\partial x^{\kappa}} \left(\frac{\partial g_{\rho\nu}}{\partial x^{\mu}} + \frac{\partial g_{\sigma\rho}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\rho}} \right) - \frac{1}{2} g_{\lambda\sigma} \frac{\partial g^{\sigma\rho}}{\partial x^{\nu}} \left(\frac{\partial g_{\rho\mu}}{\partial x^{\kappa}} + \frac{\partial g_{\rho\kappa}}{\partial x^{\mu}} - \frac{\partial g_{\mu\kappa}}{\partial x^{\rho}} \right)$$
$$+ g_{\lambda\sigma} \left(\Gamma^{\eta}_{\mu\nu}\Gamma^{\sigma}_{\kappa\eta} - \Gamma^{\eta}_{\mu\kappa}\Gamma^{\sigma}_{\nu\eta} \right) \tag{1.3.32}$$

Now define the kronecker delta$\delta^{\rho}_{\lambda} = 1$, where $\rho = \lambda$ and ,

$$g_{\lambda\sigma} \frac{\partial g^{\sigma\rho}}{\partial x^{\kappa}} = -g^{\lambda\rho} \frac{\partial g_{\lambda\sigma}}{\partial x^{\kappa}} = -g^{\sigma\rho} \left(\Gamma^{\eta}_{\kappa\lambda} g_{\eta\sigma} + \Gamma^{\eta}_{\kappa\sigma} g_{\eta\lambda} \right)$$

Therefore most of $\Gamma\Gamma$ terms cancel,then

$$R_{\lambda\mu\nu\kappa} = \frac{1}{2} \left(\frac{\partial^2 g_{\lambda\nu}}{\partial x^{\kappa} \partial x^{\mu}} - \frac{\partial^2 g_{\mu\nu}}{\partial x^{\kappa} \partial x^{\lambda}} - \frac{\partial^2 g_{\lambda\kappa}}{\partial x^{\nu} \partial x^{\mu}} + \frac{\partial^2 g_{\mu\kappa}}{\partial x^{\nu} \partial x^{\lambda}} \right) + g_{\eta\sigma} \left(\Gamma^{\eta}_{\nu\lambda}\Gamma^{\sigma}_{\mu\kappa} - \Gamma^{\eta}_{\kappa\lambda}\Gamma^{\sigma}_{\mu\nu} \right) \tag{1.3.33}$$

This is the covariant form of Riemann-Christoffel curvature tensor. The algebraic property of the curvature tensors are,

1. Symmetry

$$R_{\lambda\mu\nu\kappa} = R_{\nu\kappa\lambda\mu} \tag{1.3.34}$$

2. Antisymmetry

$$R_{\lambda\mu\nu\kappa} = -R_{\mu\lambda\nu\kappa} = -R_{\lambda\mu\kappa\nu} = R_{\nu\kappa\lambda\mu} \tag{1.3.35}$$

3. Cyclicity

$$R_{\lambda\mu\nu\kappa} + R_{\mu\kappa\lambda\nu} + R_{\lambda\kappa\mu\nu} = 0 \tag{1.3.36}$$

Therefore,the Riemann tensor in 4-dimensional space-time has only 20 independent components because of symmetries. Thus the general rule for computing the number of independent components in a N-dimension space-time is $\frac{N^2(N^2-1)}{12}$[**?**].

1.3.3 Ricci Tensor,Ricci Scalar and Einstein Field Tensor

Ricci Tensor: Obtained from the Riemann curvature tensor by contracting over two of the indices

$$\begin{aligned} R_{\mu\kappa} &= R^{\lambda}_{\mu\lambda\kappa} \\ &= g^{\lambda\nu} R_{\lambda\mu\nu\kappa} \end{aligned}$$

Which can be written as,

$$R_{\mu\kappa} = \frac{1}{2} g^{\lambda\nu} \left(\frac{\partial^2 g_{\lambda\nu}}{\partial x^\kappa \partial x^\mu} - \frac{\partial^2 g_{\mu\nu}}{\partial x^\kappa \partial x^\lambda} - \frac{\partial^2 g_{\lambda\kappa}}{\partial x^\nu \partial x^\mu} + \frac{\partial^2 g_{\mu\kappa}}{\partial x^\nu \partial x^\lambda} \right) + g^{\lambda\nu} \left(\Gamma^{\eta}_{\nu\lambda} \Gamma^{\sigma}_{\mu\kappa} - \Gamma^{\eta}_{\kappa\lambda} \Gamma^{\sigma}_{\mu\nu} \right) \tag{1.3.37}$$

and also one can write the Ricci tensor as,

$$R_{\mu\kappa} = \frac{\partial \Gamma^{\lambda}_{\mu\lambda}}{\partial x^\kappa} - \frac{\partial \Gamma^{\lambda}_{\mu\kappa}}{\partial x^\lambda} + \Gamma^{\eta}_{\mu\lambda} \Gamma^{\lambda}_{\kappa\eta} - \Gamma^{\eta}_{\mu\kappa} \Gamma^{\lambda}_{\lambda\eta} \tag{1.3.38}$$

Ricci tensor is symmetric.Therefore,it has at most ten independent components.

$$R_{\mu\kappa} = R_{\kappa\mu}$$

Ricci Scalar: By further contracting the Ricci tensor with the contravariant component of the metric, one obtains the curvature scalar,

$$R = g^{\mu\kappa} R_{\mu\kappa} = g^{\lambda\nu} g^{\mu\kappa} R_{\lambda\mu\nu\kappa} \tag{1.3.39}$$

or

$$R = R^{\mu}_{\mu} \tag{1.3.40}$$

Einstein Field Tensor: Einstein field tensor $G_{\mu\kappa}$ is constructed only from the Riemann tensor and the metric

$$G_{\mu\kappa} = R_{\mu\kappa} - \frac{1}{2} g_{\mu\kappa} R \tag{1.3.41}$$

Where, $G_{\mu\kappa}$ is a linear combination of the metric tensor and its first and second derivatives. Since the Ricci tensor and metric tensor are symmetric, so Einstein field tensor also symmetric, thus

$$G_{\mu\kappa} = G_{\kappa\mu} \tag{1.3.42}$$

1.3.4 Bianchi Identity

The Riemann curvature tensor obeys important differential identities. These can be derived at a given point,x by adopting a locally inertial coordinate system in which $\Gamma^{\lambda}_{\mu\nu}$ vanishes at x thus at x,

$$R_{\lambda\mu\nu\kappa;\eta} = \frac{1}{2} \frac{\partial}{\partial x^{\eta}} \left[\frac{\partial^2 g_{\lambda\nu}}{\partial x^{\kappa} \partial x^{\mu}} - \frac{\partial^2 g_{\mu\nu}}{\partial x^{\kappa} \partial x^{\lambda}} - \frac{\partial^2 g_{\lambda\kappa}}{\partial x^{\nu} \partial x^{\mu}} + \frac{\partial^2 g_{\mu\kappa}}{\partial x^{\nu} \partial x^{\lambda}} \right] \tag{1.3.43}$$

By permuting ν,κ and η cyclically,we obtain the Bianchi identities,

$$R_{\lambda\mu\nu\kappa;\eta} + R_{\lambda\mu\eta\nu;\kappa} + R_{\lambda\mu\kappa\eta;\nu} = 0 \tag{1.3.44}$$

Now recalling that the covariant derivatives of $g^{\lambda\nu}$ vanish,then we find on contraction of λ with ν that,

$$R_{\mu\kappa;\eta} - R_{\mu\eta;\kappa} + R^{\nu}_{\mu\kappa\eta;\nu} = 0 \tag{1.3.45}$$

Again contracting gives,

$$R_{;\eta} - R^{\mu}_{\eta;\mu} - R^{\nu}_{\eta;\nu} = 0$$

or

$$\big(R^{\mu}_{\eta} - \frac{1}{2}\delta^{\mu}_{\eta}R\big)_{;\mu} = 0 \tag{1.3.46}$$

An equivalent but more familiar form is,

$$\big(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\big)_{;\mu} = 0 \tag{1.3.47}$$

or

$$G^{\mu\nu}_{;\mu} = 0 \tag{1.3.48}$$

1.3.5 Energy-Momentum Tensor

Some time Energy Momentum called as stress-energy tensor. It describes the density and flows of the 4 momentum .

In the absence of gravity Energy-Momentum Tensor for perfect fluid is given by

$$T^{\alpha\beta} = P\eta^{\alpha\beta} + (P+\rho)U^{\alpha}U^{\beta} \tag{1.3.49}$$

The four velocity, U^{α} is define as

$$U^{\alpha} = \frac{dx^{\alpha}}{d\tau} \quad and \quad U^{\beta} = \frac{dx^{\beta}}{d\tau} \tag{1.3.50}$$

From line element

$$d\tau^2 = \eta_{\alpha\beta}dx^{\alpha}dx^{\beta} \tag{1.3.51}$$

Now using (**??**),(**??**) we can derive,

$$1 = \eta_{\alpha\beta}U^{\alpha}U^{\beta} \tag{1.3.52}$$

Using (??),(??),(??), The energy-momentum tensor of a perfect fluid therefore takes the following form in its rest frame.

$$T^{\alpha\beta} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix} \tag{1.3.53}$$

This is an important formula for applications such as stellar structure and cosmology[7,11]. In fact,one way to define T would be a(2,0) tensor with units of energy per volume,which is conserved.

In the presence of gravity Energy-Momentum Tensor for perfect fluid given by

$$T^{\mu\nu} = Pg^{\mu\nu} + (P + \rho)U^{\mu}U^{\nu} \tag{1.3.54}$$

also from normalization

$$g_{\mu\nu}U^{\mu}U^{\nu} = -1 \tag{1.3.55}$$

1.4 Einstein Field Equation

In Newtonian theory, gravity can only exist where there exists matter. However Einstein showed that matter and energy are only different faces of the same coin. This encouraged him to make the conclusion that gravity is not only created by the presence of matter, it is in fact the product of the presence of energy. General relativity must present appropriate analogues of the two parts of the dynamics,one how particles move in the response to gravity,and secondly,how particles generate gravitational effects [?]. The analogue of the poisson equation of the second idea can be,

$$\nabla^2\phi(x) = 4\pi G\rho(x) \tag{1.4.1}$$

Now we start to derive Einstein field equation under the approximation of a weak static field produced by a non-relativistic mass density ρ [?, ?]. Therefore,the energy density for non-relativistic matter is,

$$T_{00} = \rho = T^{00} \tag{1.4.2}$$

One can write the poisson equation as,

$$\nabla^2 \phi = 4\pi G T_{00} \tag{1.4.3}$$

From equation(**??**) we get,

$$\nabla^2 \phi = \frac{1}{2} \nabla^2 g_{00} \tag{1.4.4}$$

Therefore poison equation result,

$$\frac{1}{2} \nabla^2 g_{00} = 4\pi G T_{00}$$

$$\nabla^2 g_{00} = 8\pi G T_{00} \tag{1.4.5}$$

From this fact the weak field equation for a general distribution of energy and momentum $T_{\alpha\beta}$ will take the form,

$$G_{\alpha\beta} = 8\pi G T_{\alpha\beta} \tag{1.4.6}$$

Where, $G_{\alpha\beta}$ is a linear combination of the metric tensor and its first and second derivatives. The principle of equivalence that the equation which govern gravitational fields of arbitrary strength must take the form,

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{1.4.7}$$

Therefore,equation (**??**) is the approximated form of equation (**??**) in a weak static gravitational field as equivalence principle states. Here is a tensor which reduce to $G_{\alpha\beta}$ for a weak fields and since $T_{\mu\nu}$ is symmetric, $G_{\mu\nu}$ also. To go further consider the nature of $G_{\mu\nu}$;

1. By definition $G_{\mu\nu}$ is a tensor
2. By assumption $G_{\mu\nu}$ contain terms that are linear in the second derivative of the metric tensor or quadratic in the first derivative of the metric.

3. Since $T_{\mu\nu}$ is symmetric so does $G_{\mu\nu}$

4. Since $T_{\mu\nu}$ is conserved in the absence of external forces,so does $G_{\mu\nu}$.

5. For a weak stationary field produced by non-relativistic matter ,the 00 component must satisfy

$$G_{00} \cong \nabla^2 g_{00} \tag{1.4.8}$$

Hence (1) and (2) require $G_{\mu\nu}$ to take the form

$$G^{\mu\nu} = C_1 R_{\mu\nu} + C_2 g_{\mu\nu} R \tag{1.4.9}$$

Where, C_1 and C_2 are constants.Since this is symmetric condition(3) is automatically satisfied. It follows from the above relation that.

$$g^{\sigma\mu} G_{\mu\nu} = C_1 g^{\sigma\mu} R_{\mu\nu} + C_2 g^{\sigma\mu} g_{\mu\nu} R \tag{1.4.10}$$

Equivalent to,

$$G^{\sigma}_{\nu} = C_1 R^{\sigma}_{\nu} + C_2 \delta^{\sigma}_{\nu} R \tag{1.4.11}$$

This follows as

$$G^{\sigma}_{\nu;\sigma} = C_1 R^{\sigma}_{\nu;\sigma} + C_2 \delta^{\sigma}_{\nu} R_{;\sigma} \tag{1.4.12}$$

Using the result,$R^{\sigma}_{\nu;\sigma} = \frac{1}{2}\delta^{\sigma}_{\nu} R_{;\sigma}$ in to the above equation and it follows,

$$G^{\sigma}_{\nu;\sigma} = \frac{1}{2} C_1 \delta^{\sigma}_{\nu} R^{\sigma}_{;\sigma} + C_2 \delta^{\sigma}_{\nu} R_{;\sigma} \tag{1.4.13}$$

If $\nu = \sigma$

$$G^{\sigma}_{\nu;\sigma} = (\frac{C_1}{2} + C_2) R_{;\sigma} \tag{1.4.14}$$

By the conservation of $G_{\mu\nu}$ we have $G^{\sigma}_{\nu;\sigma} = 0$ and this yields the relation,

$$(\frac{C_1}{2} + C_2) R_{;\sigma} = 0$$

$$\frac{C_1}{2} = -C_2 \tag{1.4.15}$$

Therefore we can rewrite $G_{\mu\nu}$ as,

$$G_{\mu\nu} = C_1 R_{\mu\nu} - \frac{C_1}{2} g_{\mu\nu} R$$

$$G_{\mu\nu} = C_1 \big(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R\big) \tag{1.4.16}$$

To fix the constant C_1,use the property[**?**]. A non-relativistic system always has $\|T_{ij}\| << \|T_{00}\|$ and here look the case where $\|G_{ij}\| << \|G_{00}\|$ thus,

$$G_{ij} \cong 0 \tag{1.4.17}$$

From equation(1.2.20)we can be written as,

$$R_{ij} - \frac{1}{2} g_{ij} R = 0$$

$$R_{ij} = \frac{1}{2} g_{ij} R \tag{1.4.18}$$

Since we deal here with a weak field approximation ($i.e. g_{\alpha\beta} \cong \eta_{\alpha\beta}$) as well as $g_{ij} \cong \eta_{\alpha\beta}$. Therefore,this lead to write as,

$$R_{ij} \cong \frac{1}{2} \eta_{ij} R \tag{1.4.19}$$

By applying the property of metric tensor $\eta_{ij} = 1$, for $i = j = 1, 2, 3$ and taking the sum over each indices,

$$R_{ij} = \sum_{i,j=1}^{3} \frac{1}{2} \eta_{ij} R \cong \frac{3}{2} R$$

$$R_{kk} = \frac{3}{2} R \tag{1.4.20}$$

The curvature scalar is therefore given by,

$$R \cong R_{kk} - R_{00} = \frac{3}{2} R - R_{00}$$

$$R \cong 2R_{00} \tag{1.4.21}$$

Thus in the weak field approximation we have the following information,

$$R \cong 2R_{00}$$

$$g_{\alpha\beta} \cong \eta_{\alpha\beta}$$

$$G_{\mu\nu} = C_1\big(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\big)$$

For the 00 component of $G_{\mu\nu}$ equals to,

$$G_{00} = C_1\big(R_{00} - \frac{1}{2}g_{00}R\big) = C_1\big(R_{00} - \frac{1}{2}\eta_{00}\big) \tag{1.4.22}$$

Now the task is to calculate R_{00}. Recall the expression given by the Riemann curvature tensor $R_{\lambda\mu\nu\kappa}$ that is,

$$R_{\lambda\mu\nu\kappa} = \frac{1}{2}\left(\frac{\partial^2 g_{\lambda\nu}}{\partial x^\kappa \partial x^\mu} - \frac{\partial^2 g_{\mu\nu}}{\partial x^\kappa \partial x^\lambda} - \frac{\partial^2 g_{\lambda\kappa}}{\partial x^\nu \partial x^\mu} + \frac{\partial^2 g_{\mu\kappa}}{\partial x^\nu \partial x^\lambda}\right) + g_{\eta\sigma}\left(\Gamma^\eta_{\ \nu\lambda}\Gamma^\sigma_{\ \mu\kappa} - \Gamma^\eta_{\ \kappa\lambda}\Gamma^\sigma_{\ \mu\nu}\right)$$

Since we are looking for a weak field approximation,it is better to use the linear part of $R_{\lambda\mu\nu\kappa}$,given by

$$R_{\lambda\mu\nu\kappa} = \frac{1}{2}\left(\frac{\partial^2 g_{\lambda\nu}}{\partial x^\kappa \partial x^\mu} - \frac{\partial^2 g_{\mu\nu}}{\partial x^\kappa \partial x^\lambda} - \frac{\partial^2 g_{\lambda\kappa}}{\partial x^\nu \partial x^\mu} + \frac{\partial^2 g_{\mu\kappa}}{\partial x^\nu \partial x^\lambda}\right) \tag{1.4.23}$$

When the field is static all the time derivatives vanish,and the components that we need are,

$$R_{0000} \cong 0$$

$$R_{iojo} \cong \frac{1}{2}\frac{\partial^2 g_{00}}{\partial x^i \partial x^j} = \frac{1}{2}\nabla^2 g_{00} \tag{1.4.24}$$

Where $\frac{\partial^2 g_{00}}{\partial x^i \partial x^j} = \nabla^2 g_{00}$. From the contraction of curvature tensor over the two indices

$$R_{00} = g^{\lambda\nu}R_{\lambda 0\nu 0}$$

$$R_{00} = R_{iojo} - R_{0000} \tag{1.4.25}$$

By using this relation in to equation(1.2.33) for $G_{\mu\nu}$,

$$G_{00} = 2C_1\big(R_{iojo} - R_{0000}\big)$$

$$G_{00} = 2C_1\big(\frac{1}{2}\nabla^2 g_{00} - 0\big) = C_1\nabla^2 g_{00} \tag{1.4.26}$$

Comparing equation (**??**) to equation (1.4.26),

$$G_{00} = C_1\nabla^2 g_{00} = \nabla^2 g_{00} \tag{1.4.27}$$

This gives the value of $C_1 = 1$,and therefore we can write the equation for $G_{\mu\nu}$ as,

$$G_{\mu\nu} = \big(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\big) = 8\pi G T_{\mu\nu}$$

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu} \tag{1.4.28}$$

Equation (1.4.28) is Einstein field equation. This shows that the metric of space-time is dependent upon the matter present in that space-time.

1.5 Introduction of Cosmological Constant into Einstein Field Equations

After completing his theory of GR, Einstein was interested to find a static solution of his field equations with the idea of incorporating Mach's principle, for details see [**?**]. But Einstein soon noticed that his original field equations yield a non - static solution. As the consequence, Einstein himself after a year, in 1917 introduced a positive cosmological constant with the belief of constructing a static solution, (**??**). But at the same year that Einstein introduced the cosmological term, de Sitter presented solutions to static Einstein universe , which had both static and dynamic features. The de Sitter's prediction is considered as the first step towards the theoretical discovery of expanding universe. On the other hand, in 1922 Freidmann constructed a matter dominated expanding universe without a cosmological constant.Then, the possibility that the universe may be expanding led Einstein to abandon the idea of a static universe and, along with it, the cosmological

constant. However, other groups sustained supporting a model with cosmological constant. For example, Weyl in 1923 recommended de Sitter's model to explain measurements of the spectra of spiral nebulae that showed redshifted; Lemaitre constructed an expanding model which originated from such an asymptotically static state (static Einstein universe) in the distant past. Since then, the cosmological constant has remained with debate where it is being cast out at a time and reintroduced at other time.
Einstein field equation with cosmological constant Λ became,

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + g_{\mu\nu}\Lambda = 8\pi G T_{\mu\nu} \tag{1.5.1}$$

Recent observational data and results in modern cosmology revealed that the dark energy which is described in majority by the cosmological constant is of dominant importance in the dynamics of our Universe. Measurements conducted by Wilkinson Microwave Anisotropic Probe (WMAP) indicate that almost three fourth of total mass-energy in the Universe is Dark Energy and the leading theory of dark energy is based on the cosmological constant characterized by repulsive pressure which was introduced by Einstein in 1917 to obtain a static cosmological model. Later on Zeldovich[**?**] interpreted this quantity physically as a vacuum energy of quantum fluctuation whose size is of the order of $\sim 3 \times 10^{-56} cm^{-2}$

Chapter 2

Hydrodynamics in General Relativity in the Schwarzschild-de sitter space-time

2.1 Introduction

The first static, spherically symmetric perfect fluid solution with constant density was already found by Schwarzschild in 1918. In spherical symmetry Tolman [?] and Oppenheimer and Volkoff [?]reduced the field equations to the well known TOV equation.

The boundary of stellar models is defined to be where the pressure vanishes. At this surface a vacuum solution is joined on as an exterior field.In case of vanishing cosmological constant it is the Schwarzschild solution.For very simple equations of state Tolman integrated the TOV equation and discussed solutions. Although he already included the cosmological constant in his calculations he did not analyze them. He stated that the cosmological constant is too small to produce effects.

Buchdahl [?] assumed the existence of a global static solution, to show that the total mass of a fluid ball is bounded by its radius. He showed the strict inequality $M < (\frac{4}{9})R$, which holds for fluid balls in which the density does not increase outwards. It implies that radii of fluid balls are always larger than the black-hole event horizon.

Geometrical properties of constant density solutions were analyzed by Stephani[?] . He showed that they can be embedded in a five dimensional flat space and that they are conformally flat. The cosmological constant can easily be included in his calculations by redefining some variables.

Collins [?] stated that for a fixed equation of state and cosmological constant the choice of central pressure and therefore central density does not uniquely determine the solution. This is disproved.

Static perfect fluid solutions with cosmological constant were analyzed by Kriele [?] and later by Winter [?]. Both derived the analogous TOV- equation. The first one shows uniqueness of the solution for given pressure and density distributions, which already disproved Collins [?]. An analogous type of Buchdahl inequality is derived but not discussed in the context of upper and lower bounds on radii of stellar objects. Winter [?] integrates the TOV- equation from the boundary inwards to the centre, without proving the existence of that boundary. This leads to solutions with non-regular centres and is therefore not suitable for discussing stellar models.

Constant density solutions with cosmological constant were first analysed by Weyl . More than 80 years later Stuchlk [?] analysed these solutions again. He integrated the TOV-Λ equation for possible values of the cosmological constant up to the limit $\Lambda < 4\pi\rho_0$, where ρ_0 denotes this constant density. In these cases constant density solutions describe stellar models. For larger cosmological constant the pressure will vanish after the coordinate singularity. The volume of group orbits is decreasing and there one has to join the Schwarzschild-de Sitter solution containing the r = 0 singularity. Increasing the cosmological constant further leads to generalizations of the Einstein static universe. These solutions have two regular centres with monotonically decreasing or increasing pressure from the first to the second centre. Certainly the Einstein cosmos itself is a solution. Another new kind describes solutions with a regular centre and increasing divergent pressure. In this case the spacetime has a geometrical singularity. These solutions are unphysical and therefore not of great interest.

This chapter deals with the Einstein field equations with cosmological constant in the spherically symmetric and static case. For a generalized Birkhoff theorem see [?].

Perfect fluid is assumed to be the matter source. This directly leads to a Λ-extended Tolman-Oppenheimer-Volkoff equation which will be called TOV-Λ equation. The TOV-Λ equation together with the mean density equation form a system of differential equations. It can easily be integrated if a constant density is assumed

2.2 Metric tensor

The metric is a geometric tool that relates distances in spacetime, a kind of generalized pythagorean theorem where the time coordinate is included as well. The underlying physics is more important than the relative coordinates, so all equations are written in the invariant language of tensors, or multi-indexed objects. The Einstein summation convention shortens the notation by assuming an implied sum over repeated indices. With this in mind, the Schwarschild-de sitter metric for a spherically symmetric vacuum space-time (valid outside a star or black hole) in coordinates (t, r, θ, φ) is

$$g_{\mu\nu} = \begin{pmatrix} -e^{2\Phi} & 0 & 0 & 0 \\ 0 & \left(1 - 2\frac{m}{r} - r^2\frac{\Lambda}{3}\right)^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 sin^2\theta \end{pmatrix} \qquad g^{\mu\nu} = \begin{pmatrix} -e^{-2\Phi} & 0 & 0 & 0 \\ 0 & \left(1 - 2\frac{m}{r} - r^2\frac{\Lambda}{3}\right) & 0 & 0 \\ 0 & 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 0 & \frac{1}{r^2\sin^2\theta} \end{pmatrix} \tag{2.2.1}$$

which induces the following line element for measuring infinitesimal distances, for the derivation see appendix (**??**)

$$ds^2 = -\left(1 - 2\frac{m}{r} - r^2\frac{\Lambda}{3}\right)dt^2 + \left(1 - 2\frac{m}{r} - r^2\frac{\Lambda}{3}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \tag{2.2.2}$$

we can re write this in the form of some potential, Φ as,

$$ds^2 = -e^{2\Phi}dt^2 + \left(1 - 2\frac{m}{r} - r^2\frac{\Lambda}{3}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \tag{2.2.3}$$

Due to spherical symmetry and no time dependence, we only have the r component of the metric and therefore the pressure P and the energy-density ϵ will only depend on this component, and thus a calculation of this component is required, let start calculating for Γ using (??)(??). Then the non vanishing terms of Γ are,

$$\Gamma^r_{tt} = \left(1 - 2\frac{m}{r} - r^2\frac{\Lambda}{3}\right)\Phi' e^{2\Phi} \qquad \Gamma^r_{rr} = \frac{\frac{\Lambda r}{3} - \frac{m}{r^2} + \frac{m' r}{r^2}}{\left(1 - \frac{2m}{r} - \frac{r^2\Lambda}{3}\right)} \qquad \Gamma^r_{\theta\theta} = -r\left(1 - 2\frac{m}{r} - r^2\frac{\Lambda}{3}\right)$$

$$\Gamma^r_{\phi\phi} = -r\left(1 - 2\frac{m}{r} - r^2\frac{\Lambda}{3}\right) sin^2\theta \qquad \Gamma^t_{rt} = \Phi' \qquad \Gamma^\phi_{\theta\phi} = \cot\theta$$

$$\Gamma^\theta_{\phi\phi} = -\cos\theta\sin\theta \qquad \Gamma^\phi_{r\phi} = \Gamma^\theta_{r\theta} = \frac{1}{r}$$

2.2.1 Ricci Tensor and Ricci Scalar

Now lets find the component of **Ricci tensor** $(R_{tt}, R_{rr}, R_{\theta\theta}, R_{\phi\phi})$ using

$$R_{\mu\nu} = \Gamma^\lambda_{\mu\nu;\lambda} - \Gamma^\lambda_{\mu\lambda;\nu} + \Gamma^\alpha_{\mu\nu}\Gamma^\lambda_{\alpha\lambda} - \Gamma^\alpha_{\mu\lambda}\Gamma^\lambda_{\alpha\nu} \tag{2.2.4}$$

Then the R_{tt}

$$R_{tt} = \Gamma^r_{tt;r} + \Gamma^r_{tt}\left[\Gamma^t_{rt} + \Gamma^\theta_{r\theta} + \Gamma^\phi_{r\phi} + \Gamma^r_{rr}\right] - \left[\Gamma^t_{tr}\Gamma^r_{tt} + \Gamma^r_{tt}\Gamma^t_{rt}\right]$$

let assume $\left(1 - \frac{2m}{r} - \frac{r^2\Lambda}{3}\right) = K$

$$R_{tt} = \frac{d}{dr}\left[\Phi' e^{2\Phi}K\right] + \Phi' e^{2\Phi}K\left[\Phi' + \frac{2}{r} + \frac{\frac{\Lambda r}{3} - \frac{m}{r^2} + \frac{m' r}{r^2}}{\left(1 - \frac{2m}{r} - \frac{r^2\Lambda}{3}\right)}\right] - \left[2\left(\Phi'\right)^2 e^{2\Phi}K\right]$$

$$= \Phi'' e^{2\Phi}K + \Phi' e^{2\Phi}K' + \Phi'(e^{2\Phi})'K + \Phi' e^{2\Phi}K\left[\Phi' + \frac{2}{r} + \frac{\frac{\Lambda r}{3} - \frac{m}{r^2} + \frac{m' r}{r^2}}{\left(1 - \frac{2m}{r} - \frac{r^2\Lambda}{3}\right)}\right] - \left[2\left(\Phi'\right)^2 e^{2\Phi}K\right]$$

$$= \Phi'' e^{2\Phi}K + 2\Phi' e^{2\Phi}\left[\frac{m - m' r}{r^2} - \frac{r\Lambda}{3}\right] + 2(\Phi')^2 e^{2\Phi}K + \Phi' e^{2\Phi}K\left[\Phi' + \frac{2}{r} + \frac{\frac{\Lambda r}{3} - \frac{m}{r^2} + \frac{m' r}{r^2}}{\left(1 - \frac{2m}{r} - \frac{r^2\Lambda}{3}\right)}\right] - 2(\Phi')^2 e^{2\Phi}K$$

$$= \left(\Phi'' + (\Phi')^2\right) e^{2\Phi} K + 2\Phi' e^{2\Phi} \left[\frac{m - m' r}{r^2} - \frac{r\Lambda}{3}\right] + \Phi' e^{2\Phi} K \left[\frac{2}{r} + \frac{\frac{\Lambda r}{3} - \frac{m}{r^2} + \frac{m' r}{r^2}}{\left(1 - \frac{2m}{r} - \frac{r^2\Lambda}{3}\right)}\right]$$

$$= \left(\Phi'' + (\Phi')^2\right) e^{2\Phi} K + e^{2\Phi}\Phi' \left[2\left(\frac{m - m'}{r^2}\right) - \frac{2r\Lambda}{3} + \frac{2K}{r} + \frac{\Lambda r}{3} - \frac{m}{r^2} + \frac{m' r}{r^2}\right]$$

$$R_{tt} = e^{2\Phi}\left[\left(\Phi'' + (\Phi')^2\right)\left(1 - \frac{2m}{r} - \frac{r^2\Lambda}{3}\right) + \Phi'\left(\frac{2r - m' r - 3m}{r^2} - r\Lambda\right)\right] \tag{2.2.5}$$

For R_{rr} we use (**??**) then,

$$R_{rr} = -\Gamma^t_{rt;r} - \Gamma^\theta_{r\theta;r} - \Gamma^\phi_{r\phi;r} + \Gamma^r_{rr}\left[\Gamma^t_{rt} + \Gamma^\theta_{r\theta} + \Gamma^\phi_{r\phi}\right] - \left[\Gamma^t_{rt}\Gamma^t_{rt} + \Gamma^\theta_{r\theta}\Gamma^\theta_{r\theta} + \Gamma^\phi_{r\phi}\right]$$

$$= -\frac{d}{dr}(\Phi') - \frac{d}{dr}\left[\frac{1}{r}\right] - \frac{d}{dr}\left[\frac{1}{r}\right] + \frac{\left[\frac{\Lambda r}{3} - \frac{m}{r^2} + \frac{m' r}{r^2}\right]}{1 - \frac{2m}{r} - \frac{r^2\Lambda}{3}}\left[\Phi' + \frac{2}{r}\right] - \left[(\Phi')^2 + \frac{2}{r^2}\right]$$

$$= -\Phi'' + \frac{2}{r^2} + (1 - \frac{2m}{r} - \frac{r^2\Lambda}{3})^{-1}\left[\frac{\frac{\Lambda r^3}{3} - m + m' r}{r^2}\right]\left[\frac{r\Phi' + 2}{r}\right] - \left[(\Phi')^2 + \frac{2}{r^2}\right]$$

$$R_{rr} = (1 - \frac{2m}{r} - \frac{r^2\Lambda}{3})^{-1}\left[\frac{[\frac{\Lambda r^3}{3} - m + m' r][r\Phi' + 2]}{r^3}\right] - \left(\Phi'' + (\Phi')^2\right) \tag{2.2.6}$$

For $R_{\theta\theta}$ we use (**??**) then,

$$R_{\theta\theta} = \Gamma^r_{\theta\theta;r} - \Gamma^\phi_{\theta\phi;\theta} + \Gamma^r_{\theta\theta}\left[\Gamma^\theta_{r\theta} + \Gamma^t_{rt} + \Gamma^\phi_{r\phi} + \Gamma^r_{rr}\right] - \left[\Gamma^\theta_{r\theta}\Gamma^r_{\theta\theta} + \Gamma^r_{\theta\theta}\Gamma^r_{r\theta} + \Gamma^\phi_{\theta\phi}\Gamma^\phi_{\phi\theta}\right]$$

$$= \frac{d}{dr}[-r + 2m + \frac{r^3\Lambda}{3}] - \frac{d}{d\theta}[\cot\theta] + [-rK]\left[\frac{2}{r} + \Phi' + \frac{\left[\frac{\Lambda r}{3} - \frac{m}{r^2} + \frac{m' r}{r^2}\right]}{K}\right] - [-2K + \cot^2\theta]$$

$$= -1 + 2m' + r^2\Lambda + \frac{1}{\sin^2\theta} - 2K - r\Phi' K - r(\frac{\Lambda r}{3} - \frac{m}{r^2} + \frac{m' r}{r^2}) + 2K - \cot^2\theta$$

$$= 2m' + r^2\Lambda - r\Phi'(1 - \frac{2m}{r} - \frac{r^2\Lambda}{3}) - r(\frac{\Lambda r}{3} - \frac{m}{r^2} + \frac{m' r}{r^2})$$

$$= 2m' - m' + r^2\Lambda - \frac{r^2\Lambda}{3} + \frac{m}{r} + \Phi'(-r + 2m + \frac{r^3\Lambda}{3})$$

$$R_{\theta\theta} = m^{'} + \frac{m}{r} + \frac{2r^2\Lambda}{3} - \Phi^{'}(-r + 2m + \frac{r^3\Lambda}{3}) \tag{2.2.7}$$

for $R_{\phi\phi}$ we use (**??**)

$$R_{\phi\phi} = \Gamma^{\theta}_{\phi\phi;\theta} + \Gamma^{r}_{\phi\phi;r} + \Gamma^{r}_{\phi\phi}\left[\Gamma^{t}_{rt} + \Gamma^{r}_{rr} + \Gamma^{\theta}_{r\theta} + \Gamma^{\phi}_{r\phi}\right] - 2\Gamma^{r}_{\phi\phi}\Gamma^{\phi}_{r\phi} - \Gamma^{\theta}_{\phi\phi}\Gamma^{\phi}_{\theta\phi}$$

$$= \frac{d}{d\theta}(-\cos\theta\sin\theta) + \frac{d}{dr}(-rK\sin^2\theta) + (-rK\sin^2\theta)\left[\Phi' + \frac{\frac{\Lambda r}{3} - \frac{m}{r^2} + \frac{m'}{r}}{K} + \frac{2}{r}\right] + 2K\sin^2\theta + \sin\theta\cos\theta\cot\theta$$

$$= \sin^2\theta - \cos^2\theta - K\sin^2\theta - rK'\sin^2\theta - rK\Phi'\sin^2\theta - r(\frac{\Lambda r}{3} - \frac{m}{r^2} + \frac{m'}{r})\sin^2\theta - 2K\sin^2\theta + 2K\sin^2\theta + \cos^2\theta$$

$$= \sin^2\theta - K\sin^2\theta - rK'\sin^2\theta - r(\frac{\Lambda r}{3} - \frac{m}{r^2} + \frac{m'}{r})\sin^2\theta - rK\Phi'\sin^2\theta$$

$$= \left[1 - (K + rK' + r(\frac{\Lambda r}{3} - \frac{m}{r^2} + \frac{m'}{r}))\right]\sin^2\theta - rK\Phi'\sin^2\theta$$

$$= \left[1 - (1 - \frac{2m}{r} - \frac{r^2\Lambda}{3} + \frac{2m}{r} - 2m' - \frac{2\Lambda r}{3} + \frac{\Lambda r^2}{3} - \frac{m}{r} + m')\right]\sin^2\theta - rK\Phi'\sin^2\theta$$

$$= (m' + \frac{m}{r} + \frac{2\Lambda r^2}{3})\sin^2\theta - (r - 2m - \frac{\Lambda r^3}{3})\Phi'\sin^2\theta$$

$$= \sin^2\theta\left[m' + \frac{m}{r} + \frac{2\Lambda r^2}{3} - (r - 2m - \frac{\Lambda r^3}{3})\Phi'\right]$$

$$= \sin^2\theta R_{\theta\theta} \tag{2.2.8}$$

Now lets find the **Ricci scalar** R using, (**??**),(**??**),(**??**),(**??**),(**??**),(**??**),(**??**)

$$R = g^{tt}R_{tt} + g^{rr}R_{rr} + g^{\theta\theta}R_{\theta\theta} + g^{\phi\phi}R_{\phi\phi} \tag{2.2.9}$$

$$= -e^{-2\Phi}\left[e^{2\Phi}\left[\left(\Phi'' + (\Phi')^2\right)\left(1 - \frac{2m}{r} - \frac{r^2\Lambda}{3}\right) + \Phi'\left(\frac{2r - m'r - 3m}{r^2} - r\Lambda\right)\right]\right]$$

$$+(1 - \frac{2m}{r} - \frac{r^2\Lambda}{3})\left[(1 - \frac{2m}{r} - \frac{r^2\Lambda}{3})^{-1}\left[\frac{[\frac{\Lambda r^3}{3} - m + m'r][r\Phi' + 2]}{r^3}\right] - \left(\Phi'' + (\Phi')^2\right)\right]$$

$$+\frac{1}{r^2}\left[m' + \frac{m}{r} + \frac{2r^2\Lambda}{3} - \Phi'(-r + 2m + \frac{r^3\Lambda}{3})\right]$$

$$+\frac{1}{r^2 sin^2\theta}\left[sin^2\theta\left(m' + \frac{m}{r} + \frac{2r^2\Lambda}{3} - \Phi'(-r + 2m + \frac{r^3\Lambda}{3})\right)\right]$$

$$= -\left(\Phi'' + (\Phi')^2\right)(1 - \frac{2m}{r} - \frac{r^2\Lambda}{3}) - \Phi'\left(\frac{2r - m'r - 3m}{r^2} - r\Lambda\right) + \frac{[\frac{\Lambda r^3}{3} - m + m'r][r\Phi' + 2]}{r^3}$$

$$-\left(\Phi'' + (\Phi')^2\right)(1 - \frac{2m}{r} - \frac{r^2\Lambda}{3}) + \frac{2}{r^2}\left(m' + \frac{m}{r} + \frac{2r^2\Lambda}{3} - \Phi'(-r + 2m + \frac{r^3\Lambda}{3})\right)$$

$$= -2\left(\Phi'' + (\Phi')^2\right)(1 - \frac{2m}{r} - \frac{r^2\Lambda}{3}) - \Phi'\left(\frac{2r - m'r - 3m}{r^2} - r\Lambda\right) + \Phi'\left[\frac{\frac{\Lambda r^3}{3} - m + m'r}{r^2}\right]$$

$$2\left(\frac{\frac{\Lambda r^3}{3} - m + m'r}{r^3}\right) + \frac{2m'}{r^2} + \frac{2m}{r^3} + \frac{4\Lambda}{3} - \Phi'\left(\frac{2}{r} - \frac{4m}{r^2} - \frac{2r\Lambda}{3}\right)$$

$$= -2\left(\Phi'' + (\Phi')^2\right)(1 - \frac{2m}{r} - \frac{r^2\Lambda}{3}) + \Phi'\left(2\Lambda r + \frac{2m'r}{r^2} + \frac{6m}{r^2} - \frac{4r}{r^2}\right) + \frac{4m'}{r^2} + 2\Lambda$$

$$\mathbf{R} = 2\left[\frac{2m'}{r^2} + \Lambda + \Phi'\left(\frac{m'r + 3m - 2r}{r^2} + \Lambda r\right) - (\Phi'' + (\Phi')^2)\left(1 - \frac{2m}{r} - \frac{r^2\Lambda}{3}\right)\right] \tag{2.2.10}$$

2.3 Energy-momentum tensor and Field Equation

The stress energy-momentum tensor $T_{\mu\nu}$ of a perfect fluid is given in terms of pressure P and the energy-density ϵ of a given stellar object and is defined by .

$$T_{\mu\nu} = (\epsilon + P)U_\mu U_\nu + g_{\mu\nu}P \tag{2.3.1}$$

Where ϵ -is the energy of the fluid, also it's

$$\epsilon = (\rho c^2 + \varepsilon) \tag{2.3.2}$$

Where ρ-rest mass density.

Where ε-internal energy (thermal motion of particles)

Also we know that from normalization

$$g^{\mu\nu}U_\mu U_\nu = -1 \tag{2.3.3}$$

Now using (**??**),(**??**),(**??**) we can find each component of $T_{\mu\nu}$ and $T^{\mu\nu}$

$$T_{\mu\nu} = \begin{pmatrix} e^{2\Phi}\epsilon & 0 & 0 & 0 \\ 0 & P(1 - 2\frac{m}{r} - r^2\frac{\Lambda}{3})^{-1} & 0 & 0 \\ 0 & 0 & r^2 P & 0 \\ 0 & 0 & 0 & r^2 P\sin^2\theta \end{pmatrix} \tag{2.3.4}$$

the inverse of $T_{\mu\nu}$

$$T^{\mu\nu} = \begin{pmatrix} e^{-2\Phi}\epsilon & 0 & 0 & 0 \\ 0 & P(1-2\frac{m}{r}-r^2\frac{\Lambda}{3}) & 0 & 0 \\ 0 & 0 & r^{-2}P & 0 \\ 0 & 0 & 0 & r^{-2}P\csc^2\theta \end{pmatrix} \tag{2.3.5}$$

2.4 Hydrodynamics Instability and Effect of Gravitation in SDS Background

2.4.1 Tolmn-Oppenheimer-Volkoff Equations

With SDS metric, we will use Einsteins equations to determine the equations for the structure of the star, so we can calculate the limits on the size and mass of the star. First we need some parameters to describe the star itself. We will use a perfect fluid to model the stars distribution of matter, since shear stresses, viscosity, or heat conduction are negligible on a hydrodynamic time scale because all fermion states are already occupied (full electron or neutron degeneracy). Therefore the star can be described in its rest frame by just two parameters: the mass density $\rho(r)$ and isotropic pressure $P(r)$

$$G_{\mu\nu} = 8\pi T_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + g_{\mu\nu}\Lambda \tag{2.4.1}$$

then the tt component became,

$$G_{tt} = 8\pi T_{tt} = R_{tt} - \frac{1}{2}g_{tt}R + g_{tt}\Lambda \tag{2.4.2}$$

by using (**??**),(**??**),(**??**),(**??**),(**??**)

$$8\pi e^{2\Phi}\epsilon = e^{2\Phi}\left[\left(\Phi^{''}+(\Phi^{'})^2\right)\left(1-\frac{2m}{r}-\frac{r^2\Lambda}{3}\right)+\Phi^{'}\left(\frac{2r-m^{'}r-3m}{r^2}-r\Lambda\right)\right]$$

$$+e^{2\Phi}\left[\frac{2m^{'}}{r^2}+\Lambda+\Phi^{'}\left(\frac{m^{'}r+3m-2r}{r^2}+\Lambda r\right)-(\Phi^{''}+(\Phi^{'})^2)\left(1-\frac{2m}{r}-\frac{r^2\Lambda}{3}\right)\right]-e^{2\Phi}\Lambda$$

$$8\pi\epsilon = \left(\Phi^{''}+(\Phi^{'})^2\right)\left(1-\frac{2m}{r}-\frac{r^2\Lambda}{3}\right)+\Phi^{'}\left(\frac{2r-m^{'}r-3m}{r^2}-r\Lambda\right)+\frac{2m^{'}}{r^2}+\Lambda+\Phi^{'}\left(\frac{m^{'}r+3m-2r}{r^2}+\Lambda r\right)$$

$$-(\Phi^{''}+(\Phi^{'})^2)\left(1-\frac{2m}{r}-\frac{r^2\Lambda}{3}\right)-\Lambda$$

$$8\pi\epsilon = \Phi^{'}\left[\frac{2r-m^{'}r-3m+3m+m^{'}r-2r}{r^2}\right]+\frac{2m^{'}}{r^2}$$

$$4\pi\epsilon = \frac{m^{'}}{r^2}$$

$$m^{'} = 4\pi r^2\epsilon$$

$$\frac{dm}{dr} = 4\pi r^2\epsilon \qquad (2.4.3)$$

now to by using (**??**),(**??**),(**??**),(**??**) we find $\Phi^{'}$ as,

$$G_{rr} = 8\pi T_{rr} = R_{rr} - \frac{1}{2}g_{rr}R + g_{rr}\Lambda \qquad (2.4.4)$$

$$8\pi P(1-2\frac{m}{r}-r^2\frac{\Lambda}{3})^{-1} = (1-\frac{2m}{r}-\frac{r^2\Lambda}{3})^{-1}\left[\frac{[\frac{\Lambda r^3}{3}-m+m^{'}r][r\Phi^{'}+2]}{r^3}\right]+(1-2\frac{m}{r}-r^2\frac{\Lambda}{3})^{-1}\Lambda-(\Phi^{''}+(\Phi^{'})^2)$$

$$-(1-2\frac{m}{r}-r^2\frac{\Lambda}{3})^{-1}\left[\frac{2m^{'}}{r^2}+\Lambda+\Phi^{'}\left(\frac{m^{'}r+3m-2r}{r^2}+\Lambda r\right)-(\Phi^{''}+(\Phi^{'})^2)\left(1-\frac{2m}{r}-\frac{r^2\Lambda}{3}\right)\right]$$

$$8\pi P = \Phi^{'}\left[\frac{\frac{\Lambda r^3}{3}-m+m^{'}r}{r^2}\right]+2\left[\frac{\frac{\Lambda r^3}{3}-m+m^{'}r}{r^3}\right]-\Phi^{'}\left[\frac{m^{'}r+3m-2r}{r^2}+\Lambda r\right]-\frac{2m}{r^2}$$

$$8\pi P = \Phi^{'}\left(\frac{\Lambda r}{3}-\Lambda r+\frac{-m+m^{'}r-m^{'}r-3m+2r}{r^2}\right)+\frac{2\Lambda}{3}-\frac{2m}{r^3}$$

$$8\pi P = 2\Phi^{'}\left(\frac{r-2m}{r^2}-\frac{\Lambda r}{3}\right)+\frac{2\Lambda}{3}-\frac{2m}{r^3}$$

$$\Phi^{'}\left(\frac{r-2m}{r^2}-\frac{\Lambda r}{3}\right) = \frac{m-\frac{\Lambda r^3}{3}+4\pi r^3 P}{r^3}$$

$$\Phi^{'} = \frac{m - \frac{\Lambda r^3}{3} + 4\pi r^3 P}{r(r - 2m - \frac{\Lambda r^3}{3})} \tag{2.4.5}$$

$$\frac{d\Phi}{dr} = \frac{m - \frac{\Lambda r^3}{3} + 4\pi r^3 P}{r(r - 2m - \frac{\Lambda r^3}{3})}$$

Another relevant issue arises from the conservation equation, $\nabla_\nu T^{\mu\nu} = 0$, which gives an equation that relates the pressure gradient in the radial direction to the fluid density and the pressure. choosing $\mu = r$.

$$\nabla_\nu T^{\mu\nu} = \frac{\partial T^{\mu\nu}}{\partial x^\nu} + T^{\sigma\nu}\Gamma^\mu_{\sigma\nu} + T^{\mu\sigma}\Gamma^\nu_{\sigma\nu} = 0$$

$$\frac{\partial T^{rr}}{\partial r} + T^{tt}\Gamma^r_{tt} + T^{rr}\Gamma^r_{rr} + T^{\theta\theta}\Gamma^r_{\theta\theta} + T^{\phi\phi}\Gamma^r_{\phi\phi} + T^{rr}\left[\Gamma^t_{rt} + \Gamma^r_{rr} + \Gamma^\theta_{r\theta} + \Gamma^\phi_{r\phi}\right] = 0$$

$$\frac{d}{dr}P(1 - 2\frac{m}{r} - r^2\frac{\Lambda}{3}) + e^{-2\Phi}\epsilon(\Phi^{'}e^{2\Phi}(1 - 2\frac{m}{r} - r^2\frac{\Lambda}{3})) + (1 - 2\frac{m}{r} - r^2\frac{\Lambda}{3})P\left[\frac{\frac{\Lambda r}{3} - \frac{m}{r^2} + \frac{m^{'}r}{r^2}}{1 - 2\frac{m}{r} - r^2\frac{\Lambda}{3}}\right]$$

$$+r^{-2}P - r(1 - 2\frac{m}{r} - r^2\frac{\Lambda}{3}) + r^{-2}csc^2\theta P(-rK(1 - 2\frac{m}{r} - r^2\frac{\Lambda}{3})sin^2\theta)$$

$$+(1 - 2\frac{m}{r} - r^2\frac{\Lambda}{3})P\left[\Phi^{'} + \frac{2}{r} + \frac{\frac{\Lambda r}{3} - \frac{m}{r^2} + \frac{m^{'}r}{r^2}}{1 - 2\frac{m}{r} - r^2\frac{\Lambda}{3}}\right] = 0$$

$$(1 - 2\frac{m}{r} - r^2\frac{\Lambda}{3})\left[P^{'} + (\epsilon + P)\Phi^{'}\right] = 0$$

Then,

$$P^{'} = -(\epsilon + P)\Phi^{'} \tag{2.4.6}$$

$$\frac{dP}{dr} = -(\epsilon + P)\frac{m - \frac{\Lambda r^3}{3} + 4\pi r^3 P}{r(r - 2m - \frac{\Lambda r^3}{3})} \tag{2.4.7}$$

Then the TOV equation summarized using (**??**), (**??**), (**??**)

$$\frac{dm}{dr} = 4\pi r^2\epsilon$$

$$\frac{d\Phi}{dr} = \frac{m - \frac{\Lambda r^3}{3} + 4\pi r^3 P}{r(r - 2m - \frac{\Lambda r^3}{3})}$$

$$\frac{dP}{dr} = -(\epsilon + P)\frac{m - \frac{\Lambda r^3}{3} + 4\pi r^3 P}{r(r - 2m - \frac{\Lambda r^3}{3})}$$

For realistic equations of state,the equations of stellar structure can be integrated analytically for some idealized and ad hoc equations of state.We shall consider in the next section one of the most useful analytic solutions.

2.5 Equation of State

To progress toward a solution of the Equation (**??**) we assume a polytropic equation of state (EOS) for a relation between the isotropic pressure and the rest mass density

$$P = K\rho^{\Gamma} \tag{2.5.1}$$

where Γ is the adiabatic index and K is a normalization constant. For adiabatic processes we may neglect heat transfer (i.e.$dQ = 0$) so the first law of thermodynamics is simply

$$dU = -PdV \tag{2.5.2}$$

where $U = \epsilon V$ is the total energy of the fluid in a volume V , including both the rest energy and internal energy. However, we may write the rest mass density as $\rho = mN = V$, where N is the number of particles of mass m in the same volume V . In other words, the first law of thermodynamics can be written

$$d\left(\frac{\epsilon}{\rho}\right) = -Pd\left(\frac{1}{\rho}\right) = \frac{P}{\rho^2}d\rho = K\rho^{\Gamma-2}d\rho \tag{2.5.3}$$

The last equality was obtained by applying the polytropic EOS $P = K\rho^{\Gamma}$ to the star. The integrated equation is written suggestively as

$$\frac{\epsilon}{\rho} = (a+1) + \frac{K}{\Gamma-1}\rho^{\Gamma-1} \tag{2.5.4}$$

The constant of integration is placed in the equation to ensure continuity of the piecewise polytropic EOS which we will adopt later on. In fact, our primary concern is that in the low density limit all energy originates from the rest mass. Specifically, we require $\lim_{\rho\to 0}\frac{\epsilon}{\rho} = 1$

and set $a = 0$ for the stellar boundary.The form of the equation is nicer when we remember the form of the internal energy $\varepsilon = \epsilon - \rho$ and the specific enthalpy $h = \frac{(\epsilon+p)}{\rho}$. The resulting equations are

$$\varepsilon = a\rho + \frac{p}{\Gamma - 1} \qquad \textit{and} \qquad h = 1 + a + \frac{\Gamma}{\Gamma - 1}\frac{p}{\rho} \tag{2.5.5}$$

Piecewise Polytrope

A nuclear equation of state is often given as a table or a piecewise polytrope for dividing densities $\rho_0 < \rho_1 < \rho_2 < ...$ based on the various envelopes and crusts . Thus, we continue by expressing the above equations in a piecewise manner and the important fluid variables are smoothly given in each section by

$$\begin{aligned} \varepsilon &= (1 + a_i)\rho + \frac{K_i}{\Gamma_i - 1}\rho^{\Gamma_i} \\ \epsilon &= a_i\rho + \frac{K_i}{\Gamma_i - 1}\rho^{\Gamma_i} \\ h &= 1 + a_i + \frac{\Gamma_i}{\Gamma_i - 1}K_i\rho^{\Gamma_i - 1} \end{aligned} \tag{2.5.6}$$

Recall that the integration constants a_i are chosen to ensure the energy is smooth at the transitions in the piecewise function so that

$$\begin{aligned} a_0 &= 0 \\ a_i &= a_{i-1} + \frac{K_{i-1}}{\Gamma_{i-1} - 1}\rho_i^{\Gamma_{i-1}-1} - \frac{K_i}{\Gamma_i - 1}\rho_i^{\Gamma_i - 1} \end{aligned} \tag{2.5.7}$$

Now when integrating the TOV equations, it is useful to define a generalization of the Newtonian specific enthalpy

$$\eta = h - 1 \tag{2.5.8}$$

which subtracts off the contribution from the rest mass of the fluid. Furthermore, the polytropic index $n_i = \frac{1}{\Gamma_i - 1}$ is defined exactly as in the class notes. Thus we may write the fluid variables in terms of η in the following manner

$$\rho(\eta) = \left(\frac{\eta - a_i}{K_i(n_i + 1)}\right)^{n_i}$$

$$p(\eta) = K_i \left(\frac{\eta - a_i}{K_i(n_i + 1)} \right)^{n_i+1}$$

$$\epsilon(\eta) = \rho(\eta) \left(1 + \frac{a_i + n_i \eta}{n_i + 1} \right) \tag{2.5.9}$$

The TOV equations of (**??**) diverge at $r = 0$ and can be difficult to integrate numerically for $r \to 0$. A common technique in the literature to continue analytically and avoid singular equations is to change variables by defining a pseudo-enthalpy

$$y(p) = \int^{R} \frac{dp^{'}}{\epsilon(p^{'}) + P^{'}} \tag{2.5.10}$$

Indeed the TOV equation for $d\Phi = dr$ can be integrated immediately to give $e^{y+\Phi} = he^{\Phi} = \sqrt{1 - \frac{2M}{R} - \frac{\Lambda R^2}{3}}$, where M and R are the mass and radius of the star. This follows from the relation $d\Phi = dy$ and choosing the constant of integration to match the Schwarzschild spacetime beyond the surface of the star. The meaning of this new insight is that the gravitational potential Φ for the star is fully determined if we can integrate the other two TOV equations. This becomes apparent when we complete the change of variables to the Newtonian specific enthalpy η. TOV Equation becomes

$$\frac{dr}{d\eta} = -\frac{r(r-2m)}{m + 4\pi r^3 p(\eta)} \frac{1}{\eta + 1} \tag{2.5.11}$$

$$\frac{dm}{d\eta} = 4\pi r^2 \epsilon(\eta) \frac{dr}{d\eta} \tag{2.5.12}$$

which are well-behaved both at the center of the star and at the surface.

Recall that in the case of a relativistic star with $\rho = \bar{\rho}$, it is not necessary to use the unrealistic notion of an incompressible fluid. One can think of the fluids with pressure growing as radius decreases, having a composition that varies from one radius to another., and being hand-tailored [**?**].

Assuming $\rho = \bar{\rho}$, we can integrate the structure equations analytically. First, we obtain from the mass formula (**??**) that

$$m(r) = \frac{4\pi}{3}\rho r^3 \tag{2.5.13}$$

At the surface of the star $(r = R)$,we obtain the total mass of the star

$$M = m(r) = \frac{4\pi}{3}\rho R^3 \tag{2.5.14}$$

Now ,we can easily rearrange the radial component of the metric tensor using (**??**)

$$\left(1 - \frac{2m}{r} - \frac{\Lambda r^2}{3}\right)^{-1} = (1 - \frac{r^2}{\beta^2})^{-1} \tag{2.5.15}$$

where we have introduced a new parameter, β by the relation

$$\frac{1}{\beta^2} = \frac{1}{3}(8\pi\rho + \Lambda) \tag{2.5.16}$$

At the surface of the star ,there is

$$\left(1 - \frac{2M}{R} - \frac{\Lambda R^2}{3}\right)^{-1} = \left(1 - \frac{R^2}{\beta^2}\right)^{-1} \tag{2.5.17}$$

and we can see immediately that the radial metric coefficient of the interior spacetime is smoothly matched to the corresponding metric coefficient of the exterior Schwarzschild-de Sitter spacetime of the mass parameter $M = m(R)$.

If $\rho = \bar{\rho}$,the modified TOV equation (**??**) became,

$$\frac{dP}{(P + \rho)(3P + \rho - \frac{\Lambda}{4\pi})} = -\frac{4\pi}{3}\frac{rdr}{(1 - \frac{r^2}{\beta^2})} \tag{2.5.18}$$

which have to be integrated from the surface of the star $(R = r)$, where $P(R) = 0$, down to the center of the star at $r = 0$. For a non-zero cosmological constant we find the pressure at a radius r to be given by the relation

$$P(r) = \frac{\rho(\rho - \frac{\Lambda}{4\pi}\left[(1 - \frac{r^2}{\alpha^2})^{\frac{1}{2}} - (1 - \frac{R^2}{\beta^2})^{\frac{1}{2}}\right]}{3\rho(1 - \frac{R^2}{\beta^2})^{\frac{1}{2}} - (\rho - \frac{\Lambda}{4\pi})(1 - \frac{r^2}{\beta^2})^{\frac{1}{2}}} \tag{2.5.19}$$

The maximum pressure is at the center of the star, where

$$p_c = p(r = 0) = \frac{\rho(\rho - \frac{\Lambda}{4\pi}\left[1 - (1 - \frac{R^2}{\alpha^2})^{\frac{1}{2}}\right]}{3\rho(1 - \frac{R^2}{\beta^2})^{\frac{1}{2}} - (\rho - \frac{\Lambda}{4\pi})} \tag{2.5.20}$$

Chapter 3

Result and Discussion

The influence of the repulsive cosmological constant on the black-hole space-time structure can be properly represented by the dimensionless cosmological parameter $y = \frac{\Lambda}{3}M^2$. For SdS black holes admitting existence of stable circular geodesics,i.e., existence of accretion discs, the cosmological parameter $y < y_{ms,e} = 0.0002378$ [?]. The cosmological tests using the supernova magnitude-redshift relation and the Cosmic Microwave Background Radiation fluctuations measurements[?] imply $\Lambda \approx 10^{-56} cm^{-2}$, and thus very low values of y for astrophysically realistic black holes. In fact,$y \sim 10^{-24}$ for super-massive black holes[?]; strong optical observable effects could be expected for super-giant black holes [?] (or clusters of galaxies) with $M \geq 10^{15} M_{\bigodot}$.

For astrophysically realistic SdS black holes, the strong gravity near the black hole horizon $r_h \sim 2M$ weakens with distance growing and at $r >> M$ can be described quite well by the Newtonian theory. However, the Newtonian theory looses its validity near the so-called static radius $r_s \sim y^{\frac{-1}{3}} M$, where the repulsive effect of the cosmological constant starts to be relevant up to the other strong gravitation region near the cosmological horizon $r_c \sim y^{\frac{-1}{2}} M$. Therefore, the cosmological constant has relevant influence on the structure of disc configurations introducing quite naturally outer edge of the accretion discs[?, ?].Inthis chapter we are going to discuss both the interior and exterior solutions

3.1 Exterior solution

In the Schwarzschild-de Sitter space-time there may exist a black-hole event horizon and there may also exist a cosmological event horizon. It depends on M and Λ in the Schwarzschild-de Sitter metric (**??**) which of the cases occur.Where M is the mass parameter of the space-times. It is useful to introduce the dimensionless parameter (**??**) and use dimensionless coordinates $t \rightarrow \frac{t}{M}$, $r \rightarrow \frac{r}{M}$, which is equivalent to putting $M = 1$ .Singularities of the line element, i.e., black-hole and cosmological horizons, are given by the relation $1 - \frac{2}{2} - yr^2 = 0$, thus by solutions of the equation

$$y = y_h \equiv \frac{r-2}{r^3} \qquad (3.1.1)$$

which can be expressed in the form

$$r_h = \frac{2}{\sqrt{3y}} \cos\frac{\pi + \xi}{3} \quad , \quad r_c = \frac{2}{\sqrt{3y}} \cos\frac{\pi - \xi}{3} \qquad (3.1.2)$$

where $\xi = \cos^{-1}(3\sqrt{3y})$.

The pressure at any relativistic star must be finite and positive. The restrictions on (**??**)

$$\rho - \frac{\Lambda}{4\pi} \geq 0 \qquad (3.1.3)$$

$$3\rho(1 - \frac{R^2}{\beta^2})^{\frac{1}{2}} - (\rho - \frac{\Lambda}{4\pi}) \geq 0 \qquad (3.1.4)$$

yield limits on the allowed values of outer radii R of the stars. The equality in (**??**) determines limiting configurations with a divergent central pressure.Substituting for ρ and β^2 from (**??**), and (**??**), respectively, and introducing new dimensionless cosmological and radius parameters by the relations

$$y = \frac{1}{3}\Lambda M^2 \qquad (3.1.5)$$

$$x = \frac{R}{M} \tag{3.1.6}$$

the condition (**??**) can be transformed into the relation(for derivation see appendix (**??**))

$$[y - y_+][y - y_-] \leq 0 \tag{3.1.7}$$

where

$$y_\pm(x) \equiv \frac{2x - 9 \pm 3 \mid 2x - 3 \mid}{2x^4} \tag{3.1.8}$$

For the cosmological repulsion ($y > 0$) only the function

$$y_+(x) = \frac{4x - 9}{x^4} \tag{3.1.9}$$

is relevant at $x \geq \frac{9}{4}$.

If $x = \frac{9}{4}$,it is the solution for vanishing cosmological constant $\Lambda = 0$. From generalised Buchdahl inequality [**?**] one finds

$$R^2 < \frac{\frac{1}{3}\left(4 - \frac{\Lambda}{4\pi\rho}\right)}{4\pi\rho} \tag{3.1.10}$$

the boundary in this case ,

$$R^2 < \frac{1}{3\pi\rho}$$

using $M = \frac{4\pi}{3}\rho R^3$ leads to

$$M < \frac{4}{9}R \tag{3.1.11}$$

Because of the analogous inequalities of the former cases write

$$3M < \frac{4}{3}R \tag{3.1.12}$$

we arrive at the well known limit of the interior Schwarzschild solutions [**?**]. However, the validity of the condition (**??**) is restricted to the region up to the maximum of $y_+(x)$, given by (**??**). It is located at $x_{max} = 3$, where $y_{max} = \frac{1}{27}$. At $x \geq x_{max} = 3$, the relevant condition is (**??**) which determines a critical value of the cosmological constant for a given mass parameter M . In terms of the dimensionless parameters x and y , it implies the condition

$$y \leq y_{stat} \equiv \frac{1}{x^3} \tag{3.1.13}$$

For $y = y_{stat}$,were $\Lambda = 4\pi\rho$

The outer radius of the star is located just at the so called static radius r_s of the corresponding external Schwarzschild-de Sitter spacetime. At $r = r_s$, the gravitational attraction acting on a test particle is just compensated by the cosmological repulsion [?].

For $y > y_{stat}$, were $\Lambda > 4\pi\rho$.

At $r > r_s$, the repulsion prevails, and a static configuration is possible only with a surface stress acting inwards.In this case the horizons disappear and the SdS spacetimes become dynamic naked singularity spacetimes

3.2 Interior solution

For an attractive cosmological constant, $\Lambda < 0$, the relations (??) and (??) have to be satisfied again, but we obtain an other family of critical values of the cosmological constant, given by the condition $\frac{1}{\beta^2} = 0$. In terms of the dimensionless parameters x and y , we arrive at

$$y_{crit} = -\frac{2}{x^3} \tag{3.2.1}$$

in terms of the constant density ρ , the critical value of the cosmological constant is given by

$$\Lambda_{crit} = -8\pi\rho \tag{3.2.2}$$

Now, we have to distinguish the cases $y > y_{crit}, y < y_{crit}, \quad and \quad y = y_{crit}$.

If $y > y_{crit}(\frac{1}{\beta^2} > 0)$, the relations (??) and (??) are valid. Recall that at $x \geq \frac{3}{2}$, there is $y_-(x) = y_{crit}(x)$,while at $x \leq \frac{3}{2}$ there is $y_+(x) = y_{crit}(x)$. For $x = \frac{3}{2}, y_-(x) = y_+(x) =$

$-\frac{16}{27}$.Therefore, in addition to $y > y_{crit} \equiv -\frac{2}{x^3}$, there must be satisfied the condition

$$-\frac{2}{x^3} \leq y \leq \frac{4x-9}{x^4} \tag{3.2.3}$$

at $x > \frac{3}{2}$,and

$$\frac{4x-9}{x^4} \leq y \leq -\frac{2}{x^3} \tag{3.2.4}$$

at $x < \frac{3}{2}$

If $y < y_{crit}(\frac{1}{\beta^2} < 0)$ the relation (**??**) has to be replaced by

$$[y - y_+][y - y_-] \geq 0 \tag{3.2.5}$$

In addition to $y < y_{crit} \equiv -\frac{2}{x^3}$, we obtain the conditions

$$y \geq -\frac{2}{x^3} \quad or \quad y \leq \frac{4x-9}{x^4} \tag{3.2.6}$$

at $x < \frac{3}{2}$.

It follows from the conditions (**??**)-(**??**) that for $\Lambda < 0$ the static configurations are allowed at radii satisfying the condition

$$y \leq \frac{4x-9}{x^4} \tag{3.2.7}$$

At $y = y_{crit}(\frac{1}{\beta^2} = 0)$,were $\Lambda = -8\pi\rho$

In this case (**??**) can be written as,

$$\frac{dp}{dr} = -4\pi r(\rho + p)^2 \tag{3.2.8}$$

after integration the the pressure is given by

$$p(r) = \frac{2\pi\rho^2(R^2 - r^2)}{1 - 2\pi\rho(R^2 - r^2)} \tag{3.2.9}$$

In term of the dimensionless parameters, the central pressure of this special class of solutions is determined by the relation

$$p_c = p(0) = \frac{3\rho}{2x-3} = -\frac{3\Lambda_{crit}}{8\pi(2x-3)} \tag{3.2.10}$$

also(**??**) can be integrated if the constant of integration is fixed at the centre by $p(r=0)=p_c$, one obtains

$$p(r) = \frac{1}{2\pi r^2 + \frac{1}{p_c+\rho}} - \rho \tag{3.2.11}$$

as $r \to \infty$ then $p \to -\rho$. At $p(R)=0$,from this we find,

$$R^2 = \frac{1}{2\pi}\left(\frac{1}{\rho} - \frac{1}{p_c+\rho}\right) \tag{3.2.12}$$

One finds that the radius R is bounded by $\frac{1}{\sqrt{2\pi\rho}}$. Inserting the definition of the density yields to

$$R^2 < \frac{1}{2\pi\rho} = \frac{1}{2\pi}\frac{4\pi R^3}{3M} \tag{3.2.13}$$

which implies

$$3M < 2R \tag{3.2.14}$$

Clearly, the special class of static configurations with $y = y_{crit}$ is allowed for $x \geq \frac{3}{2}$ only. Values of the cosmological parameter y must be restricted by the condition

$$-\frac{16}{27} \leq y < 0 \tag{3.2.15}$$

3.3 Potential

Finally, we determine the time component of the internal metric tensor, using the boundary condition of smooth matching of the internal solution onto the external time metric coefficient at $r = R$:

$$e^{2\Phi(R)} = \left(1 - \frac{2M}{R} - \frac{1}{3}\Lambda R^2\right) \tag{3.3.1}$$

or

$$e^{\Phi(R)} = \left(1 - \frac{2M}{R} - \frac{1}{3}\Lambda R^2\right)^{\frac{1}{2}} \tag{3.3.2}$$

The function $\Phi(r)$ can be found from (**??**) by using the relation for the pressure as a function of radius(**??**). If $\frac{1}{\beta^2} \neq 0$, we arrive at the expression

$$\frac{d\Phi}{dr} = \frac{m - \frac{\Lambda r^3}{3} + 4\pi r^3 P}{r(r - 2m - \frac{\Lambda r^3}{3})}$$

$$= \frac{\frac{4\pi\rho r^3}{3} - \frac{\Lambda r^3}{3} + 4\pi r^{3p(r)}}{r^2(1 - \frac{8\pi\rho r^2}{3} - \frac{\Lambda r^2}{3})}$$

$$= \frac{\frac{4\pi}{3}(\rho - \frac{\Lambda}{4\pi})r + 4\pi r p(r)}{(1 - \frac{r^2}{\beta^2})}$$

now using (**??**),we can wright,

$$= \frac{4\pi 3(\rho - \frac{\Lambda}{4\pi})r + 4\pi r\left[\frac{\rho(\rho - \frac{\Lambda}{4\pi}\left[(1-\frac{r^2}{\beta^2})^{\frac{1}{2}} - (1-\frac{R^2}{\beta^2})^{\frac{1}{2}}\right]}{3\rho(1-\frac{R^2}{\beta^2})^{\frac{1}{2}} - (\rho-\frac{\Lambda}{4\pi})(1-\frac{r^2}{\beta^2})^{\frac{1}{2}}}\right]}{(1 - \frac{r^2}{\beta^2})}$$

$$= \frac{4\pi\rho(\rho - \frac{\Lambda}{4\pi})(1 - \frac{R^2}{\beta^2})^{\frac{1}{2}} r - \frac{4\pi}{3}(\rho - \frac{\Lambda}{4\pi})^2(1 - \frac{r^2}{\beta^2})r + 4\pi\rho(\rho - \frac{\Lambda}{4\pi})(1 - \frac{r^2}{\beta^2})^{\frac{1}{2}} r - 4\pi\rho(\rho - \frac{\Lambda}{4\pi})(1 - \frac{R^2}{\beta^2})^{\frac{1}{2}} r}{(1 - \frac{r^2}{\beta^2})(3\rho(1 - \frac{R^2}{\beta^2})^{\frac{1}{2}} - (\rho - \frac{\Lambda}{4\pi})(1 - \frac{r^2}{\beta^2})^{\frac{1}{2}})}$$

then,removing the first and the forth term, the equation became,

$$= \frac{4\pi\rho(\rho - \frac{\Lambda}{4\pi})(1 - \frac{r^2}{\beta^2})^{\frac{1}{2}} r - \frac{4\pi}{3}(\rho - \frac{\Lambda}{4\pi})^{\frac{1}{2}}(1 - \frac{r^2}{\beta^2})r}{(1 - \frac{r^2}{\beta^2})(3\rho(1 - \frac{R^2}{\beta^2})^{\frac{1}{2}} - (\rho - \frac{\Lambda}{4\pi})(1 - \frac{r^2}{\beta^2})^{\frac{1}{2}})}$$

$$= \frac{(\rho - \frac{1}{3}(\rho - \frac{\Lambda}{4\pi}))4\pi(\rho - \frac{\Lambda}{4\pi})(1 - \frac{r^2}{\beta^2})^{\frac{-1}{2}} r}{3\rho(1 - \frac{R^2}{\beta^2})^{\frac{1}{2}} - (\rho - \frac{\Lambda}{4\pi})(1 - \frac{r^2}{\beta^2})^{\frac{1}{2}}}$$

$$d\Phi = \frac{\frac{1}{\beta^2}(\rho - \frac{\Lambda}{4\pi})(1 - \frac{r^2}{\beta^2})^{\frac{-1}{2}} r dr}{3\rho(1 - \frac{R^2}{\beta^2})^{\frac{1}{2}} - (\rho - \frac{\Lambda}{4\pi})(1 - \frac{r^2}{\beta^2})^{\frac{1}{2}}}$$

then we integrate both sides,

$$\int_R^r d\Phi = \int_R^r \frac{\frac{1}{\beta^2}(\rho - \frac{\Lambda}{4\pi})(1 - \frac{r^2}{\beta^2})^{\frac{-1}{2}} dr}{3\rho(1 - \frac{R^2}{\beta^2})^{\frac{1}{2}} - (\rho - \frac{\Lambda}{4\pi})(1 - \frac{r^2}{\beta^2})^{\frac{1}{2}}}$$

using the technique of integration by substitution the above integral became,

$$\Phi(r) - \Phi(R) = \ln\left[3\rho(1 - \frac{R^2}{\beta^2})^{\frac{1}{2}} - (\rho - \frac{\Lambda}{4\pi})(1 - \frac{r^2}{\beta^2})^{\frac{1}{2}}\right] - \ln\left[3\rho(1 - \frac{R^2}{\beta^2})^{\frac{1}{2}} - (\rho - \frac{\Lambda}{4\pi})(1 - \frac{R^2}{\beta^2})^{\frac{1}{2}}\right]$$

$$\Phi(r)-\Phi(R)=\ln\left[3\rho(1-\frac{R^2}{\beta^2})^{\frac{1}{2}}-(\rho-\frac{\Lambda}{4\pi})(1-\frac{r^2}{\beta^2})^{\frac{1}{2}}\right]-\ln\left[(2\rho+\frac{\Lambda}{4\pi})(1-\frac{R^2}{\beta^2})^{\frac{1}{2}}\right]$$

$$\Phi(r)-\Phi(R)=\ln\left[\frac{3\rho(1-\frac{R^2}{\beta^2})^{\frac{1}{2}}-(\rho-\frac{\Lambda}{4\pi})(1-\frac{r^2}{\beta^2})^{\frac{1}{2}}}{(2\rho+\frac{\Lambda}{4\pi})(1-\frac{R^2}{\beta^2})^{\frac{1}{2}}}\right]$$

$$e^{\Phi(r)-\Phi(R)}=\frac{3\rho(1-\frac{R^2}{\beta^2})^{\frac{1}{2}}-(\rho-\frac{\Lambda}{4\pi})(1-\frac{r^2}{\beta^2})^{\frac{1}{2}}}{(2\rho+\frac{\Lambda}{4\pi})(1-\frac{R^2}{\beta^2})^{\frac{1}{2}}}$$

$$\frac{e^{\Phi(r)}}{e^{\Phi(R)}}=\frac{3\rho(1-\frac{R^2}{\beta^2})^{\frac{1}{2}}}{(2\rho+\frac{\Lambda}{4\pi})(1-\frac{R^2}{\beta^2})^{\frac{1}{2}}}-\frac{(\rho-\frac{\Lambda}{4\pi})(1-\frac{r^2}{\beta^2})^{\frac{1}{2}}}{(2\rho+\frac{\Lambda}{4\pi})(1-\frac{R^2}{\beta^2})^{\frac{1}{2}}}$$

now using the relation (**??**),(**??**) also for ρ we can substitute using (**??**),then we get the potential out side of the compact object

$$e^{\Phi(r)}=\frac{9M}{6M+\Lambda R^3}\left(1-\frac{2M}{R}-\frac{1}{3}\Lambda R^2\right)^{\frac{1}{2}}-\frac{9M-\Lambda R^3}{6M+\Lambda R^3}\left(1-\frac{2Mr^2}{R^3}-\frac{1}{3}\Lambda r^2\right)^{\frac{1}{2}} \tag{3.3.3}$$

This can be write as

$$\Phi(r)=\ln\left[\frac{3M}{2R}\left[\frac{1}{\frac{M}{R}+\frac{\Lambda R^2}{6}}\right]\left((1-\frac{2M}{R}-\frac{1}{3}\Lambda R^2)^{\frac{1}{2}}-(1-\frac{\Lambda R^3}{9M})\left(1-\frac{2Mr^2}{R^3}-\frac{1}{3}\Lambda R^2\right)^{\frac{1}{2}}\right)\right] \tag{3.3.4}$$

now lets interpret the potential at different conditions,

1.When $r\leq R$

The potential derived (**??**) holds at $r\leq R$ equally for both cases $y>y_{crit}$ and $y<y_{crit}$. And it became

$$\Phi(r)=\ln\left[\frac{3M}{2R}\left[\frac{1}{\frac{M}{R}+\frac{\Lambda R^2}{6}}\right]\left((1-\frac{2M}{R}-\frac{1}{3}\Lambda R^2)^{\frac{1}{2}}-(1-\frac{\Lambda R^3}{9M})\left(1-\frac{2Mr^2}{R^3}-\frac{1}{3}\Lambda R^2\right)^{\frac{1}{2}}\right)\right]$$

2.When $r=R$

At $r=R$ the relation (**??**) really reduces to (**??**) so it became

$$\Phi(R)=\frac{1}{2}\ln\left(1-\frac{2M}{R}-\frac{1}{3}\Lambda R^2\right)$$

3.When $y=y_{crit}(\frac{1}{\beta^2}=0)$

$$\Phi(r)=\ln\left(1+\frac{3M}{2R}\left(\frac{r^2}{R^2}-1\right)\right)$$

Chapter 4

Summery and Conclusion

General theory of relativity is the theory of gravitation and geometry of spacetime.It generalizes the spacial theory of relativity and Newtons law of universal gravitation. The matter and geometry of spacetime are related by the Einstein field equations ($G_{\mu\nu} + g_{\mu\nu}\Lambda = 8\pi G T_{\mu\nu}$),where $G_{\mu\nu}$ is Einsteins field tensor that tales geometry of spacetime and $T_{\mu\nu}$ is energy-momentum tensor that is matter. Techniques equivalent this energy-momentum tensor is deduce from perfect fluid that is important to stellar structure and cosmology $T^{\mu\nu} = Pg^{\mu\nu} + (P + \rho)U^{\mu}U^{\nu}$. Generally, the spacetime geometry and gravitation are described by tensors specially second rank $(0, 2)$ tensors like Metric tensor,Reiman curvature tensor,Ricci tensor,Ricci scalar,Einstein field tensor and energy-momentum tensor in addition to Affine connections.Using schwarzchild-de sitter space-time metric we derive the basic $TOV - \Lambda$ equations then by using equation of state the pressure as a function of radius considering $\rho = \rho_0 = \rho_{mean}$ derived.From $TOV - \Lambda$ equations we derived important formulas of pressure and potential.

The optical reference geometry is very useful in attempts to understand the character of spherically symmetric spacetimes. Geodesics of the optical geometry exhibit some interesting physical propertiesthey coincide with the possible trajectories of light rays, massive particles require a speed-independent orthogonal thrust in order to move along them, and gyroscopes transported along them do not precess with respect to the direction of motion

. Moreover, the optical geometry appears to be very useful in the analysis of a variety of unusual processes that take place around compact objects . Properties of the optical spaces associated with spherically symmetric spacetimes can be appropriately represented by embedding diagrams of their central planes into the 3-dimensional Euclidean space. The embedding allows an accurate treatment of some non-trivial effects. However, the embedding is possible for a limited part of the optical space only. It is interesting that in the case of Schwarzschild spacetimes the limit of embeddability of the optical geometry $r = \frac{9}{4}M$, coincides with the minimum possible radius of a static configuration of matter of uniform density with fixed mass M .For the spacetimes with an attractive cosmological constant,$y < 0$, there is no outer limit of embeddability of the optical geometry and the inner limit coincides with the limit on radius of static configurations of uniform density. However, there exists a special class ($\frac{1}{\beta^2} = 0$) of the internal solutions which corresponds to the outer limit of embeddability of the ordinary induced geometry on $t = cons$, hypersurfaces . The existence of stable circular photon orbits inside the configurations having radius $R < 3m$; surprisingly, the limiting radius is independent of the cosmological constant. The radius $r = 3M$, corresponding to the unstable circular photon orbits in the external Schwarzschild-de Sitter and Schwarzschild-anti-de Sitter spacetimes plays a crucial role in embedding of the optical reference geometry associated with these spacetimes .When the radius $r \leq R$, the potential have the same value for $y > y_{crit}$ and $y < y_{crit}$.

Generally,according to the above discussions cosmological constant has an effect at large scale as well as local size and major impact on the motion of particles around compact object which cause instability due to the in flow motion of particles .

Appendix A

Derviation of Schawrzschil-de sitter- metrc

The line element is given as:

$$ds^2 = -Adt^2 + Bdr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \tag{A.0.1}$$

$$g_{\mu\nu} = \begin{pmatrix} -A & 0 & 0 & 0 \\ 0 & B & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 sin^2\theta \end{pmatrix} \qquad g^{\mu\nu} = \begin{pmatrix} -\frac{1}{A} & 0 & 0 & 0 \\ 0 & \frac{1}{B} & 0 & 0 \\ 0 & 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 0 & \frac{1}{r^2\sin^2\theta} \end{pmatrix} \tag{A.0.2}$$

then using(**??**),Then the non vanishing terms of Γ are,

$$\Gamma^r_{tt} = \frac{\acute{A}}{2B} \qquad \Gamma^r_{rr} = \frac{\acute{B}}{2B} \qquad \Gamma^r_{\theta\theta} = -\frac{r}{B}$$
$$\Gamma^r_{\phi\phi} = -r\frac{\sin^2\theta}{B} \qquad \Gamma^t_{rt} = \frac{\acute{A}}{2A} \qquad \Gamma^\phi_{\theta\phi} = \cot\theta$$
$$\Gamma^\theta_{\phi\phi} = -\cos\theta\sin\theta \qquad \Gamma^\phi_{r\phi} = \Gamma^\theta_{r\theta} = \frac{1}{r}$$

Now using (**??**), the Ricci tensors for the corresponding component found

For $\mathbf{R_{tt}}$

$$R_{tt} = \Gamma^r_{tt;r} + \Gamma^r_{tt}\left[\Gamma^t_{rt} + \Gamma^\theta_{r\theta} + \Gamma^\phi_{r\phi} + \Gamma^r_{rr}\right] - \left[\Gamma^t_{tr}\Gamma^r_{tt} + \Gamma^r_{tt}\Gamma^t_{rt}\right]$$

then,

$$R_{tt} = \frac{A''}{2B} - \frac{1}{4}\frac{A'}{B}\left[\frac{B'}{B} + \frac{A'}{A}\right] + \frac{A'}{Br} \tag{A.0.3}$$

For $\mathbf{R_{rr}}$

$$R_{rr} = -\Gamma^{t}_{rt;r} - \Gamma^{\theta}_{r\theta;r} - \Gamma^{\phi}_{r\phi;r} + \Gamma^{r}_{rr}\left[\Gamma^{t}_{rt} + \Gamma^{\theta}_{r\theta} + \Gamma^{\phi}_{r\phi}\right] - \left[\Gamma^{t}_{rt}\Gamma^{t}_{rt} + \Gamma^{\theta}_{r\theta}\Gamma^{\theta}_{r\theta} + \Gamma^{\phi}_{r\phi}\right]$$

$$R_{rr} = -\frac{A''}{2A} + \frac{1}{4}\frac{A'}{A}\left(\frac{A'}{A} + \frac{B'}{B}\right) + \frac{B'}{Br} \tag{A.0.4}$$

For $\mathbf{R}_{\theta\theta}$

$$R_{\theta\theta} = \Gamma^{r}_{\theta\theta;r} - \Gamma^{\phi}_{\theta\phi;\theta} + \Gamma^{r}_{\theta\theta}\left[\Gamma^{\theta}_{r\theta} + \Gamma^{t}_{rt} + \Gamma^{\phi}_{r\phi} + \Gamma^{r}_{rr}\right] - \left[\Gamma^{\theta}_{r\theta}\Gamma^{r}_{\theta\theta} + \Gamma^{r}_{\theta\theta}\Gamma^{r}_{r\theta} + \Gamma^{\phi}_{\theta\phi}\Gamma^{\phi}_{\phi\theta}\right] \tag{A.0.5}$$

$$R_{\theta\theta} = -\frac{1}{B} + \frac{r}{2B}\left[\frac{B'}{B} - \frac{A'}{A}\right] + 1 \tag{A.0.6}$$

For $\mathbf{R}_{\phi\phi}$

$$R_{\phi\phi} = \Gamma^{\theta}_{\phi\phi;\theta} + \Gamma^{r}_{\phi\phi;r} + \Gamma^{r}_{\phi\phi}\left[\Gamma^{t}_{rt} + \Gamma^{r}_{rr} + \Gamma^{\theta}_{r\theta} + \Gamma^{\phi}_{r\phi}\right] - 2\Gamma^{r}_{\phi\phi}\Gamma^{\phi}_{r\phi} - \Gamma^{\theta}_{\phi\phi}\Gamma^{\phi}_{\theta\phi}$$

$$R_{\phi\phi} = \sin^2\theta\left[-\frac{1}{B} + \frac{r}{2B}\left[\frac{B'}{B} - \frac{A'}{A}\right] + 1\right] \tag{A.0.7}$$

$$R_{\phi\phi} = sin^2\theta R_{\theta\theta} \tag{A.0.8}$$

Now from (**??**) and (**??**) we get,

$$\frac{B}{A}R_{tt} + R_{rr} = \frac{A'}{Ar} + \frac{B'}{Br} \tag{A.0.9}$$

From (**??**) by taking $T_{\mu\nu} = 0$ we get

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + g_{\mu\nu}\Lambda = 0 \tag{A.0.10}$$

then multiplying both sides by $g^{\mu\nu}$ we get

$$R = 4\Lambda \tag{A.0.11}$$

Then put (??) in to (??) we get

$$R_{\mu\nu} = g_{\mu\nu}\Lambda \tag{A.0.12}$$

using (??) and (??) the components of ricci tensors are as follow

$$R_{tt} = g_{tt}\Lambda = -A\Lambda \tag{A.0.13}$$

$$R_{rr} = g_{rr}\Lambda = -B\Lambda \tag{A.0.14}$$

$$R_{\theta\theta} = g_{\theta\theta}\Lambda = -r^2\Lambda \tag{A.0.15}$$

$$R_{\phi\phi} = g_{\phi\phi}\Lambda = -r^2 \sin^2\theta\Lambda \tag{A.0.16}$$

From this equation we find the relation

$$AB = constant \tag{A.0.17}$$

Now we impose the condition on A and on B at $r \to \infty$, the metric became minkowskian in spherically closed system.

as $A \to 1, B \to 1$

$$AB = 1$$

$$A(r) = \frac{1}{B(r)} \tag{A.0.18}$$

Now put (??) and (??) into (??) then the result is

$$\frac{B'}{B} = -\frac{A'}{A} \tag{A.0.19}$$

Now substitute (??) and (??) in to (??) then,

$$\frac{1}{B} - r\frac{B'}{B^2} = 1 - r^2\Lambda \tag{A.0.20}$$

$$\frac{d}{dr}\left[\frac{r}{B}\right] = 1 - r^2\Lambda \tag{A.0.21}$$

Then by integrating both sides by

$$\frac{1}{B} = 1 - r^2\frac{\Lambda}{3} + \frac{C}{r} \tag{A.0.22}$$

Now imposing the condition on A and B at large distance if Λ is very small,so that $\frac{C}{r} \gg r^2\Lambda$ for values of r on the scale of inter galactic distance (millions of light years),then for these distance the cosmological constant term can be neglected and the requirement thatwe reclaim newton's law of gravity for those distances give us the condition $C = -2Gm$. Then using(**??**) so the metric became. ,

$$ds^2 = -(1 - r^2\frac{\Lambda}{3} - 2\frac{Gm}{r})dt^2 + \frac{dr^2}{(1 - r^2\frac{\Lambda}{3} - 2\frac{Gm}{r})} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \tag{A.0.23}$$

Appendix B

Boundry condition

$$3\rho(1-\frac{R^2}{\beta^2})^{\frac{1}{2}}-(\rho-\frac{\Lambda}{4\pi})\geq 0 \tag{B.0.1}$$

Substituting for ρ and β^2 from (**??**), and (**??**), respectively, and introducing new dimensionless cosmological and radius parameters by the relations

$$y=\frac{1}{3}\Lambda M^2 \tag{B.0.2}$$

$$x=\frac{R}{M} \tag{B.0.3}$$

$$\rho=\frac{3}{4\pi}\frac{M}{R^3} \tag{B.0.4}$$

Then,

$$3\left[\frac{3}{4\pi}\frac{M}{R^3}\right]\left(1-\frac{R^2}{3}(8\pi\rho+\Lambda)\right)^{\frac{1}{2}}-\left(\frac{3M}{4\pi R^3}-\frac{\Lambda}{4\pi}\right)\geq 0$$

$$\frac{9}{4\pi}\left(\frac{M}{R}\right)^3\frac{1}{M^2}\left(1-2\frac{M}{R}-\frac{1}{3}(\frac{R}{M})^2M^2\Lambda\right)^{\frac{1}{2}}-\left(\frac{3}{4\pi}\frac{1}{x^3}\frac{\Lambda}{3y}-\frac{\Lambda}{4\pi}\right)\geq 0$$

$$\frac{\Lambda}{4\pi}\frac{3}{x^3y}\left(1-\frac{2}{x}-x^2y\right)^{\frac{1}{2}}-\frac{\Lambda}{4\pi}\left(\frac{1}{x^3y}-1\right)\geq 0$$

$$\frac{9}{x^6y^2}\left(1-\frac{2}{x}-x^2y\right)\geq\frac{1}{x^6y^2}-\frac{2}{x^3y}+1$$

$$\frac{8}{x^6y^2} - \frac{18}{x^7y^2} - \frac{9}{x^4y} + \frac{2}{x^3y} - 1 \geq 0$$
$$y^2x^7 + y(9x^3 - 2x^4) + 18 - 8x \leq 0 \quad \text{(B.0.5)}$$

then using quadratic formula

$$y_{\pm}(x) = \frac{2x^4 - 9x^3 \pm \sqrt{9x^6(9 - 12x + 4x^2)}}{2x^7}$$

then ,

$$y_{\pm}(x) = \frac{2x - 9 \pm 3 \mid 2x - 3 \mid}{2x^4}$$
$$y_{+}(x) = \frac{4x - 9}{x^4} \quad , \quad y_{-}(x) = \frac{-2}{x^3} \quad \text{(B.0.6)}$$

then (**??**) can be written as,

$$[y - y_{+}(x)]\,[y - y_{-}(x)] \leq 0 \quad \text{(B.0.7)}$$

then the solution set became

$$y_{-}(x) < y < y_{+}(x) \quad \text{(B.0.8)}$$

for

$$[y - y_{+}(x)]\,[y - y_{-}(x)] \geq 0 \quad \text{(B.0.9)}$$

the solution set is

$$y < y_{-}(x) \qquad and \qquad y > y_{+}(x) \quad \text{(B.0.10)}$$

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We the undersigned, number of the Board of Examiners of the final open defense by **Nehemiah Estifanos Worku** have read and evaluated his/her thesis entitled

"**Hydrodynamics Instability Around Compact Object In SdS Background** " and examined the candidate. This is therefore to certify that the thesis has been accepted in partial fulfilment of the requirements for the degree Master of Science in **Physics(Astrophysics)**.

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Course Code	Course Title	Cr. hr	Number Grade	Rank **	Remark
Phys699	**MSc. Thesis**	**6**	**83**	**Very Good**	

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