GENERAL RELATIVISTIC ORBITAL PRECESSION OF PLANETS

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To My mother

Table of Contents

List of Tables

List of Figures

Abstract

Since the discovery of General theory of relativity, a great deal of astrophysical issues have been studied. Among which the precession of planetary orbits has given attention. Yet, the study is incomplete. Here, we studied the precession of solar planetary system using GR field equations. The results are in agreement with previous studies, including the anomalies. We have suggested some future perspective for the anomalies for progress of the issues.

Keywords: General Relativity, precession,perihelion, aphelion.

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Chapter 1

General Introduction

1.1 Background

One of the first phenomenon which was elucidated by Einstein′s General Theory of Relativity was anomalous precession of the perihelion of Mercury[1, 2]. The theory owes its success to the numerical value provided by Einstein for the perihelion precession of Mercury similar to observation value [3, 4]. This resulted in changes in the apprehension of astronomers and physicists about the concepts of space and time and a different way of viewing the problems [5]. Jean Joseph Le Verrier (1811-1877), a French mathematician, reported the perihelion precession for the first time in 1859, investigation of the solar system appealed to astronomers and theorists more than ever. What attracted Le Verriers attention to the advance of the perihelion of Mercury was its unusual orbital motion [6]. This was associated with an unknown planet that was never found, which he referred as Vulcan. The value he obtained for the precession of the perihelion using Newtonian mechanics was 38 arc seconds per century [7]. The results obtained by Le Verrier were corrected in 1895 by Simon Newcomb (1835-1909)[8], whose theory confirmed Le Varrie′s finding about the advance of the perihelion of Mercury. Also following the Newtonian method with a slight changes in the planetary masses, Newcomb obtained the 42.95 arc seconds per century for the advance of Mercury, close to the actual value.

However, the planets cannot advance when only the gravitational force between the planet and the Sun is taken into account [9]. Einstein′s General Theory of Relativity[10] finally provided explanation for the phenomenon.

Today, it is well known that, not only Mercury but all the planets and other systems with high central gravitating object precess. Since the discovery of General Relativity ,there is a great deal of progress in the subject both observationally and theoretically. Observational data extracted for all the planets. The inner planets observational value and General Relativity theoretical prediction are very good agreement. But,the fitting to the outer planets differ from slight to high deviation as we go outward and towards the Kiuper .

1.2 Statement of the problem

Since the discovery of General theory of Relativity theory,a number of astrophysical relativistic effects have gained research attention. The theory has successfully passing observational tests. For example, deflection of light and precession of orbits around gravitating central force fields. In our solar system, Planetary orbit precession is one of the areas where General Relativity has extensively used with great success to the inner planets. But there are discrepancies in a outer planets with respect to observation. So it is important to re-examine the precession of planets for further developments of the subject.

Research questions

- In what way gravity affect the orbit of planet around the sun?
- What is effective potential of a point mass in General Relativity theory?
- What are the characteristics of closed orbits in General Relativity?

1.3 Objectives

1.3.1 General objective

The general objective of this thesis will be to study the General relativistic orbital precession of planets.

1.3.2 Specific objectives

1. To derive General Relativity orbit equations for Planetary system.
2. To develop General Relativity effect in potential of a point mass.
3. To characterize orbits in strong gravity with general relativistic effect.

1.4 Methodology

General Relativity field equations one used to derive the Effective potential and closed Orbit equations for Solar planetary system. Then the potential and the Orbit equations are used to characterize the Orbits. Observational data is used to quantify the effect of Gravity on the precession of the planets orbit. The result Will be discussed and commented.

Chapter 2

Planetary Orbits

2.1 Keplerian Orbits

In celestial mechanics, a Kepler orbit is the motion of one body relative to another, as an ellipse, parabola, or hyperbola, which forms a two-dimensional orbital plane in three-dimensional space. A Kepler orbit can also form a straight line. It considers only the point-like gravitational attraction of two bodies, neglecting perturbations due to gravitational interactions with other objects, atmospheric drag, solar radiation pressure, a non-spherical central body, and so on. It is thus said to be a solution of a special case of the two-body problem, known as the Kepler problem. As a theory in classical mechanics, it also does not take into account the effects of general relativity.

2.1.1 Kepler's Laws of Planetary Motion

The First successful description of planetary motion in agreement with observational data was put forth by the German mathematician Johannes Kepler. After the death of Tycho Brahe, the foremost naked-eye-observer at the time, Kepler inherited his massive amount of data and over a period of about 20 years put forth three laws of planetary motion famously attributed to him. These three laws, known as Kepler's Laws of Planetary Motion, are at the foundation of modern astronomy. This motion is summed up in three simple laws:

1. The orbit of every planet is an ellipse with the Sun at one of the two foci.

2. A line joining a planet and the Sun sweeps out equal areas during equal intervals of time.

3. The square of the orbital period of each planet is proportional to the cube of its orbital major radii.

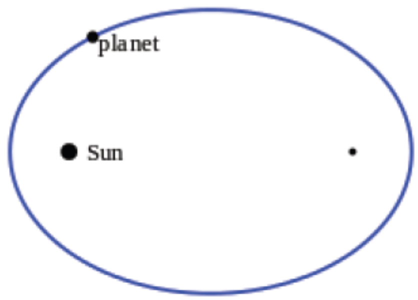
2.1.2 Keplers first law

The orbit of every planet is an ellipse with the Sun at one of the two foci. Mathematically, an ellipse can be represented by the formula:

$$r = \frac{p}{1 + \varepsilon \cos\theta} \tag{2.1.1}$$

where p is the semi-latus rectum,ε is the eccentricity of the ellipse, r is the distance from the Sun to the planet, and θ is the angle to the planet's current position from its closest approach, as seen from the Sun. So (r, θ) are polar coordinates.[11]

Figure 2.1: Kepler's first law placing the Sun at the focus of an elliptical orbit



For an ellipse $0 < \varepsilon < 1$; in the limiting case $\varepsilon = 0$, the orbit is a circle with the Sun at the centre (i.e. where there is zero eccentricity). At $\theta = 0$, perihelion, the distance is minimum

$$r_{min} = \frac{p}{1+\varepsilon} \tag{2.1.2}$$

At θ= 90° and at θ =270° the distance is equal to p. At θ= 180°, aphelion, the distance is maximum (by definition, aphelion is invariably perihelion plus 180°). The semi-major axis a is the arithmetic mean between r_{min} and r_{max} :

$$r_{max} = \frac{p}{1-\varepsilon} \tag{2.1.3}$$

$$r_{max} - a = a - r_{min}$$

$$a = \frac{p}{1-\varepsilon^2} \tag{2.1.4}$$

The semi-minor axis b is the geometric mean between r_{min} and r_{max} :

$$\frac{r_{max}}{b} = \frac{b}{r_{min}}$$

$$b = \frac{p}{\sqrt{1-\varepsilon^2}} \tag{2.1.5}$$

The semi-latus rectum p is the harmonic mean between r_{min} and r_{max}:

$$\frac{1}{r_{min}} - \frac{1}{p} = \frac{1}{p} - \frac{1}{r_{max}}$$

$$pa = r_{max} r_{min} = b^2 \tag{2.1.6}$$

The eccentricity ε is the coefficient of variation between r_{min} and r_{max}:

$$\varepsilon = \frac{r_{max} - r_{min}}{r_{max} + r_{min}} \tag{2.1.7}$$

The area of the ellipse is

$$A = \pi ab$$

The special case of a circle is $\varepsilon = 0$, resulting in r = p = r_{min} = r_{max} = a = b and A = πr^2.

2.1.3 Keplers second law

A line joining a planet and the Sun sweeps out equal areas during equal intervals of time. The orbital radius and angular velocity of the planet in the elliptical orbit will vary. This is shown in the animation: the planet travels faster when closer to the Sun, then slower when farther from the Sun. Kepler's second law states that the blue sector has constant area.[12] In a small time dt the planet sweeps out a small triangle having base line r and height $rd\theta$ and area $dA = \frac{1}{2} \cdot r \cdot rd\theta$ so the constant areal velocity is

$$\frac{dA}{dt} = \frac{r^2}{2}\frac{d\theta}{dt} \tag{2.1.8}$$

The area enclosed by the elliptical orbit is π ab. So the period P satisfies

$$P \cdot \frac{r^2}{2}\frac{d\theta}{dt} = \pi ab. \tag{2.1.9}$$

and the mean motion of the planet around the Sun

$$n = \frac{2\pi}{P} \tag{2.1.10}$$

satifies

$$r^2 d\theta = abndt. \tag{2.1.11}$$

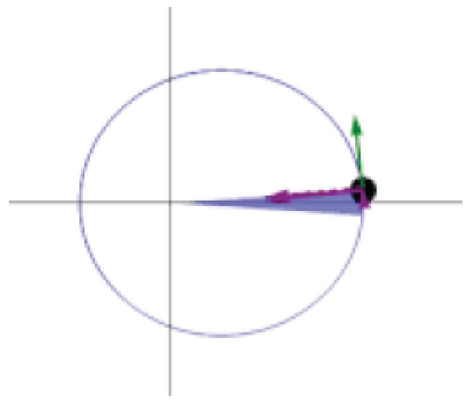


Figure 2.2: The same (blue) area is swept out in a fixed time period. The green arrow is velocity. The purple arrow directed towards the Sun is the acceleration.

2.1.4 Keplers Third Law

The square of the orbital period of a planet is directly proportional to the cube of the semi-major axis of its orbit. This captures the relationship between the distance of planets from the Sun, and their orbital periods. Kepler enunciated in 1619 this third law in a laborious attempt to determine what he viewed as the "music of the spheres" according to precise laws, and express it in terms of musical notation[13]. So it was known as the harmonic law. Using Newton's Law of gravitation (published 1687), this relation can be found in the case of a circular orbit by setting the centripetal force equal to the gravitational force:

$$mr\omega^2 = G\frac{mM}{r^2} \tag{2.1.12}$$

Then, expressing the angular velocity in terms of the orbital period and then rearranging, we find Kepler's Third Law:

$$mr\left(\frac{2\pi}{T}\right) = G\frac{mM}{r^2} \rightarrow T^2 = \left(\frac{4\pi^2}{GM}\right)r^3 \rightarrow T^2 \alpha r^3 \tag{2.1.13}$$

A more detailed derivation can be done with general elliptical orbits, instead of circles, as well as orbiting the center of mass, instead of just the large mass. This is replacing a circular radius,r , with the semi-major axis, a , of the elliptical relative motion of one mass relative to the other, as well as replacing the large mass M with M+m. However, with planet masses being so much smaller than the Sun, this correction is often ignored. The full corresponding formula is:

$$\frac{a^3}{T^2} = \frac{GM + m}{4\pi^2} \approx \frac{GM}{4\pi^2} \approx 7.469 \times 10^{-6} \left(AU^3/days^2\right) \tag{2.1.14}$$

is constant. where M is the mass of the Sun, m is the mass of the planet, and G is the gravitational constant, T is the orbital period and a is the elliptical semi-major axis.

2.2 Newtonian Gravity

The force which maintains the Planets in orbit around the Sun is called gravity, and was first correctly described by Isaac Newton (in 1687). According to Newton, any two point mass objects (or spherically symmetric objects of finite extent) exert a force of attraction on one another. This force points along the line of centers joining the objects, is directly proportional to the product of the objects masses, and inversely proportional to the square of the distance between them. Suppose that the first object is the Sun, which is mass M, and is located at the origin of our coordinate system. Let the second object be some planet, of mass m, which is located at position vector r. The gravitational force exerted on the planet by the Sun is thus written

$$f = -\frac{GMm}{r^2} \tag{2.2.1}$$

The constant of proportionality, G, is called the gravitational constant, and takes the value $G = 6.67300 \times 10^{-11} m^3 kg^{-1} s^{-2}$.

An equal and opposite force to Eq. (2.1.1) acts on the Sun. In Newtons second law, the gravitational force acting on an object is directly proportional to its inertial mass. According to Equation (2.1.1), and Newtons second law, the equation of motion of our planet takes the form

$$\frac{d^2r}{dt^2} = -\frac{GM}{r^2} \tag{2.2.2}$$

Note that the planetary mass, m, has canceled out on both sides of the above equation.

Chapter 3

General Relativistic Orbit Equations for planetary Motion

In general relativity, a geodesic generalizes the notion of a "straight line" to curved space time. Importantly, the world line of a particle free from all external, non-gravitational forces is a particular type of geodesic. In other words, a freely moving or falling particle always moves along a geodesic.

In general relativity, gravity can be regarded as not a force but a consequence of a curved space time geometry where the source of curvature is the stress energy tensor (representing matter, for instance). Thus, for example, the path of a planet orbiting a star is the projection of a geodesic of the curved four-dimensional (4-D) space time geometry around the star on to three-dimensional (3-D) space. The full geodesic equation is

$$\frac{d^2\mu}{ds^2} + \Gamma^{\mu}_{\alpha\beta}\frac{dx^{\alpha}}{ds}\frac{dx^{\beta}}{ds} = 0 \tag{3.0.1}$$

where s is a scalar parameter of motion (e.g. the proper time), and $\Gamma^{\mu}_{\alpha\beta}$ are Christoffel symbols (sometimes called the affine connection coefficients or Levi-Civita connection coefficients) symmetric in the two lower indices. Greek indices may take the values: 0, 1, 2,3 and the summation convention is used for repeated indices α and β. The quantity on the left-hand-side of this equation is the acceleration of a particle, so this equation is analogous to Newton's laws of motion, which like wise provide formula for the acceleration of a particle. This equation of motion employs the Einstein notation, meaning that repeated

indices are summed (i.e. from zero to three). The Christoffel symbols are functions of the four space-time coordinates and so are independent of the velocity or acceleration or other characteristics of a test particle whose motion is described by the geodesic equation.
So far the geodesic equation of motion has been written in terms of a scalar parameter s. It can alternatively be written in terms of the time coordinate, (here we have used the triple bar to signify a definition). The geodesic equation of motion then becomes:

$$\frac{d^2\mu}{dt^2} = -\Gamma^{\mu}_{\alpha\beta}\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt} + \Gamma^{0}_{\alpha\beta}\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt}\frac{dx^\mu}{dt} \tag{3.0.2}$$

This formulation of the geodesic equation of motion can be useful for computer calculations and to compare General Relativity with Newtonian Gravity [14]. It is straight forward to derive this form of the geodesic equation of motion from the form which uses proper time as a parameter using the chain rule. Notice that both sides of this last equation vanish when the μ index is set to zero.

3.1 Einstein field equations and the schwarzschild Solution

The Schwarzschild metric is a solution of Einstein's field equations in empty space, that it is only outside the gravitating body. That is, for a spherical body of radius R the solution is valid for $r > R$.The gravitational field both inside and outside the gravitating body the Schwarzschild solution must be matched with some suitable interior solution at $r = R$.In Schwarzschild coordinates (t, r, θ, ϕ) the Schwarzschild metric has the form

$$g = -c^2 d\tau^2 = -(1 - \frac{r_s}{r})c^2 dt^2 + (1 - \frac{r_s}{r})^{-1}(dr^2 + r^2 g_\Omega) \tag{3.1.1}$$

where g_Ω is the metric on the two sphere,i.e $g_\Omega = (d\theta^2 + \sin^2\theta d\phi^2)$. The Schwarzschild metric has a singularity for $r = 0$ which is an intrinsic curvature singularity.It also seems to have a singularity on the event horizon $r = r_s$.

3.2 Euler Lagrangian Equation of Motion

Nearly spherically symmetric, the Sun has a very small radius as compared to the position of the planets. The space time around it may thus be considered to be in the form of the solution to Einsteins vacuum equations, quite famous as the Schwarzschild space time with the line element

$$ds^2 = c^2(1 - \frac{2\mu}{r})dt^2 - (1 - \frac{2\mu}{r})^{-1}dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 \qquad (3.2.1)$$

where

$$\mu = GM/c^2$$

Defining the squared Lagrangian for massive bodies as

$$L = g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu \qquad (3.2.2)$$

$$= c^2$$

where

$$\dot{x}^\mu = \frac{dx^\mu}{d\tau}$$

and is the affine time parameter, we get, using (3.2.1),

$$ds^2 = c^2(1 - \frac{2\mu}{r})\dot{t}^2 - (1 - \frac{2\mu}{r})^{-1}\dot{r}^2 - r^2\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right) \qquad (3.2.3)$$

The Euler-Lagrange equations are

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{x}^\mu}\right) - \frac{\partial L}{\partial x^\mu} = 0 \qquad (3.2.4)$$

$$(\mu = 0, 1, 2, 3)$$

By means of (3.2.3) and (3.2.4), we get the four space time geodesic equations

$$(1 - \frac{2\mu}{r})\dot{t} = k \qquad (3.2.5)$$

$$\left(1 - \frac{2\mu}{r}\right)^{-1}\ddot{r} + \frac{\mu c^2}{r^2}\dot{t}^2 - \left(1 - \frac{2\mu}{r}\right)^{-2}\frac{\mu}{r^2}\dot{r}^2 - r\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right) = 0 \qquad (3.2.6)$$

$$\ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta} - \sin\theta\cos\theta\dot{\phi}^2 = 0, \qquad (3.2.7)$$

$$r^2\sin^2\theta\dot{\phi} = h \qquad (3.2.8)$$

where h, the angular momentum per unit rest mass, and k, the total energy per unit rest energy, are constants. For a celestial massive body moving in the plane

$$\theta = \frac{\pi}{2}$$

the set of the third space time geodesic equations reduces to

$$(1 - \frac{2\mu}{r})\dot{t} = k \tag{3.2.9}$$

$$\left(1 - \frac{2\mu}{r}\right)^{-1} \ddot{r} + \frac{\mu c^2}{r^2}\dot{t}^2 - \left(1 - \frac{2\mu}{r}\right)^{-2} \frac{\mu}{r^2}\dot{r}^2 - r\dot{\phi}^2 = 0, \tag{3.2.10}$$

$$r^2\dot{\phi} = h \tag{3.2.11}$$

These equations are valid for both null and non null affinity parameterized geodesics. For a non-null geodesic the first integral is simply

$$g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} = x^2 \tag{3.2.12}$$

where x is some constant. For a null geodesic it is as follows,

$$g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} = 0 \tag{3.2.13}$$

3.2.1 Massive particles

The trajectory of a massive particle is a time like geodesic. Considering motion in the equatorial plane,we replace the geodesic equation(3.2.10)by equation(3.2.12), where $g_{\mu\nu}$ is taken from Eq.(3.2.13)with $\theta = \frac{\pi}{2}$. Moreover, since we are considering a time like geodesic we can choose our affine parameter λ to be the proper time λ along the path. Thus we find that the world line x^{μ} (τ) of a massive particle moving in the equatorial plane of the Schwarzschild geometry must satisfy the equations,

$$(1 - \frac{2\mu}{r})\dot{t} = k \tag{3.2.14}$$

$$c^2(1 - \frac{2GM}{c^2 r})\dot{t}^2(1 - \frac{2GM}{c^2 r})r^{-1}\dot{r}^2 - r^2\dot{\phi}^2 = x^2 \tag{3.2.15}$$

$$r^2\dot{\phi} = h \tag{3.2.16}$$

Substituting Eq.(3.2.14) and Eq.(3.2.16) in to Eq.(3.2.15), we obtain the combined Energy equation for the r coordinate.

$$\dot{r}^2 + \frac{h^2}{c^2}(1 - \frac{2GM}{c^2 r}) - \frac{2GM}{r} = x^2(k^2 - 1) \tag{3.2.17}$$

We use this Energy equation to discuss radial free fall and the stability of orbits. Note that the right hand side of Eq.(3.2.17) is a constant of the motion. The constant of proportionality is fixed by requiring that, for a particle at rest at $r = \infty$, we have $E = m_O c^2$. Letting $r = \infty$ and $\dot{r} = 0$,in Eq.(3.2.17) we get $k^2 = 1$.

Hence we must have $k = \frac{E}{m_o c^2}$ where E- is the total energy of the particle in its orbit. A second useful equation which help us to determine the shape of a particle orbit(i.e r as a function of ϕ) we found by using $h = r^2\dot{\phi}$ to express $\dot{r}$ in the energy equation (3.2.17) as,

$$\frac{dr}{d\tau} = \frac{dr}{d\phi}\frac{d\phi}{d\tau} = \frac{h}{r^2}\frac{dr}{d\phi}$$

Thus we obtain

$$(\frac{h}{r^2}\frac{dr}{d\phi})^2 + \frac{h^2}{r^2} = c^2(k^2 - 1) + \frac{2GM}{r} + \frac{2GMh^2}{c^2 r^3}$$

Let $u = \frac{1}{r}$, that is usually employed in Newtonian orbit calculation, we find that:

$$(\frac{du}{d\phi})^2 + u = c^2(k^2 - 1) + \frac{2GMu}{h^2} + \frac{2GMu^3}{c^2}$$

By differentiating this equation with respect to ϕ finally we get,

$$(\frac{d^2u}{d\phi^2}) + u = \frac{GM}{h^2} + \frac{3GM}{c^2}u^2 \tag{3.2.18}$$

In Newtonian gravity, the equations of motion of a particle of mass m in the equatorial plane $\theta = \frac{\pi}{2}$ may be determined from the Lagrangian as,

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) + \frac{GMm}{r}$$

From the Euler-Lagrangian equations we have

$$r^2\dot{\phi} = h,$$

$$\ddot{r} = \frac{h}{r^3} - \frac{GM}{r^2},$$

where the integration constant h is the specific angular momentum of the particle. If we now substitute $u = \frac{1}{r}$ and eliminate the time variable the Newtonian equation of motion for planetary orbit is obtained,

$$\frac{d^2u}{d\phi^2} + u = \frac{GM}{h^2}$$

In this equation $u = \frac{1}{r}$ where r is the radial distance from the mass, where as in Eq.(3.2.18) r is a radial coordinate that is related to distance through the metric.

3.2.2 Circular motion of massive particles

For circular motion in the equatorial plane, r=constant and so $\dot{r} = \ddot{r} = 0$ setting $u = \frac{1}{r} =$ constant in the "shape" Eq.(3.2.18)we have the following,

$$u = \frac{GM}{h^2} + \frac{3GM}{c^2}u^2$$

From which

$$h^2 = \frac{3GMr^2}{r - \frac{3GM}{c^2}}.$$

Putting $\dot{r} = 0$ in the energy equation (3.2.17) and substituting the above expression for h^2 allows us to identify the constant k,

$$k = \left(\frac{1 - \frac{2GM}{c^2r}}{1 - \frac{3GM}{c^2r}} \right)^2$$

The energy of a particle of rest mass m_o in a circular of radius r is given by; $E = km_0c^2$. We use this result to determine which circular orbits are bound. For this we require $E < m_oc^2$, so the limits on r for the orbit to be bound are given by k = 1. This gives : $(1 - \frac{2GM}{c^2r})^2 = 1 - \frac{3GM}{c^2r}$,Which is satisfied when $r = \frac{4GM}{c^2}$ or $r \to \infty$. Thus over the range $r = \frac{4GM}{c^2} < r < \infty$, circular orbits are bound.

3.2.3 Stability of massive particle orbits

The above analysis appears to suggest that the closest bound circular orbit around a massive spherical body is at $r = \frac{4GM}{c^2}$. However, we have not yet determine whether this orbit is stable or not. In Newtonian dynamics the equation of motion of a particle in a central potential can be written

$$\frac{1}{2}(\frac{dr}{dt})^2 + V_{eff}(r) = E,$$

where: V_{eff} (r)- is the effective potential and

E- is the total energy of the particle per unit mass. For an orbit around a spherical mass M, the effective potential is:

$$V_{eff}(r) = \frac{-GM}{r} + \frac{h^2}{2r^2} \tag{3.2.19}$$

Where h is the specific angular momentum of the particle. In general relativity, the energy equation (3.2.17) for the motion of a particle around a central mass can be written as,

$$\frac{1}{2}(\frac{dr}{d\tau})^2 + \frac{h^2}{2r^2}(1 - \frac{2GM}{c^2 r}) - \frac{GM}{r} = \frac{c^2}{2}(k^2 - 1)$$

where the constant $k = \frac{E}{m_o c^2}$. Thus in general relativity we identify the effective potential per unit mass as follows;

$$V_{eff}(r) = \frac{-GM}{r} + \frac{h^2}{2r^2} - \frac{GMh^2}{c^2 r^3} \tag{3.2.20}$$

Differentiating Eq.(3.2.20) gives,

$$\frac{dV_{eff}}{dr} = \frac{GM}{r^2} - \frac{h^2}{r^3} + \frac{3GMh^2}{c^2 r^4},$$

and so the exterema of the effective potential are located at the solutions of the quadratic equation.

$$GMr^2 - h^2 r + \frac{3GMh^2}{c^2} = 0$$

Which occur at

$$r = \frac{h}{2GM}(h \pm \sqrt{h^2 - \frac{12G^2M^2}{c^2}})$$

We note that if $h = \sqrt{\frac{12GM}{c}} = 2\sqrt{3\frac{GM}{c}}$ then there is only one extremum and, there are no turning points in the orbit for lower values of h. The significance of this result is that the inner most stable circular orbit has

$$r_{min} = \frac{6GM}{c^2}$$

This orbit, with $r = \frac{6GM}{c^2}$ and $\frac{hc}{GM} = h\sqrt{3}$, is unique in satisfying both

$$\frac{dV_{eff}}{dr} = 0$$

and

$$\frac{d^2V_{eff}}{dr^2} = 0$$

If $d^2V_{eff}/dr^2 > 0$ the the curvature of the effective potential is positive and it is a minimum. This means the orbit is stable. On the other hand if $d^2V_{eff}/dr^2 < 0$,it means there is a maximum in the effective potential at that point, and the orbit is unstable.

3.3 Equation of Orbits

In discussing the exact solutions for the orbital motion in the equatorial plane by considering r as a function of ϕ instead of τ we get,

$$(\frac{dr}{d\phi})^2 = (E^2 - 1)\frac{r^4}{h^2} + \frac{2M}{h^2}r^3 - r^2 + 2Mr \quad (3.3.1)$$

If we introduce the variable $u = \frac{1}{r}$, as in the analysis of the Keplerian orbits in the Newtonian theory. Now by replacing this the fundamental equation becomes;

$$(\frac{du}{d\phi})^2 = 2GMu^3 - u^2 + \frac{2M}{h^2}u - \frac{1 - E^2}{h^2} \quad (3.3.2)$$

This equation determines the geometry of the geodesics in the invariant plane. Once it have been solved for $u = u(\phi)$.

$$\frac{d\tau}{d\phi} = \frac{1}{hu^2} \quad (3.3.3)$$

$$\frac{dt}{d\phi} = \frac{E}{hu^2(1 - 2Mu)} \quad (3.3.4)$$

3.3.1 Bound Orbits

This solutions of Eq.(3.3.2) will depend on $E^1 < 1$ or $E^2 \geq 1$. This distinction are between bound orbits and unbound orbits. Bound orbits are governed by an equation:

$$\frac{du}{d\phi} = f(u) \quad (3.3.5)$$

where $f(u)$ is given by,

$$f(u) = 2Mu^3 - u^2 + \frac{2M}{h^2}u - \frac{1-E^2}{h^2} \tag{3.3.6}$$

It is clear that the geometry of geodesics will be determined by the positions of the roots f(u)=0. Since f(u) is cubic in u, there are two possibilities: either all roots are all, or one of them is real and the two remaining are complex conjugate ones. Let u_1, u_2, u_3 denote the roots of f(u) = 0. Then we have,

$$u_1 u_2 u_3 = \frac{(1-E^2)}{2Mh^2} \tag{3.3.7}$$

and

$$u_1 + u_2 + u_3 = \frac{1}{2}M \tag{3.3.8}$$

Since$1 - E^2 > 0$, it must allow for one positive real root. From the further facts that $f < 0$ for $f(u) \rightarrow \pm\infty$, for $u \rightarrow \infty$

Case

$\Rightarrow$ If the three roots are all different,

There exists two distinct orbits confined to the interval $u_1 < u < u_3$ and $u > u_3$, i.e an orbit that oscillates between two extreme values for r and an orbit , starting at a certain aphelion distance given by $\frac{1}{u^3}$ plunges in to the singularity at r = 0, i.e $u \rightarrow \infty$. These two classes of orbits are called orbits of the first kind and the second kinds. Orbits of both kinds are most conveniently parameterized by an eccentricity e and a latus rectum l, similar to Newtonian orbits.

3.3.2 Orbits of the first kind

For this all three roots are positive, and we can write them as;

$$u_1 = \frac{1}{l}(1-e) \tag{3.3.9}$$

$$u_2 = \frac{1}{l}(1+e) \tag{3.3.10}$$

$$u_3 = \frac{1}{2M} - \frac{2}{l} \tag{3.3.11}$$

The semilatus rectum l is some positive constant and the eccentricity $e < 1$ for $u_1 > 0$, as required by the condition $E^2 < 1$. The conformity with the ordering $u_1 < u_2 < u_3$ requires

$$\frac{1}{2M} - \frac{2}{l} \geq \frac{1+e}{l}$$

$$l \geq 2M(3+e) \tag{3.3.12}$$

let $\mu \equiv \frac{M}{l}$. The inequality becomes,

$$\mu \leq \frac{1}{2(3+e)}, or, 1 - 6\mu - 2\mu e \geq 0 \tag{3.3.13}$$

this parameter now f(u) is written as

$$f(u) = 2M(u - \frac{1+e}{l})(u - \frac{1}{2M} + \frac{2}{l}) \tag{3.3.14}$$

For a Keplerian ellipse, the semilatus rectum l is the distance l measured from a focus such that;

$$\frac{1}{l} = \frac{1}{2}(\frac{1}{r_+} + \frac{1}{r_-}) \tag{3.3.15}$$

where $r_+ = a(1+e)$ and $r_- = a(1-e)$ are the aphelion and perihelion positions of the orbit respectively. Substituting the values of r_+ and r_- in to Eq.(3.3.15) for l it gives:

$$\frac{1}{l} = \frac{1}{a(1-e^2)} \tag{3.3.16}$$

The values of the two becomes,

$$r_+ = \frac{l}{1-e}, and, r_- = \frac{l}{1+e} \tag{3.3.17}$$

This justifies for the roots u_1 and u_2. This has to agree with the original form of the function, giving the relations;

$$\frac{M}{h^2} = \frac{1}{l^2}[l - M](3+e^2)] \tag{3.3.18}$$

$$\frac{1-E^2}{h^2} = \frac{1}{l^3}[(l-4M)(3-e^2)] \tag{3.3.19}$$

If expressed in terms of μ

$$\frac{1}{h^2} = \frac{1}{lM}[1 - \mu(3+e^2)] \tag{3.3.20}$$

$$\frac{1-E^2}{h^2} = \frac{1}{h^2}[(1-4\mu)(1-e^2)] \tag{3.3.21}$$

From this equation it follows that $\mu < \frac{1}{3+e^2}$ and $\mu < \frac{1}{4}$ As in the Keplerian problem, we now make

$$u = \frac{1}{l}(1 + e\cos\chi) \tag{3.3.22}$$

χ is now a kind of relativistic anomaly.
At aphelion,$\chi = \pi$, we found $u = \frac{(1-e)}{l}$ and
At perihelion, $\chi = 0$,$u = \frac{(1+e)}{l}$
This substitution leads to the equation;

$$(\frac{d\chi}{d\phi})^2 = 1 - 2\mu(3 + e\cos\chi)$$

$$= (1 - 6\mu + 2\mu e) - 4\mu e \cos^2(\frac{\chi}{2}) \tag{3.3.23}$$

or,

$$\pm\frac{d\chi}{d\phi} = \sqrt{1-6\mu+2\mu e}\sqrt{1-k^2\cos^2(\frac{\chi}{2})}$$

where

$$k^2 = \frac{4\mu e}{1-6\mu e+2\mu e}$$

The solution for ϕ can be expressed in terms of the Jacobian integral as,

$$F(\psi, k) = \int^{\psi_o} \frac{d\gamma}{\sqrt{1-k^2\sin^2\gamma}} \tag{3.3.24}$$

where $\psi = \frac{1}{2}(\pi - \chi)$, thus finally written as

$$\phi = \frac{2}{\sqrt{1-6\mu+2\mu e}} F(\frac{\pi}{2}, \frac{\chi}{2}, k) \tag{3.3.25}$$

where the origin of ϕ has been chosen at aphelion passage where $\chi = \pi$. The perihelion passage occurs at $\chi = 0$, where $\psi = \frac{\chi}{2}$. The solution can be completed by the expressions for the proper time and the coordinate time as;

$$\tau = \frac{1}{h}\int \frac{d\phi}{u^2} = \frac{1}{h}\int \frac{d\phi}{d\chi}\frac{d\chi}{u^2} \tag{3.3.26}$$

and

$$t = \frac{E}{h}\int \frac{d\phi}{d\chi}\frac{d\chi}{u^2(1-2Mu)} \tag{3.3.27}$$

The first order corrections to the Keplerian orbits of the Newtonian theory can readily be deduced from Eq.(3.3.23)
Under normal conditions, the parameter $\frac{\mu}{h}$ is a very small quantity. It is essentially the ratio of the gravitational radius M to the major axis of a planetary orbit or a binary star orbit. So expanding Eq.(3.3.23) to the first order in μ to obtain

$$-d\phi = d\chi(1 + 3\mu + \mu e \cos\chi) \tag{3.3.28}$$

integrating this gives

$$-\phi = (1 + 3\mu)\chi + \mu e \sin\chi) \tag{3.3.29}$$

From this we understand that the change in ϕ after one complete revolution during which χ changes by 2π is $2\pi(1 + 3\mu)$. Therefore,the advance of the perihelion $\Delta\phi$, per revolution is,

$$\Delta\phi = \frac{6\pi M}{l} = \frac{6\pi GM}{a(1-e^2)c^2} \tag{3.3.30}$$

where
a- is the semi major axis of the particles orbit
l- is semilatus rectum and
e- is eccentricity of particles orbit
From Eq.(3.2.43) replacing $u = \frac{1}{r}$ one can have,

$$\frac{1}{r} = \frac{1}{l}(1 + e\cos\phi)$$

from $l = r(1 + e\cos\phi)$ and
$l = a(1 - e^2)$ finally this gives

$$r(\phi) = \frac{a(1-e^2)}{(1 + e\cos(\phi))} \tag{3.3.31}$$

3.4 Precession of orbits

The function the sn and sn^2 have periods of 4K and 2K, respectively, where K is defined by the equation

$$K = \int_0^1 \frac{dy}{\sqrt{(1-y^2)(1-k^2y^2)}} \tag{3.4.1}$$

where,k- is elliptic modulus function.

Therefore, the change in φ over one oscillation of u (or, equivalently, one oscillation of r) equals

$$\Delta\varphi = \frac{4k}{\sqrt{r_s(u_3 - u_1)}} \tag{3.4.2}$$

In the classical limit, u_3 approaches $\frac{1}{r_s}$ and is much larger than u_1 or u_2. Hence, k^2 is approximately

$$k^2 = \frac{u_2 - u_1}{u_3 - u_1} \approx r_s(u_2 - u_1) << 1 \tag{3.4.3}$$

For the same reasons, the denominator of $\Delta\varphi$ is approximately

$$\frac{1}{\sqrt{r_s(u_3 - u_1)}} = \frac{1}{\sqrt{1 - r_s(2u_1 + u_2)}} \approx 1 + \frac{1}{2} r_s(2u_1 + u_2) \tag{3.4.4}$$

Since the modulus k is close to zero, the period K can be expanded in powers of k; to lowest order, this expansion yields

$$k \approx \int_0^1 \frac{dy}{\sqrt{1 - y^2}} \left(1 + \frac{1}{2} k^2 y^2\right) = \frac{\pi}{2} \left(1 + \frac{k^2}{4}\right) \tag{3.4.5}$$

Substituting these approximations into the formula for $\Delta\varphi$ yields a formula for angular advance per radial oscillation

$$\delta\varphi = \Delta\varphi - 2\pi \approx \frac{3}{2} \pi r_s(u_1 + u_2) \tag{3.4.6}$$

For an elliptical orbit, u_1 and u_2 represent the inverses of the longest and shortest distances, respectively. These can be expressed in terms of the ellipse's semi-major axis a and its orbital eccentricity e,

$$r_{max} = \frac{1}{u_1} = a(1 + e)$$

$$r_{min} = \frac{1}{u_2} = a(1 - e)$$

those equations gives

$$u_1 + u_2 = \frac{2}{a(1 - e^2)} \tag{3.4.7}$$

Substituting the definition of r_s gives the final equation

$$\delta\varphi \approx \frac{6\pi GM}{c^2 a(1 - e^2)} \tag{3.4.8}$$

Chapter 4

Result and Discussion

4.1 General relativistic effect on particle orbit

Here to characterize we use the effective potential derived in chapter 3. Since we are interested in the precession of solar planetary precessions, here we focus on the bound orbit characterization with the aid of the effective potential. There are in general two classes of bound orbits: circular and elliptical bound orbits where the circular orbit will further divided into stable and unstable circular orbits.

i) Circular orbits

In Newtonian gravity, we found that circular orbits appear at $r_c = \frac{h^2}{GM}$. In general relativity the situation is different, only for r is sufficiently small. Since the difference resides in the term $\frac{-GMh^2}{r^3}$, as $r \to \infty$, the behaviors are identical in the two cases. But as $r \to 0$, the potential goes to 1, as in the Newtonian case. At $r = \frac{2GM}{c^2}$ the potential is always zero. For massive particles depending on the angular momentum:-

$$r_c = \frac{h^2 \pm \sqrt{h^4 - 12G^2M^2h^2}}{2GM}$$

For large h,there are two circular orbits, one stable and one unstable.

$$r = \frac{h^2 \pm h^2 \frac{(1-6GM^2)}{h^2}}{2GM} = (\frac{h^2}{GM}, 3GM)$$

In this limit, the stable circular orbit becomes further and further away, while the unstable one approaches 3GM. As we decrease h, the two circular orbits come closer together,they coincide when the discriminant vanishes, i.e $h = \sqrt{12GM}$ for which $r_{min} = 3R_s$. It disappears entirely for smaller h. Thus $\frac{6GM}{c^2}$ is the smallest possible radius of a stable circular orbit in the Schwarzschild metric. There also unbound orbits, which come in from infinity and turn around, and bound but non circular ones, which oscillate around the stable circular radius. Therefore, Schwarzschild solution possesses stable circular orbits for $r > \frac{6GM}{c^2}$ and unstable circular orbits for $\frac{3GM}{c^2} < r < \frac{6GM}{c^2}$. For massless particles there are no circular orbits. Massless particles actually move in straight line, since the Newtonian gravitational force on a massless particle is zero. In terms of the effective potential a photon with a given energy will come in from $r \to \infty$ and gradually slow down, but the speed of light is not changing until it reaches the turning point, then it will start moving away back to $r \to \infty$. The smallest value of h for which the photon will come closer before it starts moving away, is those trajectories which are initially aimed closer to the gravitating body.

ii) Elliptical orbits

In this case once a particle is trapped in the potential well of the gravitating system, and if it lacks sufficient energy either to jump into the attractor or go away to infinity, the particle will orbit in elliptical path bounded by perihelion and aphelion points.

In the proceeding discussion we characterize the orbits in relation to the general effective potential as shown in 4.1

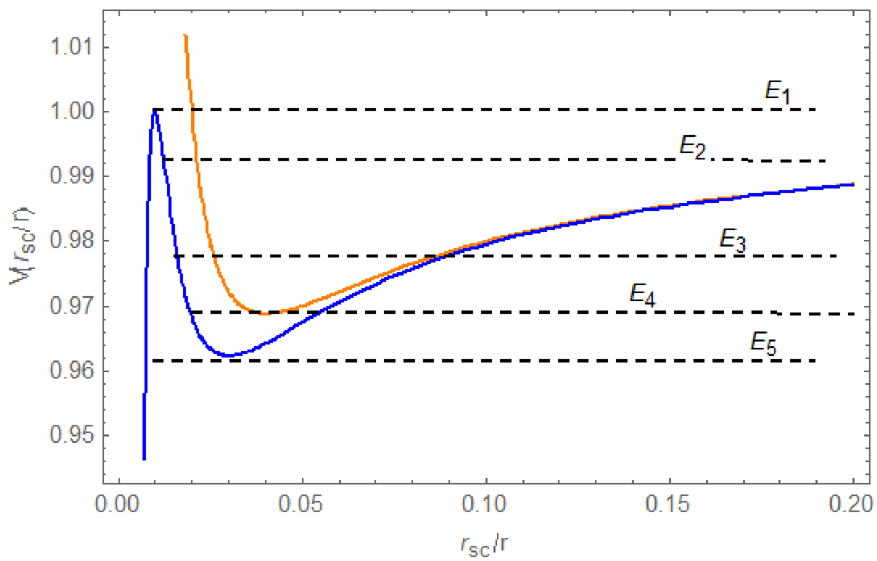


Figure 4.1: Effective potential as a function of radius ($\frac{r_{sc}}{r}$) for various values of angular momentum

From the above graph there are five cases as follows:

(i) If $E > E_1$ (where E is the energy of incoming test particle), the particle will fall directly into the singularity starting from rest. In such cases we expect a plunge orbit in which the particle comes from infinity, moves part way around the central mass and then plunges into the center.

(ii)If $E = E_1$, the particle has an unstable circular orbit, it may fall into the singularity beyond this, depending on initial energy conditions of the particle.

(iii) If $E = E_2$, the particle will have a fly by orbit, i.e. the particle comes from infinity, moves towards the center and after approaching a minimum distance , it flies again back towards the infinity.

(iv) If $E = E_3$, the particle shows a bounded circular motion between the turning points which represent aphelion and perihelion distances respectively(in Newtonian case)and If $E = E_4$, the particle shows a bounded circular motion between the turning points which represent aphelion and perihelion distances respectively(in Schwarzschild).

(v) If $E = E_5$,with this energy particle strikes the minima of potential energy curve. This represents the possibility of bounded circular motion with radius equivalent to the distance of the minima.

Generally from the graphs :

There are bound orbits with energy $E < 0$, and unbound orbits, with energy $E \geq 0$. And there are orbits with $E = E_{min}$ are circles, those with $E_{min} < E < 0$ are ellipses, those with $E = 0$ are parabolas, and those with $E > 0$ are hyperbolas.

When we come to the particles trajectory of bounded orbits round the black hole, since the eccentricity we have taken during our work is $0 \leq e < 1$, the trajectory that happened while the particle rotates round the black hole is ellipse. The points in the trajectory which are closest to the focus and furthest away from from the focus are called the perihelion and aphelion respectively.

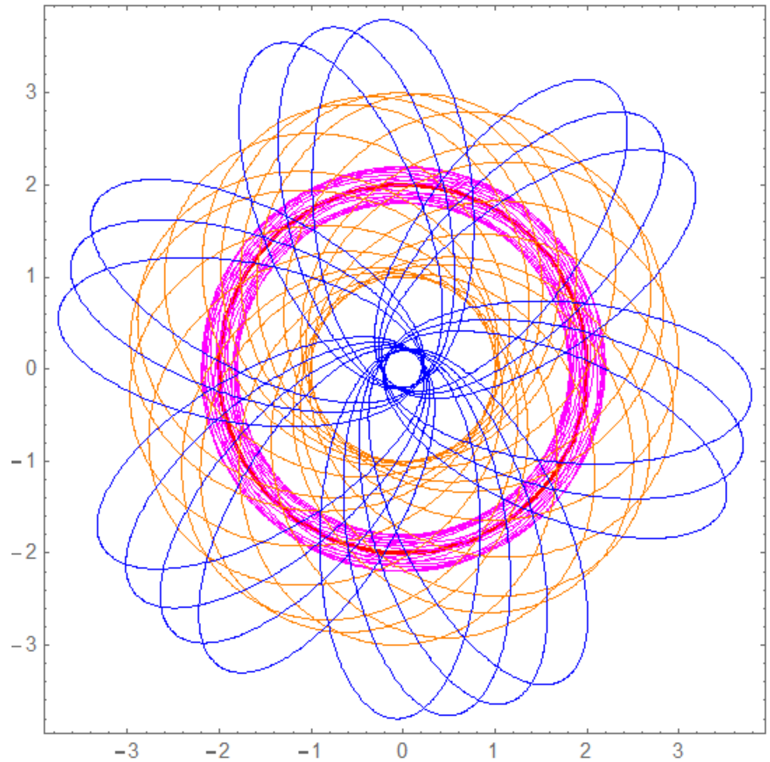


Figure 4.2: Ellipses with different eccentricities for all in Schwarzschild

For some values of eccentricities ($e < 1$), i.e (e=0, 0.1, 0.5, and 0.9) the particles have different elliptical shapes. The particle is closest to the black hole when $\phi = 0$ and this minimum distance is a $r_+ = \frac{l}{1+e}$ and the motion is slow, again the greatest distance occurs at aphelion when $\phi = \pi r_+ = \frac{l}{1-e}$. When farthest away the motion is fast and it follows planetary orbit.

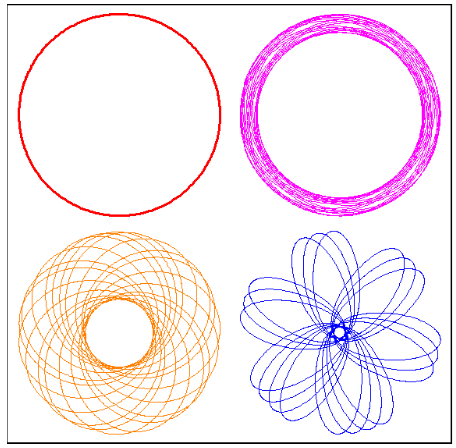


Figure 4.3: Ellipses with different eccentricities in Schwarzschild cases

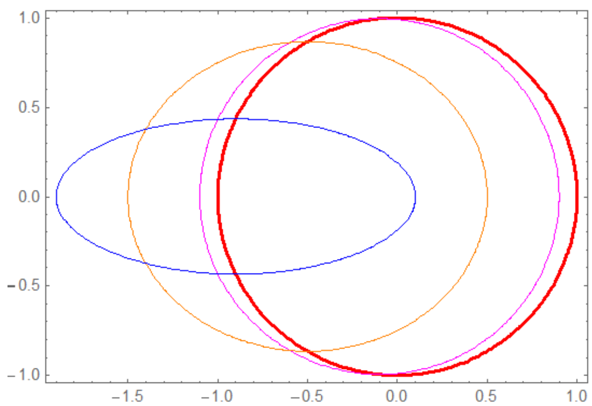


Figure 4.4: Ellipses with different eccentricities for all in Newtonian cases

From figure 4.4 All Four planets share the same radial motion, but move at different angular speeds. The Magenta planet feels only an inverse-square force. The blue planet moves angularly fast as the orange planet. The Orange planets slow angularly than the blue planet. The red planet illustrates purely radial motion with no angular motion.

Table 4.1: Perihelion precession of the planetary systems

System	$a(m) \times 10^{11}$	T(d)	e	$\Delta\phi(\frac{sec}{century})$
Mercury	0.579	88	0.206	42.842
Venus	1.082	224.7	0.007	8.5979
Earth	1.496	365.25	0.017	3.8374
Mars	2.279	687	0.093	1.3467
Jupiter	7.783	4331.865	0.048	0.0621
Saturn	14.27	10760.25	0.056	0.0136
Uranus	28.709	30676.8	0.047	0.00237
Neptune	44.97	60266.25	0.009	0.0007
Pluto	59.00	90582	0.254	0.00041

Data source:
https:www.info.please.com(Basic planetary data)

According to some data and the perihelion precession of final equation(3.4.8), we prepare a table that represents the results for eight planets in the solar system (Mercury,Venus, Earth, Mars, Jupiter, Saturn, Uranus and Neptune). There is a conversion in our table that gives us different values for advance of perihelion that we express the (sec/century), represented by $\delta\varphi = \frac{6\pi GM}{c^2 a(1-e^2)}$

Chapter 5

Summary and conclusion

One of the astrophysical systems that requires GR is the precession of planets which has motivated Einstein to develop GR itself. Since then attention has given for the study of precession of orbits around strong gravity sources. Yet, the it remains an active research area including planetary orbits around the sun. We studied about the Perihelion Precession of Planetary Systems on the basis of general relativity. Also starting from geodesic equation in connection with Lagrangian equation we derived equations for the trajectories of both massive and massless particles, like equation for effective potential of massive and massless particles, polar equations of ellipse and the behavior of the trajectories have been studied. In our analytical derivations particles motion would be considered and using these equations we generate the numerical data by MATHEMATICA and produce different graphs(figures). As a result during the motion of particles around Elliptical Orbit those particles with weak gravity (Newtonian)have stable circular orbits while particles with strong gravity have unstable orbits. when these particles approach to the Schwarzschild radius they will trapped in to a black hole. There are also no stable circular photon orbits in the Schwarzschild geometry.

Bibliography

[1] S Weinberg. 1972gravitation and cosmology.

[2] Michael P Price and William F Rush. Nonrelativistic contribution to mercury's perihelion precession. *American Journal of Physics*, 47(6):531–534, 1979.

[3] MG Stewart. Precession of the perihelion of mercury's orbit. *American journal of physics*, 73(8):730–734, 2005.

[4] Irwin I Shapiro, Gordon H Pettengill, Michael E Ash, Richard P Ingalls, DB Campbell, and RB Dyce. Mercury's perihelion advance: Determination by radar. *Physical Review Letters*, 28(24):1594, 1972.

[5] S Habib Mazharimousavi, Mustafa Halilsoy, and Tayebeh Tahamtan. Solutions for f (r) gravity coupled with electromagnetic field. *The European Physical Journal C*, 72(1):1–5, 2012.

[6] Roger A Rydin. Le verrier's 1859 paper on mercury, and possible reasons for mercury's anomalous precession. *The general science*, 13, 2009.

[7] Takeshi Inoue. An excess motion of the ascending node of mercury in the observations used by le verrier. In *Qualitative and Quantitative Behaviour of Planetary Systems*, pages 69–69. Springer, 1993.

[8] S Newcomb. Astronomical constants (the elements of the four inner planets and the fundamental constants of astronomy). *Supplement to the American Ephemeris and Nautical Almanac for*, 1897.

[9] John Knight Fotheringham. Relativity: Note on the motion of the perihelion of mercury. *Monthly Notices of the Royal Astronomical Society*, 91:1001, 1931.

[10] JD Giorgini, DK Yeomans, AB Chamberlin, PW Chodas, RA Jacobson, MS Keesey, JH Lieske, SJ Ostro, EM Standish, and RN Wimberly. Jpl's on-line solar system data service. In *DPS*, pages 25–04, 1996.

[11] Gerald James Holton and Stephen G Brush. *Physics, the human adventure: From Copernicus to Einstein and beyond*. Rutgers University Press, 2001.

[12] Jeff Bryant and Oleksandr Pavlyk. Kepler's second law. *Wolfram Demonstrations Project. Retrieved December*, 27, 2009.

[13] Michael Dickreiter. Der musiktheoretiker johannes kepler. *Journal for the History of Astronomy*, 7:198, 1976.

[14] Clifford M Will. *Theory and experiment in gravitational physics*. Cambridge university press, 2018.