Existence and Uniqueness of Positive Solutions for Caputo Fractional Order Boundary Value Problems

A RESEARCH SUBMITTED TO THE DEPARTMENT OF MATHEMATICS IN PARTIAL FULFILLMENT FOR THE REQUIREMENTS OF THE DEGREE OF MASTERS OF SCIENCE IN MATHEMATICS

By: Abebech Yifru

Advisor : Dr. Wosen Legese (Ph.D.)
Co-Advisor : Mr. Girma Kebede (M.Sc.)

July, 2021
Jimma, Ethiopia

Declaration

I, the undersigned declare that, this research paper entitled ”Existence and Uniqueness of Positive for Caputo Fractional Boundary Value Problems” is my own original work and it has not been submitted for the award of any academic degree or the like in any other institution or university, and that all the sources I have used or quoted have been indicated and acknowledged.

Name: Abebech Yifru

Signature: ....

Date:..

The work has been done under the supervision of:

Name: Wosen Legese (Ph.D.)

Signature:..................................

Date:..

Name: Girma Kebede (M.Sc.)

Signature: ....

Date: ....

Acknowledgment

First of all, I am indebted to God who gave me long life and helped me to pass through different ups and downs to reach this time. Next, my special heartfelt thanks go to my advisor, Dr. Wosen Legesse and co-advisor Mr. Girma Kebede for their unreserved support, advice and guidance throughout the work of this research. Also, I would like to thank Jimma University College of Natural Science Department of Mathematics for their support. Lastly, I express my deepest gratitude to all my friends and my teachers specially Dr. Kidane Koyas for their guidance and comment throughout the work of this research.

Abstract

A class of two-point boundary value problems whose highest-order term is a Caputo fractional derivative of order $\alpha \in (1,2)$ with a reaction term is considered. It focused on constructing Green's function for corresponding homogeneous equation by using Laplace transform and Mittag-Leffler function. Under the suitable conditions, we established the existence of positive solution for Caputo fractional order differential equation BVPs by applying fixed point index theorem.
Finally, we established the uniqueness of positive solution for Caputo fractional order differential equation BVPs by applying Banach contraction principle.

Acronym

Throughout this research, we denote the following notation.

- $\mathbb{N}$ is the set of positive integers.
- $\mathbb{R}$ is the set of real numbers.
- $\mathbb{R}^+$ is the set of non-negative real numbers.
- $\mathbb{C}$ is the set of complex numbers.
- L and L^{-1} is Laplace transform and Laplace inverse transform respectively.
- $G(t,s)$ is Green's function
- ${}^{C}D^{\alpha}_{0^+}$ and ${}^{RL}D^{\alpha}_{0^+}$ is Caputo fractional and Riemann-Liouville derivative of order α respectively.
- $\partial\Omega$ is boundary of omega.
- BVPs is boundary value problems.

Contents

Declaration i
Acknowledgment ii
Abstract iii
Acronym iv

1 Introduction **1**
1.1 Background of the Study 1
1.2 Statements of the Problem 4
1.3 Objectives of the Study 5
1.3.1 General Objective 5
1.3.2 Specific Objectives 5
1.4 Significance of the Study 6
1.5 Delimitation of the Study 6

2 Review of Related Literatures **7**
2.1 Preliminaries 9

3 Research Design and Methodology **11**
3.1 Study Area and Period 11
3.2 Study Design 11
3.3 Source of Information 11
3.4 Mathematical Procedure of the Study 12

4 Main Result and Discussion **13**
4.1 Construction of Green's Function 13
4.2 Bounds of Green's Function 17
4.3 Existence of Positive Solutions 20
4.4 Uniqueness of Positive Solutions 25

5 Conclusion and Future scope **26**
5.1 Conclusion 26
5.2 Future scope 27
References 28

Chapter 1

Introduction

1.1 Background of the Study

The concept of the differential operator $\frac{d}{dx}$ is familiar to all who have studied the elementary calculus. And the suitable function f, the n^{th} derivative of f, namely $D^n f(x) = \frac{d^n f(x)}{dx^n}$ is well defined provided that n is a positive integer.
In 1698 L'Hospital inquired of Leibniz what meaning could be ascribed to $D^n f$ if n is a fraction. But it was not until 1884 that the theory of generalized operators achieved a level in its development suitable as point of departure for the modern mathematician. The theory had been extended to include operators D^α where α could be rational or irrational, positive or negative, real or complex. Thus the name fractional calculus can the meaning of a derivative of integer order $\frac{d^n y}{dx^n}$ be extended to have meaning when n is any number: fractional, irrational or complex. Because the latter question was answered affirmatively, the name fractional calculus has become a misnomer and might better be called integration and differentiation to an arbitrary order.

Leibniz invented the notation $\frac{d^n y}{dx^n}$. Perhaps it was a naive play with symbols that prompted L'Hospital in 1695 to asked "Leibniz what if n be $\frac{1}{2}$"? Leibniz 1697 states that differential calculus might have been used to achieve this result. He used the notation $d^{1/2}$ to denote the derivative of order $\frac{1}{2}$.
In 1819 the first mention of derivative of arbitrary order appears in a text. S. F. Lacroix 1819, developed a more mathematical exercise generalizing from a case of integer order. Starting with $y = x^m$ where m is positive integer, Lacroix easily developed the n^{th} derivative.

$$\frac{d^n y}{dx^n} = \frac{m!}{(m-n)!} x^{m-n} \text{ where } m \geq n.$$

Using Lagrange's symbol for the generalized factorial (the Gamma function), he gets

$$\frac{d^n y}{dx^n} = \frac{\Gamma(m+1)}{\Gamma(m-n+1)} x^{m-n}.$$

He then gives the example for $y = x$ and $n = \frac{1}{2}$, obtains

$$\frac{d^{1/2} x}{dx^{1/2}} = \frac{2\sqrt{x}}{\sqrt{\pi}}.$$

There are also well-known approaches to fractional derivative, by Riemann-Liouville, Grunwald-Letnikov and Caputo. The most commonly used definitions are Riemann-Liouville and Caputo. Caputo introduced a definition which has the advantage of defining integer order initial conditions for fractional order. Applied problems require definitions of fractional derivatives allowing the utilization of physically interpretable initial conditions, which contains $y(0), y'(0)$, etc.
The Caputo fractional order derivative is important because it allows traditional initial conditions to be included in the formulation of the problems with integer order initial conditions.

Boundary value problems associated with linear as well as non-linear ordinary differential equations or finite difference equations have great deal of interest and play an important role in many fields of applied mathematics such as Engineering and Technology, major industries like automobile, aerospace, optimization theory, electromagnetic potential and heat power transmission theory are few on the boundary value problems to simulate complex phenomena at different scales for designing and manufacturing of heat-technological products.

By using various fixed point theorem methods existence of positive solutions studied by different researchers such as L.H and Haiyan Wang 1994, Erbe, Hu and Wang 1994, Lian, Wong and Yah 1996, Henderson and Wang in 1997, Karakostas and Tsamatos in 2002, Hederson, Ntouyas and Purnaras in 2008, Dang Quang and Ngo Thi Kim Quy in 2018, Zhanbing Bai, Sujing Sun, Zengji Du, Yang Quan Chen in 2020.
Wang, Y. and Liu, L. (2017). Studied positive properties of Green's function for

two term fractional differential equation and its application.

$$-{}^{RL}D_{0+}^{\alpha}u(t)+bu(t)=f(t,u(t)),$$
$$u(0)=0,\ u(1)=0,$$

where $1<\alpha<2$ and $0<t<1$.

Meng, X. and Stynes, M., (2018). Studied Green's function and a maximum principle for a Caputo two-point BVPs with a convection term by using two parameter Mittag- Leffler functions.

$$-{}^{C}D_{0+}^{\alpha}u(t)+bu'(t)=f(t),\ t\in(0,1),$$
$$u(0)-\beta_0 u'(0)=\gamma_0,\ u(1)+\beta_1 u'(1)=\gamma_1,$$

where the parameter α satisfies $1<\alpha<2$, the constants $b,\beta_0,\beta_1,\gamma_0,\gamma_1$ and the function $f\in C[0,1]$ are given.

Bai, Z., *et al*, (2020). Studied Green's function for a class of Caputo fractional differential equations with a convection term by using Laplace transform.

$$-{}^{c}D_{0+}^{\alpha}u(t)+bu'(t)=h(t),\ 0<t<1,$$
$$u(0)-\beta_0 u'(0)=0,\ u(1)+\beta_1 u'(1)=0,$$

where the parameter α satisfies $1<\alpha<2$, the constants b,β_0,β_1 and the function $h\in C[0,1]$ are given.
Motivated by the above mentioned results, in this thesis, we investigated the existence and uniqueness of positive solutions for the following Robin type Caputo fractional order differential equation of BVPs with parameter α satisfying $1<\alpha<2$ and $t\in[0.1]$.

$$-{}^{C}D_{0+}^{\alpha}u(t)+bu(t)=f(t,u(t)) \tag{1.1}$$

$$\alpha_0 u(0)-\beta_0 u'(0)=0, \tag{1.2}$$

$$\gamma u(1)+\delta u'(1)=0 \tag{1.3}$$

where $\alpha_0,\gamma>0$; $\delta,\beta_0\geq 0$ and b a constant and the function $f\in L^1[0,1]$ by applying fixed point index theorem and Banach contraction principle.

1.2 Statements of the Problem

Wang,Y. and Liu, L. (2017). Studied positive properties of the Green function for two term fractional differential equation and its application.

$$-{}^{RL}D_{0^+}^{\alpha}u(t)+bu(t)=f(t,u(t)),\ 0<t<1,$$
$$u(0)=0,\ u(1)=0,$$

where 1< $\alpha<2$ and $t\in(0,1)$. And Bai, Z., *et al*, (2020). Studied the Green function for a class of Caputo fractional differential equations with a convection term by using Laplace transform.

$$-{}^{C}D_{0^+}^{\alpha}u(t)+bu'(t)=h(t),\ 0<t<1,$$

$$u(0)-\beta_0u'(0)=0,\ u(1)+\beta_1u'(1)=0,$$

where the parameter α satisfies $1<\alpha<2$; b,β_0,β_1 are constants and the function $h\in C[0,1]$ are given. In this Research, the Author was studied the existence and uniqueness of positive solutions for Caputo fractional order BVPs (1.1)- (1.3).

1.3 Objectives of the Study

1.3.1 General Objective

The main objective of this study is to establish the existence and uniqueness of positive solutions for Caputo fractional order differential equation of BVPs (1.1)-(1.3) by applying fixed point index theorem and Banach contraction principle respectively.

1.3.2 Specific Objectives

This study has the following specific objectives:

- To construct the Green function for corresponding homogeneous equation.
- To formulate the problem in the form of integral equation with considered condition.
- To prove the existence of positive solution by using fixed point index theorem.
- To prove the uniqueness of positive solution by using Banach contraction principle.

1.4 Significance of the Study

The outcome of this study have the following importance:

- It may built the research skill and scientific communication skill of the researcher.
- It may provide some background information for other researchers who want to conduct a research on related topics.
- It may help to show existence and uniqueness of positive solution for some fractional order differentiation.

1.5 Delimitation of the Study

This study was delimited to finding the existence and uniqueness of positive solution for Caputo fractional order differential equation of BVPs (1.1)- (1.3) by applying fixed point index theorem and Banach contraction principle.

Chapter 2

Review of Related Literatures

Fractional differential equations have become important in recent years as mathematical models of phenomena in engineering, chemistry, physics, and other sciences by using mathematical tools from the theory of derivatives and integrals of fractional or non-integer order. Different Authors have proved the positive solutions of fractional BVPs by using different methods and conditions. From this some of them are listed as follow.

Bai, Z. and Lu, H., (2005). Studied positive solutions for BVP of nonlinear fractional differential equation.

$$\begin{aligned} {}^{RL}D^{\alpha}_{0^+}u(t) + f(t,u(t)) = 0,\ 0 < t < 1, \\ u(0) = u(1) = 0, \end{aligned}$$

where $1 < \alpha \leq 2$ and $f : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous function by means of some fixed-point theorems on cone.

Zhang, S. (2006). Studied positive solutions for BVPs of nonlinear fractional differential equations.

$$\begin{aligned} {}^{C}D^{\alpha}_{0^+}u(t) = f(t,u(t)),\ 0 < t < 1, \\ u(0) + u'(0) = 0,\ u(1) + u'(1) = 0 \end{aligned}$$

where $1 < \alpha \leq 2$, and $f : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous by means of a fixed-point theorem on cones.

Benchohra, M., and Hedia, B. (2013). Studied multiple positive solutions for boundary value problems with fractional order by using the Krasnosel'skii fixed-point theorem in cones.

$$
\begin{gathered}
{}^{C}D_{0^+}^{\alpha} y(t) + \varphi(t) f(t, y(t)) = 0,\, t \in J = [0,1],\ 0 < \alpha \leq 1. \\
ay(0) + by(1) = c,
\end{gathered}
$$

where $f : J \times \mathbb{R} \to [0, +\infty)$ is continuous function, a, b, c are real constants with $a + b \neq 0$ and $\varphi : [0,1] \to \mathbb{R}$ is a given function.

Meng, X. and Stynes, M., (2019). Studied Green's functions, positive solutions and Lyapunov's inequality for a Caputo fractional-derivative BVPs by using Guo-Krasnoseleskii fixed point theorem.

$$
\begin{gathered}
{}^{c}D_{0^+}^{\alpha} u(t) + q(t) f(t) = 0,\ a < t < b, \\
y(a) - \beta_0 y'(a) = 0,\ y(b) + \beta_1 y'(b) = 0,
\end{gathered}
$$

where $q \in C[a,b],\ f$ is continuous function and $1 < \alpha < 2.$

2.1 Preliminaries

First we recall some known definitions that we used in the proof of our main results.

Definition 2.1.1 *Let X be a non-empty set. A map $T : X \to X$ is said to be a self-map with domain of $T = D(T) = X$ and range of $T = R(T) \subset X$.*

Definition 2.1.2 *Let $T : X \to X$ be a self-map. A point x in X is called fixed point of T if $Tx = x$.*

Definition 2.1.3 *Let $X = (X,d)$ be a metric space. A mapping $T : X \to X$ is called a contraction on X if there is a positive real number $\alpha \in [0,1)$ such that*
$d(Tx,Ty) \leq \alpha d(x,y)$ for all $x,\ y \in X$.

Theorem 2.1.1 *(Contraction Mapping Theorem) If T is a contraction mapping on a Banach space X with contraction constant α, with $0 \leq \alpha < 1$, then T has a unique fixed point $\bar{x} \in X$.*

Definition 2.1.4 *A linear space X is called a normed linear space provided there is a function $\|.\| : X \to \mathbb{R}$, called a norm, satisfying*

i. $\|x\| \geq 0$, *for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$.*

ii. $\|x+y\| \leq \|x\| + \|y\|$, *for all $x, y \in X$.*

iii. $\|\alpha x\| = |\alpha|\|x\|$, *for all $x \in X$ and α being a scalar.*

Definition 2.1.5 *We say that x_n subset of normed space X is a Cauchy sequence provided given any $\varepsilon > 0$ there is a positive integer N such that $\|x_n - x_m\| < \varepsilon$ for all $n,m \geq N$.*

Definition 2.1.6 *A normed linear space X is said to be complete, if every Cauchy sequence in X converges to a point in X.*

Definition 2.1.7 *A Banach space is a complete normed linear space.*

Definition 2.1.8 *Let E be a real Banach space and P a subset of E, P is called a cone if and only if:*

i. *P is closed, nonempty and* $P \neq \{0\}$*;*

ii. $a, b \in \mathbb{R},\ a, b \geq 0,\ x, y \in P$ *implies that* $ax + by \in P$*;*

iii. $x \in P$ *and* $-x \in P$ *wich implies* $x = 0$*.*

Definition 2.1.9 *Mittag-Leffler function is a generalization of the exponential function.*
(Mittag-Leffler, 1903) For $\alpha > 0,\ \alpha \in \mathbb{R}$ *and* $z \in \mathbb{C}$*, the series representation*

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}$$

is called one parameter Mittag-Leffler function.

Definition 2.1.10 *(Wiman, 1905) Two parameter Mittag-Leffler function is defined by the series,*

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} \text{ where } \alpha,\ \beta > 0,\ \alpha,\ \beta \in \mathbb{R},\ z \in \mathbb{C}.$$

Definition 2.1.11 *Suppose that* $\alpha > 0,\ t > a$ *and for all* $\alpha,\ a,\ t \in \mathbb{R}$*. The fractional operator*

$$D_{0^+}^{\alpha} f(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau \text{ where } n-1 < \alpha < n \in \mathbb{N} \\ \\ \dfrac{d^n}{dt^n} f(t) \text{ where } \alpha = n \in \mathbb{N} \end{cases}$$

is called the Caputo fractional derivatives of order α*.*

Definition 2.1.12 *Let* $f = f(t)$ *be a function of* $\mathbb{R}^+$*. The Laplace transform* $F(s)$ *is given by the integral*

$$F(s) = L[f(t)] = \int_0^{\infty} e^{-st} f(t) dt, \text{ for } s \in \mathbb{R}.$$

Chapter 3

Research Design and Methodology

This chapter contains study period and site, study design, source of information and mathematical procedures.

3.1 Study Area and Period

The study was conducted at Jimma University under the department of mathematics from September, 2020 G.C. to July, 2021 G.C.

3.2 Study Design

In order to achieve the objective of the study we followed analytical method of design.

3.3 Source of Information

The relevant sources of information for this study are books, published articles related to the area of the study and Internet.

3.4 Mathematical Procedure of the Study

In this study we followed the procedures stated below:

- Defining the Caputo fractional order BVPs.
- Constructing the Green function for the corresponding homogeneous equation of Caputo fractional order differential equation.
- Formulating equivalent operator equation for the BVPs.
- Determining the existence of positive solution by using fixed point index theorem.
- Proving the uniqueness of positive solution by using Banach contraction principle.

Chapter 4

Main Result and Discussion

4.1 Construction of Green's Function

In this section, we construct Green's function for the homogeneous problems corresponding to (1.1)-(1.3).
For the constants α and b from (1.1) we let the auxiliary function

$$F_\lambda(x) = x^{\lambda-1}E_{\alpha,\,\lambda}(bx^\alpha), \text{ for } \lambda \geq 0 \text{ and } x > 0. \tag{4.1}$$

(Podlubny, 1999). Derivative of the series representation of two parameter Mittag-Leffler function, for $\alpha > 0,\ \lambda \geq 0,\ b \in \mathbb{R}$ and β is an arbitrary,

$$D^\beta_{0^+}[x^{\lambda-1}E_{\alpha,\,\lambda}(bx^\alpha)] = x^{\lambda-\beta-1}E_{\alpha,\,\lambda-\beta}(bx^\alpha). \tag{4.2}$$

By the use of particular case of Formula (4.2) with $\beta \in \mathbb{N}$, certain properties of F_λ are list as follows:

$$(P1) : [F_{\lambda+1}(x)]' = F_\lambda(x), \text{ for } \lambda \geq 0 \text{ and } x \geq 0;$$
$$(P2) : F_1(0) = 1, \text{ and } F_\lambda(0) = 0 \text{ for } \lambda > 1,\ F_1(1) > 0,\ F_2(1) > 0;$$
$$(P3) : F_1(x) > 0 \text{ for } x > 0, \text{ and } F_2(x) \text{ is increasing for } x \geq 0;$$
$$(P4) : F_{\alpha-1}(x) \geq 0 \text{ for } x > 0, \text{ and } F_\alpha(x) \text{ is increasing for } x > 0.$$

Now we recall the fractional order BVP (1.1)-(1.3)

$$-{}^CD^\alpha u(t) + bu(t) = f(t, u(t))$$

$$\alpha_0 u(0) - \beta_0 u'(0) = 0, \qquad \gamma u(1) + \delta u'(1) = 0$$

and we construct its Green's function by using Laplace transform method. The Laplace transform of the Caputo fractional derivative is given by,

$$L\{{}^{C}D_{0^+}^{\alpha}u(t)\} = s^{\alpha}U(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1}u^{(k)}(0), \text{ where } n-1 < \alpha \leq n. \tag{4.3}$$

We apply the Laplace transform on both sides of Equation (1.1) and obtain

$$\begin{aligned} L\{-{}^{C}D_{0^+}^{\alpha}u(t) + bu(t)\} &= L\{f(t,u(t))\} \\ \text{which implies that, } L\{-{}^{C}D_{0^+}^{\alpha}u(t)\} + bL\{u(t)\} &= L\{f(t,u(t))\} \end{aligned}$$

in (4.3) $n = 2$, we have

$$-s^{\alpha}U(s) + \sum_{k=0}^{1} s^{\alpha-k-1}u^{(k)}(0) + bU(s) = F(s,u(s))$$

by expanding the series we get that,

$$-s^{\alpha}U(s) + s^{\alpha-1}u(0) + s^{\alpha-2}u'(0) + bU(s) = F(s,u(s)).$$

When we collect like terms, we have,

$$U(s)[b - s^{\alpha}] = F(s,u(s)) - s^{\alpha-1}u(0) - s^{\alpha-2}u'(0)$$

So, dividing both side by $b - s^{\alpha}$ we obtain,

$$\begin{aligned} U(s) &= \frac{F(s,u(s)) - s^{\alpha-1}u(0) - s^{\alpha-2}u'(0)}{b - s^{\alpha}} \\ &= \frac{1}{b - s^{\alpha}}F(s,u(s)) - \frac{s^{\alpha-1}}{b - s^{\alpha}}u(0) - \frac{s^{\alpha-2}}{b - s^{\alpha}}u'(0). \\ &= \frac{1}{b - s^{\alpha}}F(s,u(s)) - \frac{s^{\alpha-1}}{b - s^{\alpha}}u(0) - \frac{s^{\alpha-2}}{b - s^{\alpha}}u'(0). \end{aligned}$$

Applying Laplace inverse on both sides,

$$\begin{aligned} u(t) &= -L^{-1}\left[\frac{1}{s^{\alpha} - b}F(s,u(s)) - \frac{s^{\alpha-1}}{(s^{\alpha} - b)}u(0) - \frac{s^{\alpha-2}}{s^{\alpha} - b}u'(0)\right]. \\ &= -L^{-1}\left[\frac{1}{s^{\alpha} - b}\right] * L^{-1}\left[F(s,u(s))\right] + L^{-1}\left[\frac{s^{\alpha-1}}{(s^{\alpha} - b)}u(0)\right] + L^{-1}\left[\frac{s^{\alpha-2}}{s^{\alpha} - b}u'(0)\right]. \end{aligned}$$

$$\text{Since, } L^{-1}\left[\frac{s^{\alpha-\lambda}}{s^{\alpha}\pm b}\right] = t^{\lambda-1}E_{\alpha,\lambda}(\pm bt^{\alpha}),$$

$$L^{-1}\left[\frac{1}{s^{\alpha}\pm b}\right] = t^{\alpha-1}E_{\alpha,\alpha}(\pm bt^{\alpha}).$$

Consequently,

$$\begin{aligned}
-L^{-1}\{\frac{1}{s^{\alpha}-b}\}*L^{-1}\{F(s,u(s))\}+L^{-1}\{\frac{s^{\alpha-1}}{(s^{\alpha}-b)}u(0)\}+L^{-1}\{\frac{s^{\alpha-2}}{s^{\alpha}-b}u'(0)\}\\
&= -t^{\alpha-1}E_{\alpha,\alpha}(bt^{\alpha})*f(t,u(t))+t^{\lambda-1}E_{\alpha,\lambda}(bt^{\alpha})u(0)+t^{\lambda-1}E_{\alpha,\lambda}(bt^{\alpha})u'(0),\\
&= -t^{\alpha-1}E_{\alpha,\alpha}(bt^{\alpha})*f(t,u(t))+t^{1-1}E_{\alpha,1}(bt^{\alpha})u(0)+t^{2-1}E_{\alpha,2}(bt^{\alpha})u'(0),\\
&= -t^{\alpha-1}E_{\alpha,\alpha}(bt^{\alpha})*f(t,u(t))+E_{\alpha,1}(bt^{\alpha})u(0)+tE_{\alpha,2}(bt^{\alpha})u'(0)\\
&= -F_{\alpha}(t)*f(t,u(t))+F_1(t)u(0)+F_2u'(0).
\end{aligned}$$

Whence, by Convolution theorem,

$$u(t) = -\int_0^t F_{\alpha}(t-s)f(s,u(s))ds+F_1(t)u(0)+F_2(t)u'(0) \tag{4.4}$$

From equation (4.4), the properties (P_1), (P_2) and $F_1(t)=E_{\alpha,1}(bt^{\alpha})$ is derivative invariant, we obtain that,

$$u'(t) = -\int_0^t F_{\alpha-1}(t-s)f(s,u(s))ds+bF_1(t)u(0)+F_1(t)u'(0). \tag{4.5}$$

Letting $t=1$ in equation (4.4) and (4.5), we get

$$u(1) = -\int_0^1 F_{\alpha}(1-s)f(s,u(s))ds+F_1(1)u(0)+F_2(1)u'(0) \tag{4.6}$$

$$u'(1) = -\int_0^1 F_{\alpha-1}(1-s)f(s,u(s))ds+bF_1(1)u(0)+F_1(1)u'(0). \tag{4.7}$$

Substituting equation (4.6) and (4.7) into the boundary condition

$$\gamma u(1)+\delta u'(1)=0, \text{ we obtain,}$$

$$\gamma[-\int_0^1 F_\alpha(1-s)f(s,u(s))ds+F_1(1)u(0)+F_2(1)u'(0)]+\delta[-\int_0^1 F_{\alpha-1}(1-s)f(s,u(s))ds$$
$$+bF_1(1)u(0)+F_1(1)u'(0)]=0.$$

$$u(0)[\gamma F_1(1)+\delta bF_1(1)]+u'(0)[\gamma F_2(1)+\delta F_1(1)]=\int_0^1[\gamma F_\alpha(1-s)+\delta F_{\alpha-1}(1-s)]f(s,u(s))ds. \tag{4.8}$$

Taking in to account $\alpha_0,\ \gamma>0$ and $\beta_0,\ \delta\geq 0$.
Combining equation (4.8) with the boundary condition $\alpha_0 u(0)-\beta_0 u'(0)=0$, we have a system of equation

$$[\gamma F_1(1)+\delta bF_1(1)]u(0)+[\gamma F_2(1)+\delta F_1(1)]u'(0)=\int_0^1[\gamma F_\alpha(1-s)+\delta F_{\alpha-1}(1-s)]f(s,u(s))ds,$$
$$\alpha_0 u(0)-\beta_0 u'(0)=0. \tag{4.9}$$

Solving the system we get,

$$u'(0)=\frac{\alpha_0\int_0^1[\gamma F_\alpha(1-s)+\delta F_{\alpha-1}(1-s)]f(s,u(s))ds}{(\beta_0\gamma+\delta\beta_0 b+\alpha_0\delta)F_1(1)+\alpha_0\gamma F_2(1)}.$$

And

$$u(0)=\frac{\beta_0\int_0^1[\gamma F_\alpha(1-s)+\delta F_{\alpha-1}(1-s)]f(s,u(s))ds}{(\beta_0\gamma+\delta\beta_0 b+\alpha_0\delta)F_1(1)+\alpha_0\gamma F_2(1)}.$$

Where $(\beta_0\gamma+\delta\beta_0 b+\alpha_0\delta)F_1(1)+\alpha_0\gamma F_2(1)\neq 0$.
Consequently, the solution of (1.1)-(1.3) expressed as ,

$$u(t)=-\int_0^t F_\alpha(t-s)f(s,u(s))ds+F_1(t)u(0)+F_2(t)u'(0), \tag{4.10}$$

as mentioned on (Kilbas, et al. 2002 and Podlubny, 1999). Substituting the result of $u(0)$ and $u'(0)$ on (4.10) we obtain,

$$u(t)=-\int_0^t F_\alpha(t-s)f(s,u(s))ds+F_1(t)\frac{\beta_0\int_0^1[\gamma F_\alpha(1-s)+\delta F_{\alpha-1}(1-s)]f(s,u(s))ds}{(\beta_0\gamma+\delta\beta_0 b+\alpha_0\delta)F_1(1)+\alpha_0\gamma F_2(1)}$$
$$+F_2(t)\frac{\alpha_0\int_0^1[\gamma F_\alpha(1-s)+\delta F_{\alpha-1}(1-s)]f(s,u(s))ds}{(\beta_0\gamma+\delta\beta_0 b+\alpha_0\delta)F_1(1)+\alpha_0\gamma F_2(1)}$$

Rearranging the terms we obtain

$$u(t) = \int_0^t \sigma(t)\Big[[\gamma F_\alpha(1-s) + \delta F_{\alpha-1}(1-s)] - F_\alpha(t-s)\Big] f(s,u(s))ds \\ + \int_t^1 \sigma(t)[\gamma F_\alpha(1-s) + \delta F_{\alpha-1}(1-s)] f(s,u(s))ds. \tag{4.11}$$

where

$$\sigma(t) = \frac{\beta_0 F_1(t) + \alpha_0 F_2(t)}{(\alpha_0\delta + \beta_0\gamma + \delta\beta_0 b)F_1(1) + \alpha_0\gamma F_2(1)}.$$

By setting

$$G(t,s) = \begin{cases} \sigma(t)[\gamma F_\alpha(1-s) + \delta F_{\alpha-1}(1-s)] - F_\alpha(t-s), & 0 \le s \le t. \\ \\ \sigma(t)[\gamma F_\alpha(1-s) + \delta F_{\alpha-1}(1-s)] & 0 \le t \le s \le 1. \end{cases} \tag{4.12}$$

Thus, (4.11) is rewritten as

$$u(t) = \int_0^1 G(t,s) f(s,u(s))ds. \tag{4.13}$$

Here, $G(t,s)$ is the Green function for the corresponding homogeneous BVPs (1.1)-(1.3).

4.2 Bounds of Green's Function

Lemma 4.2.1 *The Green function $G(t,s)$ in Equation* (4.12) *is nonnegative for all $t,s \in [0,1]$.*

Lemma 4.2.2 *The Green function $G(t,s)$ which is given in Equation* (4.12) *satisfies the following inequalities*

i. $G(t,s) \le LG(s,s)$ *for* $t,s \in [0,1]$, *where* $L = \max\left\{1, \frac{\beta_0 F_1(1)+\alpha_0 F_2(1)}{\beta_0}\right\}$;

ii. $G(t,s) \ge MG(s,s)$ *for* $t,s \in [0,1]$, *where* $M = min\left\{1, \frac{\beta_0}{\beta_0 F_1(1)+\alpha_0 F_2(1)}\right\}$.

Proof: (i). First we prove (i) by two cases.

Case 1: Let $t \leq s$ then,

$$\frac{G(t,s)}{G(s,s)} = \frac{\sigma(t)[\gamma F_\alpha(1-s)+\delta F_{\alpha-1}(1-s)]}{\sigma(s)[\gamma F_\alpha(1-s)+\delta F_{\alpha-1}(1-s)]} = \frac{\sigma(t)}{\sigma(s)} \leq 1.$$

Therefore, $G(t,s) \leq G(s,s)$.

Case 2: Let $s \leq t$, then,

$$\frac{G(t,s)}{G(s,s)} = \frac{\sigma(t)[\gamma F_\alpha(1-s)+\delta F_{\alpha-1}(1-s)]-F_\alpha(t-s)}{\sigma(s)[\gamma F_\alpha(1-s)+\delta F_{\alpha-1}(1-s)]-F_\alpha(s-s)}$$

$$= \frac{\sigma(t)[\gamma F_\alpha(1-s)+\delta F_{\alpha-1}(1-s)]-F_\alpha(t-s)}{\sigma(s)[\gamma F_\alpha(1-s)+\delta F_{\alpha-1}(1-s)]}$$
$$\leq \frac{\sigma(t)[\gamma F_\alpha(1-s)+\delta F_{\alpha-1}(1-s)]}{\sigma(s)[\gamma F_\alpha(1-s)+\delta F_{\alpha-1}(1-s)]} = \frac{\sigma(t)}{\sigma(s)}$$

since, $\sigma(t) = \dfrac{\beta_0 F_1(t)+\alpha_0 F_2(t)}{(\alpha_0\delta+\beta_0\gamma+\delta\beta_0)F_1(1)+\alpha_0\gamma F_2(1)}$ and

$$\sigma(s) = \frac{\beta_0 F_1(s)+\alpha_0 F_2(s)}{(\alpha_0\delta+\beta_0\gamma+\delta\beta_0)F_1(1)+\alpha_0\gamma F_2(1)}$$

Then, ratio of the two is,

$$\frac{\sigma(t)}{\sigma(s)} = \left(\frac{\beta_0 F_1(t)+\alpha_0 F_2(t)}{(\alpha_0\delta+\beta_0\gamma+\delta\beta_0)F_1(1)+\alpha_0\gamma F_2(1)}\right)\cdot\left(\frac{(\alpha_0\delta+\beta_0\gamma+\delta\beta_0)F_1(1)+\alpha_0\gamma F_2(1)}{\beta_0 F_1(s)+\alpha_0 F_2(s)}\right)$$

$$= \frac{\beta_0 F_1(t)+\alpha_0 F_2(t)}{\beta_0 F_1(s)+\alpha_0 F_2(s)} = \frac{\beta_0 F_1(1)+\alpha_0 F_2(1)}{\beta_0}.$$

This implies that, $\dfrac{G(t,s)}{G(s,s)} \leq \dfrac{\beta_0 F_1(1)+\alpha_0 F_2(1)}{\beta_0}$.

Therefore, $G(t,s) \leq LG(s,s)$ where $L = \max\left\{1,\ \frac{\beta_0 F_1(1)+\alpha_0 F_2(1)}{\beta_0}\right\}$

This completes the prove of (i).

Also, we can proof (ii) by two cases.

Case 1: Let $s \le t$ then,

$$\frac{G(t,s)}{G(s,s)} = \frac{\sigma(t)[\gamma F_\alpha(1-s)+\delta F_{\alpha-1}(1-s)] - F_\alpha(t-s)}{\sigma(s)[\gamma F_\alpha(1-s)+\delta F_{\alpha-1}(1-s)] - F_\alpha(s-s)}$$

$$= \frac{\sigma(t)[\gamma F_\alpha(1-s)+\delta F_{\alpha-1}(1-s)] - F_\alpha(t-s)}{\sigma(s)[\gamma F_\alpha(1-s)+\delta F_{\alpha-1}(1-s)]}$$
$$\ge \frac{\sigma(t)}{\sigma(s)} \ge 1.$$

Therefore, $G(t,s) \ge G(s,s)$.
Case 2: If $t \le s$ then,

$$\frac{G(t,s)}{G(s,s)} = \frac{\sigma(t)[\gamma F_\alpha(1-s)+\delta F_{\alpha-1}(1-s)]}{\sigma(s)[\gamma F_\alpha(1-s)+\delta F_{\alpha-1}(1-s)]}$$
$$= \frac{\sigma(t)}{\sigma(s)} = \frac{\beta_0 F_1(t)+\alpha_0 F_2(t)}{\beta_0 F_1(s)+\alpha_0 F_2(s)} \ge \frac{\beta_0}{\beta_0 F_1(1)+\alpha_0 F_2(1)} = M$$

Therefore, $G(t,s) \ge MG(s,s)$ when $M = min\left\{1, \frac{\beta_0}{\beta_0 F_1(1)+\alpha_0 F_2(1)}\right\}$.
Hence, the proof of Lemma (4.2.2) is complete. □

Definition 4.2.1 *An operator T is completely continuous if T is continuous and compact, i.e., T maps bounded sets into precompact sets.*
Let X be a Banach Space and P be a cone in X. For $k > 0$, define

$$P_k = \{x \in P : \|x\| < k\} \text{ and } \partial P_k = \{x \in P : \|x\| = k\}.$$

The following well-known fixed point index theorem will be the fundamental tool to prove our main results.

Theorem 4.2.3 *(Krasnoselskii, M., 1964). Let X be a Banach space and P be a cone in X. Assume that $T : P_k \longrightarrow P$ is completely continuous such that $Tx \ne x$ for $x \in \partial P_k$*

i. If $\|Tx\| > \|x\|$ for $x \in \partial P_k$, then $i(T, P_k, P) = 0$.

ii. If $\|Tx\| < \|x\|$ for $x \in \partial P_k$, then $i(T, P_k, P) = 1$.

Here $i(T, P_k, P)$ is called the fixed point index of T on P_k with respect to P.

4.3 Existence of Positive Solutions

In this section, we establish the existence of positive solution for Caputo fractional order BVPs (1.1)-(1.3) by using fixed point index theory.
Let $X = \{u(t) : u(t) \in L^1[0,1] \text{ for } t \in [0,1]\}$ be a Banach space with norm

$$\|u\| = \max_{t\in[0,1]} |u(t)| \text{ and let}$$

$P = \{\ u \in X : u(t) > 0,\ t \in [0,1],\ \min_{t\in I} |u(t)| \geq M\|u\|\ \}$ where, $M = \min\left\{1,\ \frac{\beta_0}{\beta_0 F_1(1) + \alpha_0 F_2(1)}\right\}$.
We note that P is a cone in X. Let the operator $T : P \longrightarrow X$ be defined as,

$$Tu(t) = \int_0^1 G(t,s) f(s,u(s))ds. \tag{4.14}$$

To obtain a positive solution of (1.1)-(1.3) we shall find a fixed point of the operator T in the cone P.
Assume the following conditions hold throughout this thesis.

A1. $0 < \int_0^1 G(t,s) f(s,u(s))ds < \infty$,

A2. $f(t,u(t))$ is a nondecreasing function.

Define the nonnegative extended real numbers f_0, f^0, f_∞ and f^∞ by,

$$f_0 = \lim_{u\to 0^+} min_{t\in[0,1]} \frac{f(t,u(t))}{u(t)}, \qquad f^0 = \lim_{u\to 0^+} max_{t\in[0,1]} \frac{f(t,u(t))}{u(t)},$$

$$f_\infty = \lim_{u\to\infty} min_{t\in[0,1]} \frac{f(t,u(t))}{u(t)}, \qquad f^\infty = \lim_{u\to\infty} max_{t\in[0,1]} \frac{f(t,u(t))}{u(t)}.$$

Assuming that they will exist. When $f^0 = 0$ and $f_\infty = \infty$ is called super-linear case and $f_0 = \infty$ and $f^\infty = 0$ is called the sub-linear case.

Lemma 4.3.1 *The operator $T : P \longrightarrow X$ defined by* (4.14) *is a self-map on P.*

Proof: from $(A1)$ and the positivity of the Green function of $G(t,s)$ in Lemma (4.2.1) that for $u(t) \in P$, $Tu(t) \geq 0$ on $t \in [0,1]$.

Now for $u(t) \in P$ and by Lemma (4.2.2) we have,

$$\begin{aligned} Tu(t) &= \int_0^1 G(t,s)f(s,u(s))ds \\ &\leq L\int_0^1 G(s,s)f(s,u(s))ds. \end{aligned}$$

So that, $\|Tu(t)\| \leq L\int_0^1 G(s,s)f(s,u(s))ds$ where $L = \max\left\{1,\ \dfrac{\beta_0 F_1(1)+\alpha_0 F_2(1)}{\beta_0}\right\}$.

Then, by Lemma (4.2.2) for $u(t) \in P$ that

$$\begin{aligned} minTu(t) &= min_{t\in I}\int_0^1 G(t,s)f(s,u(s))ds \\ &\geq \int_0^1 G(t,s)f(s,u(s))ds \\ &\geq M\int_0^1 G(s,s)f(s,u(s))ds = M\|Tu(t)\| \text{ where } M = \min\left\{1,\ \frac{\beta_0}{\beta_0 F_1(1)+\alpha_0 F_2(1)}\right\}. \end{aligned}$$

Therefore, $T : P \longrightarrow P$ and hence the proof is complete . i.e., the map T is a self-map. □

Theorem 4.3.2 *Assume that* $(A1)$ *and* $(A2)$ *are satisfied. If* $f^0 = 0$ *and* $f_\infty = \infty$, *then the BVPs* (1.1)-(1.3) *have a positive solution that lies in P.*

Proof: Let T be the cone preserving, completely continuous operator defined by (4.14).

since, $f^0 = 0$, we choose $\zeta_1 > 0$ and $\eta_1 > 0$ such that ,

$$f(t,u(t)) \leq \zeta_1 u(t) \text{ for } 0 \leq u(t) < \eta_1 \text{ where } \zeta_1 \text{ satisfies,}$$

$$\zeta_1 \int_0^1 G(s,s)f(s,u(s))ds < 1. \tag{4.15}$$

Now, Let $u(t) \in P$ with $\|u(t)\| = \eta_1$. Then by Lemma (4.2.2) and for $t \in [0,1]$

$$\begin{aligned} \text{we have, } Tu(t) &= \int_0^1 G(t,s)f(s,u(s))ds \\ &\leq L\int_0^1 G(s,s)f(s,u(s))ds \\ &\leq \zeta_1\int_0^1 G(s,s)f(s,u(s))ds\|u(t)\| \leq \|u(t)\|. \end{aligned}$$

Therefore, $\|Tu(t)\| \leq \|u(t)\|$. If we set $\Omega_1 = \{u(t) \in X : \|u(t)\| < \eta_1\}$ then,

$$\|Tu(t)\| \leq \|u(t)\|, \; u(t) \in P \cap \partial\Omega_1. \tag{4.16}$$

By Theorem (4.2.3) we have $i(T, P \cap \partial\Omega_1, P) = 1$.
Furthermore, since $f_\infty = \infty$, there exist $\zeta_2 > 0$ and $\eta_2 > 0$ such that,
$f(t, u(t)) > \zeta_2 u(t)$ for $u(t) \geq \eta_2$ where ζ_2 satisfies

$$\zeta_2 \int_{s \in I} G(s,s) f(s, u(s)) ds \geq 1. \tag{4.17}$$

Let $\eta_2 = max\left\{2\eta_1, \; \dfrac{\eta_2}{M}\right\}$, choose $u(t) \in P$ and $\|u(t)\| = \eta_2$, then, $min_{t \in I} u(t) \geq M\|u(t)\| \geq \eta_2$.

Where $M = \min\left\{1, \; \dfrac{\beta_0}{\beta_0 F_1(1) + \alpha_0 F_2(1)}\right\}$ and from Lemma (4.2.2), for $t \in [0,1]$ we have,

$$\begin{aligned}
\|Tu(t)\| &= \int_0^1 G(t,s) f(s, u(s)) ds \\
&\geq M \int_0^1 G(s,s) f(s, u(s)) ds \\
&\geq \zeta_2 \int_0^1 G(s,s) f(s, u(s)) ds \\
&\geq \zeta_2 \int_0^1 G(s,s) f(s, u(s)) ds \|u(t)\| \\
&\geq \|u(t)\|.
\end{aligned}$$

This implies that, $\|Tu(t)\| \geq \|u(t)\|$. So, if we set $\Omega_2 = \{u(t) \in X : \|u(t)\| < \eta_2\}$ then,

$$\|Tu(t)\| \geq \|u(t)\|, \; u(t) \in P \cap \partial\Omega_2. \tag{4.18}$$

By Theorem (4.2.3), we have $i(T, P \cap \Omega_2, P) = 0$.
If $\eta_1 < \eta_2$, then $i(T, P \cap (\Omega_2 \backslash \Omega_1), P) = i(T, P \cap \Omega_2, P) - i(T, P \cap \Omega_1, P) = 0 - 1 = -1$. It follows that T has a fixed point $u \in P \cap (\Omega_2 \backslash \Omega_1)$ and that u is a positive solution of the BVPs (1.1)-(1.3) .
If $\eta_1 > \eta_2$ then, $i(T, P \cap (\Omega_1 \backslash \Omega_2, P)) = i(T, P \cap \Omega_1, P) - i(T, P \cap \Omega_2, P) = 1 - 0 = 1$. It follows that, T has a fixed point $u \in P \cap (\Omega_1 \backslash \Omega_2)$ and that u is the positive solution of the BVPs (1.1)-(1.3). $\square$

Now, we establish the existence of positive solution of the BVPs (1.1)-(1.3) for sub-linear case.

Theorem 4.3.3 *Assume that the condition* $(A1),(A2)$ *are satisfied. If* $f_0 = \infty$ *and* $f^{\infty} = 0$*, then the BVPs* (1.1)-(1.3) *have a positive solution that lies in P.*

Proof: Let T be the cone preserving completely continuous operator defined by (4.14). Since $f_0 = \infty$ there exist $\bar{\zeta}_1 > 0$ and $\bar{\eta}_1 > 0$ such that,

$$\|f(t,u(t))\| \geq \bar{\zeta}_1 u(t),\ 0 < u(t) < \bar{\eta}_1,$$

where, $\bar{\zeta}_1 > \zeta_2$ and ζ_2 is given in (4.17), then for $u(t) \in P$ and $\|u(t)\| = \bar{\eta}_1$, we have,

$$\begin{aligned} Tu(t) &= \int_0^1 G(t,s)f(s,u(s))ds \\ &\geq min_{t\in I}\int_0^1 G(t,s)f(s,u(s))ds \\ &\geq \bar{\zeta}_1 \int_0^1 G(s,s)f(s,u(s))dsM\|u(t)\| \geq \|u(t)\| \end{aligned}$$

Therefore, $\|Tu(t)\| \geq \|u(t)\|$.

Now if we set, $\Omega_3 = \{\ u(t) \in X : \|u(t)\| < \bar{\eta}_1\ \}$, then $\|Tu(t)\| \geq \|u(t)\|$, for $t \in [0,1]$ and $u(t) \in P \cap \partial\Omega_3$.
Hence by Theorem (4.2.3), we have $i(T, P\cap\Omega_3, P) = 0$.
Furthermore, since $f^{\infty} = 0$, then there exist $\bar{\zeta}_2 > 0$ and $\bar{\eta}_2 > 0$ such that $f(t,u(t)) \leq \bar{\zeta}_2 u(t)$ for $u(t) \geq \bar{\eta}_2$ where $\bar{\zeta}_2 \leq \zeta_1$ and ζ_1 is given in (4.15).
Now, we consider two cases, f is either bounded or unbounded.

Case (i): Suppose that f is bounded. Then there exist $N > 0$ such that $f(t,u(t)) \leq N$ for $0 < u(t) < \infty$, in this case, we may choose,

$$\bar{\eta}_2 = max\{2\eta_1,\ L\int_0^1 G(s,s)f(s,u(s))ds\}$$

then, for $u(t) \in P$ and $\|u(t)\| = \bar{\eta}_2$ we have

$$
\begin{aligned}
Tu(t) &= \int_0^1 G(t,s)f(s,u(s))ds \\
&\leq L\int_0^1 G(s,s)f(s,u(s))ds \leq \bar{\eta_2} = \|u(t)\|.
\end{aligned}
$$

Therefore, $\|Tu(t)\| \leq \|u(t)\|$.

Case (ii): Suppose that f is unbounded, choose $\bar{\eta_2} > max\{2\eta_1, \eta_2\}$ such that

$f(t,u(t)) \leq f(t,\bar{\eta_2})$ for $0 < u(t) < \bar{\eta_2}$ then, for $u(t) \in P$ and $\|u(t)\| = \bar{\eta_2}$ we have,

$$
\begin{aligned}
Tu(t) &= \int_0^1 G(t,s)f(s,u(s))ds \\
&\leq \int_0^1 G(s,s)f(s,u(s))ds \\
&\leq L\int_0^1 G(s,s)f(s,u(s))ds \\
&\leq \bar{\eta_2} = \|u(t)\| \text{ . Which implies that, } \|Tu(t)\| \leq \|u(t)\|.
\end{aligned}
$$

In either case by setting $\Omega_4 = \{u(t) : \|u(t)\| < \bar{\eta_2}\}$ we have $\|Tu(t)\| \leq \|u(t)\|$, for $u(t) \in P \cap \partial\Omega_4$. Hence by Theorem (4.2.3), $i(T, P \cap \partial\Omega_4, P) = 1$.

If $\bar{\eta_1} < \bar{\eta_2}$ then, $i(T, P \cap (\Omega_4 \backslash \Omega_3), P) = i(T, P \cap \Omega_4, P) - i(T, P \cap_3, P) = 1 - 0 = 1$.

It follows from Theorem (4.2.3) that T has a fixed point $u(t) \in P \cap (\Omega_4 \backslash \Omega_3)$ and $u(t)$ is positive solution of the BVPs (1.1)-(1.3) and for $\bar{\eta_1} > \bar{\eta_2}$ we have,

$$
i(T, P \cap (\Omega_3 \backslash \Omega_4), P) = i(T, P \cap \Omega_3, P) - i(T, P \cap \Omega_4, P) = 0 - 1 = -1.
$$

It follows from Theorem (4.2.3) that T has a fixed point $u(t) \in P \cap (\Omega_3 \backslash \Omega_4)$ and $u(t)$ is a positive solution of the BVPs (1.1)-(1.3). $\square$

4.4 Uniqueness of Positive Solutions

In this section by applying Banach Contraction Principle we verify the uniqueness of the positive solution for BVPs (1.1)-(1.3).

Lemma 4.4.1 *Assume $f(t,u(t))$ satisfies Lipschitz conditions with respect to the second variable with Lipschitz constant k for all $t \in [0,1]$, then the BVPs* (1.1)-(1.3) *has a unique solution when* $0 < k\int_0^1 G(t,s)ds < 1$.

Proof: We consider the operator T defined on cone P given by,

$$Tu(t) = \int_0^1 G(t,s)f(s,u(s))ds, u \in P.$$

Since, T is self-map on cone P. Then we prove that T satisfies the Banach contraction, for all $u,\ v \in P$ and $t \in [0,1]$.
Where,

$$\begin{aligned}
Tu(t) &= \int_0^1 G(t,s)f(s,u(s))ds, u \in P. \\
Tv(t) &= \int_0^1 G(t,s)f(s,v(s))ds, v \in P. \\
|Tu - Tv| &= |\int_0^1 G(t,s)f(s,u)ds - \int_0^1 G(t,s)f(s,v)ds| \\
&= |\int_0^1 G(t,s)[f(s,u) - f(s,v)]ds| \\
&\leq \int_0^1 |G(t,s)[f(s,u) - f(s,v)]ds| \\
&\leq \int_0^1 G(t,s)k|u-v|ds \\
&= k\int_0^1 G(t,s)ds|u-v| \\
&= \alpha|u-v|.
\end{aligned}$$

Therefore, $|Tu(t) - Tv(t)| \leq \alpha|u(t) - v(t)|$.

Notice that when $\alpha = k\int_0^1 G(t,s)ds < 1$, the operator T is a contraction. Hence, by Banach contraction principle, T has a unique fixed point which is a unique positive solution of BVP (1.1)-(1.3) in Cone P. □

Chapter 5

Conclusion and Future scope

5.1 Conclusion

In this thesis we considered Caputo fractional order differential equations with Robin boundary conditions. By the use of the Laplace transform, the Green's function for the corresponding homogeneous Caputo fractional order BVPs with a reaction term was obtained in terms of two-parameter Mittag-Leffler functions. After these we formulated equivalent operator for the BVPs (1.1)-(1.3) in the given interval and established the existence of positive solution for Caputo fractional order BVPs by applying fixed point index theorem.
Finally, we established, the uniqueness of positive solution for Caputo fractional order BVPs by using Banach contraction principle.

5.2 Future scope

This study focused on existence and uniqueness of positive solution for Caputo fractional order $1 < \alpha < 2$ with reaction term and Robin type boundary condition. So, any interested Researchers may conduct the research on:

- Existence and uniqueness of positive solution for Caputo fractional order derivative differential equation by expanding the value of order α.
- Existence and uniqueness of positive solution for Caputo fractional order derivative differential equation by changing the boundary condition.
- Existence and uniqueness of positive solution for Caputo fractional order derivative by using another term.

References

Abbas, S. (2011). Existence of solutions to fractional order ordinary and delay differential equations and applications. *Electronic Journal of Differential Equations,* 2011(09), 1-11.

Abdulla, A. B., Al-Refai, M. and Al-Rawashdeh, A. (2016). On the existence and uniqueness of solutions for a class of non-linear fractional boundary value problems.*Journal of King Saud University-Science,* 28(1), 103-110.

Bagley, R. L. and Torvik, P. J. (1986). On the fractional calculus model of viscoelastic behavior. *Journal of Rheology,* 30(1), 133-155.

Bai, Z. and L, H. (2005). Positive solutions for boundary value problem of nonlinear fractional differential equation. *Journal of mathematical analysis and applications,* 311(2), 495-505.

Bai, Z., Sun, S., Du, Z. and Chen, Y. (2020). The green function for a class of Caputo fractional differential equations with a convection term. *Fractional Calculus and Applied Analysis,* 23(3), 787-798.

Benchohra, M. and Hedia, B. (2013). Multiple positive solutions for boundary value problem with fractional order. *Electronic Journal of Mathematical Analysis and Applications,* 2, 138-148.

Cheng,C., Feng, Z. and Su,Y. (2012). Positive solutions of fractional differential equations with derivative terms. *Electronic Journal of differential Equations,* 1, 1-27.

Hilfer, R. (Ed.). (2000). Applications of fractional calculus in physics. *World scientific.*

Hu, L. and Zhang, S., (2017). Existence of positive solutions to a periodic boundary value problems for nonlinear fractional differential equations at resonance. *J. Fract. Calc. Appl.,* 8(2), 19-31.

Kilbas, A.A. and Trujillo, J.J., (2002). Differential equations of fractional order: methods, results and problems. *II. Applicable Analysis,* 81(2), 435-493.

Krasnoselskii, M. (1964). Positive solutions of operator equations. P. Noordhoff, Groningen, The Netherlands. *English translation of Russian original.*

Mahto, L. and Abbas, S. (2012, September). Existence and uniqueness of solution of Caputo fractional differential equations. *In AIP Conference Proceedings* (Vol. 1479, No. 1, pp. 896-899).

Mathieu, B *et al* (2003). Fractional differentiation for edge detection. *Signal Processing,* 83(11), 2421-2432.

Meng, X. and Stynes, M., (2018). The Green's function and a maximum principle for a Caputo two-point boundary value problem with a convection term. *Journal of Mathematical Analysis and Applications,* 461(1), 198-218.

Meng, X. and Stynes, M., (2019). Greens functions, positive solutions, and a Lyapunov inequality for a Caputo fractional-derivative boundary value problem. *Fractional Calculus and Applied Analysis,* 22(3), 750-766.

Miller, K. S. and Ross, B. (1993). An Introduction to The Fractional Calculus and Fractional Differential Equations, John-Wily and Sons. *Inc. New York.*

Mittag-Leffler, G. M. (1903). Une gnralisation de lintgrale de Laplace-Abel. *CR Acad. Sci. Paris (Ser. II),* 137, 537-539.

Prasad, K. R. and Krushna, B. M. B. (2015). Lower and upper solutions for general two-point fractional order boundary value problems. *TWMS Journal of Applied and Engineering Mathematics,* 5(1), 80-87.

Podlubny, I. (1999). An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications. *Mathematics in science and engineering,* 198(6), 312-340.

Smart, D. R. (1980). Fixed point Theorems, Cambridge Uni. *Press.,Cambridge.*

Wang, Y. and Liu, L. (2017). Positive properties of the Green function for two-term fractional differential equations and its application. *J. Nonlinear Sci. Appl,* 10(4), 2094-2102.

Wiman, A. (1905). ber den Fundamentalsatz in der Teorie der Funktionen E a (x). *Acta Mathematica,* 29(1), 191-201.

Zhang, S. (2006). Positive solutions for boundary-value problems of nonlinear fractional differential equations. *Electronic Journal of Differential Equations (EJDE)[electronic only],* (36)2006, 1-12.