Dynamics of Nondegenerate Three-Level Laser with a Closed Cavity Containing N-Three-Level Atoms and Coupled to Two-Mode Vacuum Reservoir

A Thesis Submitted to the School of Graduate Studies Department of Physics

Jimma University

In Partial Fulfilment of

the Requirement of the Degree of

Masters of Science in Physics

(**Quantum Optics**)

Fekadu Dejene

Jimma, Ethiopia

February, 2021

DECLARATION

I hereby declare that this thesis is my original work and has not been presented for a degree in any other university, and that all sources of material used for the thesis have been duly acknowledged.

Name: Fekadu Dejene

Signature: ——————

This Masters thesis has been submitted for examination with my approval as university advisor.

Name: Dr. Tamirat Abebe

Signature: ———————

Place and date of submission:

Jimma University

Department of Physics

February, 2021

Jimma University
Department of Physics

Dynamics of Nondegenerate Three-Level Laser with a Closed Cavity Containing N-Three-Level Atoms and Coupled to Two-Mode Vacuum Reservoir

by :

Fekadu Dejene

Approved by the Examination Committee

Dr. Deribe Hirpo, External Examiner: ________________

Dr. Solomon Negash, Internal Examiner: ________________

Dr. Tamirat Abebe, Advisor: ________________

Dr. Solomon Negash, Chairman : ________________

Acknowledgement

First of all, I would like to thank the almighty God for letting me accomplished this thesis. Secondly, I would like to express my deepest heartfelt gratitude and respect to my advisor and instructor, Dr. Tamirat Abebe, for formulating this MSc thesis and for inspirational guidance, his unreserved support and effort, critical comments and helpful discussion through out my research work. Next, I would also like to thank Limmu Kossa administration office for sponsoring my MSc studies. I would like to thank my beloved wife Degitu Alemu for her sincere encouragement and moral support. Finally, I would like to thank my friends for their support and encouragement.

Abstract

In this research we have studied the squeezing, entanglement and statistical properties of the cavity light beams produced by a nondegenerate three-level laser with a closed cavity and coupled to a two-mode vacuum reservoir via a single-port mirror. We have carried out our analysis by putting the noise operators associated with the vacuum reservoir in normal order. Applying the solutions of the equations of evolution for the expectation values of the atomic operators, the interaction Hamiltonian and the quantum Langavin equations for the cavity mode operators, we have calculated the mean and variance of the photon number as well as the quadrature squeezing, photon and atom entanglement of the cavity light. We have found that the mean photon number and its fluctuation, quadrature squeezing, and photon entanglement within the laser cavity are enhanced by increasing the stimulated emission decay constant. In addition, proper choice of the amplitude of the driving coherent light increases these non-classical features. The mean photon number and its fluctuation also become more intense for large frequency interval. Moreover, the squeezing and entanglement of the two-mode light are directly related and their maximum strength is found to be 43%.

Contents

List of Figures

1

Introduction

Light has played a special role in our attempt to understand nature quantum mechanically. Squeezing is one of the nonclassical features of light that has attracted a great deal of interest [1-8]. In squeezed light the noise in one quadrature is below the vacuum-state level at the expense of enhanced fluctuations in the other quadrature, with the product of the uncertainties in the two quadratures satisfying the uncertainty relation [7,8]. Squeezed light has potential applications in low-noise optical communications and weak signal detection [8-11].

The other important quantum feature that is attributed to the correlation between two or more group of quantum particles is quantum entanglement, which is the non-locality aspect of quantum correlations with no classical similarity. This wonderful feature is a very key resource for many fascinating applications such as quantum memories for quantum computers, quantum information and communication, quantum dense coding, quantum teleportation for secure communication, and atom clocks and interferometers for quantum sensing and metrology [10-12].

There has been a considerable interest in the analysis of the squeezing, entanglement, and statistical properties of the light generated by three-level lasers [11-26]. When a three-level atom in a cascade configuration makes a transition from the top to the bottom level via the intermediate level, two photons are generated. If the two photons have the same frequency, then the three-level atom is called degenerate three-level atom otherwise it is called nondegenerate. The squeezing and statistical properties of the light produced by three-level lasers when the atoms are initially prepared in a coherent superposition of the top and bottom levels or when these levels are coupled by a strong coherent light have been studied by several authors [27-29]. These authors have found that these quantum optical systems can generate squeezed light under certain conditions.

Moreover, Fesseha [27, 29] has studied the squeezing and the statistical properties of the light produced by a three-level laser with the atoms in a closed cavity and pumped by electron bombardment. He has shown that the maximum quadrature squeezing of the light generated by the laser, operating below threshold, is found to be 50% below the vacuum-state level. In addition, he has also found that the quadrature squeezing of the output light is equal to that of the cavity light. On the other hand, this study shows that the local quadrature squeezing is greater than the global quadrature squeezing. He has also found that a large part of the total mean photon number is confined in a relatively small frequency interval. Moreover, T. Abebe [30] has studied the entanglement of the light produced by a degenerate three-level laser coupled to a noiseless vacuum reservoir by employing coherent light for pumping process. He has shown that the maximum quadrature squeezing is 43% below the vacuum-state level, which is slightly less than the result found with electron bombardment.

In this thesis we wish to study the squeezing, entanglement and statistical properties of the light generated by nondegenerate three-level laser with a closed cavity containing N three-level atoms and coupled to a two-mode vacuum reservoir via a single-port mirror. We carry out our calculation by putting the noise operators associated with the vacuum reservoir in normal order. We thus first determine the interaction Hamiltonian for a dynamics of nondegenerate three-level laser coupled to a two-mode vacuum reservoir and obtain the quantum Langevin equations for the cavity mode operators. In addition, employing the interaction Hamiltonian and the large-time approximation scheme, we obtain equations of evolution of the expectation values of atomic operators. Moreover, we determine the solutions of the equations of evolution of the expectation values of the atomic operators and the quantum Langevin equations for cavity mode operators. Then applying the resulting solutions, we calculate the photon statistics and the quadrature variances of the two-mode cavity light beams. Furthermore, applying the same solutions, we obtain the mean and variance of the two-mode cavity light, the intracavity quadrature squeezing and photon and atom entanglement of the system under consideration.

2

Operator Dynamics

In this chapter we consider a nondegenerate three-level laser driven by coherent light and with the cavity modes coupled to a two-mode vacuum reservoir via a single-port mirror as shown in Figure (2.1). We first set up the interaction Hamiltonian for a coherently driven nondegenerate three-level atom with the cavity modes and the quantum Langevin equations for the cavity mode operators. In addition, employing the Hamiltonian and the Heisenberg equation, we drive the equations of evolution of the expectation values of the atomic operators. Finally, we determine the steady-state solutions of the resulting equations of evolution. Here we carry out our calculation by putting the noise operators associated with the two-mode thermal reservoir in normal order.

2.1 The Interaction Hamiltonian

We consider here the case in which N nondegenerate three-level atoms in cascade configuration are available in a closed cavity. We denote the top, intermediate, and bottom levels of the three-level atom by $|a\rangle_k$, $|b\rangle_k$, and $|c\rangle_k$, respectively. As shown in Figure (2.1) for nondegenerate cascade configuration, when the atom makes a transition from level $|a\rangle_k$ to $|b\rangle_k$ and from levels $|b\rangle_k$ to $|c\rangle_k$ two photons with different frequencies are emitted. The emission of light when the atom makes the transition from the top level to the intermediate level is light mode a and the emission of light when the atom makes the transition from the intermediate level to the bottom level is light mode b. We assume that the cavity mode a is at resonance with transition $|a\rangle_k \rightarrow |b\rangle_k$ and the cavity mode b is at resonance with the transition $|b\rangle_k \rightarrow |c\rangle_k$, with top and bottom levels of the three-level atom coupled by coherent light. The cou-

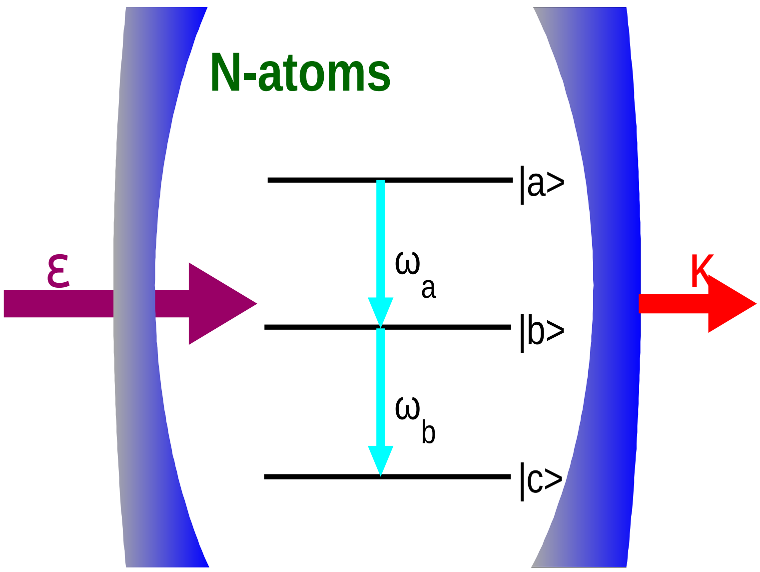

Figure 2.1: Schematic representation of a coherently driven nondegenerate three-level laser coupled to a two-mode vacuum reservoir.

pling of the top and bottom levels of a nondegenerate three-level atom by coherent light can be described by the Hamiltonian [27]

$$\hat{H}' = \frac{i\Omega}{2}\left[\hat{\sigma}_c^{\dagger k} - \hat{\sigma}_c^k\right], \tag{2.1}$$

where

$$\hat{\sigma}_c^k = |c\rangle_{k\,k}\langle a| \tag{2.2}$$

is lowering atomic operator and

$$\Omega = 2\varepsilon\lambda. \tag{2.3}$$

Here ε, considered to be real and constant, is the amplitude of the driving coherent light and λ is the coupling constant between the driving coherent light and the three-level atom. In addition, the interaction of a three-level atom with the cavity modes can be described by the Hamiltonian [29]

$$\hat{H}'' = ig\left[\hat{\sigma}_a^{\dagger k}\hat{a} - \hat{a}^\dagger\hat{\sigma}_a^k + \hat{\sigma}_b^{\dagger k}\hat{b} - \hat{b}^\dagger\hat{\sigma}_b^k\right], \tag{2.4}$$

where

$$\hat{\sigma}_a^k = |b\rangle_{k\,k}\langle a|, \tag{2.5}$$

$$\hat{\sigma}_b^k = |c\rangle_{k\,k}\langle b|, \tag{2.6}$$

are atomic operators, g is the coupling constant between the atom and cavity mode a or b, and $\hat{a}$ and $\hat{b}$ are the annihilation operators for light modes a and b. Thus up on combining Eqs. (2.1) and (2.4), the interaction of the three-level atom with the driving coherent light and cavity mode a and b is described by the Hamiltonian

$$\hat{H}_S(t) = ig\big[\hat{\sigma}_a^{\dagger k}\hat{a} - \hat{a}^\dagger\hat{\sigma}_a^k + \hat{\sigma}_b^{\dagger k}\hat{b} - \hat{b}^\dagger\hat{\sigma}_b^k\big] + \frac{i\Omega}{2}\big[\hat{\sigma}_c^{\dagger k} - \hat{\sigma}_c^k\big]. \tag{2.7}$$

2.2 Quantum Langevin Equations

We recall that the laser cavity is coupled to a two-mode vacuum reservoir via a single-port mirror. In addition, we carry out our calculation by putting the noise operators associated with the vacuum reservoir in normal order. Thus the noise operators will not have any effect on the dynamics of the cavity mode operators [7,8]. We can therefore drop the noise operators and write the quantum Langevin equations for the operators $\hat{a}$ and $\hat{b}$ as

$$\frac{d\hat{a}}{dt} = -\frac{\kappa}{2}\hat{a} - i[\hat{a}, \hat{H}], \tag{2.8}$$

$$\frac{d\hat{b}}{dt} = -\frac{\kappa}{2}\hat{b} - i[\hat{b}, \hat{H}], \tag{2.9}$$

where κ is the cavity damping constant. Then in view of Eq. (2.7), the quantum Langevin equations for cavity mode operators $\hat{a}$ and $\hat{b}$ turns out to be

$$\frac{d\hat{a}}{dt} = -\frac{\kappa}{2}\hat{a} - g\hat{\sigma}_a^k, \tag{2.10}$$

$$\frac{d\hat{b}}{dt} = -\frac{\kappa}{2}\hat{b} - g\hat{\sigma}_b^k. \tag{2.11}$$

2.3 Equations of Evolution of the Atomic Operators

Here we seek to derive the equations of evolution of the expectation values of the atomic operators by applying the Heisenberg equation. Moreover, we find the steady-state solutions of the equations of evolution of the atomic operators. To this end, employing the relation

$$\frac{d}{dt}\langle\hat{A}\rangle = -i\langle[\hat{A},\hat{H}]\rangle \tag{2.12}$$

along with the interaction Hamiltonian of the system described by (2.7), one can readily establish that

$$\frac{d}{dt}\langle\hat{\sigma}_a^k\rangle = g[\langle\hat{\eta}_b^k\hat{a}\rangle - \langle\hat{\eta}_a^k\hat{a}\rangle + \langle\hat{b}^\dagger\hat{\sigma}_c^k\rangle] + \frac{\Omega}{2}\langle\hat{\sigma}_b^{\dagger k}\rangle, \tag{2.13}$$

$$\frac{d}{dt}\langle\hat{\sigma}_b^k\rangle = g[\langle\hat{\eta}_c^k\hat{b}\rangle - \langle\hat{a}^\dagger\hat{\sigma}_c^k\rangle - \langle\hat{\eta}_b^k\hat{b}\rangle] - \frac{\Omega}{2}\langle\hat{\sigma}_a^{\dagger k}\rangle, \tag{2.14}$$

$$\frac{d}{dt}\langle\hat{\sigma}_c^k\rangle = g[\langle\hat{\sigma}_b^k\hat{a}\rangle - \langle\hat{\sigma}_a^k\hat{b}\rangle] + \frac{\Omega}{2}\big[\langle\hat{\eta}_c^k\rangle - \langle\hat{\eta}_a^k\rangle\big], \tag{2.15}$$

$$\frac{d}{dt}\langle\hat{\eta}_a^k\rangle = g[\langle\hat{\sigma}_a^{\dagger k}\hat{a}\rangle + \langle\hat{a}^\dagger\hat{\sigma}_a^k\rangle] + \frac{\Omega}{2}\big[\langle\hat{\sigma}_c^k\rangle + \langle\hat{\sigma}_c^{\dagger k}\rangle\big], \tag{2.16}$$

$$\frac{d}{dt}\langle\hat{\eta}_b^k\rangle = g[\langle\hat{\sigma}_b^\dagger\hat{b}\rangle + \langle\hat{b}^\dagger\hat{\sigma}_b^k\rangle - \langle\hat{\sigma}_a^{\dagger k}\hat{a}\rangle - \langle\hat{a}^\dagger\hat{\sigma}_a^k\rangle], \tag{2.17}$$

$$\frac{d}{dt}\langle\hat{\eta}_c^k\rangle = -g(\langle\hat{\sigma}_b^\dagger\hat{a}\rangle + \langle\hat{b}^\dagger\hat{\sigma}_b^k\rangle) - \frac{\Omega}{2}\big[\langle\hat{\sigma}_c^k\rangle + \langle\hat{\sigma}_c^{\dagger k}\rangle\big] \tag{2.18}$$

where

$$\hat{\eta}_a^k = |a\rangle_{k\,k}\langle a|, \tag{2.19}$$

$$\hat{\eta}_b^k = |b\rangle_{k\,k}\langle b|, \tag{2.20}$$

$$\hat{\eta}_c^k = |c\rangle_{k\,k}\langle c|. \tag{2.21}$$

We see that Eqs. (2.13) - (2.18) are nonlinear differential equations and hence it is not possible to find exact time-dependent solutions of these equations. We intend to overcome this problem by applying the large-time approximation [7,8]. Therefore, employing this approximation scheme, we get from Eqs. (2.10) and (2.11) the approximately valid relations

$$\hat{a} = -\frac{2g}{\kappa}\hat{\sigma}_a^k, \tag{2.22}$$

$$\hat{b} = -\frac{2g}{\kappa}\hat{\sigma}_b^k. \tag{2.23}$$

Evidently, these turn out to be exact relations at steady-state. Now introducing Eqs. (2.22) and (2.23) into Eqs. (2.13) - (2.18), the equations of evolution of the atomic operators take the form

$$\frac{d}{dt}\langle\hat{\sigma}_a^k\rangle = -\gamma_c\langle\hat{\sigma}_a^k\rangle + \frac{\Omega}{2}\langle\hat{\sigma}_b^{\dagger k}\rangle, \tag{2.24}$$

$$\frac{d}{dt}\langle\hat{\sigma}_b^k\rangle = -\frac{\gamma_c}{2}\langle\hat{\sigma}_b^k\rangle - \frac{\Omega}{2}\langle\hat{\sigma}_a^{\dagger k}\rangle, \tag{2.25}$$

$$\frac{d}{dt}\langle\hat{\sigma}_c^k\rangle = -\frac{\gamma_c}{2}\langle\hat{\sigma}_c^k\rangle + \frac{\Omega}{2}\big[\langle\hat{\eta}_c^k\rangle - \langle\hat{\eta}_a^k\rangle\big], \tag{2.26}$$

$$\frac{d}{dt}\langle\hat{\eta}_a^k\rangle = -\gamma_c\langle\hat{\eta}_a^k\rangle + \frac{\Omega}{2}\big[\langle\hat{\sigma}_c^k\rangle + \langle\hat{\sigma}_c^{\dagger k}\rangle\big], \tag{2.27}$$

$$\frac{d}{dt}\langle\hat{\eta}_b^k\rangle = -\gamma_c[\langle\hat{\eta}_b^k\rangle - \langle\hat{\eta}_a^k\rangle], \tag{2.28}$$

$$\frac{d}{dt}\langle\hat{\eta}_c\rangle = -\gamma_c\langle\hat{\eta}_b\rangle - \frac{\Omega}{2}\big[\langle\hat{\sigma}_c\rangle + \langle\hat{\sigma}_c^{\dagger}\rangle\big], \tag{2.29}$$

where

$$\gamma_c = \frac{4g^2}{\kappa} \tag{2.30}$$

is the stimulated emission decay constant.

We next sum Eqs. (2.24) - (2.29) over the N three-level atoms, so that

$$\frac{d}{dt}\langle\hat{m}_a\rangle = -\gamma_c\langle\hat{m}_a\rangle + \frac{\Omega}{2}\langle\hat{m}_b^{\dagger}\rangle, \tag{2.31}$$

$$\frac{d}{dt}\langle\hat{m}_b\rangle = -\frac{\gamma_c}{2}\langle\hat{m}_b\rangle - \frac{\Omega}{2}\langle\hat{m}_a^{\dagger}\rangle, \tag{2.32}$$

$$\frac{d}{dt}\langle\hat{m}_c\rangle = -\frac{\gamma_c}{2}\langle\hat{m}_c\rangle + \frac{\Omega}{2}\big[\langle\hat{N}_c\rangle, \tag{2.33}$$

$$\frac{d}{dt}\langle\hat{N}_a\rangle = -\gamma_c\langle\hat{N}_a\rangle + \frac{\Omega}{2}\big[\langle\hat{m}_c\rangle + \langle\hat{m}_c^{\dagger}\rangle\big], \tag{2.34}$$

$$\frac{d}{dt}\langle\hat{N}_b\rangle = -\gamma_c[\langle\hat{N}_b\rangle - \langle\hat{N}_a\rangle], \tag{2.35}$$

$$\frac{d}{dt}\langle\hat{N}_c\rangle = -\gamma_c\langle\hat{N}_b\rangle - \frac{\Omega}{2}\big[\langle\hat{m}_c\rangle + \langle\hat{m}_c^{\dagger}\rangle\big], \tag{2.36}$$

in which

$$\hat{m}_a = \sum_{k=1}^{N} \hat{\sigma}_a^k, \tag{2.37}$$

$$\hat{m}_b = \sum_{k=1}^{N} \hat{\sigma}_b^k, \tag{2.38}$$

$$\hat{m}_c = \sum_{k=1}^{N} \hat{\sigma}_c^k, \tag{2.39}$$

$$\hat{N}_a = \sum_{k=1}^{N} \hat{\eta}_a^k, \tag{2.40}$$

$$\hat{N}_b = \sum_{k=1}^{N} \hat{\eta}_b^k, \tag{2.41}$$

$$\hat{N}_c = \sum_{k=1}^{N} \hat{\eta}_c^k, \tag{2.42}$$

with the operators $\hat{N}_a$, $\hat{N}_b$, and $\hat{N}_c$ representing the number of atoms in the top, intermediate, and bottom levels, respectively. In addition, employing the completeness relation

$$\hat{\eta}_a^k + \hat{\eta}_b^k + \hat{\eta}_c^k = \hat{I}, \tag{2.43}$$

we easily arrive at

$$\langle \hat{N}_a \rangle + \langle \hat{N}_b \rangle + \langle \hat{N}_c \rangle = N. \tag{2.44}$$

Furthermore, using the definition given by Eq. (2.5) and setting for any k

$$\hat{\sigma}_a^k = |b\rangle\langle a|, \tag{2.45}$$

we have

$$\hat{m}_a = N|b\rangle\langle a|. \tag{2.46}$$

Following the same procedure, one can also easily establish that

$$\hat{m}_b = N|c\rangle\langle b|, \tag{2.47}$$

$$\hat{m}_c = N|c\rangle\langle a|, \tag{2.48}$$

$$\hat{N}_a = N|a\rangle\langle a|, \tag{2.49}$$

$$\hat{N}_b = N|b\rangle\langle b|, \tag{2.50}$$

$$\hat{N}_c = N|c\rangle\langle c|. \tag{2.51}$$

Using the definition

$$\hat{m} = \hat{m}_a + \hat{m}_b \tag{2.52}$$

and taking into account Eqs. (2.46)-(2.51), it can be readily established that [8]

$$\hat{m}^\dagger \hat{m} = N(\hat{N}_a + \hat{N}_b), \tag{2.53}$$

$$\hat{m}\hat{m}^\dagger = N(\hat{N}_b + \hat{N}_c), \tag{2.54}$$

$$\hat{m}^2 = N\hat{m}_c. \tag{2.55}$$

In the presence of N three-level atoms, we rewrite Eqs. (2.10) and (2.11) as [8]

$$\frac{d\hat{a}}{dt} = -\frac{\kappa}{2}\hat{a} + \lambda\hat{m}_a, \tag{2.56}$$

$$\frac{d\hat{b}}{dt} = -\frac{\kappa}{2}\hat{b} + \beta\hat{m}_b, \tag{2.57}$$

in which λ and β are constants whose values remain to be fixed. We note that the steady-state solutions of Eqs. (2.10) and (2.11) are

$$\hat{a} = -\frac{2g}{\kappa}\hat{\sigma}_a^k, \tag{2.58}$$

$$\hat{b} = -\frac{2g}{\kappa}\hat{\sigma}_b^k. \tag{2.59}$$

Now employing Eqs. (2.58) and (2.59), the commutation relations for the cavity mode operators are found to be

$$[\hat{a}, \hat{a}^\dagger]_k = \frac{\gamma_c}{\kappa}\big[\hat{\eta}_b^k - \hat{\eta}_a^k\big], \tag{2.60}$$

$$[\hat{b}, \hat{b}^\dagger]_k = \frac{\gamma_c}{\kappa}\big[\hat{\eta}_c^k - \hat{\eta}_b^k\big], \tag{2.61}$$

and on summing over all atoms, we have

$$[\hat{a}, \hat{a}^\dagger] = \frac{\gamma_c}{\kappa}\big[\hat{N}_b - \hat{N}_a\big], \tag{2.62}$$

$$[\hat{b}, \hat{b}^\dagger] = \frac{\gamma_c}{\kappa}\big[\hat{N}_c - \hat{N}_b\big], \tag{2.63}$$

where

$$[\hat{a}, \hat{a}^\dagger] = \sum_{k=1}^{N}[\hat{a}, \hat{a}^\dagger]_k, \tag{2.64}$$

$$[\hat{b}, \hat{b}^\dagger] = \sum_{k=1}^{N}[\hat{b}, \hat{b}^\dagger]_k. \tag{2.65}$$

We note that Eqs. (2.64) and (2.65) stand for the commutators $\hat{a}$ and $\hat{a}^{\dagger}$, and for $\hat{b}$ and $\hat{b}^{\dagger}$ when the light modes a and b are interacting with all the N three-level atoms. On the other hand, using the steady-state solutions of Eqs. (2.56) and (2.57), one can easily verify that

$$[\hat{a}, \hat{a}^{\dagger}] = N\left(\frac{2\lambda}{\kappa}\right)^2\left(\hat{N}_b - \hat{N}_a\right), \tag{2.66}$$

$$[\hat{b}, \hat{b}^{\dagger}] = N\left(\frac{2\beta}{\kappa}\right)^2\left(\hat{N}_c - \hat{N}_b\right). \tag{2.67}$$

Thus on account of Eqs. (2.62) and (2.66), we see that

$$\lambda = \pm\frac{g}{\sqrt{N}}. \tag{2.68}$$

Similarly, inspection of Eqs. (2.63) and (2.67) shows that

$$\beta = \pm\frac{g}{\sqrt{N}}. \tag{2.69}$$

Hence in view of these two results, the equations of evolution of the light modes a and b operators given by Eqs. (2.56) and (2.57) can be written as [8]

$$\frac{d\hat{a}}{dt} = -\frac{\kappa}{2}\hat{a} + \frac{g}{\sqrt{N}}\hat{m}_a, \tag{2.70}$$

$$\frac{d\hat{b}}{dt} = -\frac{\kappa}{2}\hat{b} + \frac{g}{\sqrt{N}}\hat{m}_b. \tag{2.71}$$

Now adding Eqs. (2.62) and (2.63) as well as Eqs. (2.70) and (2.71), we get

$$[\hat{c}, \hat{c}^{\dagger}] = \frac{\gamma_c}{\kappa}\left[\hat{N}_c - \hat{N}_a\right] \tag{2.72}$$

and

$$\frac{d\hat{c}}{dt} = -\frac{\kappa}{2}\hat{c} + \frac{g}{\sqrt{N}}\hat{m}, \tag{2.73}$$

in which

$$\hat{c} = \hat{a} + \hat{b}. \tag{2.74}$$

We next proceed to obtain the expectation value of the cavity mode operators. One can rewrite Eq. (2.30) and the adjoint of (2.31) as

$$\frac{d}{dt}\langle\hat{m}_a\rangle = -\gamma_c\langle\hat{m}_a\rangle + \frac{\Omega}{2}\langle\hat{m}_b^{\dagger}\rangle, \tag{2.75}$$

$$\frac{d}{dt}\langle\hat{m}_b^{\dagger}\rangle = -\frac{\Omega}{2}\langle\hat{m}_a\rangle - \frac{\gamma_c}{2}\langle\hat{m}_b^{\dagger}\rangle. \tag{2.76}$$

To solve the coupled differential equations (2.75) and (2.76), we write the single-matrix equation

$$\frac{d}{dt}\begin{pmatrix} \langle \hat{m}_a(t)\rangle \\ \langle \hat{m}_b^\dagger(t)\rangle \end{pmatrix} = M \begin{pmatrix} \langle \hat{m}_a(t)\rangle \\ \langle \hat{m}_b^\dagger(t)\rangle \end{pmatrix}, \tag{2.77}$$

with

$$M = \begin{pmatrix} -\gamma_c & \frac{\Omega}{2} \\ -\frac{\Omega}{2} & -\frac{\gamma_c}{2} \end{pmatrix}. \tag{2.78}$$

In order to solve Eq. (2.77), we need the eigenvalues and eigenvectors of M such that

$$MV_i = \lambda_i V_i, \tag{2.79}$$

with $i = 1, 2$, and the eigenvectors

$$V_i = \begin{pmatrix} x_i \\ y_i \end{pmatrix}, \tag{2.80}$$

subject to the normalization condition

$$x_i^2 + y_i^2 = 1. \tag{2.81}$$

The eigenvalue equation (2.79) has nontrivial solution, provided that

$$det(M - \lambda I) = 0, \tag{2.82}$$

so that applying Eq. (2.77), the eigenvalues are found to be

$$\lambda_1 = -\frac{3}{4}\gamma_c + \frac{1}{2}p, \tag{2.83}$$

$$\lambda_2 = -\frac{3}{4}\gamma_c - \frac{1}{2}p, \tag{2.84}$$

$$p = \sqrt{\frac{1}{4}\gamma_c^2 - \Omega^2}. \tag{2.85}$$

We next seek to obtain the eigenvectors of M. To this end, the eigenvector corresponding to λ_1 is expressible as

$$V_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}. \tag{2.86}$$

Then employing Eqs. (2.78) and (2.79), we write the matrix equation

$$\begin{pmatrix} -\gamma_c & \frac{\Omega}{2} \\ -\frac{\Omega}{2} & -\frac{1}{2}\gamma_c \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \lambda_1 \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}. \tag{2.87}$$

Taking into account this equation and the normalization condition

$$x_1^2 + y_1^2 = 1, \tag{2.88}$$

we get

$$V_1 = \frac{1}{\sqrt{\frac{\Omega^2}{4} + (\lambda_1 + \gamma_c)^2}} \begin{pmatrix} \frac{\Omega}{2} \\ \lambda_1 + \gamma_c \end{pmatrix}. \tag{2.89}$$

The eigenvector corresponding to λ_2 can also be established following a similar procedure that

$$V_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \frac{1}{\sqrt{\frac{\Omega^2}{4} + (\lambda_2 + \gamma_c)^2}} \begin{pmatrix} \frac{\Omega}{2} \\ \lambda_2 + \gamma_c \end{pmatrix}. \tag{2.90}$$

Finally, we construct a matrix V consisting of the eigenvectors of the matrix M as column matrices

$$V = \begin{pmatrix} \frac{\frac{\Omega}{2}}{\sqrt{\frac{\Omega^2}{4}+(\lambda_1+\gamma_c)^2}} & \frac{\frac{\Omega}{2}}{\sqrt{\frac{\Omega^2}{4}+(\lambda_2+\gamma_c)^2}} \\ \frac{\lambda_1+\gamma_c}{\sqrt{\frac{\Omega^2}{4}+(\lambda_1+\gamma_c)^2}} & \frac{\lambda_2+\gamma_c}{\sqrt{\frac{\Omega^2}{4}+(\lambda_2+\gamma_c)^2}} \end{pmatrix}. \tag{2.91}$$

We next proceed to determine the inverse of the matrix V . To this end, it can be readily verified that the characteristic equation

$$det(V - \lambda I) = 0 \tag{2.92}$$

has explicit form

$$\begin{aligned} \lambda^2 \quad &- \quad \left[\frac{\frac{\Omega}{2}}{\sqrt{\frac{\Omega^2}{4} + (\lambda_1 + \gamma_c)^2}} + \frac{\lambda_2 + \gamma_c}{\sqrt{\frac{\Omega^2}{4} + (\lambda_2 + \gamma_c)^2}}\right]\lambda \\ &- \quad \frac{\frac{\Omega}{2}(\lambda_1 - \lambda_2)}{\sqrt{\frac{\Omega^2}{4} + (\lambda_1 + \gamma_c)^2}\sqrt{\frac{\Omega^2}{4} + (\lambda_2 + \gamma_c)^2}} I = 0. \end{aligned} \tag{2.93}$$

Thus applying the Cayley-Hamilton theorem that a matrix satisfies its own characteristic equation, we have

$$\begin{aligned} V^2 \quad &- \quad \left[\frac{\frac{\Omega}{2}}{\sqrt{\frac{\Omega^2}{4} + (\lambda_1 + \gamma_c)^2}} + \frac{\lambda_2 + \gamma_c}{\sqrt{\frac{\Omega^2}{4} + (\lambda_2 + \gamma_c)^2}}\right]V \\ &- \quad \frac{\frac{\Omega}{2}(\lambda_1 - \lambda_2)}{\sqrt{\frac{\Omega^2}{4} + (\lambda_1 + \gamma_c)^2}\sqrt{\frac{\Omega^2}{4} + (\lambda_2 + \gamma_c)^2}} I = 0. \end{aligned} \tag{2.94}$$

In view of this, we obtain

$$V^{-1} = \frac{1}{\lambda_1 - \lambda_2} \begin{pmatrix} \frac{-(\lambda_2+\gamma_c)}{\frac{\Omega}{2}} \sqrt{\frac{\Omega^2}{4} + (\lambda_1+\gamma_c)^2} & \sqrt{\frac{\Omega^2}{4} + (\lambda_1+\gamma_c)^2} \\ \frac{(\lambda_1+\gamma_c)}{\frac{\Omega}{2}} \sqrt{\frac{\Omega^2}{4} + (\lambda_2+\gamma_c)^2} & -\sqrt{\frac{\Omega^2}{4} + (\lambda_2+\gamma_c)^2} \end{pmatrix}. \tag{2.95}$$

Using the fact that $VV^{-1} = I$, Eq. (2.95) can be rewritten as

$$\frac{d}{dt}\langle \hat{U}(t)\rangle = VV^{-1}MVV^{-1}\langle \hat{U}(t)\rangle, \tag{2.96}$$

in which

$$\langle \hat{U}(t)\rangle = \begin{pmatrix} \langle \hat{m}_a(t)\rangle \\ \langle \hat{m}_b^\dagger(t)\rangle \end{pmatrix}. \tag{2.97}$$

Multiplying Eq. (2.96) by V^{-1} from the left, we get

$$\frac{d}{dt}(V^{-1}\langle \hat{U}(t)\rangle) = DV^{-1}\langle \hat{U}(t)\rangle, \tag{2.98}$$

where

$$D = V^{-1}MV = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \tag{2.99}$$

in which λ_1 and λ_2 are the eigenvalues of the matrix M. We note that Eq. (2.98) has a well defined solution for $\lambda_1 > 0$ and $\lambda_2 > 0$. The formal solution of Eq. (2.98) can be written as

$$V^{-1}\langle \hat{U}(t)\rangle = e^{Dt}V^{-1}\langle \hat{U}(0)\rangle, \tag{2.100}$$

from which follows

$$\langle \hat{U}(t)\rangle = Ve^{Dt}V^{-1}\langle \hat{U}(0)\rangle. \tag{2.101}$$

In view of the fact that D is diagonal, we have

$$e^{Dt} = \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix}. \tag{2.102}$$

Therefore, on account of Eq. (2.101) along with (2.91), (2.95), (2.99), and (2.102), we obtain

$$Ve^{Dt}V^{-1}\langle \hat{U}(0)\rangle = \frac{1}{p} \begin{bmatrix} [ae^{\lambda_2 t} - be^{\lambda_1 t}]\langle \hat{m}_a(0)\rangle & \frac{\Omega}{2}(e^{\lambda_1 t} - e^{\lambda_2 t})\langle \hat{m}_b^\dagger(0)\rangle \\ -\frac{\Omega}{2}(e^{\lambda_1 t} - e^{\lambda_2 t})\langle \hat{m}_a(0)\rangle & [ae^{\lambda_1 t} - be^{\lambda_2 t}]\langle \hat{m}_b^\dagger(0)\rangle \end{bmatrix}, \tag{2.103}$$

where

$$p = \lambda_1 - \lambda_2, \tag{2.104}$$

$$a = \lambda_1 + \gamma_c, \tag{2.105}$$

$$b = \lambda_2 + \gamma_c. \tag{2.106}$$

In view of Eqs. (2.97) and (2.103) along with (2.101), we see that

$$\begin{pmatrix} \langle \hat{m}_a(t) \rangle \\ \langle \hat{m}_b^\dagger(t) \rangle \end{pmatrix} = \frac{1}{p} \begin{bmatrix} [ae^{\lambda_2 t} - be^{\lambda_1 t}]\langle \hat{m}_a(0) \rangle & \frac{\Omega}{2}(e^{\lambda_1 t} - e^{\lambda_2 t})\langle \hat{m}_b^\dagger(0) \rangle \\ -\frac{\Omega}{2}(e^{\lambda_1 t} - e^{\lambda_2 t})\langle \hat{m}_a(0) \rangle & [ae^{\lambda_1 t} - be^{\lambda_2 t}]\langle \hat{m}_b^\dagger(0) \rangle \end{bmatrix}. \tag{2.107}$$

It then follows that

$$\langle \hat{m}_a(t) \rangle = \frac{1}{p}[ae^{\lambda_2 t} - be^{\lambda_1 t}]\langle \hat{m}_a(0) \rangle + \frac{\Omega}{2p}(e^{\lambda_1 t} - e^{\lambda_2 t})\langle \hat{m}_b^\dagger(0) \rangle \tag{2.108}$$

$$\langle \hat{m}_b^\dagger(t) \rangle = \frac{1}{p}[ae^{\lambda_1 t} - be^{\lambda_2 t}]\langle \hat{m}_b^\dagger(0) \rangle - \frac{\Omega}{2p}(e^{\lambda_1 t} - e^{\lambda_2 t})\langle \hat{m}_a(0) \rangle. \tag{2.109}$$

Furthermore, the adjoint of Eq. (2.109) can be written as

$$\langle \hat{m}_b(t) \rangle = \frac{1}{p}[ae^{\lambda_1 t} - be^{\lambda_2 t}]\langle \hat{m}_b(0) \rangle - \frac{\Omega}{2p}(e^{\lambda_1 t} - e^{\lambda_2 t})\langle \hat{m}_a^\dagger(0) \rangle. \tag{2.110}$$

With the atoms considered to be initially in the bottom level, Eqs. (2.108) and (2.110) reduce to

$$\langle \hat{m}_a(t) \rangle = 0, \tag{2.111}$$

$$\langle \hat{m}_b(t) \rangle = 0. \tag{2.112}$$

The expectation value of the solution of Eq. (2.70) is expressible as

$$\langle \hat{a}(t) \rangle = \langle \hat{a}(0) \rangle e^{-\kappa t/2} + \frac{g}{\sqrt{N}} \int_0^t e^{\kappa t'/2} \langle \hat{m}_a(t') \rangle dt'. \tag{2.113}$$

With the help of Eq. (2.111) and the assumption that the cavity light is initially in a vacuum state, Eq. (2.113) turns out to be

$$\langle \hat{a}(t) \rangle = 0. \tag{2.114}$$

In view of the linear equation described by Eq. (2.70) and the result given by Eq. (2.114), we claim that $\hat{a}(t)$ is a Gaussian variable with zero mean. Following a similar procedure, one can readily obtain the expectation value of the solution of Eq. (2.71) to be

$$\langle \hat{b}(t) \rangle = 0. \tag{2.115}$$

Then on account of the linear equation described by Eq. (2.71) and the result given by Eq. (2.115), we realize $\hat{b}(t)$ to be a Gaussian variable with zero mean. Now with the aid of Eqs. (2.114) and (2.115) together with (2.74), we have

$$\langle \hat{c}(t) \rangle = 0. \tag{2.116}$$

In view of Eqs. (2.116) and (2.73), we see that $\hat{c}(t)$ is a Gaussian variable with zero mean.

Finally, we seek to determine the steady-state solutions of the expectation values of the atomic operators. We note that the steady-state solutions of Eqs. (2.33), (2.34), and (2.35) are given by

$$\langle \hat{m}_c \rangle = \frac{\Omega}{\gamma_c}\big[\langle \hat{N}_c \rangle - \langle \hat{N}_a \rangle\big], \tag{2.117}$$

$$\langle \hat{N}_a \rangle = \frac{\Omega}{2\gamma_c}\big[\langle \hat{m}_c \rangle + \langle \hat{m}_c^\dagger \rangle\big], \tag{2.118}$$

$$\langle \hat{N}_b \rangle = \langle \hat{N}_a \rangle. \tag{2.119}$$

Furthermore, with the help of Eq. (2.44) together with (2.119), we see that

$$\langle \hat{N}_c \rangle = N - 2\langle \hat{N}_a \rangle. \tag{2.120}$$

With the aid of Eq. (2.120), Eq. (2.117) can be written as

$$\langle \hat{m}_c \rangle = \frac{\Omega}{\gamma_c}\big[N - 3\langle \hat{N}_a \rangle\big] \tag{2.121}$$

and in view of Eq. (2.121), we observe that

$$\langle \hat{m}_c \rangle = \langle \hat{m}_c^\dagger \rangle. \tag{2.122}$$

Now taking into consideration this result, Eq. (2.118) can be put in the form

$$\langle \hat{N}_a \rangle = \frac{\Omega}{\gamma_c}\langle \hat{m}_c \rangle. \tag{2.123}$$

Using Eqs. (2.121) and (2.123), one readily gets

$$\langle \hat{N}_a \rangle = \left[\frac{\Omega^2}{\gamma_c^2 + 3\Omega^2} \right] N. \tag{2.124}$$

Substitution of Eq. (2.124) into Eqs. (2.119), (2.120), and (2.121) results in

$$\langle \hat{N}_a \rangle = \left[\frac{\Omega^2}{\gamma_c^2 + 3\Omega^2} \right] N, \tag{2.125}$$

$$\langle \hat{N}_b \rangle = \left[\frac{\Omega^2}{\gamma_c^2 + 3\Omega^2} \right] N, \tag{2.126}$$

$$\langle \hat{N}_c \rangle = \left[\frac{\gamma_c^2 + \Omega^2}{\gamma_c^2 + 3\Omega^2} \right] N, \tag{2.127}$$

$$\langle \hat{m}_c \rangle = \left[\frac{\gamma_c \Omega}{\gamma_c^2 + 3\Omega^2} \right] N. \tag{2.128}$$

These equations represent the steady-state solutions of the equations of evolution of the atomic operators for a coherently driven nondegenerate three-level atom in an open cavity and coupled to a two-mode reservoir. In addition, we note that for $\Omega \gg \gamma_c$, Eqs. (2.125)-(2.128) reduce to

$$\langle \hat{N}_a \rangle = \frac{1}{3} N, \tag{2.129}$$

$$\langle \hat{N}_b \rangle = \frac{1}{3} N, \tag{2.130}$$

$$\langle \hat{N}_c \rangle = \frac{1}{3} N, \tag{2.131}$$

$$\langle \hat{m}_c \rangle = 0. \tag{2.132}$$

Finally, in the absence of the deriving coherent light, when $\Omega = 0$, Eqs. (2.125)-(2.128) turns out to be

$$\langle \hat{N}_a \rangle = 0, \tag{2.133}$$

$$\langle \hat{N}_b \rangle = 0, \tag{2.134}$$

$$\langle \hat{N}_c \rangle = N, \tag{2.135}$$

$$\langle \hat{m}_c \rangle = 0. \tag{2.136}$$

These results show initially, when the deriving coherent light $(\Omega = 0)$, all the atoms are found to be in bottom level.

3

Photon Statistics

In this chapter we seek to study the statistical properties of the light produced by the coherently driven non degenerate three-level laser with a closed cavity and coupled to a two-mode vacuum reservoir via a single-port mirror. Applying the solutions of the equations of evolution of the expectation values of the atomic operators and the quantum Langavin equations for the cavity mode operators, we obtain the global and local photon statistics for light modes a and b. In addition, we determine the global photon statistics of the two-mode cavity light.

3.1 Single-Mode Photon Statistics

In this section we obtain the global mean and variance of the photon numbers for light modes a and b. Moreover, we determine the local mean and variance of the photon numbers for light modes a and b.

3.1.1 Global Mean Photon Number

Here we seek to calculate the global mean photon numbers of light modes a and b, produced by the coherently driven non degenerate three-level laser with a closed cavity and coupled to a two-mode vacuum reservoir.

A. Global mean photon number of light mode a

We now proceed to obtain the mean photon number of light mode a in the entire frequency interval. The mean photon number of light mode a, represented by the operators $\hat{a}$ and $\hat{a}^\dagger$, is defined by

$$\bar{n}_a = \langle \hat{a}^\dagger \hat{a} \rangle. \tag{3.1}$$

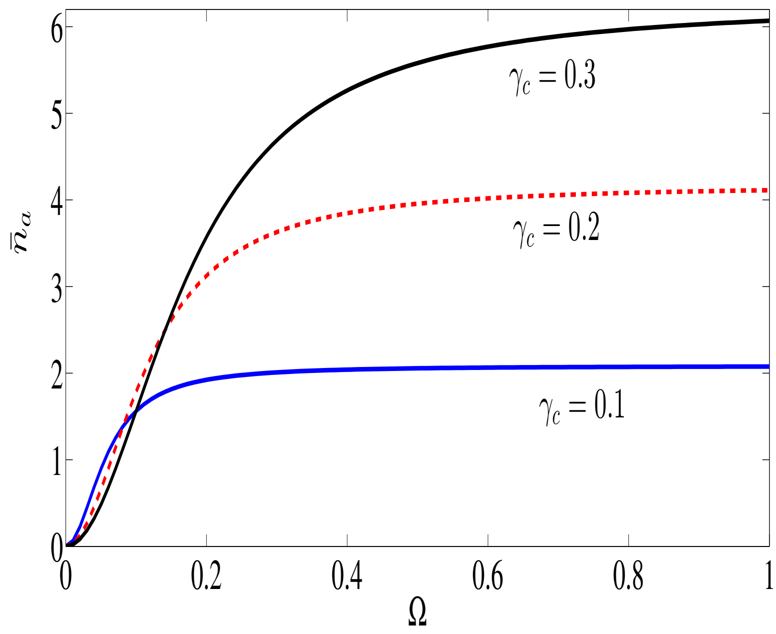

Figure 3.1: Plots of the global mean photon number of light mode a [Eq. (3.6)] versus Ω for $\kappa = 0.8$, $N = 50$, and for different values of γ_c.

We note that the steady-state solution of Eq. (2.70) is

$$\hat{a} = \frac{2g}{\kappa\sqrt{N}}\hat{m}_a, \tag{3.2}$$

so that introducing Eq. (3.2) and its adjoint into (3.1), we see that

$$\bar{n}_a = \frac{\gamma_c}{\kappa N}\langle \hat{m}_a^\dagger \hat{m}_a\rangle. \tag{3.3}$$

With the help of Eq. (2.46), one can write

$$\hat{m}_a^\dagger \hat{m}_a = N\hat{N}_a. \tag{3.4}$$

On account of Eq. (3.4), Eq. (3.3) can be expressed as

$$\bar{n}_a = \frac{\gamma_c}{\kappa}\langle \hat{N}_a\rangle. \tag{3.5}$$

In view of Eq. (2.124), there follows

$$\bar{n}_a = \frac{\gamma_c}{\kappa}N\left[\frac{\Omega^2}{\gamma_c^2 + 3\Omega^2}\right]. \tag{3.6}$$

This is the steady-state mean photon number of light mode a produced by the coherently driven non-degenerate three-level laser with a closed cavity and coupled to a two-mode vacuum reservoir. In addition, we note that for $\Omega \gg \gamma_c$, Eq. (3.6) reduces to

$$\bar{n}_a = \frac{\gamma_c}{3\kappa}N. \tag{3.7}$$

Figure 3.1 shows that the mean photon number of light mode a increases with the stimulation emission decay constant γ_c, particularly for $\Omega \geq 0.1$.

B. Global mean photon number of light mode b

Here we seek to determine the mean photon number of light mode b in the entire frequency interval produced by the system under consideration. The mean photon number of light mode b, represented by the operators $\hat{b}$ and $\hat{b}^\dagger$, is defined by

$$\bar{n}_b = \langle \hat{b}^\dagger \hat{b} \rangle. \tag{3.8}$$

We note that the steady-state solution of Eq. (2.71) is

$$\hat{b} = \frac{2g}{\kappa\sqrt{N}}\hat{m}_b, \tag{3.9}$$

so that introducing Eq. (3.9) and its adjoint into (3.8), we see that

$$\bar{n}_b = \frac{\gamma_c}{\kappa N}\langle \hat{m}_b^\dagger \hat{m}_b \rangle. \tag{3.10}$$

With the help of Eq. (2.47), one can write

$$\hat{m}_b^\dagger \hat{m}_b = N\hat{N}_b. \tag{3.11}$$

On account of Eq. (3.11), Eq. (3.10) can be expressed as

$$\bar{n}_b = \frac{\gamma_c}{\kappa}\langle \hat{N}_b \rangle. \tag{3.12}$$

Now on substituting Eq. (2.125) into (3.12), the mean photon number of light mode b takes, at steady-state, the form

$$\bar{n}_b = \frac{\gamma_c}{\kappa}N\left[\frac{\Omega^2}{\gamma_c^2 + 3\Omega^2}\right]. \tag{3.13}$$

This is the steady-state mean photon number of light mode b produced by the coherently driven non-degenerate three-level laser in a closed cavity and coupled to a two-mode vacuum reservoir.

We would like to point out that this result is exactly the same as that described by Eq. (3.6). In addition, we note that for $\Omega \gg \gamma_c$, Eq. (3.13) reduces to

$$\bar{n}_b = \frac{\gamma_c}{3\kappa} N. \tag{3.14}$$

3.1.2 Local Mean Photon Number

Here we seek to determine the local mean photon numbers of light modes a and b, produced by the coherently driven nondegenerate three-level laser with a closed cavity and coupled to a two-mode vacuum reservoir.

A. Local mean photon number of light mode a

We now proceed to obtain the mean photon number of light mode a in a given frequency interval. To determine the local mean photon number of light mode a, we need to consider the power spectrum of light mode a. The power spectrum of light mode a with central frequency ω_0 is expressible as [8]

$$P_a(\omega) = \frac{1}{\pi} Re \int_0^\infty d\tau e^{i(\omega-\omega_0)\tau} \langle \hat{a}^\dagger(t)\hat{a}(t+\tau)\rangle_{ss}. \tag{3.15}$$

Upon integrating both sides of Eq. (3.15) over ω, we readily get

$$\int_{-\infty}^{\infty} P_a(\omega) d\omega = \bar{n}_a, \tag{3.16}$$

in which $\bar{n}_a$ is the steady-state mean photon number of light mode a. From this result, we observe that $P_a(\omega)d\omega$ is the steady-state mean photon number of light mode a in the frequency interval between ω and $\omega + d\omega$ [8].

We now proceed to determine the two-time correlation function that appears in Eq. (3.15). To this end, we realize that the solution of Eq. (2.70) can be written as

$$\hat{a}(t+\tau) = \hat{a}(t)e^{-\kappa\tau/2} + \frac{g}{\sqrt{N}} e^{-\kappa\tau/2} \int_0^\tau e^{\kappa\tau'/2} \hat{m}_a(t+\tau') d\tau'. \tag{3.17}$$

Applying the large time approximation on Eq. (2.76), we have

$$\langle \hat{m}_b^\dagger \rangle = -\frac{\Omega}{\gamma_c} \langle \hat{m}_a \rangle. \tag{3.18}$$

Employing this result, Eq. (2.75) takes the form

$$\frac{d}{dt}\langle \hat{m}_a \rangle = -\frac{1}{2}\eta \langle \hat{m}_a \rangle. \tag{3.19}$$

On the basis of Eq. (3.19), we see that

$$\frac{d}{dt}\hat{m}_a(t) = -\frac{1}{2}\eta\hat{m}_a(t) + \hat{F}_a(t), \tag{3.20}$$

in which $\hat{F}_a(t)$ is a noise operator with a vanishing mean and η is given by

$$\eta = \left[\frac{\Omega^2 + 2\gamma_c^2}{\gamma_c}\right]. \tag{3.21}$$

The solution of Eq. (3.20) can be put in the form

$$\hat{m}_a(t+\tau) = \hat{m}_a(t)e^{-\eta\tau/2} + e^{-\eta\tau/2}\int_0^\tau e^{\eta\tau'/2}\hat{F}_a(t+\tau')d\tau', \tag{3.22}$$

so that on introducing this into Eq. (3.17), there follows

$$\begin{aligned}\hat{a}(t+\tau) &= \hat{a}(t)e^{-\kappa\tau/2} + \frac{g}{\sqrt{N}}e^{-\kappa\tau/2}\hat{m}_a(t)\int_0^\tau e^{(\kappa-\eta)\tau'/2}d\tau' \\ &+ \frac{g}{\sqrt{N}}e^{-\kappa\tau/2}\int_0^\tau d\tau'\int_0^{\tau'} d\tau'' e^{[(\kappa-\eta)\tau'+\eta\tau'']/2}\hat{F}_a(t+\tau'').\end{aligned} \tag{3.23}$$

Thus on carrying out the first integration, we arrive at

$$\begin{aligned}\hat{a}(t+\tau) &= \hat{a}(t)e^{-\kappa\tau/2} + \frac{2g\hat{m}_a(t)}{\sqrt{N}(\kappa-\eta)}\left[e^{-\eta\tau/2} - e^{-\kappa\tau/2}\right] \\ &+ \frac{g}{\sqrt{N}}e^{-\kappa\tau/2}\int_0^\tau d\tau'\int_0^{\tau'} d\tau'' e^{[(\kappa-\eta)\tau'+\eta\tau'']/2}\hat{F}_a(t+\tau'').\end{aligned} \tag{3.24}$$

Now multiplying on the left by $\hat{a}^\dagger(t)$ and taking the expectation value of the resulting expression, we have

$$\begin{aligned}\langle\hat{a}^\dagger(t)\hat{a}(t+\tau)\rangle &= \langle\hat{a}^\dagger(t)\hat{a}(t)\rangle e^{-\kappa\tau/2} + \frac{2g\langle\hat{a}^\dagger(t)\hat{m}_a(t)\rangle}{\sqrt{N}(\kappa-\eta)}\left[e^{-\eta\tau/2} - e^{-\kappa\tau/2}\right] \\ &+ \frac{g}{\sqrt{N}}e^{-\kappa\tau/2}\int_0^\tau d\tau'\int_0^{\tau'} d\tau'' e^{[(\kappa-\eta)\tau'+\eta\tau'']/2}\langle\hat{a}^\dagger(t)\hat{F}_a(t+\tau'')\rangle\end{aligned} \tag{3.25}$$

Applying the large-time approximation scheme, one gets from Eq. (2.70)

$$\hat{a}(t) = \frac{2g}{\kappa\sqrt{N}}\hat{m}_a(t), \tag{3.26}$$

so that in view of this result, we get

$$\hat{m}_a(t) = \frac{\kappa\sqrt{N}}{2g}\hat{a}(t). \tag{3.27}$$

Thus substitution of Eq. (3.27) into Eq. (3.25) results in

$$\begin{aligned}\langle\hat{a}^\dagger(t)\hat{a}(t+\tau)\rangle &= \langle\hat{a}^\dagger(t)\hat{a}(t)\rangle\left[\frac{\kappa}{\kappa-\eta}e^{-\eta\tau/2} - \frac{\eta}{\kappa-\eta}e^{-\kappa\tau/2}\right] \\ &+ \frac{g}{\sqrt{N}}e^{-\kappa\tau/2}\int_0^\tau d\tau'\int_0^{\tau'} d\tau'' e^{[(\kappa-\eta)\tau'+\eta\tau'']/2}\langle\hat{a}^\dagger(t)\hat{F}_a(t+\tau'')\rangle\end{aligned} \tag{3.28}$$

Since a noise operator at a certain time should not affect a light mode operator at an earlier time [8], we note that

$$\langle \hat{a}^\dagger(t)\hat{F}_a(t+\tau'')\rangle = 0. \tag{3.29}$$

It then follows that

$$\langle \hat{a}^\dagger(t)\hat{a}(t+\tau)\rangle = \langle \hat{a}^\dagger(t)\hat{a}(t)\rangle \left[\frac{\kappa}{\kappa-\eta}e^{-\eta\tau/2} - \frac{\eta}{\kappa-\eta}e^{-\kappa\tau/2}\right] \tag{3.30}$$

and at steady-state, we have

$$\langle \hat{a}^\dagger(t)\hat{a}(t+\tau)\rangle_{ss} = \bar{n}_a\left[\frac{\kappa}{\kappa-\eta}e^{-\eta\tau/2} - \frac{\eta}{\kappa-\eta}e^{-\kappa\tau/2}\right]. \tag{3.31}$$

Thus on combining Eq. (3.31) with Eq. (3.15), the power spectrum of light mode a with central frequency ω_0 is expressible as

$$P_a(\omega) = \frac{1}{\pi}\left[\frac{\bar{n}_a}{\kappa-\eta}\right] Re\left[\kappa\int_0^\infty d\tau e^{-[\eta/2-i(\omega-\omega_0)]\tau} - \eta\int_0^\infty d\tau e^{-[\kappa/2-i(\omega-\omega_0)]\tau}\right], \tag{3.32}$$

so that on carrying out the integration, we readily arrive at

$$P_a(\omega) = \frac{1}{\pi}\left[\frac{\bar{n}_a}{\kappa-\eta}\right] Re\left[\frac{\kappa}{[\eta/2-i(\omega-\omega_0)]} - \frac{\eta}{[\kappa/2-i(\omega-\omega_0)]}\right]. \tag{3.33}$$

This can be rewritten as

$$P_a(\omega) = \frac{\kappa\bar{n}_a}{\kappa-\eta}\left[\frac{\eta/2\pi}{[\eta/2]^2+(\omega-\omega_0)^2}\right] - \frac{\eta\bar{n}_a}{\kappa-\eta}\left[\frac{\kappa/2\pi}{[\kappa/2]^2+(\omega-\omega_0)^2}\right]. \tag{3.34}$$

We realize that the mean photon number of light mode a in the interval between $\omega' = -\lambda$ and $\omega' = \lambda$ is expressible as [8]

$$\bar{n}_{a\pm\lambda} = \int_{-\lambda}^{\lambda} P_a(\omega')d\omega', \tag{3.35}$$

in which $\omega' = \omega - \omega_0$. Therefore, upon substituting Eq. (3.34) into Eq. (3.35) and carrying out the integration by employing the relation

$$\int_{-\lambda}^{\lambda}\frac{dx}{x^2+a^2} = \frac{2}{a}\tan^{-1}\left(\frac{\lambda}{a}\right), \tag{3.36}$$

the local mean photon number of light mode a produced by the coherently driven non-degenerate three-level laser with a closed cavity and coupled to a two-mode vacuum reservoir is found to be

$$\bar{n}_{a\pm\lambda} = \bar{n}_a z_a(\lambda), \tag{3.37}$$

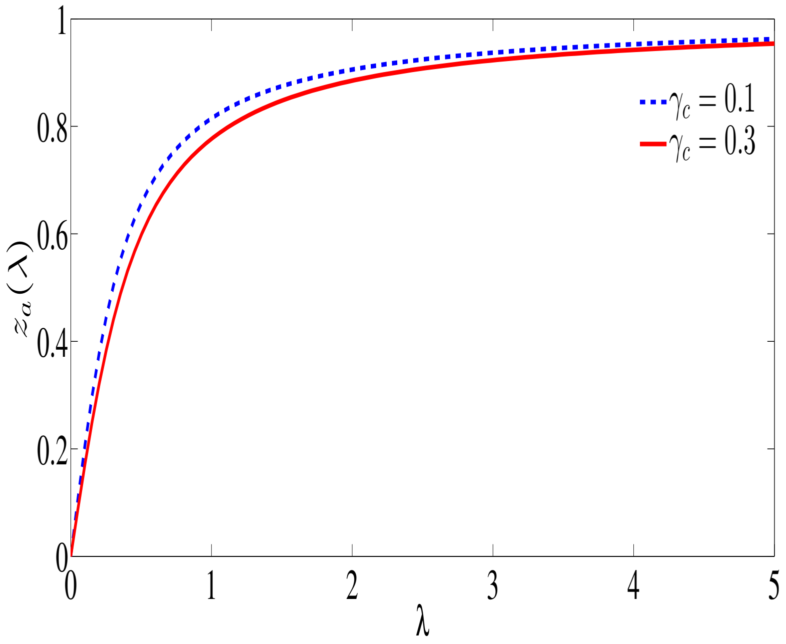

Figure 3.2: Plots z_a [Eq. (3.37)] versus λ for $\kappa = 0.8$, $N = 50$, $\Omega = 0.2$, and for different values of γ_c.

where $z_a(\lambda)$ is given by

$$z_a(\lambda) = \frac{2\kappa/\pi}{\kappa - \eta}\tan^{-1}\left(\frac{2\lambda}{\eta}\right) - \frac{2\eta/\pi}{\kappa - \eta}\tan^{-1}\left(\frac{2\lambda}{\kappa}\right). \tag{3.38}$$

We see from Eq. (3.37) along with the plot of $z_a(\lambda)$ that $\bar{n}_{a\pm\lambda}$ increases with λ until it reaches the maximum value of the global mean photon number. From the plots in Figure 3.2, we find the values indicated in table below.

Table 3.1: Numerical Values of $z_a(\lambda)$ for $N = 50$, $\kappa = 0.8$, and $\Omega = 0.2$.

γ_c	$z_a(0.5)$	$z_a(1)$	$z_a(2)$	$z_a(4)$
0.3	0.6005	0.7787	0.8861	0.944
0.1	0.6591	0.8166	0.9066	0.9527

Moreover, using the above results of $z_a(\lambda)$ and on account of Eq. (3.37), we have the following result indicated below.

Table 3.2: Numerical Values of $\bar{n}_{a\pm\lambda}$ for for $N = 50$, $\kappa = 0.8$ and $\Omega = 0.2$.

γ_c	$\bar{n}_{a\pm0.5}$	$\bar{n}_{a\pm1}$	$\bar{n}_{a\pm2}$	$\bar{n}_{a\pm4}$
0.3	2.145	2.781	3.165	3.365
0.1	1.268	1.57	1.743	1.832

We therefore observe that the total mean photon number of mode a increases with the frequency interval to reach its global value.

B. Local mean photon number of light mode b

We now proceed to obtain the mean photon number of a light mode b in a given frequency interval produced by the system under consideration. To determine the local mean photon number of light mode b, we need to consider the power spectrum of light mode b. The power spectrum of light mode b with central frequency ω_0 is expressible as

$$P_b(\omega) = \frac{1}{\pi} Re \int_0^\infty d\tau e^{i(\omega-\omega_0)\tau} \langle \hat{b}^\dagger(t)\hat{b}(t+\tau)\rangle_{ss}. \tag{3.39}$$

Upon integrating both sides of Eq. (3.39) over ω, we readily get

$$\int_{-\infty}^{\infty} P_b(\omega) d\omega = \bar{n}_b, \tag{3.40}$$

in which $\bar{n}_b$ is the steady-state mean photon number of light mode b. From this result, we observe that $P_b(\omega)d\omega$ is the steady-state mean photon number of light mode b in the frequency interval between ω and $\omega + d\omega$. We now proceed to calculate the two-time correlation function that appears in Eq. (3.40). To this end, we realize that the solution of Eq. (2.71) can be written as

$$\hat{b}(t+\tau) = \hat{b}(t)e^{-\kappa\tau/2} + \frac{g}{\sqrt{N}} e^{-\kappa\tau/2} \int_0^\tau e^{\kappa\tau'/2} \hat{m}_b(t+\tau') d\tau'. \tag{3.41}$$

Applying the large time approximation on Eq. (2.75), we have

$$\langle \hat{m}_a \rangle = \frac{\Omega}{\gamma_c} \langle \hat{m}_b^\dagger \rangle. \tag{3.42}$$

Employing this result, Eq. Eq. (2.76) takes the form

$$\frac{d}{dt} \langle \hat{m}_b^\dagger \rangle = -\frac{1}{2}\mu \langle \hat{m}_b^\dagger \rangle. \tag{3.43}$$

On the basis of Eq. (3.43), we see that

$$\frac{d}{dt} \hat{m}_b(t) = -\frac{1}{2}\mu \hat{m}_b(t) + \hat{F}_b(t), \tag{3.44}$$

in which $\hat{F}_b(t)$ is a noise operator with a vanishing mean and η is given by

$$\mu = \left[\frac{\Omega^2 + 2\gamma_c^2}{2\gamma_c}\right]. \tag{3.45}$$

The solution of equation (3.44) can be put in the form

$$\hat{m}_b(t+\tau') = \hat{m}_b(t)e^{-\mu\tau'/2} + e^{-\mu\tau'/2}\int_0^{\tau'} e^{-\mu\tau''/2}\hat{F}_b(t+\tau'')d\tau'', \tag{3.46}$$

so that on introducing this into Eq. (3.41), we have

$$\begin{aligned}\hat{b}(t+\tau) &= \hat{b}(t)e^{-\kappa\tau/2} + \frac{g}{\sqrt{N}}e^{-\kappa\tau/2}\hat{m}_b(t)\int_0^{\mu} e^{(\kappa-\mu)\tau'/2}d\tau' \\ &+ \frac{g}{\sqrt{N}}e^{-\kappa\tau/2}\int_0^{\tau} d\tau' \int_0^{\tau'} d\tau'' e^{[(\kappa-\mu)\tau'+\mu\tau'']/2}\hat{F}_b(t+\tau''). \end{aligned} \tag{3.47}$$

Thus on carrying out the first integration, we arrive at

$$\begin{aligned}\hat{b}(t+\tau) &= \hat{b}(t)e^{-\kappa\tau/2} + \frac{2g\hat{m}_b(t)}{\sqrt{N}(\kappa-\mu)}\left[e^{-\mu\tau/2} - e^{-\kappa\tau/2}\right] \\ &+ \frac{g}{\sqrt{N}}e^{-\kappa\tau/2}\int_0^{\tau} d\tau' \int_0^{\tau'} d\tau'' e^{[(\kappa-\mu)\tau'+\mu\tau'']/2}\hat{F}_b(t+\tau''). \end{aligned} \tag{3.48}$$

Now multiplying both sides on the left by $\hat{b}^\dagger(t)$ and taking the expectation value of the resulting equation, we have

$$\begin{aligned}\langle\hat{b}^\dagger(t)\hat{b}(t+\tau)\rangle &= \langle\hat{b}^\dagger(t)\hat{b}(t)\rangle e^{-\kappa\tau/2} + \frac{2g\langle\hat{b}^\dagger(t)\hat{m}_b(t)\rangle}{\sqrt{N}(\kappa-\mu)}\left[e^{-\mu\tau/2} - e^{-\kappa\tau/2}\right] \\ &+ \frac{g}{\sqrt{N}}e^{-\kappa\tau/2}\int_0^{\tau} d\tau' \int_0^{\tau'} d\tau'' e^{[(\kappa-\mu)\tau'+\mu\tau'']/2}\langle\hat{b}^\dagger(t)\hat{F}_b(t+\tau'')\rangle. \end{aligned} \tag{3.49}$$

Applying the large-time approximation scheme, one gets from Eq. (2.71)

$$\hat{b}(t) = \frac{2g}{\kappa\sqrt{N}}\hat{m}_b(t). \tag{3.50}$$

In view of Eq. (3.50), we see that

$$\hat{m}_b(t) = \frac{\kappa\sqrt{N}}{2g}\hat{b}(t). \tag{3.51}$$

With this substituted into Eq. (3.49), there follows

$$\begin{aligned}\langle\hat{b}^\dagger(t)\hat{b}(t+\tau)\rangle &= \langle\hat{b}^\dagger(t)\hat{b}(t)\rangle\left[\frac{\kappa}{\kappa-\mu}e^{-\mu\tau/2} - \frac{\mu}{\kappa-\mu}e^{-\kappa\tau/2}\right] \\ &+ \frac{g}{\sqrt{N}}e^{-\kappa\tau/2}\int_0^{\tau} d\tau' \int_0^{\tau'} d\tau'' e^{[(\kappa-\mu)\tau'+\mu\tau'']/2}\langle\hat{b}^\dagger(t)\hat{F}_b(t+\tau'')\rangle \end{aligned} \tag{3.52}$$

and taking into account the fact that

$$\langle \hat{b}^\dagger(t)\hat{F}_b(t+\tau'')\rangle = 0, \tag{3.53}$$

we arrive at

$$\langle \hat{b}^\dagger(t)\hat{b}(t+\tau)\rangle = \langle \hat{b}^\dagger(t)\hat{b}(t)\rangle \left[\frac{\kappa}{\kappa-\mu}e^{-\mu\tau/2} - \frac{\mu}{\kappa-\mu}e^{-\kappa\tau/2}\right]. \tag{3.54}$$

Therefore, at steady-state, Eq. (3.54) takes the form

$$\langle \hat{b}^\dagger(t)\hat{b}(t+\tau)\rangle_{ss} = \bar{n}_b \left[\frac{\kappa}{\kappa-\mu}e^{-\mu\tau/2} - \frac{\mu}{\kappa-\mu}e^{-\kappa\tau/2}\right]. \tag{3.55}$$

Thus on combining Eq. (3.55) with Eq. (3.39), the power spectrum of light mode b with central frequency ω_0 can be put in the form

$$P_b(\omega) = \frac{1}{\pi}\left[\frac{\bar{n}_b}{\kappa-\mu}\right] Re\left[\kappa \int_0^\infty d\tau e^{-[\mu/2-i(\omega-\omega_0)]\tau} - \mu \int_0^\infty d\tau e^{-[\kappa/2-i(\omega-\omega_0)]\tau}, \tag{3.56}$$

so that on carrying out the integration, we readily arrive at

$$P_b(\omega) = \frac{\kappa\bar{n}_b}{\kappa-\mu}\left[\frac{\mu/2\pi}{[\mu/2]^2+(\omega-\omega_0)^2}\right] - \frac{\mu\bar{n}_b}{\kappa-\mu}\left[\frac{\kappa/2\pi}{[\kappa/2]^2+(\omega-\omega_0)^2}\right]. \tag{3.57}$$

We realize that the mean photon number of light mode b in the interval between $\omega' = -\lambda$ and $\omega' = \lambda$ is expressible as

$$\bar{n}_{b\pm\lambda} = \int_{-\lambda}^{\lambda} P(\omega')d\omega', \tag{3.58}$$

in which $\omega' = \omega - \omega_0$. Therefore, upon substituting Eq. (3.57) into Eq. (3.58), and performing the integration by using the relation given by Eq. (3.36), we readily get

$$\bar{n}_{b\pm\lambda} = \bar{n}_b z_b(\lambda), \tag{3.59}$$

where $z_b(\lambda)$ is given by

$$z_b(\lambda) = \frac{2\kappa/\pi}{\kappa-\mu}\tan^{-1}\left(\frac{2\lambda}{\mu}\right) - \frac{2\mu/\pi}{\kappa-\mu}\tan^{-1}\left(\frac{2\lambda}{\kappa}\right). \tag{3.60}$$

We see from Eq. (3.59) along with the plot of $z_b(\lambda)$ that $\bar{n}_{b\pm\lambda}$ increases with λ until it reaches the maximum value of the global mean photon number.

From the plots in Figure (3.3), we find the values indicated in table below.

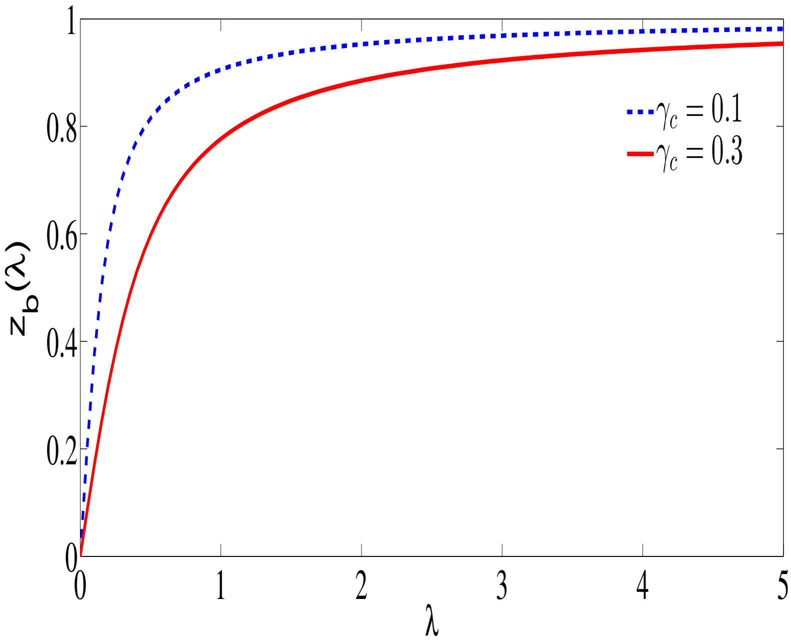

Figure 3.3: Plots of $z_b(\lambda)$[Eq. 3.60] versus λ for $\kappa = 0.8$, $N = 50$, $\Omega = 0.2$, and for different values of γ_c.

Table 3.3: Numerical Values of $z_b(\lambda)$ for $N = 50$, $\kappa = 0.8$ and $\Omega = 0.2$.

γ_c	$z_b(0.5)$	$z_b(1)$	$z_b(2)$	$z_b(4)$
0.3	0.6005	0.7787	0.886	0.9421
0.1	0.8166	0.84	0.9533	0.9766

The result indicated in Figure 3.3 demonstrates that $z_b(\lambda)$ decreases as γ_c becomes larger. Furthermore, using the result presented in the figure of $z_b(\lambda)$ and on account of Eq. (3.59), we have tabulated the following result.

Table 3.4: Numerical Values of $\bar{n}_{b\pm\lambda}$ for $N = 50$, $\kappa = 0.8$ and $\Omega = 0.2$.

γ_c	$\bar{n}_{b\pm0.5}$	$\bar{n}_{b\pm1}$	$\bar{n}_{b\pm2}$	$\bar{n}_{b\pm4}$
0.3	2.145	2.781	3.165	3.365
$\gamma_c = 0.1$	1.57	1.743	1.833	1.878

From Tables 3.2 and 3.4, we therefore observe that a large part of the mean photon number is confined in a relatively large frequency interval.

3.1.3 Global Photon-Number Variance

Here we seek to obtain the global photon number variance of light modes a and b, produced by the coherently driven nondegenerate three-level laser with a closed cavity and coupled to a two-mode thermal reservoir.

A. Global photon-number variance of light mode a

We now proceed to calculate the photon number variance of light mode a in the entire frequency interval. The photon number variance of light mode a is expressible as

$$(\Delta n)_a^2 = \langle \hat{a}^\dagger \hat{a} \hat{a}^\dagger \hat{a} \rangle - \langle \hat{a}^\dagger \hat{a} \rangle^2. \tag{3.61}$$

Applying the fact that $\hat{a}$ is a Gaussian variable with zero mean, we arrive at

$$(\Delta n)_a^2 = \langle \hat{a}^\dagger \hat{a} \rangle \langle \hat{a} \hat{a}^\dagger \rangle + \langle \hat{a}^{\dagger 2} \rangle \langle \hat{a}^2 \rangle. \tag{3.62}$$

In view of Eq. (3.2), we see that

$$\langle \hat{a}^2 \rangle = 0, \tag{3.63}$$

$$\langle \hat{a} \hat{a}^\dagger \rangle = \frac{\gamma_c}{\kappa} \langle \hat{N}_b \rangle. \tag{3.64}$$

Thus on account of Eqs. (3.5), (3.63) and (3.64), the photon number variance (3.62) turns out to be

$$(\Delta n)_a^2 = \left(\frac{\gamma_c}{\kappa} \right)^2 \langle \hat{N}_a \rangle \langle \hat{N}_b \rangle. \tag{3.65}$$

With the aid of Eqs. (2.125) and (2.126), the photon number variance of light mode a takes, at steady-state, the form

$$(\Delta n)_a^2 = \left[\frac{\gamma_c}{\kappa} N \right]^2 \left[\frac{\Omega^2}{\gamma^2 + 3\Omega^2} \right]^2. \tag{3.66}$$

This is the global photon number variance of light mode a, produced by the coherently driven non-degenerate three-level laser with a closed cavity and coupled to a two-mode vacuum reservoir. Moreover, in view of Eq. (3.6), we have

$$(\Delta n)_a^2 = \bar{n}_a^2, \tag{3.67}$$

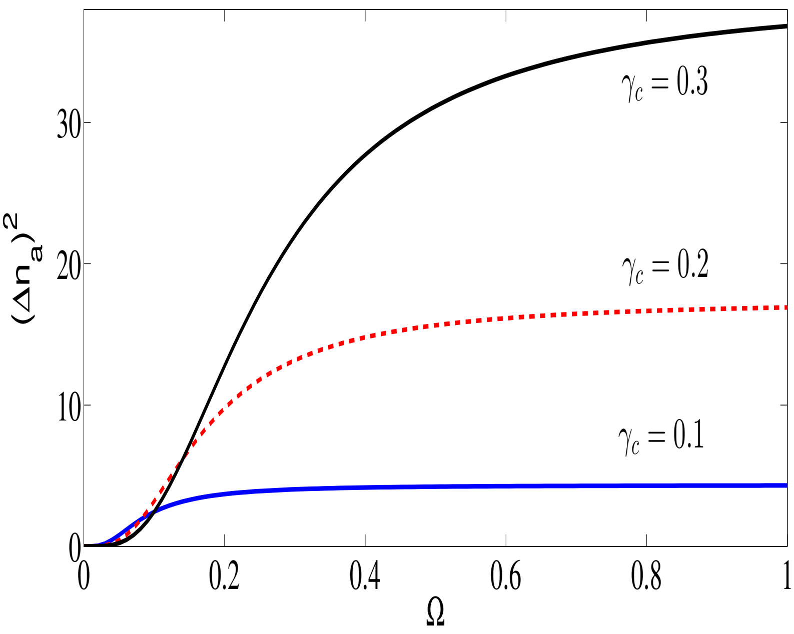

Figure 3.4: Plots of the photon number variance of light mode a [Eq. 3.66] versus Ω for $\kappa = 0.8$, $N = 50$, and for different values of γ_c.

which represents the normally-ordered variance of the photon number for chaotic light.

In addition, we note that for $\Omega \gg \gamma_c$, Eq. (3.66) reduces to

$$(\Delta n)_a^2 = \left[\frac{\gamma_c}{3\kappa}N\right]^2, \tag{3.68}$$

so that with the aid of Eq. (3.7), we see that

$$(\Delta n)_a^2 = \bar{n}_a^2. \tag{3.69}$$

Figure 3.4 shows that the photon number variance of light mode a increases with Ω and γ_c.

B. Global photon-number variance of light mode b

Here we seek to obtain the photon number variance of light mode b in the entire frequency interval. The photon number variance of light mode b is defined as

$$(\Delta n)_b^2 = \langle \hat{b}^\dagger \hat{b} \hat{b}^\dagger \hat{b} \rangle - \langle \hat{b}^\dagger \hat{b} \rangle^2 \tag{3.70}$$

and using the fact that $\hat{b}$ is a Gaussian variable with zero mean, we readily get

$$(\Delta n)_b^2 = \langle \hat{b}^\dagger \hat{b} \rangle \langle \hat{b} \hat{b}^\dagger \rangle + \langle \hat{b}^{\dagger 2} \rangle \langle \hat{b}^2 \rangle. \tag{3.71}$$

In view of Eq. (3.9), we have

$$\langle \hat{b}^2 \rangle = 0, \tag{3.72}$$

$$\langle \hat{b} \hat{b}^\dagger \rangle = \frac{\gamma_c}{\kappa} \langle \hat{N}_c \rangle. \tag{3.73}$$

Thus on account of Eqs. (3.12), (3.72) and (3.73), the photon number variance (3.79) turns out to be

$$(\Delta n)_b^2 = \left(\frac{\gamma_c}{\kappa} \right)^2 \langle \hat{N}_c \rangle \langle \hat{N}_b \rangle, \tag{3.74}$$

from which follows

$$(\Delta n)_b^2 = \bar{n}_b \left[\frac{\gamma_c}{\kappa} N - 2\bar{n}_b \right]. \tag{3.75}$$

With the aid of Eq. (3.13), the photon number variance of light mode b takes, at steady-state, the form

$$(\Delta n)_b^2 = \left(\frac{\gamma_c}{\kappa} N \right)^2 \left[\frac{\Omega^2 (\gamma_c^2 + \Omega^2)}{(\gamma_c^2 + 3\Omega^2)^2} \right]. \tag{3.76}$$

This is the steady-state the global photon number variance of of light mode b produced by the coherently driven non-degenerate three-level laser with a closed cavity and coupled to a two-mode vacuum reservoir.

Furthermore, we note that for $\Omega \gg \gamma_c$, Eq. (3.76) reduces to

$$(\Delta n)_b^2 = \left[\frac{\gamma_c}{3\kappa} N \right]^2 \tag{3.77}$$

and in view of Eq. (3.14), there follows

$$(\Delta n)_b^2 = \bar{n}_b^2, \tag{3.78}$$

which represents the normally-ordered variance of the photon number for chaotic light.

We readily observe from the plots in Figure 3.5 that the photon number variance of light mode b grows with the γ_c, for instance, $(\Delta n)_b^2 = 44.573$ for $\gamma_c = 0.3$ and $\Omega = 0.276$.

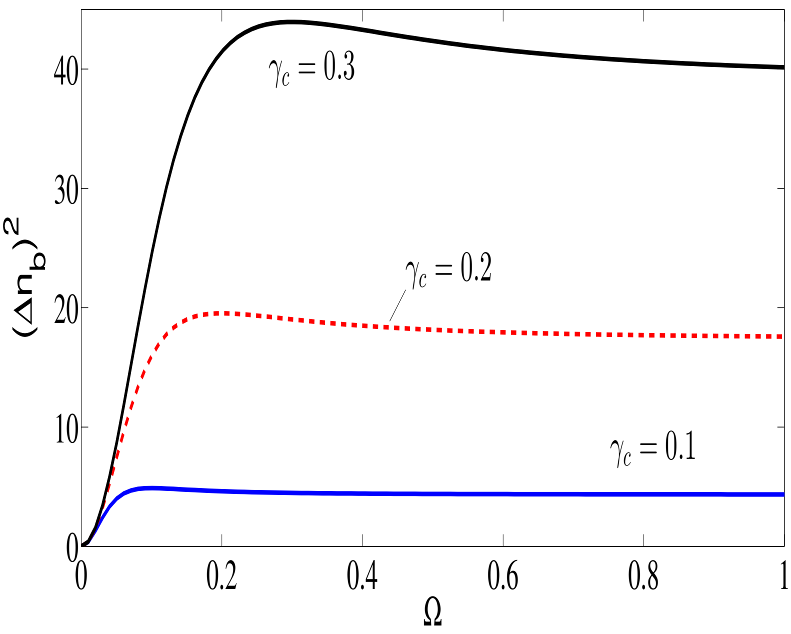

Figure 3.5: Plots of the photon number variance of light mode b [Eq. 3.76] versus Ω for $\kappa = 0.8$, $N = 50$, and for different values of γ_c.

3.1.4 Local Photon-Number Variance

Here we seek to study the local photon number variance of light modes a and b, produced by the coherently driven non degenerate three-level laser with a closed cavity and coupled to a two-mode vacuum reservoir.

A. Local photon-number variance of light mode a

We now proceed to obtain the photon number variance of light mode a in a given frequency interval. To determine the local photon number variance of light mode a, we need to consider the spectrum of photon number fluctuations of light mode a. The spectrum of photon number fluctuations of light mode a with central frequency ω_0 is expressible as [4]

$$S_a(\omega) = Re\frac{1}{\pi}\int_0^\infty d\tau e^{i(\omega-\omega_0)\tau}\langle\hat{n}_a(t), \hat{n}_a(t+\tau)\rangle_{ss}, \tag{3.79}$$

where

$$\hat{n}_a(t) = \hat{a}^\dagger(t)\hat{a}(t) \tag{3.80}$$

and

$$\hat{n}_a(t+\tau) = \hat{a}^\dagger(t+\tau)\hat{a}(t+\tau). \tag{3.81}$$

Upon integrating both sides of Eq. (3.79) over ω, we find

$$\int_{-\infty}^{\infty} S_a(\omega)d\omega = (\Delta n)_a^2, \tag{3.82}$$

in which $(\Delta n)_a^2$ is the steady-state global photon number variance of light mode a. From this result, we realize that $S_a(\omega)d\omega$ is the photon number variance of light mode a in the frequency interval between ω and $\omega + d\omega$ [4].

We now proceed to evaluate the two-time correlation function that appears in Eq. (3.79). Applying the notation [9]

$$\langle \hat{A}, \hat{B} \rangle = \langle \hat{A}\hat{B} \rangle - \langle \hat{A} \rangle \langle \hat{B} \rangle, \tag{3.83}$$

we see that

$$\langle \hat{n}_a(t), \hat{n}_a(t+\tau) \rangle = \langle \hat{n}_a(t)\hat{n}_a(t+\tau) \rangle - \langle \hat{n}_a(t) \rangle \langle \hat{n}_a(t+\tau) \rangle. \tag{3.84}$$

On account of Eqs. (3.80) and (3.81) and using the fact that a is a Gaussian variable with zero mean given by Eq. (2.114), we have

$$\begin{aligned} \langle \hat{n}_a(t)\hat{n}_a(t+\tau) \rangle &= \langle \hat{a}^\dagger(t)\hat{a}(t) \rangle \langle \hat{a}^\dagger(t+\tau)\hat{a}(t+\tau) \rangle \\ &+ \langle \hat{a}(t)\hat{a}(t+\tau) \rangle \langle \hat{a}^\dagger(t)\hat{a}^\dagger(t+\tau) \rangle \\ &+ \langle \hat{a}^\dagger(t)\hat{a}(t+\tau) \rangle \langle \hat{a}(t)\hat{a}^\dagger(t+\tau) \rangle. \end{aligned} \tag{3.85}$$

Thus substitution of Eq. (3.85) into Eq. (3.84) results in

$$\begin{aligned} \langle \hat{n}_a(t), \hat{n}_a(t+\tau) \rangle &= \langle \hat{a}^\dagger(t)\hat{a}^\dagger(t+\tau) \rangle \langle \hat{a}(t)\hat{a}(t+\tau) \rangle \\ &+ \langle \hat{a}^\dagger(t)\hat{a}(t+\tau) \rangle \langle \hat{a}(t)\hat{a}^\dagger(t+\tau) \rangle. \end{aligned} \tag{3.86}$$

With the help of Eq. (3.24), one can readily establish that

$$\langle \hat{a}^\dagger(t)\hat{a}^\dagger(t+\tau) \rangle = \langle \hat{a}^{\dagger 2}(t) \rangle \left[\frac{\kappa}{\kappa-\eta} e^{-\eta\tau/2} - \frac{\eta}{\kappa-\eta} e^{-\kappa\tau/2} \right], \tag{3.87}$$

$$\langle \hat{a}(t)\hat{a}(t+\tau) \rangle = \langle \hat{a}^2(t) \rangle \left[\frac{\kappa}{\kappa-\eta} e^{-\eta\tau/2} - \frac{\eta}{\kappa-\eta} e^{-\kappa\tau/2} \right], \tag{3.88}$$

$$\langle \hat{a}(t)\hat{a}^\dagger(t+\tau) \rangle = \langle \hat{a}(t)\hat{a}^\dagger(t) \rangle \left[\frac{\kappa}{\kappa-\eta} e^{-\eta\tau/2} - \frac{\eta}{\kappa-\eta} e^{-\kappa\tau/2} \right], \tag{3.89}$$

$$\langle \hat{a}^\dagger(t+\tau) \rangle = 0. \tag{3.90}$$

Now employing Eqs. (3.30), (3.87), (3.88), and (3.89), we obtain

$$\begin{aligned}\langle\hat{n}_a(t)\hat{n}_a(t+\tau)\rangle &= \Big[\langle\hat{a}^\dagger(t)\hat{a}(t)\rangle\langle\hat{a}(t)\hat{a}^\dagger(t)\rangle+\langle\hat{a}^2(t)\rangle\langle\hat{a}^{\dagger 2}(t)\rangle\Big]\\ &\times \Big[\Big(\frac{\eta}{\kappa-\eta}\Big)^2 e^{-\kappa\tau}+\Big(\frac{\kappa}{\kappa-\eta}\Big)^2 e^{-\eta\tau}-\frac{2\kappa\eta}{(\kappa-\eta)^2}e^{-(\kappa+\eta)\tau/2}\Big]. \qquad (3.91)\end{aligned}$$

This can be rewritten as

$$\langle\hat{n}_a(t)\hat{n}_a(t+\tau)\rangle_{ss}=\frac{(\Delta n)_a^2}{(\kappa-\eta)^2}\Big[\eta^2e^{-\kappa\tau}+\kappa^2e^{-\eta\tau}-2\kappa\eta e^{-(\kappa+\eta)\tau/2}\Big], \qquad (3.92)$$

in which $(\Delta n)_a^2$ is the steady-state photon number variance of light mode a given by Eq. (3.66). Therefore, in view of Eq. (3.92), the spectrum of photon number fluctuations can be put in the form

$$\begin{aligned}S_a(\omega) &= \frac{(\Delta n)_a^2}{\pi(\kappa-\eta)^2}Re\Big[\eta^2\int_0^\infty d\tau e^{-[\kappa-i(\omega-\omega_0)]\tau}\\ &+ \kappa^2\int_0^\infty d\tau e^{-[\eta-i(\omega-\omega_0)]\tau}-2k\eta\int_0^\infty d\tau e^{-[\frac{(\kappa+\eta)}{2}-i(\omega-\omega_0)]\tau}\Big]. \qquad (3.93)\end{aligned}$$

Thus on carrying out the integration, the spectrum of photon number fluctuations of light mode a turns out to be

$$S_a(\omega)=\frac{(\Delta n)_a^2}{(\kappa-\eta)^2}\Big[\frac{\eta^2\kappa/\pi}{\kappa^2+(\omega-\omega_0)^2}+\frac{\kappa^2\eta/\pi}{\eta^2+(\omega-\omega_0)^2}-\frac{2\kappa\eta(\kappa+\eta)/2\pi}{(\frac{\kappa+\eta}{2})^2+(\omega-\omega_0)^2}\Big]. \qquad (3.94)$$

Now we realize that the photon number variance in the frequency interval between $\omega'=-\lambda$ and $\omega'=\lambda$ is expressible as [4]

$$(\Delta n)_{a\pm\lambda}^2=\int_{-\lambda}^{\lambda}S_a(\omega')d\omega', \qquad (3.95)$$

in which $\omega'=\omega-\omega_0$. Therefore, substituting Eq. (3.94) into Eq. (3.95) leads to

$$(\Delta n)_{a\pm\lambda}^2=\frac{(\Delta n)_a^2}{\pi(\kappa-\eta)^2}\Big[\int_{-\lambda}^{\lambda}\frac{\eta^2\kappa d\omega'}{\kappa^2+\omega'^2}-\int_{-\lambda}^{\lambda}\frac{2\kappa\eta(\kappa+\eta)d\omega'}{(\frac{\kappa+\eta}{2})^2+\omega'^2}+\int_{-\lambda}^{\lambda}\frac{\eta\kappa^2d\omega'}{\eta^2+\omega'^2}\Big]. \qquad (3.96)$$

Employing the relation given by Eq. (3.36), the local photon number variance of light mode a produced by the coherently driven nondegenerate three-level laser with a closed cavity and coupled to a two-mode vacuum reservoir is found to be

$$(\Delta n)_{a\pm\lambda}^2=(\Delta n)_a^2z_a^{'}(\lambda), \qquad (3.97)$$

where $z_a^{'}(\lambda)$ is given by

$$z_a^{'}(\lambda)=\frac{2\eta^2/\pi}{(\eta-\kappa)^2}tan^{-1}\Big(\frac{\lambda}{\kappa}\Big)+\frac{2\kappa^2/\pi}{(\kappa-\eta)^2}tan^{-1}\Big(\frac{\lambda}{\eta}\Big)-\frac{4\kappa\eta/\pi}{(\kappa-\eta)^2}tan^{-1}\Big(\frac{2\lambda}{\kappa+\eta}\Big). \qquad (3.98)$$

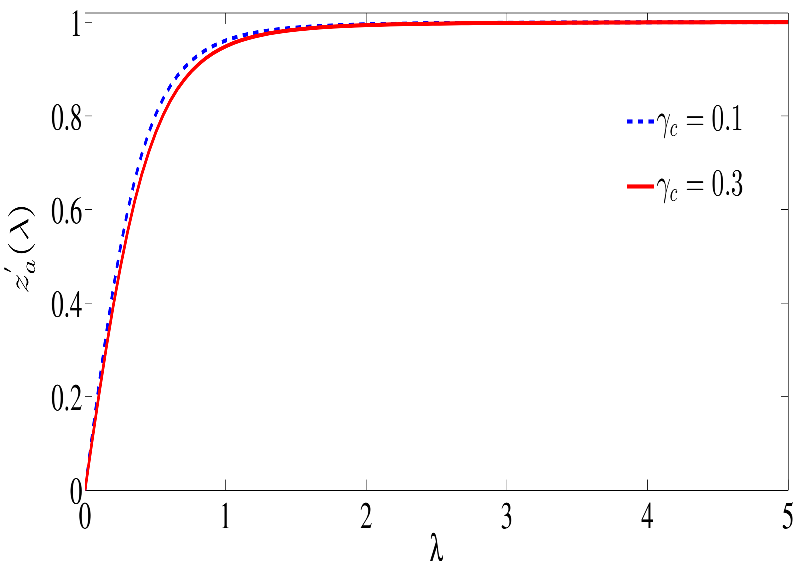

Figure 3.6: Plots of $z_a'(\lambda)$[Eq. 3.98] versus λ, for $\kappa = 0.8$, $N = 50$, $\Omega = 0.2$, and for different values of γ_c.

We see from Eq. (3.97) along with the plot $z_a'(\lambda)$ (given in Figure (3.6) that $(\Delta n)^2_{a\pm\lambda}$ increases with λ until it reaches the maximum value of the global photon number variance. From the plots in Figure (3.6), we find the values indicated in table below.

Table 3.5: Numerical Values of $z_a'(\lambda)$ for $N = 50$, $\kappa = 0.8$, and $\Omega = 0.2$.

γ_c	$z_a'(0.5)$	$z_a'(1)$	$z_a'(2)$	$z_a'(4)$
0.3	0.7661	0.9499	0.9941	0.9997
0.1	0.8044	0.9621	0.9958	0.9999

We see from these results that $z_a'(\lambda)$ increases rapidly for smaller values of λ but saturates for larger values. Moreover, using the above results of $z_a'(\lambda)$ and on account of Eq. (3.98), we have indicated in Table 3.6.

Table 3.6: Numerical Values of $(\Delta n)^2_{a\pm\lambda}$ for $N = 50$, $\kappa = 0.8$, and $\Omega = 0.2$.

γ_c	$(\Delta n)^2_{a\pm0.5}$	$(\Delta n)^2_{a\pm1}$	$(\Delta n)^2_{a\pm2}$	$(\Delta n)^2_{a\pm4}$
0.3	9.772	12.12	12.68	12.75
0.1	2.975	3.558	3.683	3.698

We also see that the variance of photon number is very larger compared with the mean photon number for the same system parameters.

B. Local photon-number variance for light mode b

We now proceed to obtain the photon number variance of light mode b in a given frequency interval produced by the system under consideration. To determine the local photon number variance of light mode b, we need to consider the spectrum of photon number fluctuations of light mode b. We define the spectrum of photon number fluctuations of light mode b with central frequency ω_0 by

$$S_b(\omega) = \frac{1}{\pi} Re \int_0^\infty d\tau e^{i(\omega-\omega_0)\tau} \langle \hat{n}_b(t), \hat{n}_b(t+\tau)\rangle_{ss}, \tag{3.99}$$

where

$$\hat{n}_b(t) = \hat{b}^\dagger(t)\hat{b}(t), \tag{3.100}$$

$$\hat{n}_b(t+\tau) = \hat{b}^\dagger(t+\tau)\hat{b}(t+\tau). \tag{3.101}$$

Upon integrating both sides of Eq. (3.99) over ω, we easily find

$$\int_{-\infty}^{\infty} S_b(\omega) d\omega = (\Delta n)_b^2, \tag{3.102}$$

in which $(\Delta n)_b^2$ is the steady-state photon number variance of the light mode b. We can then assert that $S_b(\omega)d\omega$ is the steady-state photon number variance of light mode b in the frequency interval between ω and $\omega + d\omega$.

We now proceed to evaluate the two-time correlation function that appears in Eq. (3.102). Applying the relation given by Eq. (3.83), we see that

$$\langle \hat{n}_b(t), \hat{n}_b(t+\tau)\rangle = \langle \hat{n}_b(t)\hat{n}_b(t+\tau)\rangle - \langle \hat{n}_b(t)\rangle\langle \hat{n}_b(t+\tau)\rangle. \tag{3.103}$$

On account of Eqs. (3.100) and (3.101), we have

$$\begin{aligned} \langle \hat{n}_b(t)\hat{n}_b(t+\tau)\rangle &= \langle \hat{b}^\dagger(t)\hat{b}(t)\rangle\langle \hat{b}^\dagger(t+\tau)\hat{b}(t+\tau)\rangle \\ &+ \langle \hat{b}(t)\hat{b}(t+\tau)\rangle\langle \hat{b}^\dagger(t)\hat{b}^\dagger(t+\tau)\rangle \\ &+ \langle \hat{b}^\dagger(t)\hat{b}(t+\tau)\rangle\langle \hat{b}(t)\hat{b}^\dagger(t+\tau)\rangle. \end{aligned} \tag{3.104}$$

Thus substitution of Eq. (3.103) into Eq. (3.104) results in

$$\begin{aligned} \langle \hat{n}_b(t), \hat{n}_b(t+\tau)\rangle &= \langle \hat{b}^\dagger(t)\hat{b}^\dagger(t+\tau)\rangle\langle \hat{b}(t)\hat{b}(t+\tau)\rangle \\ &+ \langle \hat{b}^\dagger(t)\hat{b}(t+\tau)\rangle\langle \hat{b}(t)\hat{b}^\dagger(t+\tau)\rangle. \end{aligned} \tag{3.105}$$

With the help of Eq. (3.48), one can readily obtain the following equations

$$\langle\hat{b}^\dagger(t)\hat{b}^\dagger(t+\tau)\rangle = \langle\hat{b}^{\dagger 2}(t)\rangle\left[\frac{\kappa}{\kappa-\mu}e^{-\mu\tau/2} - \frac{\mu}{\kappa-\mu}e^{-\kappa\tau/2}\right], \tag{3.106}$$

$$\langle\hat{b}(t)\hat{b}(t+\tau)\rangle = \langle\hat{b}^{2}(t)\rangle\left[\frac{\kappa}{\kappa-\mu}e^{-\mu\tau/2} - \frac{\mu}{\kappa-\mu}e^{-\kappa\tau/2}\right], \tag{3.107}$$

$$\langle\hat{b}(t)\hat{b}^\dagger(t+\tau)\rangle = \langle\hat{b}(t)\hat{b}^\dagger(t)\rangle\left[\frac{\kappa}{\kappa-\mu}e^{-\mu\tau/2} - \frac{\mu}{\kappa-\mu}e^{-\kappa\tau/2}\right], \tag{3.108}$$

$$\langle\hat{b}^\dagger(t+\tau)\rangle = 0. \tag{3.109}$$

Hence on account of Eqs. (3.54), (3.106), (3.107), and (3.108), Eq. (3.105) can be put in the form

$$\begin{aligned}\langle\hat{n}_b(t)\hat{n}_b(t+\tau)\rangle &= \left[\langle\hat{b}^\dagger(t)\hat{b}(t)\rangle\langle\hat{b}(t)\hat{b}^\dagger(t)\rangle + \langle\hat{b}^2(t)\rangle\langle\hat{b}^{\dagger 2}(t)\rangle\right]\\ &\times \left[\left(\frac{\mu}{\kappa-\mu}\right)^2 e^{-\kappa\tau} + \left(\frac{\kappa}{\kappa-\mu}\right)^2 e^{-\mu\tau} - \frac{2\kappa\mu}{(\kappa-\mu)^2}e^{-(\kappa+\mu)\tau/2}\right].\end{aligned} \tag{3.110}$$

This can be rewritten as

$$\langle\hat{n}_b(t)\hat{n}_b(t+\tau)\rangle_{ss} = \frac{(\Delta n)_b^2}{(\kappa-\mu)^2}\left[\mu^2 e^{-\kappa\tau} + \kappa^2 e^{-\mu\tau} - 2\kappa\mu e^{-(\kappa+\mu)\tau/2}\right], \tag{3.111}$$

in which $(\Delta n)_b^2$ is the steady-state photon number variance of light mode b given by Eq. (3.76). With the help of Eq. (3.111), the spectrum of photon number fluctuations can be put in the form

$$\begin{aligned}S_b(\omega) &= \frac{(\Delta n)_b^2}{\pi(\kappa-\mu)^2}Re\left[\mu^2\int_0^\infty d\tau e^{-[\kappa - i(\omega-\omega_0)]\tau}\right.\\ &+ \kappa^2\int_0^\infty d\tau e^{-[\mu - i(\omega-\omega_0)]\tau}\\ &- \left.2\kappa\mu\int_0^\infty d\tau e^{-[\frac{(\kappa+\mu)}{2} - i(\omega-\omega_0)]\tau}\right]\end{aligned} \tag{3.112}$$

and carrying out the integration, we obtain

$$S_b(\omega) = \frac{(\Delta n)_b^2}{(\kappa-\mu)^2}\left[\frac{\mu^2\kappa/\pi}{\kappa^2+(\omega-\omega_0)^2} - \frac{2\kappa\mu(\kappa+\mu)/2\pi}{(\frac{\kappa+\mu}{2})^2+(\omega-\omega_0)^2} + \frac{\kappa^2\mu/\pi}{\mu^2+(\omega-\omega_0)^2}\right]. \tag{3.113}$$

Now we realize that the photon number variance in the frequency interval between $\omega' = -\lambda$ and $\omega' = \lambda$ is expressible as

$$(\Delta n)_{b\pm\lambda}^2 = \int_{-\lambda}^{\lambda} S_b(\omega')d\omega', \tag{3.114}$$

in which $\omega' = \omega - \omega_0$. Therefore, substitution of Eq. (3.113) into Eq. (3.114) leads to

$$(\Delta n)_{b\pm\lambda}^2 = \frac{(\Delta n)_b^2}{\pi(\kappa-\mu)^2}\left[\int_{-\lambda}^{\lambda}\frac{\mu^2\kappa d\omega'}{\kappa^2+\omega'^2} - \int_{-\lambda}^{\lambda}\frac{2\kappa\mu(\kappa+\mu/2)d\omega'}{(\frac{\kappa+\mu}{2})^2+\omega'^2} + \int_{-\lambda}^{\lambda}\frac{\kappa^2\mu d\omega'}{\mu^2+\omega'^2}\right]. \tag{3.115}$$

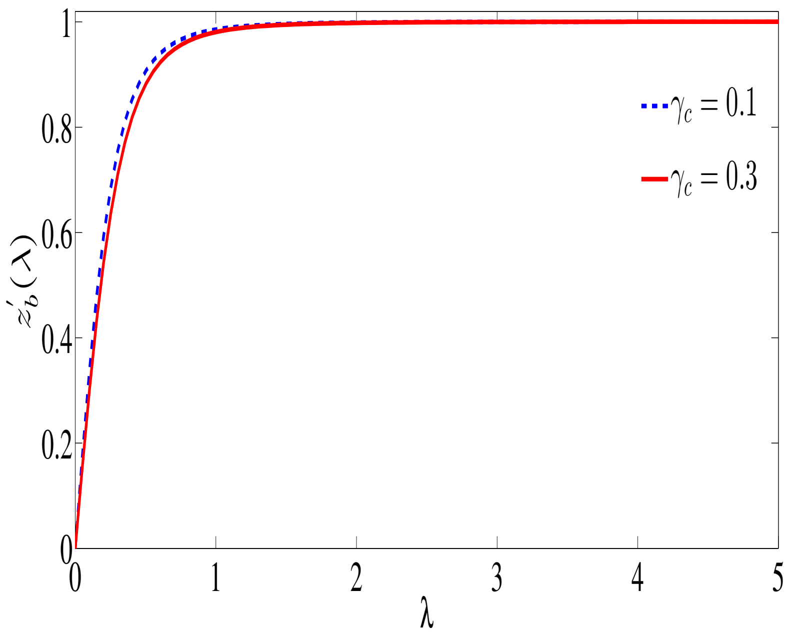

Figure 3.7: Plots of $z_b'(\lambda)$[Eq. 3.117] versus λ, for $\kappa = 0.8$, $N = 50$, $\Omega = 0.2$, and for different values of γ_c.

Employing the relation given by Eq. (3.36), the local photon number variance of light mode b produced by the coherently driven nondegenerate three-level laser with an open cavity and coupled to a two-mode vacuum reservoir is found to be

$$(\Delta n)^2_{b\pm\lambda} = (\Delta n)^2_b z_b'(\lambda), \tag{3.116}$$

where $z_b'(\lambda)$ is given by

$$z_b'(\lambda) = \frac{2\mu^2/\pi}{(\mu-\kappa)^2} tan^{-1}\left(\frac{\lambda}{\kappa}\right) + \frac{2\kappa^2/\pi}{(\kappa-\mu)^2} tan^{-1}\left(\frac{\lambda}{\mu}\right) - \frac{4\kappa\mu/\pi}{(\kappa-\mu)^2} tan^{-1}\left(\frac{2\lambda}{\kappa+\mu}\right). \tag{3.117}$$

We see from Eq. (3.116) along with the plot $z_b'(\lambda)$ that $(\Delta n)^2_{b\pm\lambda}$ increases with λ until it reaches the maximum value of the global photon number variance. From the plots in Figure (3.7), we indicated in table below:

Table 3.7: Numerical Values of $z_b'(\lambda)$ for $N = 50$, $\kappa = 0.8$ and $\Omega = 0.2$.

γ_c	$z_b'(0.5)$	$z_b'(1)$	$z_b'(2)$	$z_b'(4)$
0.3	0.8835	0.9813	0.9982	1
0.1	0.908	0.9859	0.9987	1

We see from these results, $z_b'(\lambda)$ in the absence of spontaneous emission ($\gamma = 0$) is less than in the presence of spontaneous emission ($\gamma \neq 0$). Moreover, using the above results of $z_b'(\lambda)$ and on account of Eq. (3.116), we have indicated in Table 3.8.

Table 3.8: Numerical Values of $(\Delta n)^2_{b\pm\lambda}$ for $N = 50$, $\kappa = 0.8$, and $\Omega = 0.2$.

γ_c	$(\Delta n)^2_{b\pm 0.5}$	$(\Delta n)^2_{b\pm 1}$	$(\Delta n)^2_{b\pm 2}$	$(\Delta n)^2_{b\pm 4}$
0.3	36.63	40.68	41.38	41.46
0.1	4.198	4.558	4.617	4.624

Tables 3.7 and 3.8 show that the total variance of photon number increases with increasing the frequency interval just similar to the mean photon number of the system.

3.2 Two-Mode Photon Statistics

In this section, applying the steady-state solutions of the equations of evolution of the expectation values of the atomic operators and the quantum Langavin equations for the cavity mode operators, we seek to obtain the mean and variance of the photon numbers for the two-mode light beam.

3.2.1 Two-Mode Mean Photon Number

Here we seek to calculate the steady-state mean photon number of the two-mode cavity light beam. The mean photon number of the two-mode light beam, represented by the operators $\hat{c}$ and $\hat{c}^\dagger$, is defined by

$$\bar{n} = \langle \hat{c}^\dagger \hat{c} \rangle. \tag{3.118}$$

The steady-state solution of Eq. (2.73) is found to be

$$\hat{c} = \frac{2g}{\kappa\sqrt{N}}\hat{m}. \tag{3.119}$$

Hence at steady state the mean photon number goes over into

$$\bar{n} = \frac{\gamma_c}{\kappa}\left[\langle \hat{N}_a \rangle + \langle \hat{N}_b \rangle\right]. \tag{3.120}$$

We see from Eq. (3.120) that the mean photon number of the two-mode light beam is the sum of the mean photon numbers of the separate single-mode light beams

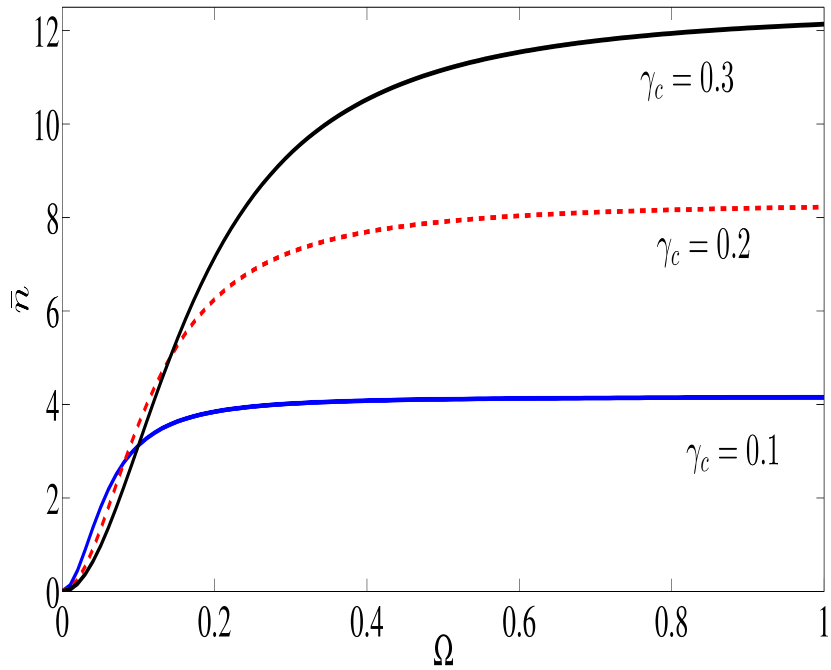

Figure 3.8: Plots of the two-mode mean photon number [Eq. (3.121)] versus Ω for $\kappa = 0.8$, $N = 50$, and for different values of γ_c.

given by Eqs. (3.5) and (3.13). Therefore, on account of Eqs. (2.125) and (2.126), Eq. (3.120) turns out to be

$$\bar{n} = \frac{\gamma_c}{\kappa} N \left[\frac{2\Omega^2}{\gamma_c^2 + 3\Omega^2} \right]. \tag{3.121}$$

This is the steady-state mean photon number of a two-mode cavity light produced by the coherently driven non-degenerate three-level laser in a closed cavity and coupled to a two-mode vacuum reservoir. Figure 3.8 shows the variation of the total mean photon number with γ_c and Ω. We observe that the mean photon number of the two-mode light beam is greater when $\gamma_c = 0.3$ than when $\gamma_c = 0.1$ for $\Omega > 0.1$? Furthermore, we note that for $\Omega \gg \gamma_c$, Eq. (3.121) reduces to

$$\bar{n} = \frac{2\gamma_c}{3\kappa} N. \tag{3.122}$$

3.2.2 Two-Mode Photon-Number Variance

Here we proceed to study the steady-state photon number variance of the two-mode light beam, produced by the coherently driven nondegenerate three-level laser in a closed cavity and coupled to a two-mode vacuum reservoir. The photon number variance for the two-mode cavity light is expressible as

$$(\Delta n)^2 = \langle \hat{c}^\dagger \hat{c} \hat{c}^\dagger \hat{c} \rangle - \langle \hat{c}^\dagger \hat{c} \rangle^2. \tag{3.123}$$

Since $\hat{c}$ is Gaussian variable with zero mean, the variance of the photon number can be written as

$$(\Delta n)^2 = \langle \hat{c}^\dagger \hat{c} \rangle \langle \hat{c} \hat{c}^\dagger \rangle + \langle \hat{c}^{\dagger 2} \rangle \langle \hat{c}^2 \rangle. \tag{3.124}$$

With the aid of Eq. (3.119), one can easily establish that

$$\langle \hat{c} \hat{c}^\dagger \rangle = \frac{\gamma_c}{\kappa} \left[\langle \hat{N}_b \rangle + \langle \hat{N}_c \rangle \right] \tag{3.125}$$

$$\langle \hat{c}^2 \rangle = \frac{\gamma_c}{\kappa} \langle \hat{m}_c \rangle. \tag{3.126}$$

Since $\langle \hat{m}_c \rangle$ is real, then $\langle \hat{c}^2 \rangle = \langle \hat{c}^{\dagger 2} \rangle$. Therefore, with the aid of Eqs. (3.120), (3.125) and (3.126), the variance of the photon number for the two-mode cavity light turns out to be

$$(\Delta n)^2 = \left(\frac{\gamma_c}{\kappa} \right)^2 \left[\left(\langle \hat{N}_a \rangle + \langle \hat{N}_b \rangle \right) \left(\langle \hat{N}_b \rangle + \langle \hat{N}_c \rangle \right) + \langle \hat{m}_c \rangle^2 \right]. \tag{3.127}$$

We observe from Eq. (3.127) that the photon number variance of the two-mode light beam does not happen to be the sum of the photon number variance of the separate single-mode light beams given by Eqs. (3.65) and (3.74). Furthermore, upon substituting of Eqs. (2.125)-(2.128) into Eq. (3.127), the steady-state variance of the photon number goes over into

$$(\Delta n)^2 = \left(\frac{\gamma_c}{\kappa} N \right)^2 \left[\frac{4\Omega^4 + 3\Omega^2 \gamma_c^2}{(\gamma_c^2 + 3\Omega^2)^2} \right]. \tag{3.128}$$

This is the steady-state photon number variance of the two-mode light beam, produced by the coherently driven nondegenerate three-level laser with a closed cavity and coupled to a two-mode vacuum reservoir. Figure 3.9 indicates that the photon number variance increases with γ_c and Ω.

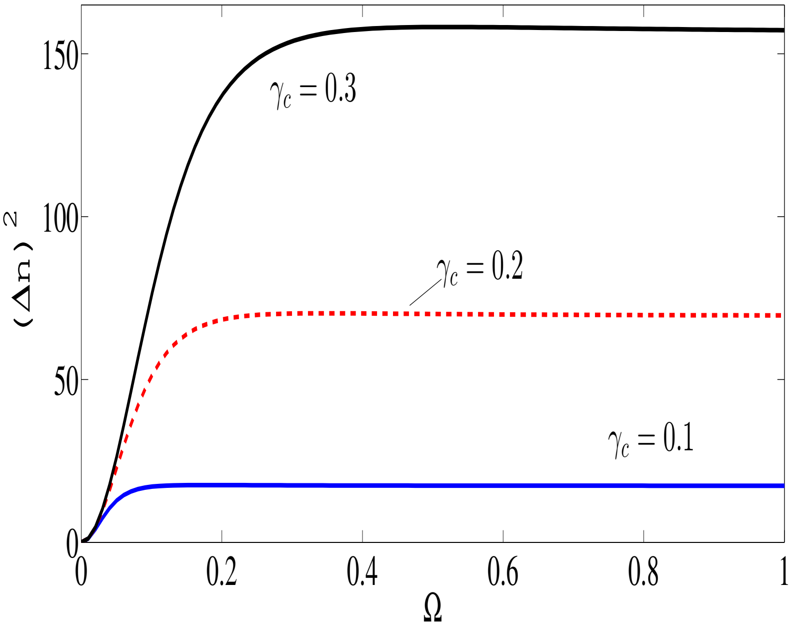

Figure 3.9: Plots of the two-mode photon number variance of [Eq. 3.128] versus Ω for $\kappa = 0.8$, $N = 50$, and for different values of γ_c.

Furthermore, we note that for $\Omega \gg \gamma_c$, Eq. (3.128) reduces to

$$(\Delta n)^2 = \left[\frac{2\gamma_c}{3\kappa}N\right]^2 \tag{3.129}$$

and in view of Eq. (3.124), we have

$$(\Delta n)^2 = \bar{n}^2, \tag{3.130}$$

which represents the normally-ordered variance of the photon number for chaotic light.

4

Quadrature Squeezing

In this chapter we seek to study the quadrature variance and the quadrature squeezing of the light produced by coherently driven nondegenerate three-level laser with a closed cavity and coupled to a two-mode vacuum reservoir via a single-port mirror. Applying the steady-state solutions of the equations of evolution of the expectation values of the atomic operators and the quantum Langevin equations for the cavity mode operators, we obtain the global quadrature variances for light modes a and b. In addition, we determine the global quadrature squeezing of the two-mode cavity light.

4.1 Single-Mode Quadrature Variance

In this section we obtain the global quadrature variances of light modes a and b, produced by the system under consideration.

A. Global quadrature variance of light mode a

We now proceed to calculate the quadrature variance of light mode a in the entire frequency interval. The squeezing properties of light mode a are described by two quadrature operators

$$\hat{a}_{+} = \hat{a}^{\dagger} + \hat{a} \tag{4.1}$$

and

$$\hat{a}_{-} = i(\hat{a}^{\dagger} - \hat{a}), \tag{4.2}$$

where $\hat{a}_{+}$ and $\hat{a}_{-}$ are Hermitian operators representing physical quantities called plus and minus quadratures, respectively, while $\hat{a}^{\dagger}$ and $\hat{a}$ are the creation and annihilation operators for light mode a. With the help of Eqs. (4.1) and (4.2), we can show that the two quadrature operators satisfy the commutation relation

$$[\hat{a}_-, \hat{a}_+] = 2i\frac{\gamma_c}{\kappa}\left[\hat{N}_a - \hat{N}_b\right]. \tag{4.3}$$

In view of this result, the uncertainty relation for the plus and minus quadrature operators of mode a is expressible as

$$\Delta a_+ \Delta a_- \geq \frac{1}{2}\left|\langle[\hat{a}_+, \hat{a}_-]\rangle\right|$$

$$\Delta a_+ \Delta a_- \geq \left|\langle\hat{a}\hat{a}^\dagger\rangle - \langle\hat{a}^\dagger\hat{a}\rangle\right|, \tag{4.4}$$

so that using Eqs. (4.3) and (4.4), there follows

$$\Delta a_+ \Delta a_- \geq \frac{\gamma_c}{\kappa}\left|\langle\hat{N}_a\rangle - \langle\hat{N}_b\rangle\right|. \tag{4.5}$$

On account of Eq. (4.5), the uncertainty relation for the quadrature operators can be put in the form

$$\Delta a_+ \Delta a_- \geq 0. \tag{4.6}$$

Next we proceed to calculate the quadrature variance of light mode a. The variance of the plus and minus quadrature operators are defined by

$$(\Delta\hat{a}_+)^2 = \langle\hat{a}_+^2\rangle - \langle\hat{a}_+\rangle^2 \tag{4.7}$$

and

$$(\Delta\hat{a}_-)^2 = \langle\hat{a}_-^2\rangle - \langle\hat{a}_-\rangle^2. \tag{4.8}$$

With the aid of Eq. (4.1), Eq. (4.7) can be expressed in terms of the creation and annihilation operators as

$$(\Delta\hat{a}_+)^2 = \langle\hat{a}\hat{a}^\dagger\rangle + \langle\hat{a}^\dagger\hat{a}\rangle + \langle\hat{a}^2\rangle + \langle\hat{a}^{\dagger 2}\rangle - \langle\hat{a}\rangle^2 - \langle\hat{a}^\dagger\rangle^2 - 2\langle\hat{a}\rangle\langle\hat{a}^\dagger\rangle. \tag{4.9}$$

In addition, on account of Eqs. (4.2) and (4.8), we get

$$(\Delta\hat{a}_-)^2 = \langle\hat{a}\hat{a}^\dagger\rangle + \langle\hat{a}^\dagger\hat{a}\rangle - \langle\hat{a}^2\rangle - \langle\hat{a}^{\dagger 2}\rangle + \langle\hat{a}\rangle^2 + \langle\hat{a}^\dagger\rangle^2 + 2\langle\hat{a}\rangle\langle\hat{a}^\dagger\rangle, \tag{4.10}$$

so that inspection of Eqs. (4.9) and (4.10) shows that

$$(\Delta\hat{a}_{\pm})^2 = \langle\hat{a}\hat{a}^{\dagger}\rangle + \langle\hat{a}^{\dagger}\hat{a}\rangle \pm \langle\hat{a}^2\rangle \pm \langle\hat{a}^{\dagger 2}\rangle \mp \langle\hat{a}\rangle^2 \mp \langle\hat{a}^{\dagger}\rangle^2 \mp 2\langle\hat{a}\rangle\langle\hat{a}^{\dagger}\rangle. \tag{4.11}$$

Moreover, with the help of Eqs. (4.9) and (4.10), we have

$$(\Delta\hat{a}_{\pm})^2 = \langle\hat{a}\hat{a}^{\dagger}\rangle + \langle\hat{a}^{\dagger}\hat{a}\rangle \tag{4.12}$$

and in view of Eqs. (4.5) and (4.12), there follows

$$(\Delta\hat{a}_{\pm})^2 = \frac{\gamma_c}{\kappa}\left[\langle\hat{N}_a\rangle + \langle\hat{N}_b\rangle\right]. \tag{4.13}$$

On account of Eq. (4.13), we see that

$$(\Delta\hat{a}_{\pm})^2 = \frac{2\gamma_c}{\kappa}\langle\hat{N}_a\rangle. \tag{4.14}$$

Now substitution of Eq. (4.14) into Eq. (4.13) results in

$$(\Delta\hat{a}_{\pm})^2 = \left(\frac{\gamma_c}{\kappa}N\right)\left[\frac{2\Omega^2}{\gamma_c^2 + 3\Omega^2}\right]. \tag{4.15}$$

This represents the quadrature variance of light mode a, produced by the coherently driven nondegenerate three-level laser with a closed cavity and coupled to a two-mode vacuum reservoir.

In addition, we note that for $\Omega \gg \gamma_c$, Eq. (4.15) reduces to

$$(\Delta\hat{a}_{\pm})^2 = \frac{2\gamma_c}{3\kappa}N. \tag{4.16}$$

It is not difficult to express Eq. (4.16) in the form

$$(\Delta\hat{a}_{\pm})^2 = 2\bar{n}_a, \tag{4.17}$$

which is the normally ordered quadrature variance for chaotic light.

B. Global quadrature variance of light mode b

Here we wish to calculate the quadrature variance of light mode b in the entire frequency interval, produced by the system under consideration. The squeezing properties of light mode b are described by two quadrature operators

$$\hat{b}_{+} = \hat{b}^{\dagger} + \hat{b} \tag{4.18}$$

and

$$\hat{b}_{-} = i(\hat{b}^{\dagger} - \hat{b}), \tag{4.19}$$

where $\hat{b}_{+}$ and $\hat{b}_{-}$ are Hermitian operators representing physical quantities called plus and minus quadratures, respectively, while $\hat{b}^{\dagger}$ and $\hat{b}$ are the creation and annihilation operators for light mode b. With the help of Eqs. (4.18) and (4.19), we can show that the two quadrature operators satisfy the commutation relation

$$[\hat{b}_{-}, \hat{b}_{+}] = 2i\frac{\gamma_c}{\kappa}\bigg[\hat{N}_b - \hat{N}_c\bigg]. \tag{4.20}$$

In view of this result, the uncertainty relation for the plus and minus quadrature operators of mode b is expressible as

$$\Delta b_{+}\Delta b_{-} \geq \frac{1}{2}\bigg|\langle[\hat{b}_{+}, \hat{b}_{-}]\rangle\bigg|$$

$$\geq \bigg|\langle\hat{b}\hat{b}^{\dagger}\rangle - \langle\hat{b}^{\dagger}\hat{b}\rangle\bigg|, \tag{4.21}$$

so that using Eqs. (4.20) and (4.21), there follows

$$\Delta b_{+}\Delta b_{-} \geq \frac{\gamma_c}{\kappa}\bigg|\langle\hat{N}_b\rangle - \langle\hat{N}_c\rangle\bigg|, \tag{4.22}$$

On account of Eqs. (4.21) and (4.22), the uncertainty relation of the quadrature operators can be put the form

$$\Delta b_{+}\Delta b_{-} \geq \frac{\gamma_c}{\kappa}N\bigg|\frac{\gamma_c^2}{\gamma_c^2 + 3\Omega^2}\bigg|. \tag{4.23}$$

Moreover, we consider the case in which the deriving coherent light is absent. Thus upon setting $\Omega = 0$ in Eq. (4.23), we readily get

$$\Delta b_{+}\Delta b_{-} \geq \frac{\gamma_c}{\kappa}N. \tag{4.24}$$

Next we proceed to calculate the quadrature variance of light mode b. The variance of the plus and minus quadrature operators for light mode b are defined by

$$(\Delta\hat{b}_{+})^2 = \langle\hat{b}_{+}^2\rangle - \langle\hat{b}_{+}\rangle^2 \tag{4.25}$$

and

$$(\Delta\hat{b}_{-})^2 = \langle\hat{b}_{-}^2\rangle - \langle\hat{b}_{-}\rangle^2. \tag{4.26}$$

On account of Eq. (4.18), Eq. (4.25) can be expressed in terms of the creation and annihilation operators as

$$(\Delta\hat{b}_+)^2 = \langle\hat{b}\hat{b}^\dagger\rangle + \langle\hat{b}^\dagger\hat{b}\rangle + \langle\hat{b}^2\rangle + \langle\hat{b}^{\dagger 2}\rangle - \langle\hat{b}\rangle^2 - \langle\hat{b}^\dagger\rangle^2 - 2\langle\hat{b}\rangle\langle\hat{b}^\dagger\rangle. \tag{4.27}$$

In addition, with the help of Eqs. (4.19) and (4.26), we see that

$$(\Delta\hat{b}_-)^2 = \langle\hat{b}\hat{b}^\dagger\rangle + \langle\hat{b}^\dagger\hat{b}\rangle - \langle\hat{b}^2\rangle - \langle\hat{b}^{\dagger 2}\rangle + \langle\hat{b}\rangle^2 + \langle\hat{b}^\dagger\rangle^2 + 2\langle\hat{b}\rangle\langle\hat{b}^\dagger\rangle, \tag{4.28}$$

so that inspection of Eqs. (4.27) and (4.28) shows that

$$(\Delta\hat{b}_\pm)^2 = \langle\hat{b}\hat{b}^\dagger\rangle + \langle\hat{b}^\dagger\hat{b}\rangle \pm \langle\hat{b}^2\rangle \pm \langle\hat{b}^{\dagger 2}\rangle \mp \langle\hat{b}\rangle^2 \mp \langle\hat{b}^\dagger\rangle^2 \mp 2\langle\hat{b}\rangle\langle\hat{b}^\dagger\rangle. \tag{4.29}$$

Moreover, with the aid of Eqs. (4.27) and (4.28), we get

$$(\Delta\hat{b}_\pm)^2 = \langle\hat{b}\hat{b}^\dagger\rangle + \langle\hat{b}^\dagger\hat{b}\rangle \tag{4.30}$$

and in view of Eqs. (4.22) and (4.30), there follows

$$(\Delta\hat{b}_\pm)^2 = \frac{\gamma_c}{\kappa}\left[\langle\hat{N}_b\rangle + \langle\hat{N}_c\rangle\right]. \tag{4.31}$$

Now on account of Eqs. (4.23) and (4.31), the quadrature variance of light mode b takes, at steady-state, the form

$$(\Delta\hat{b}_\pm)^2 = \frac{\gamma_c}{\kappa}N\left[\frac{\gamma_c^2 + 2\Omega^2}{\gamma_c^2 + 3\Omega^2}\right]. \tag{4.32}$$

This represents the quadrature variance of light mode b, produced by the coherently driven nondegenerate three-level laser with a closed cavity and coupled to a two-mode vacuum reservoir. We therefore notice that the product of the uncertainties in the two quadratures satisfies the minimum uncertainty relation In addition, we note that for $\Omega \gg \gamma_c$, Eq. (4.32) reduces to

$$(\Delta\hat{b}_\pm)^2 = \frac{2\gamma_c}{3\kappa}N. \tag{4.33}$$

In view of Eq. (4.33), this can be expressed as

$$(\Delta\hat{b}_\pm)^2 = 2\bar{n}_b, \tag{4.34}$$

which is the normally ordered quadrature variance for chaotic light. For $\Omega = 0$, Eq. (4.32) reduces to Eq. (4.24), implying that the product of the uncertainties of the plus and minus quadrature operators of mode b satisfies the minimum uncertainty principle.

4.2 Two-Mode Quadrature Squeezing

In this section we proceed to study the quadrature variance and the quadrature squeezing of the two-mode light beam produced by the coherently driven nondegenerate three-level laser with a closed cavity and coupled to a two-mode vacuum reservoir.

Now we seek to determine the quadrature variances of the two-mode light beam. The squeezing properties of the two-mode cavity light are described by two quadrature operators

$$\hat{c}_+ = \hat{c}^\dagger + \hat{c} \tag{4.35}$$

and

$$\hat{c}_- = i(\hat{c}^\dagger - \hat{c}), \tag{4.36}$$

where $\hat{c}_+$ and $\hat{c}_-$ are Hermitian operators representing the physical quantities called plus and minus quadratures, respectively while $\hat{c}^\dagger$ and $\hat{c}$ are the creation and annihilation operators of the two-mode cavity light. With the aid of Eqs. (4.35) and (4.36), we show that the two quadrature operators satisfy the commutation relation

$$[\hat{c}_-, \hat{c}_+] = 2i\frac{\gamma_c}{\kappa}\left[\hat{N}_a - \hat{N}_c\right]. \tag{4.37}$$

In view of this result, the uncertainty relation for the plus and minus quadrature operators of the two-mode cavity light is expressible as

$$\begin{aligned}\Delta c_+ \Delta c_- &\geq \frac{1}{2}\left|\langle[\hat{c}_+, \hat{c}_-]\rangle\right| \\ &\geq \left|\langle\hat{c}\hat{c}^\dagger\rangle - \langle\hat{c}^\dagger\hat{c}\rangle\right|,\end{aligned} \tag{4.38}$$

so that using Eqs. (4.37) and (4.38), there follows

$$\Delta c_+ \Delta c_- \geq \frac{\gamma_c}{\kappa}\left|\langle\hat{N}_a\rangle - \langle\hat{N}_c\rangle\right|. \tag{4.39}$$

On account of Eqs. (4.38) and (4.39), the uncertainity relation for the plus and minus quadrature operators is found to be

$$\Delta c_+ \Delta c_- \geq \frac{\gamma_c}{\kappa}N\left|\frac{\gamma_c^2}{\gamma_c^2 + 3\Omega^2}\right|. \tag{4.40}$$

Moreover, we consider the case in which the deriving coherent light is absent. Thus upon setting $\Omega = 0$ in Eq. (4.40), we readily get

$$\Delta c_+ \Delta c_- \geq \frac{\gamma_c}{\kappa} N. \tag{4.41}$$

Next we proceed to calculate the quadrature variance of the two-mode cavity light. The variance of the plus and minus quadrature operators of the two-mode cavity light are defined by

$$(\Delta \hat{c}_+)^2 = \langle \hat{c}_+^2 \rangle - \langle \hat{c}_+ \rangle^2 \tag{4.42}$$

and

$$(\Delta \hat{c}_-)^2 = \langle \hat{c}_-^2 \rangle - \langle \hat{c}_- \rangle^2. \tag{4.43}$$

On account of Eqs. (4.35) and (4.42), the plus quadrature variance can be expressed in terms of the creation and annihilation operators as

$$(\Delta \hat{c}_+)^2 = \langle \hat{c}\hat{c}^\dagger \rangle + \langle \hat{c}^\dagger \hat{c} \rangle + \langle \hat{c}^2 \rangle + \langle \hat{c}^{\dagger 2} \rangle - \langle \hat{c} \rangle^2 - \langle \hat{c}^\dagger \rangle^2 - 2\langle \hat{c} \rangle \langle \hat{c}^\dagger \rangle \tag{4.44}$$

and with the help of Eqs. (4.36) and (4.43), we get

$$(\Delta \hat{c}_-)^2 = \langle \hat{c}\hat{c}^\dagger \rangle + \langle \hat{c}^\dagger \hat{c} \rangle - \langle \hat{c}^2 \rangle - \langle \hat{c}^{\dagger 2} \rangle + \langle \hat{c} \rangle^2 + \langle \hat{c}^\dagger \rangle^2 - 2\langle \hat{c} \rangle \langle \hat{c}^\dagger \rangle, \tag{4.45}$$

so that inspection of Eqs. (4.44) and (4.45) shows that

$$(\Delta \hat{c}_\pm)^2 = \langle \hat{c}\hat{c}^\dagger \rangle + \langle \hat{c}^\dagger \hat{c} \rangle \pm \langle \hat{c}^2 \rangle \pm \langle \hat{c}^{\dagger 2} \rangle \mp \langle \hat{c} \rangle^2 \mp \langle \hat{c}^\dagger \rangle^2 - 2\langle \hat{c} \rangle \langle \hat{c}^\dagger \rangle. \tag{4.46}$$

In view of Eqs. (4.46), we see that

$$(\Delta \hat{c}_\pm)^2 = \langle \hat{c}\hat{c}^\dagger \rangle + \langle \hat{c}^\dagger \hat{c} \rangle \pm \langle \hat{c}^2 \rangle \pm \langle \hat{c}^{\dagger 2} \rangle. \tag{4.47}$$

In addition, with the aid of Eqs. (4.45), (4.44) and (4.46), expression (4.47) goes over into

$$(\Delta \hat{c}_\pm)^2 = \frac{\gamma_c}{\kappa} \left[\langle \hat{N}_a \rangle + 2\langle \hat{N}_b \rangle + \langle \hat{N}_c \rangle \pm \langle \hat{m}_c \rangle \pm \langle \hat{m}_c^\dagger \rangle \right]. \tag{4.48}$$

Now using Eqs. (4.47) and (4.48), the quadrature variance of the two-mode cavity light is found to be

$$(\Delta \hat{c}_\pm)^2 = \frac{\gamma_c}{\kappa} \left[N + \langle \hat{N}_b \rangle \pm 2\langle \hat{m}_c \rangle \right]. \tag{4.49}$$

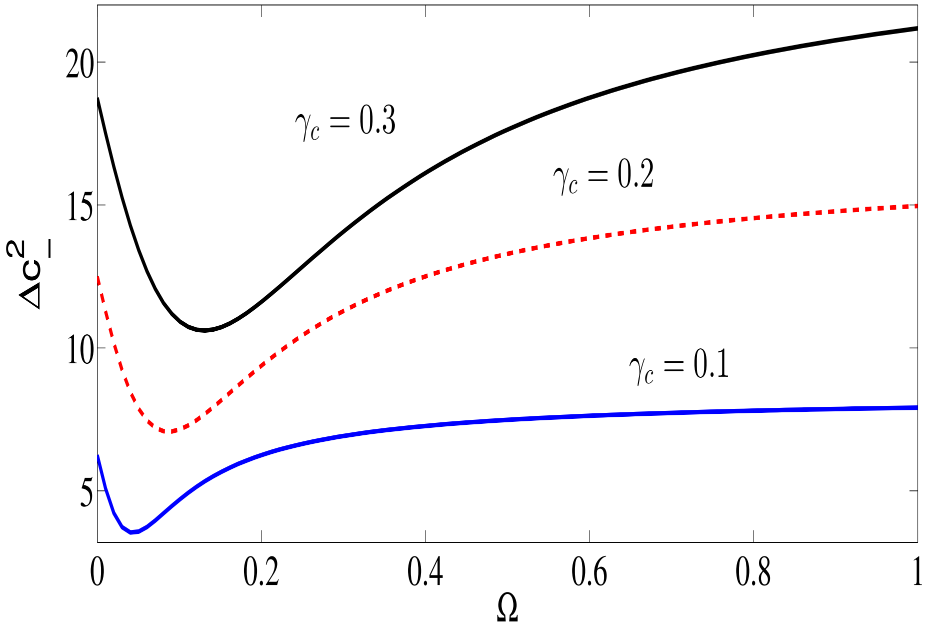

Figure 4.1: Plots of $(\Delta c_-)^2$ [Eq. (4.50)] versus Ω for $\kappa = 0.8$, $N = 50$ and for different values of γ_c.

Finally, on account of Eqs. (4.48) and (4.49), the quadrature variance of the two-mode cavity light takes, at steady-state, the form

$$(\Delta \hat{c}_\pm)^2 = \frac{\gamma_c}{\kappa} N \left[\frac{4\Omega^2 + \gamma_c^2 \pm 2\Omega\gamma_c}{\gamma_c^2 + 3\Omega^2} \right]. \tag{4.50}$$

This represents the quadrature variance of the two-mode cavity light produced by the coherently driven nondegenerate three-level laser with a closed cavity and coupled to a two-mode vacuum reservoir. therefore, the minimum uncertainty relation for the two-mode cavity vacuum state. We therefore notice that the uncertainties in the two quadratures are equal and their product satisfies the minimum uncertainty relation. In addition, we note that for $\Omega \gg \gamma_c$, Eq. (4.50) reduces to

$$(\Delta \hat{c}_\pm)^2 = \frac{4\gamma_c}{3\kappa} N. \tag{4.51}$$

This can be rewritten as

$$(\Delta \hat{c}_\pm)^2 = 2\bar{n}, \tag{4.52}$$

where $\bar{n}$ is given by Eq. (4.52). We see that Eq. (4.52) represents the normally ordered quadrature variance for chaotic light. Moreover, we consider the case in which the driving coherent light is absent. Thus upon setting $\Omega = 0$ in Eq. (4.50), we get

$$(\Delta \hat{c}_+)_v^2 = (\Delta \hat{c}_-)_v^2 = \frac{\gamma_c}{\kappa} N, \tag{4.53}$$

which is the normally ordered quadrature variance of the two-mode cavity vacuum state. We note that for $\Omega = 0$ the uncertainty in the plus and minus quadratures are equal and satisfy the minimum uncertainty relation.

The result presented in Figure 4.1), in which the minimum value of the quadrature variance can be obtained, are appeared below.

Table 4.1: Values of Δc_-^2 for $N = 50$ and $\kappa = 0.8$.

γ_c	Δc_-^2	Ω
0.1	3.876	0.0532
0.2	7.095	0.0989
0.3	10.225	0.1526

Next we proceed to calculate the quadrature squeezing of the two-mode cavity light in the entire frequency interval relative to the quadrature variance of the two-mode vacuum state. We then define the quadrature squeezing of the two-mode cavity light by [8]

$$S = \frac{(\Delta \hat{c}_-)_v^2 - (\Delta \hat{c}_-)^2}{(\Delta \hat{c}_-)_v^2}. \tag{4.54}$$

It then follows that

$$S = 1 - \frac{(\Delta \hat{c}_-)^2}{(\Delta \hat{c}_-)_v^2}. \tag{4.55}$$

In view of Eqs. (4.50) and (4.53), the quadrature squeezing of the two-mode cavity light takes, at steady-state, the form

$$S = \left[\frac{2\Omega\gamma_c - \Omega^2}{\gamma_c^2 + 3\Omega^2}\right]. \tag{4.56}$$

This represents the quadrature squeezing of the two-mode cavity light produced by the coherently driven nondegenerate three-level laser with a closed cavity and coupled to a two-mode vacuum reservoir.

We observe from this equation that unlike the mean photon number and the

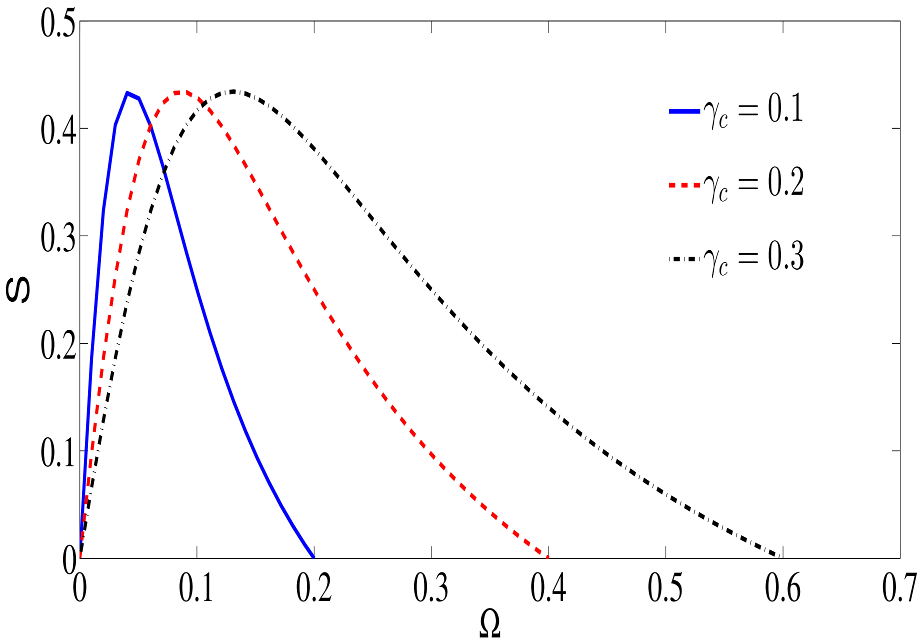

Figure 4.2: Plot of the quadrature squeezing [Eq. (4.56)] versus Ω for $N = 50$, $\kappa = 0.8$, and for different values of γ_c.

quadrature variance, the quadrature squeezing does not depend on the number of atoms. This implies that the quadrature squeezing of the two-mode cavity light is independent of the number of photons.

Applying Eqs. (4.55) and (4.56), we find

$$\langle \hat{b}\hat{a} \rangle = \frac{\gamma_c}{\kappa} \langle \hat{m}_c \rangle. \tag{4.57}$$

Since $\langle \hat{b} \rangle = \langle \hat{a} \rangle = 0$, we see that light modes a and b are correlated. The squeezing of the two-mode cavity light is due to this correlation. The two-mode light can be used in experiments involving entangled light modes.

It is shown in Figure (4.2) that the maximum quadrature squeezing of the same strength (42.976%) is produced for the considered stimulated emission decay constants, $\gamma_c = 0.1$, $\gamma_c = 0.2$ and $\gamma_c = 0.3$, and occurs when the laser is operating at $\Omega = 0.0532$, $\Omega = 0.0989$, and $\Omega = 0.1526$, respectively.

5

Entanglement Properties of the Two-mode Light

In this chapter we study the photon entanglement as well as atom entanglement of a two-mode laser light beams produced by the coherently driven nondegenerate three-level lasers with a closed cavity and coupled to the two-mode vacuum reservoir. Applying the solutions of the equations of evolution of the expectation values of the atomic operators and the quantum Langavin equations for the cavity mode operators, we obtain the entanglement of the two-mode light beams.

5.1 Photon Entanglement

Entanglement is an important manifestation of quantum mechanics. Highly entangled states play a key role in an efficient realization of quantum information processing [31]. Quantum entanglement is a physical phenomenon that occurs when pairs or groups of particles cannot be described independently instead, a quantum state may be given for the system as a whole. Measurements of physical properties such as position, momentum, spin, polarization, etc. performed on entangled particles are found to be appropriately correlated. A pair of particles is taken to be entangled in quantum theory, if its states cannot be expressed as a product of the states of its individual constituents. The preparation and manipulation of these entangled states that have nonclassical and nonlocal properties lead to a better understanding of the basic quantum principles. It is in this spirit that this section is devoted to the analysis of the entanglement of the two-mode photon states. In other words, it is a well-known fact that a quantum system is said to be entangled, if it is not separable. That is, if the density operator for the combined state cannot be described as a combination of the product of the density operators of the constituents,

$$\hat{\rho} \neq \sum_k p_k \hat{\rho}_k^{(1)} \otimes \hat{\rho}_k^{(2)}, \tag{5.1}$$

in which $p_k \gg 0$ and $\sum_k p_k = 1$ to verify the normalization of the combined density states. On the other hand, a maximally entangled CV state can be expressed as a coeigenstate of a pair of EPR-type operators [32] such as $\hat{x}_a - \hat{x}_b$ and $\hat{P}_a - \hat{P}_b$.The total variance of these two operators reduces to zero for maximally entangled CV states. According to the inseparable criteria given by Duan et al [33], cavity photon-states of a system are entangled, if the sum of the variance of a pair of EPR-like operators,

$$\hat{s} = \hat{x}_a - \hat{x}_b, \tag{5.2}$$

$$\hat{t} = \hat{p}_a + \hat{p}_b, \tag{5.3}$$

where

$$\hat{x}_a = \frac{1}{\sqrt{2}}\left(\hat{a} + \hat{a}^\dagger\right), \tag{5.4}$$

$$\hat{x}_b = \frac{1}{\sqrt{2}}\left(\hat{b} + \hat{b}^\dagger\right), \tag{5.5}$$

$$\hat{p}_a = \frac{i}{\sqrt{2}}\left(\hat{a}^\dagger - \hat{a}\right), \tag{5.6}$$

$$\hat{p}_b = \frac{i}{\sqrt{2}}\left(\hat{b}^\dagger - \hat{b}\right), \tag{5.7}$$

are quadrature operators for modes a and b, satisfy

$$\Delta s^2 + \Delta t^2 < 2N \tag{5.8}$$

and recalling the cavity mode operators $\hat{a}$ and $\hat{b}$ are Gaussian variables with zero mean, we readily get

$$\Delta s^2 + \Delta t^2 = \left[\langle\hat{a}^\dagger\hat{a}\rangle + \langle\hat{a}\hat{a}^\dagger\rangle + \langle\hat{b}^\dagger\hat{b}\rangle + \langle\hat{b}\hat{b}^\dagger\rangle\right] - \left[\langle\hat{a}\hat{b}\rangle + \langle\hat{a}^\dagger\hat{b}^\dagger\rangle + \langle\hat{b}\hat{a}\rangle + \langle\hat{b}^\dagger\hat{a}^\dagger\rangle\right]. \tag{5.9}$$

Thus with the aid of Eqs. (3.2) and (3.11), we see that

$$\Delta s^2 + \Delta t^2 = \frac{2\gamma_c}{\kappa}\left[N + \langle\hat{N}_b\rangle - 2\langle\hat{m}_c\rangle\right]. \tag{5.10}$$

It then follows that

$$\Delta s^2 + \Delta t^2 = 2\Delta c_-^2. \tag{5.11}$$

where Δc_-^2 is given by Eq. (4.53). One can readily see from this result that the degree of entanglement is directly proportional to the degree of squeezing of the two-mode light. Eq. (5.11) can easily be expressed at steady-state in the form

$$\Delta s^2 + \Delta t^2 = \frac{2\gamma_c}{\kappa} N \left[\frac{\gamma_c^2 + 4\Omega^2 - 2\Omega\gamma_c}{\gamma_c^2 + 3\Omega^2} \right]. \tag{5.12}$$

This is the steady-state the photon entanglement of a two-mode cavity light produced by the system under consideration.

One can immediately see from Eq. (5.11) that, this particular entanglement measure is directly related the two-mode squeezing. This direct relationship shows that whenever there is a two-mode squeezing in the system there will be entanglement in the system as well. It is noted that the entanglement disappears when the squeezing vanishes. This is due to the fact that the entanglement is directly related to the squeezing as given by Eq. (5.11). It also follows that like the mean photon number and quadrature variance the degree of entanglement depends on the number of atom. With the help of the criterion [33] that a significant entanglement between the states of the light generated in the cavity. This is due to the strong correlation between the radiation emitted when the atoms decay from the upper energy level to the lower via the intermediate level.

We note that Eq. (5.12) reduces, for $\Omega \gg \gamma_c$, to the form

$$\Delta s^2 + \Delta t^2 = \frac{8\gamma_c}{3\kappa} N. \tag{5.13}$$

This can also be rewritten as

$$\Delta s^2 + \Delta t^2 = 4\bar{n}, \tag{5.14}$$

where $\bar{n}$ is given by Eq. (3.122).

Figure 5.1 along with Table 5.1 show that as the stimulated emission decay constant increases the photon entanglement. It is also worth noting that the stimulated emission decay constant increases the squeezing and mean photon number of the quantum system. From the plots of Figure 5.1 along with Eq. (5.12), we have indicated the minimum values of the sum of the variances of the photon operators.

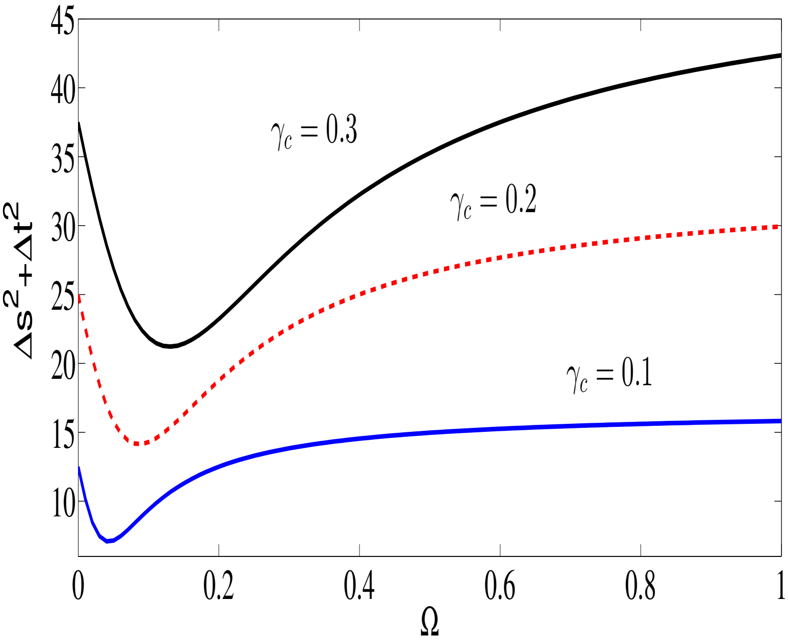

Figure 5.1: Plots of the photon entanglement of the two-mode cavity light [Eq. (5.12)] versus Ω for $N = 50$, $\kappa = 0.8$, and for different values of γ_c.

Table 5.1: The minimum values of $\Delta u^2 + \Delta v^2$, which correspond to the maximum degree of photon entanglement, for $N = 50$ and $\kappa = 0.8$.

γ_c	$\Delta s^2 + \Delta t^2$	Ω
0.1	7.74	0.0532
0.2	14.213	0.0989
0.3	20.45	0.1526

5.2 Cavity Atomic-States Entanglement

The quantum states of the atoms can also produce entanglement according to the criterion developed by Duan-Giedke-Cirac-Zoller (DGCZ) [32], which is a sufficient condition for entangled quantum states. According to DGCZ, a quantum state of the atoms is said to be entangled if the sum of the variances of the EPR-like quadrature operators, $\hat{u}$ and $\hat{v}$, satisfy the inequality condition

$$\Delta u^2 + \Delta v^2 < 2N^2. \tag{5.15}$$

in which

$$\hat{u} = \hat{x}'_a - \hat{x}'_b, \tag{5.16}$$

$$\hat{v} = \hat{p}'_a + \hat{p}'_b, \tag{5.17}$$

with

$$\hat{x}'_a = \frac{1}{\sqrt{2}}\left(\hat{m}_a + \hat{m}_a^\dagger\right), \tag{5.18}$$

$$\hat{x}'_b = \frac{1}{\sqrt{2}}\left(\hat{m}_b + \hat{m}_b^\dagger\right), \tag{5.19}$$

$$\hat{p}'_a = \frac{i}{\sqrt{2}}\left(\hat{m}_a^\dagger - \hat{m}_a\right), \tag{5.20}$$

$$\hat{p}'_b = \frac{i}{\sqrt{2}}\left(\hat{m}_b^\dagger - \hat{m}_b\right). \tag{5.21}$$

Since $\hat{m}_a$ and $\hat{m}_b$ are Gaussian variables with zero means, one can easily verify that

$$\Delta u^2 + \Delta v^2 = \left[\langle\hat{m}_a^\dagger\hat{m}_a\rangle + \langle\hat{m}_a\hat{m}_a^\dagger\rangle + \langle\hat{m}_b^\dagger\hat{m}_b\rangle + \langle\hat{m}_b\hat{m}_b^\dagger\rangle - \langle\hat{m}_b^\dagger\hat{m}_a^\dagger\rangle - \langle\hat{m}_a\hat{m}_b\rangle\right]. \tag{5.22}$$

Now with the aid of Eqs. (2.53) and (2.54), Eq. (5.22) takes the form

$$\Delta u^2 + \Delta v^2 = N\big[N + \langle\hat{N}_a\rangle - 2\langle\hat{m}_c\rangle\big], \tag{5.23}$$

which can be described at the steady state in the form

$$\Delta u^2 + \Delta v^2 = N^2\left[\frac{\gamma_c^2 + 4\Omega^2 - 2\Omega\gamma_c}{\gamma_c^2 + 3\Omega^2}\right]. \tag{5.24}$$

This is the cavity atomic-states entanglement of the two-mode cavity light produced by the coherently driven non-degenerate three-level laser with a closed cavity and coupled to a two-mode vacuum reservoir. When we observe Eq. (5.24), the cavity atomic-states entanglement of the two-mode cavity light highly depends on the number of atoms.

In addition, we note that for $\Omega \gg \gamma_c$, Eq. (5.24) reduces to

$$\Delta u^2 + \Delta v^2 = \frac{4}{3}N^2. \tag{5.25}$$

Furthermore, when $\Omega = 0$, Eq. (5.24) turns out to be

$$\Delta u^2 + \Delta v^2 = N^2. \tag{5.26}$$

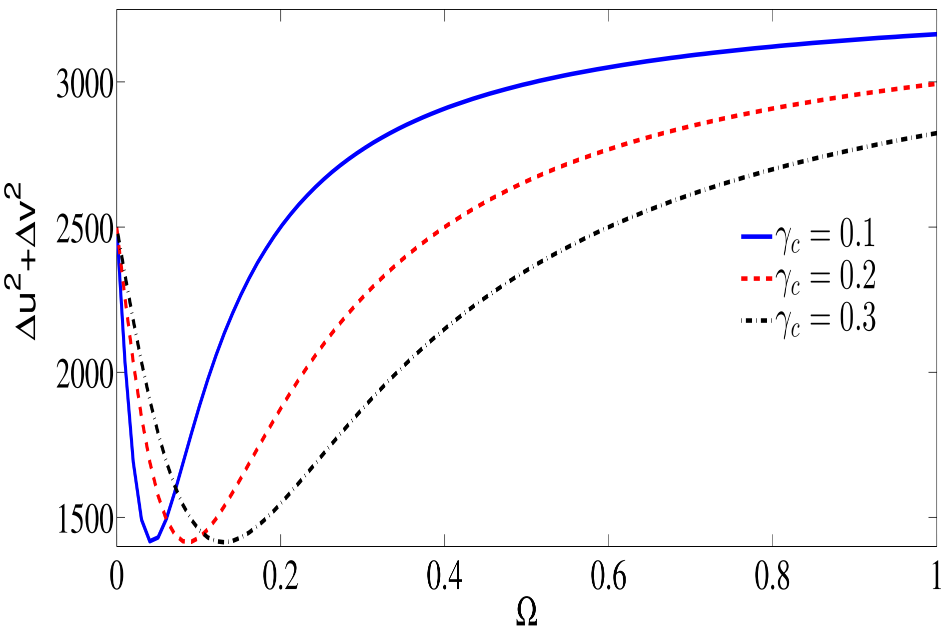

Figure 5.2: Plots of the atom entanglement of the two-mode cavity light [Eq. (5.24)] versus Ω for $N = 50$, $\kappa = 0.8$, and for different values of γ_c.

It is shown in Figure (5.2) that as the stimulated emission decay constant does not alter the maximum attainable degree of atom entanglement. However, the maximum degree of atom entanglement, corresponding to different stimulated emission decay constant, occurs at different values of Ω as indicated in table 3.2.

Table 5.2: The minimum values of $\Delta s^2 + \Delta t^2$, which correspond to the maximum degree of atom entanglement, for $N = 50$ and $\kappa = 0.8$.

γ_c	$\Delta u^2 + \Delta v^2$	Ω
0.1	1414	0.025479
0.2	1414	0.0501
0.3	1414	0.06356

6

Conclusion

In this thesis, the squeezing, entanglement and photon statistics of a non-degenerate three-level laser driven by coherent light, with a closed laser cavity containing N non-degenerate three-level atoms, and coupled to two-mode vacuum reservoir have thoroughly been analyzed using the quantum Langevin equations and large time approximations. We have carried out the analysis by putting the noise operators associated with the vacuum reservoir in normal order and without considering the interaction of the three-level atoms with the vacuum reservoir outside the cavity. The interaction Hamiltonian and the quantum Langevin equations for the cavity light are obtained. Applying these equations, the equations of evolution of the cavity mode and the atomic operators are solved. Making use of the steady-state solutions of atomic and cavity mode operators, the various quantities that help to investigate the quantum and statistical properties of interest are determined.

The analysis has shown that the global and local mean photon numbers and their fluctuations, quadrature squeezing, and photon entanglement within the laser cavity are enhanced by increasing the stimulated emission decay constant. On the other hand, the driving coherent light increases the intracavity global and local mean photon numbers and their fluctuations. Moreover, the mean photon number and its fluctuation becomes more intense for large frequency interval. On the other hand, the photon and atom entanglement, and squeezing are enhanced for relatively small value of the driving coherent light but declined for large values. We have also found that the squeezing and entanglement of the two-mode light are directly related. As a result, an increase in the degree of squeezing directly implies the improvement of the strength of the two-mode entanglement and vise versa. The quantum system has exhibited 43% degree of entanglement and squeezing for which the mean photon number is also significantly intense.

In general, it is possible to infer from this study that the proposed scheme can be utilized as a source of the quantum and statistical properties which are easily manageable by the various parameters involved in it. Therefore, the considered scheme can be a potential resource to be used in quantum optical experiments for the development of the quantum optics, quantum information and quantum technology.

7

Reference

[1] M. O. Scully, and M. S. Zubairy, *Opt. Commun.* **66**, 303 (1988).

[2] N. Lu, and Y. Zhu, *Phys. Rev. A* **41**, 2865 (1988).

[3] K. Fesseha, *Fundamentals of Quantum Optics.* (Lulu, North Carolina, 2008).

[4] M. O. Scully, Z. K. Wedkiewic, M. S. Zubairy, J. Bergou, L. Meyerter, and J. Veh, *Phys. Rev. Lett.* **60**, 1832 (1988).

[5] J. Anwar and M. S. Zubairy, *Phys. Rev.* **49**, 481 (1994).

[6] N. Lu and S. Y. Zhu, *Phys. Rev. A* **40**, 5735 (1989).

[7] T. Abebe, and C. G. Feyisa, *Braz. J. Phys.* **50**, 495-508 (2020).

[8] F. Kassahun, *Refined Quantum Analysis of Light.* (Create Space Independent Publishing Platform, 2014).

[9] N. A. Ansary, J. Gea-Banacloche, and M. S. Zubairy, *Phys. Rev. A* **41**, 5179 (1990).

[10] Ch. Gashu, T. Abebe, *Phys. Scr.*, **95** 075105 (2020).

[11] T. Abebe, N. Gemechu, Ch. Gashu, K. Shogile, S. Hailemariam, Sh. Adisu, *Int. J. Opt.* **2020**, 11, (2020).

[12] G. S. Agrawal and G. Adam, *Phys. Rev. A* **39**, 6259 (1989).

[13] Vyas and S. Singh, *Phys. Rev. A* **40**, 5147 (1989).

[14] B. Daniel and K. Fesseha, *Opt. Commun.* **151**, 384 (1998).

[15] M. J. Collet and C, W. Gadner, *Phys. Rev. A* **30**, 1386 (1984).

[16] G. J. Milburn and D. F. Walls, *Phys. Rev. A* **27**, 392 (1983).

[17] L. Wu, M. Xiao, and H. J. Kimble, *Opt. Soc. Am. B* **4**, 1465 (1987).

[18] M. Xiao, M. Wu La, and H. J. Kimble, *Phys. Rev. Lett.* **59**, 278 (1992).

[19] K. Fesseha, *Phys. Rev. A* **63**, 033811 (2001).

[20] E. Alebachew and K. Fesseha, *Opt. commun.* **265**, 314 (2006).

[21] S. M. Barnt and P. M. Radmore, *Methods In Theoretical Quantum Optics.* (Oxford University Press, New York, 1997).

[22] E. Alebachew, *Opt. commun.* **280**, 133 (2007).

[23] D. T. Smithey, M. Beck, J. Cooper, and M. J. Raymer, *Phys. Rev. A* **48**, 3159 (1993).

[24] A. Sumairi, S. N. Hazmin, and C. R. Ooi, *J. M. Opt.* **60**, 589-597 (2013).

[25] S. Tesfa, *arXiv preprint arXiv:0708.2815*, (2007).

[26] N. A. Ansari, *Phys Rev A* **48**, 4686 (1993).

[27] F. Kassahun, *Opt. Commun.* **284**, 1357 (2011).

[28] T. Abebe, Ch. Gashu, and N. Gemechu, *Adv. in Math. Phys.* **2020**, 10 (2020).

[29] F. Kassahun, *arXiv:1105.1438 [quant-ph]*, (2012).

[30] T. Abebe, *Ukr. J. Phys.*, **63**, 969-969 (2018).

[31] A. Barenco, D. Dutch, A. Ekert, and R Jozsa, *Phys. Rev. Lett. A* **74**, 4083 (1995).

[32] A. Einstein, B. Podolsky, and R. Rosen, *Phys. Rev.* **47**, 777 (1935).

[33] L. M. Duan, G. Giedke, J. J. Cirac, and P. Zoller, *Phys. Rev. Lett. A* **84**, 2722 (2000).