GENERALIZED VOLTERRA TYPE INTEGRAL OPERATORS ACTING BETWEEN GENERALIZED FOCK SPACES

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Declaration

I, Gobena Dugassa Abetu, with student ID number RM0234/11, declare that this thesis entitled ”Generalized Volterra-type integral operators acting between generalized Fock spaces” is my own original work and it has not been submitted to any institution or University elsewhere for the award of any academic degree and sources of information that I have been used or quoted are indicated and acknowledged.

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The work has been done under the supervision and approval of:

Name: Mafuz Humer Worku (PhD)
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Acknowledgment

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Abstract

The theory and study of integral operators is a wide history. Specially, boundedness and compactness properties of different integral operators have been widely studied on several spaces. Due to this, there is a big interest to study this properties for the generalized Volterra-type integral operator also and have been studied by many researchers acting between different spaces. In this thesis, we studied the and compactness properties of generalized Volterra-type integral operator acting on generalized Fock spaces $\mathcal{F}_p^\phi$, where $0 < p \leq \infty$ and ϕ is a faster growing weight when compared with the Gaussian weight function $\frac{|z|^2}{2}$.

Contents

Chapter 1

Introduction

1.1 Background of the study

The study of boundedness and compactness of different linear operators on spaces of analytic functions defined over a domain $U \subseteq \mathbb{C}$ is a rich history, where many authors are participated and many papers and books are written on. Let X and Y be Banach spaces and $T : X \to Y$ is a linear operator. If there is a constant $c > 0$ such that $\|Tx\| \leq c\|x\|$, $x \in X$, then we say that T is a bounded linear operator. Moreover, if $\|Tx_n\| \to 0$ whenever $x_n \to 0$ weakly in X, then we say that T is compact. For a given space $\mathcal{H}(U)$ of holomorphic or analytic functions on U, the Volterra-type integral operator on $\mathcal{H}(U)$ induced by a holomorphic symbol g,

$$V_g f(z) = \int_0^z f(w)g'(w)dw,$$

is among the linear operators studied a lot acting between different spaces. The operator is first introduced by (Pommerenke, 1977) and studied a lot by other authors with the aim to explore the connection between their operator theoretic behaviors with the function-theoretic properties of the symbols g. (Pommerenke, 1977) studied continuity of the operator on the Hilbert space of Hardy space H^2 and this result is extended to $H^p, 0 < p < \infty$, in general by (Aleman and Siskakis, 1995) and furthermore they studied compactness property also. Later (Aleman and Siskakis, 1997), gave the analogous characterization on the Bergman space. But, those studies are considered on spaces of analytic functions defined over a disk. (Constantin, 2012) and (Mengestie, 2013) considered the problem over a space defined over the whole complex plane , namely the classical Fock spaces .
(Li and Stevic, 2008) raised an idea to extend the Volterra-type integral operator V_g by considering its product with composition operator $C_\psi f = f(\psi)$ and they studied their operator theoretic properties in terms of the inducing pair of symbols on some spaces of analytic functions on the unit disk. They eventually considered the following operator

induced by analytic functions g and ψ

$$V_{(g,\psi)}f(z) = \int_0^z f(\psi(w))g'(w)dw.$$

Since a particular choice of $\psi(z) = z$ reduce $V_{(g,\psi)}$ to the Volterra-type integral operator V_g, the operator $V_{(g,\psi)}$ is called the generalized Volterra-type integral operator. Boundedness and compactness of this operator have been studied on different spaces and the characterization of these properties on the classical Fock space have been given by (Mengestie, 2014) and later by (Mengestie and Worku, 2018). The aim of this thesis is to characterize boundedness and compactness of $V_{(g,\psi)}$ on generalized Fock spaces.

1.1.1 Generalized Fock Spaces

Notation:- $\mathbb{C}$ denotes the set of complex numbers.

$\mathbb{R}^+ = [0, \infty) \to \mathbb{R}^+$ represents the set of all non-negative real numbers.

Let $0 < p \leq \infty$ and $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ be a twice continuously differentiable function, which can be extended to $\mathbb{C}$ by setting $\phi(z) = \phi(|z|)$, $z \in \mathbb{C}$. Then the generalized Fock spaces $\mathcal{F}_\phi^p$ is given by,

$$\mathcal{F}_\phi^p = \{f \in \mathcal{H}(\mathbb{C}) : \|f\|_{\mathcal{F}_\Phi^p}^p = \int_{\mathbb{C}} |f(z)|^p e^{-p\phi(z)} dm(z) < \infty\}$$

and

$$\mathcal{F}_\phi^\infty = \{f \in \mathcal{H}(\mathbb{C}) : \|f\|_{\mathcal{F}_\Phi^\infty} = \sup_{z \in \mathbb{C}} |f(z)| e^{-\Phi(z)} < \infty\}$$

where dm denotes the Lebesgue area measure in $\mathbb{C}$. In particular, for the weight $\phi(z) = \frac{|z|^2}{2}$ called the Gaussian weight, we get the classical Fock spaces $\mathcal{F}_p$, which is the space of entire functions such that

$$\|f\|_p = \begin{cases} \left(\frac{p}{2\pi} \int_{\mathbb{C}} |f(z)|^p e^{-\frac{p}{2}|z|^2} dm(z) \right)^{\frac{1}{p}} < \infty, & 0 < p < \infty \\ \sup_{z \in \mathbb{C}} |f(z)| e^{-\frac{|z|^2}{2}} < \infty, & p = \infty. \end{cases}$$

The space is named after the Soviet physicist Vladmir Aleksandrovich Fock (1898-1974) and has an application in quantum physics, harmonic analysis on the Heisenberg group and partial differential equations.

We assume that the Laplacian of ϕ is positive and [1]

$$\tau(z) \simeq \begin{cases} 1, & if \quad 0 \leq |z| < 1 \\ (\Delta\phi(|z|))^{-\frac{1}{2}}, & if \ |z| \geq 1 \end{cases}$$

where $\tau(z)$ is a radial positive differentiable function that decreases to zero as $|z| \to \infty$ and $\lim_{r\to\infty} \tau'(r) = 0$. We suppose also that either there exists a constant $\alpha > 0$ such that $\tau(r)r^\alpha$ increases for large r or

$$\lim_{r\to\infty} \tau'(r) \log \frac{1}{\tau(r)} = 0.$$

The functions $\phi_1(r) = r^m, m > 2$, $\phi_2(r) = e^{\beta r}, \beta > 0$ and $\phi_3(r) = e^{e^r}$ are some examples of such weights with the above assumptions. This type of space have been introduced over the complex plane $\mathbb{C}$ by (Constantin and Peláez, 2015) for finite exponent and by (Mengestie and Ueki, 2015) for the infinite case, by imposing similar assumptions posed over generalized Bergman spaces.

For a subharmonic functions ϕ and f, from Lemma 7 of (Constantin and Peláez, 2015), we have a pointwise estimate

$$|f(z)|^p e^{-\beta\phi(z)} \lesssim \frac{1}{\sigma^2\tau(z)^2} \int_{D(z,\sigma\tau(z))} |f(w)|^p e^{-\beta\phi(w)} dm(w) \tag{1.1.1}$$

for all finite exponent p, any real number β, and a small positive number σ where $D(z, \sigma\tau(z))$ is a disc with center z and radius $\sigma\tau(z)$. The estimate implies that point evaluation functionals are bounded on $\mathcal{F}^2_\phi$ and hence $\mathcal{F}^2_\phi$ is a reproducing kernel Hilbert space, but an explicit formula for the kernel function is an open problem. We note that, an explicit formula for the kernel function for the classical Fock space $\mathcal{F}_2$ is known, which is given by

$$K_w(z) = e^{z\overline{w}}.$$

This is one of the difference between the classical Fock spaces and generalized Fock spaces with the above assumptions and makes the study of operators on $\mathcal{F}^p_\phi$ difficult, since many estimates are based on the kernel function. In (Constantin and Peláez, 2015) test functions are constructed to overcome this problem and play the role of kernel function. For large R, there exists a number $\eta(R)$ such that for any $w \in \mathbb{C}$ with $|w| > \eta(R)$, there exists an entire function $F_{(w,R)}$ such that

$$|F_{(w,R)}(z)|e^{-\phi(z)} \leq C \min\left\{1, \left(\frac{\min\{\tau(w), \tau(z)\}}{|z-w|}\right)^{\frac{R^2}{2}}\right\}$$

[1]The notation $S(z) \simeq T(z)$ means both $S(z) \lesssim T(z)$ and $T(z) \lesssim S(z)$, where $S(z) \lesssim T(z)$ (or equivalently $T(z) \gtrsim S(z)$) means that there is a constant C such that $S(z) \leq CT(z)$ holds.

for all $z \in \mathbb{C}$ and for some constant C that depends on ϕ and R. In particular, the above inequality shows that $|F_{(w,R)}(z)|e^{-\phi(z)} \simeq 1$ whenever $z \in D(w, R\tau(w))$. Furthermore, $F_{(w,R)} \in \mathcal{F}_\phi^p$, for all p with norm estimated by

$$\|F_{(w,R)}\|_{\mathcal{F}_\phi^p} \simeq \begin{cases} \tau^{\frac{2}{p}}(w), & \eta(R) \leq |w|,\ 0 < p < \infty \\ 1, & p = \infty. \end{cases}$$

Thus, the normalized test function, $F^*_{(w,R)}$, is given by,

$$F^*_{(w,R)} \simeq \begin{cases} \frac{F_{(w,R)}}{\tau(w)^{\frac{2}{p}}}, & \eta(R) \leq |w|,\ 0 < p < \infty \\ F_{(w,R)}, & p = \infty, \end{cases}$$

and it converges uniformly to zero on compact subsets of $\mathbb{C}$ as $|w| \to \infty$.
Another big difference between the two space is the inclusion property, which in the case of classical Fock space is given by, $\mathcal{F}_p \subseteq \mathcal{F}_q$ for $p \leq q$, whereas the family of generalized Fock space $(\mathcal{F}_\phi^p)_p$ with the above assumptions is not nested. In fact, $\mathcal{F}_\phi^p \setminus \mathcal{F}_\phi^q \neq \emptyset$ and $\mathcal{F}_\phi^q \setminus \mathcal{F}_\phi^p \neq \emptyset$, for all $p \neq q$.
In (Constantin and Peláez, Mengestie and Ueki, 2015) the space $\mathcal{F}_\phi^p$, for $0 < p \leq \infty$, have been characterized in terms of the following Littlewood-Paley type formula

$$\|f\|^p_{\mathcal{F}_\phi^p} \simeq \begin{cases} |f(0)|^p + \int_{\mathbb{C}} \frac{|f'(z)|^p}{(1+\phi'(z))^p} e^{-p\phi(z)} dm(z), & 0 < p < \infty \\ |f(0)| + \sup_{z\in\mathbb{C}} \frac{|f'(z)|}{1+\phi'(z)} e^{-\phi(z)}, & p = \infty. \end{cases} \tag{1.1.2}$$

The description is very important in the study of integral operators and plays a big role in this thesis.

1.2 Statement of the problem

As noted in the background of the study, boundedness and compactness of generalized Volterra-type integral operator on the classical Fock space was studied by (Mengestie, 2014) in terms of Berezin type integral transforms. Later, (Mengestie and Worku, 2018) simplified the Berezin type characterization to a new simpler function to apply. But, the characterization of boundedness and compactness of $V_{(g,\psi)}$ acting between generalized Fock spaces is not studied yet, except for the case $\psi(z) = z$, which is studied in (Constantin and Peláez, Mengestie and Ueki, 2015). Therefore, this thesis studies boundedness and compactness of $V_{(g,\psi)}$ on generalized Fock spaces $\mathcal{F}_\phi^p$, extending the results of (Constantin and Peláez, Mengestie and Ueki, 2015).

1.3 Objectives of the study

1.3.1 General objectives

The general objective of this thesis is to study boundedness and compactness properties of generalized Volterra type integral operators acting between generalized Fock Spaces.

1.3.2 Specific objectives

The specific objectives of this study is;

- to describe boundedness of the generalized Volterra type integral operators and give sufficient and necessary conditions for boundedness.
- to establish sufficient and necessary conditions for compactness of the generalized Volterra type integral operators.
- to find a condition for which boundedness and compactness are equivalent.
- to provide examples that support the main results.

1.4 Significance of the study

The result of this study have the following importance:

1. It generalizes study of the operators into a more general space.
2. It may be used as a base for any researcher who is interested to study other properties of the operators on the space.
3. Help the graduate students to acquire research skills and scientific procedures.

1.5 Delimitation of the study

This study was delimated to studying bounded and compact generalized Volterra type integral operators acting between generalized Fock spaces.

Chapter 2

Review of Related Literature

Ever since introduced by (Pommerenke, 1977) and after the works of (Aleman and Siskakis, 1995), a number of researchers were motivated to study different properties of the Volterra-type integral operator V_g on different spaces. (Constantin, 2012) studied bounded, compact and other properties of V_g on the classical Fock spaces $\mathcal{F}_p$. Then the study was continued by (Mengestie, 2014) on the growth type classical Fock space $\mathcal{F}_\infty$. We state the two results by the following theorems.

Theorem 2.0.1 (Constantin, 2012 and Mengestie, 2014).
Let $0 < p \leq q \leq \infty$. Then $V_g : \mathcal{F}_p \to \mathcal{F}_q$ is

(i) bounded if and only if $g(z) = az^2 + bz + c$ for $a, b, c \in \mathbb{C}$.

(ii) compact if and only if $g(z) = az + b$ for $a, b \in \mathbb{C}$.

For the case when the operator maps from larger space to the smaller, there is a stronger condition in which boundedness and compactness are equivalent.

Theorem 2.0.2 (Constantin, 2012 and Mengestie, 2014).
Let $0 < q < p \leq \infty$. Then the following are equivalent.

a) $V_g : \mathcal{F}_p \to \mathcal{F}_q$ is bounded,

b) $V_g : \mathcal{F}_p \to \mathcal{F}_q$ is compact,

c) $q > \begin{cases} \frac{2p}{p+2}, & p < \infty \\ 2, & p = \infty \end{cases}$ *and $g(z) = az + b$ for some $a, b \in \mathbb{C}$.*

(Mengestie, 2013) studied the extended operator, namely the generalized Volterra type integral operators, on the classical Fock spaces $\mathcal{F}_p$. Recently, (Mengestie and Worku, 2018) studied also bounded and compact generalized Volterra type integral operator $V_{(g,\psi)}$ with simpler characterization on classical Fock spaces $\mathcal{F}_p$. They obtained the following results.

Theorem 2.0.3 (Mengestie and Worku, 2018).
Let $0 < p \leq q \leq \infty$ and (g, ψ) be pairs of nonconstant entire functions. Then

i) $V_{(g,\psi)} : \mathcal{F}_p \to \mathcal{F}_q$ *is bounded if and only if* $\frac{|g'(z)|}{1+|z|} e^{\frac{1}{2}(|\psi(z)|^2 - |z|^2)} \in L^\infty(\mathbb{C}, dm)$.

ii) $V_{(g,\psi)} : \mathcal{F}_p \to \mathcal{F}_q$ *is compact if and only if* $\lim_{|z|\to\infty} \frac{|g'(z)|}{1+|z|} e^{\frac{1}{2}(|\psi(z)|^2 - |z|^2)} = 0$.

Their result is different for the cases $p \leq q$ and $q < p$. For the latter case, we have a stronger condition under which the boundedness implies compactness as stated below.

Theorem 2.0.4 (Mengestie and Worku, 2018).
Let $0 < q < p \leq \infty$ and (g, ψ) be pairs of nonconstant entire functions. Then the following statements are equivalent.

i) $V_{(g,\psi)} : \mathcal{F}_p \to \mathcal{F}_q$ *is bounded;*

ii) $V_{(g,\psi)} : \mathcal{F}_p \to \mathcal{F}_q$ *is compact;*

iii) $\frac{|g'(z)|}{1+|z|} e^{\frac{1}{2}(|\psi(z)|^2 - |z|^2)} \in \begin{cases} L^{\frac{pq}{p-q}}(\mathbb{C}, dm), & p < \infty \\ L^q(\mathbb{C}, dm), & p = \infty. \end{cases}$

The purpose of this thesis is to find an analogous characterization for $V_{(g,\psi)}$ on $\mathcal{F}_\phi^p$, which is studied only for the Volterra-type integral operator in (Constantin and Peláez, Mengestie and Ueki, 2015) and stated as follows.

Theorem 2.0.5 (Constantin and Peláez, Mengestie and Ueki, 2015).
Let $0 < p \leq q \leq \infty$. Then $V_g : \mathcal{F}_\phi^p \to \mathcal{F}_\phi^q$ is

(i) bounded if and only if

$$\begin{cases} \sup_{z\in\mathbb{C}} \frac{|g'(z)|\Delta\phi(z)^{\frac{q-p}{pq}}}{1+\phi'(z)} < \infty, & q < \infty \\ \sup_{z\in\mathbb{C}} \frac{|g'(z)|\Delta\phi(z)^{\frac{1}{p}}}{1+\phi'(z)} < \infty, & p < q = \infty \\ \sup_{z\in\mathbb{C}} \frac{|g'(z)|}{1+\phi'(z)} < \infty, & p = q = \infty. \end{cases}$$

(ii) compact if and only if

$$\begin{cases} \lim_{|z|\to\infty} \frac{|g'(z)|\Delta\phi(z)^{\frac{q-p}{pq}}}{1+\phi'(z)} < \infty, & q < \infty \\ \lim_{|z|\to\infty} \frac{|g'(z)|\Delta\phi(z)^{\frac{1}{p}}}{1+\phi'(z)} < \infty, & p < q = \infty \\ \sup_{|z|\to\infty} \frac{|g'(z)|}{1+\phi'(z)} < \infty, & p = q = \infty. \end{cases}$$

Similarly, for the case $q < p$ and the operator V_g maps from $\mathcal{F}_\phi^p$ into $\mathcal{F}_\phi^q$, we have the following.

Theorem 2.0.6 (Constantin and Peláez, Mengestie and Ueki, 2015)**.**
Let $0 < q < p \leq \infty$. Then the following are equivalent.

(i) $V_g : \mathcal{F}_\phi^p \to \mathcal{F}_\phi^q$ *is bounded;*

(ii) $V_g : \mathcal{F}_\phi^p \to \mathcal{F}_\phi^q$ *is compact;*

(iii) $\frac{|g'(z)|}{1+\phi'(z)} \in L^r(\mathbb{C}, dm)$ *where* $r = \begin{cases} \frac{pq}{p-q}, & p < \infty \\ q, & p = \infty. \end{cases}$.

Our results in Chapter 4 extends those results mentioned above either in terms of the operator or in terms of the working space.

Chapter 3

Methodology of the study

3.1 Study area and Period

The study was conducted in Jimma University department of mathematics under the functional analysis stream from September, 2020 G.C. to July, 2021 G.C. Conceptually, the study focused on generalized Volterra-type integral operators acting between generalized Fock spaces.

3.2 Study design

In this research work we employed analytical method of design.

3.3 Source of information

The relevant sources of information for this study were journals, books, published articles and related studies from Internet.

3.4 Mathematical Procedure of the study

The mathematical procedure that the researcher followed for this research work were the following:

- Establishing theorems.
- Providing sufficient and necessary conditions for boundedness and compactness of the operators.
- Characterizing boundedness and compactness of Volterra type integral operator.
- Giving conclusion based on the main findings.

Chapter 4

Main Result and Discussion

We start the chapter by defining a function $M_{(g,\psi,\phi)}$ to be

$$M_{(g,\psi,\phi)}(z) := \frac{|g'(z)|}{1+\phi'(z)}e^{\phi(\psi(z))-\phi(z)}$$

for simplicity and our main results are also expressed in terms of this function. We then, state some properties related to the function.

Lemma 4.0.1. *Let (g,ψ) be a pair of nonconstant entire functions. Then*

(i) if $M_{(g,\psi,\phi)}(z)$ is bounded, then $\psi(z) = az + b$ for some $a, b \in \mathbb{C}$ with $|a| \leq 1$.

(ii) if $\Delta\phi(z)^{\frac{1}{p}} M_{(g,\psi,\phi)}(z)$ for $0 < p < \infty$ is bounded, then $\psi(z) = az+b$ for some $a, b \in \mathbb{C}$ with $|a| \leq 1$.

(iii) if $\Delta\phi(z)^{\frac{q-p}{pq}} M_{(g,\psi,\phi)}(z)$ for $0 < p < q < \infty$ is bounded, then $\psi(z) = az + b$ for some $a, b \in \mathbb{C}$ with $|a| \leq 1$.

Proof. The boundedness of $M_{(g,\psi,\phi)}$ implies that,

$$|g'(z)| \lesssim \frac{1+\phi'(z)}{e^{\phi(\psi(z))-\phi(z)}}$$

and since g is nonconstant, we must have

$$\phi(\psi(z)) - \phi(z) \leq 0.$$

Otherwise, $\frac{1+\phi'(z)}{e^{\phi(\psi(z))-\phi(z)}}$ goes to zero as $|z| \to \infty$, which implies g is constant and it is a contradiction. Since ϕ is radial, we have

$$\phi(|\psi(z)|) \leq \phi(|z|).$$

Therefore, $|\psi(z)| \leq |z|$ and by Liouville's theorem we get $\psi(z) = az + b$ for some $a, b \in \mathbb{C}$ with $|a| \leq 1$.
The proofs for part (ii) and (iii) of the lemma follow from part (i), $0 < \frac{1}{p}, \frac{q-p}{pq} < \infty$ and unboundedness of the Laplacian of ϕ, which is from the assumption that $\tau(z)$ decreases to zero as $|z| \to \infty$. □

Proposition 4.0.2. *Let $0 < p \leq q \leq \infty$ and (g, ψ) be a pair of nonconstant entire functions. If $V_{(g,\psi)} : \mathcal{F}_\phi^p \to \mathcal{F}_\phi^q$ is bounded (respectively, compact), then*

$$\begin{cases} \Delta\phi(z)^{\frac{q-p}{pq}} M_{(g,\psi,\phi)}(z), & \text{for } p \leq q < \infty \\ \Delta\phi(z)^{\frac{1}{p}} M_{(g,\psi,\phi)}(z), & \text{for } p < q = \infty \\ M_{(g,\psi,\phi)}(z), & \text{for } p = q = \infty \end{cases} \tag{4.0.1}$$

is bounded (respectively, the function in (4.0.1) *goes to zero as $|z| \to \infty$).*

Proof. We consider different cases.

Case 1: $0 < p \leq q < \infty$.
Applying $V_{(g,\psi)}$ to the test function $F^*_{(w,R)}$ and using the estimate in (1.1.2),

$$\begin{aligned} \|V_{(g,\psi)}\| \geq \|V_{(g,\psi)} F^*_{(w,R)}\|_{\mathcal{F}_\phi^q} &\simeq \frac{1}{\tau(w)^{\frac{2}{p}}} \left(\int_{\mathbb{C}} \frac{|F_{(w,R)}(\psi(z))|^q |g'(z)|^q}{(1+\phi'(z))^q} e^{-q\phi(z)} dm(z) \right)^{\frac{1}{q}} \\ &\geq \frac{1}{\tau(w)^{\frac{2}{p}}} \left(\int_{D(w,\delta\tau(w))} \frac{|F_{(w,R)}(\psi(z))|^q e^{-q\phi(\psi(z))} |g'(z)|^q}{(1+\phi'(z))^q} e^{q\phi(\psi(z)) - q\phi(z)} dm(z) \right)^{\frac{1}{q}} \\ &\gtrsim \frac{\tau(w)^{\frac{2}{q}}}{\tau(w)^{\frac{2}{p}}} \left(\frac{|g'(w)|^q}{(1+\phi'(w))^q} e^{q\phi(\psi(w)) - q\phi(w)} \right)^{\frac{1}{q}} = \Delta\phi(w)^{\frac{q-p}{pq}} M_{(g,\psi,\phi)}(w). \end{aligned} \tag{4.0.2}$$

Thus, if $V_{(g,\psi)}$ is bounded, then $\Delta\phi(w)^{\frac{q-p}{pq}} M_{(g,\psi,\phi)}(w)$ is bounded. Using the fact that $F^*_{(w,R)}$ converges to zero uniformly on compact subsets of $\mathbb{C}$ as $|w| \to \infty$, if $V_{(g,\psi)}$ is compact, then $\|V_{(g,\psi)} F^*_{(w,R)}\|_{\mathcal{F}_\phi^q} \to 0$ as $|w| \to \infty$. From which and the above estimate, the conclusion $\Delta\phi(w)^{\frac{q-p}{pq}} M_{(g,\psi,\phi)}(w) \to 0$ as $|w| \to \infty$ follows.

Case 2: $0 < p < q = \infty$.
Following similar procedure as in the above case and using the Littlewood-Paley type

estimate in (1.1.2) we obtain

$$\begin{aligned}\|V_{(g,\psi)}\| &\gtrsim \|V_{(g,\psi)}F^*_{(w,R)}\|_{\mathcal{F}^\infty_\phi}\\ &\simeq \tau(w)^{-\frac{2}{p}}\sup_{z\in\mathbb{C}}\frac{|F_{(w,R)}(\psi(z))||g'(z)|}{1+\phi'(z)}e^{-\phi(z)}\\ &\geq \tau(w)^{-\frac{2}{p}}\frac{|F_{(w,R)}(\psi(z))|e^{-\phi(\psi(z))}|g'(z)|}{1+\phi'(z)}e^{\phi(\psi(z))-\phi(z)}\\ &\simeq \tau(z)^{-\frac{2}{p}}\frac{|g'(z)|}{1+\phi'(z)}e^{\phi(\psi(z))-\phi(z)} \simeq \Delta\phi(z)^{\frac{1}{p}}M_{(g,\psi,\phi)}(z). \quad (4.0.3)\end{aligned}$$

The last estimate above is obtained by putting $w = z$ and using the estimate

$$|F_{(z,R)}(\psi(z))|e^{-\phi(\psi(z))} \simeq 1.$$

Thus, if $V_{(g,\psi)}$ is bounded, then (4.0.3) impies that $\Delta\phi(z)^{\frac{1}{p}}M_{(g,\psi,\phi)}(z)$ is bounded. If $V_{(g,\psi)}$ is compact, then from (4.0.3) we conclude that $\Delta\phi(z)^{\frac{1}{p}}M_{(g,\psi,\phi)}(z) \to 0$ as $|z| \to \infty$.

Case 3: $p = q = \infty$.
Again by similar procedure as above, we have

$$\begin{aligned}\|V_{(g,\psi)}\| \gtrsim \|V_{(g,\psi)}F^*_{(w,R)}\|_{\mathcal{F}^\infty_\phi} &\simeq \sup_{z\in\mathbb{C}}\frac{|F_{(w,R)}(\psi(z))||g'(z)|}{1+\phi'(z)}e^{-\phi(z)}\\ &\geq \frac{|F_{(w,R)}(\psi(z))|e^{-\phi(\psi(z))}|g'(z)|}{1+\phi'(z)}e^{\phi(\psi(z))-\phi(z)} \simeq M_{(g,\psi,\phi)}(z), \quad (4.0.4)\end{aligned}$$

which by similar argument as above gives the conclusion. □

From Lemma 4.0.1 and Proposition 4.0.2 we remark that, if the generalized Volterra-type integral operator $V_{(g,\psi)} : \mathcal{F}^p_\phi \to \mathcal{F}^q_\phi$ for $0 < p \leq q \leq \infty$ is bounded, then $\psi(z) = az + b$ for some $a, b \in \mathbb{C}$.

Theorem 4.0.3. *Let $0 < p \leq q \leq \infty$ and (g, ψ) be a pair of nonconstant entire functions. Then*

(i) $V_{(g,\psi)} : \mathcal{F}^p_\phi \to \mathcal{F}^q_\phi$ *is bounded if and only if*

$$\begin{cases}\sup_{z\in\mathbb{C}}\Delta\phi(z))^{\frac{q-p}{pq}}M_{(g,\psi,\phi)}(z) < \infty, & p \leq q < \infty\\ \sup_{z\in\mathbb{C}}\Delta\phi(z))^{\frac{1}{p}}M_{(g,\psi,\phi)}(z) < \infty, & p < q = \infty\\ \sup_{z\in\mathbb{C}}M_{(g,\psi,\phi)}(z) < \infty, & p = q = \infty.\end{cases}$$

(ii) $V_{(g,\psi)} : \mathcal{F}^p_\phi \to \mathcal{F}^q_\phi$ *is compact if and only if*

$$\begin{cases} \lim_{|z|\to\infty} \Delta\phi(z))^{\frac{q-p}{pq}} M_{(g,\psi,\phi)}(z) = 0, & p \leq q < \infty \\ \lim_{|z|\to\infty} \Delta\phi(z))^{\frac{1}{p}} M_{(g,\psi,\phi)}(z) = 0, & p < q = \infty \\ \lim_{|z|\to\infty} M_{(g,\psi,\phi)}(z) = 0, & p = q = \infty. \end{cases}$$

Proof. For the case $0 < p \leq q < \infty$, we notice that, for any entire function f, using the Littlewood-Paley type formula in (1.1.2), we have

$$\begin{aligned} \|V_{(g,\psi)} f\|^q_{\mathcal{F}^q_\phi} &\simeq \int_{\mathbb{C}} \frac{|f(\psi(z))|^q |g'(z)|^q}{(1+\phi'(z))^q} e^{-q\phi(z)} dm(z) \\ &= \int_{\mathbb{C}} |f(z)|^q d\mu_{(q,\psi)}(z) \end{aligned}$$

where $\mu_{(q,\psi)}$ is a pull-back measure given by

$$\mu_{(q,\psi)}(E) = \int_{\psi^{-1}(E)} \frac{|g'(w)|^q}{(1+\phi'(w))^q} e^{-q\phi(w)} dm(w)$$

for every Borel subset E of $\mathbb{C}$. Thus, $V_{(g,\psi)} : \mathcal{F}^p_\phi \to \mathcal{F}^q_\phi$ is bounded (resp. compact) if and only if the embedding operator $I_d : \mathcal{F}^p_\phi \to L^q(\mu_{(q,\psi)})$ is bounded (resp. compact). By Theorem 1 of (Constantin and Peláez, 2015), the embedding operator $I_d : \mathcal{F}^p_\phi \to L^q(\mu_{(q,\psi)})$ is bounded if and only if for some $\delta > 0$

$$\sup_{w\in\mathbb{C}} \frac{1}{\tau(w)^{\frac{2q}{p}}} \int_{D(w,\delta\tau(w))} e^{q\phi(z)} d\mu_{(q,\psi)}(z) < \infty.$$

Substituting back $d\mu_{(q,\psi)}$, the above condition is equivalent to

$$\sup_{w\in\mathbb{C}} \frac{1}{\tau(w)^{\frac{2q}{p}}} \int_{D(w,\delta\tau(w))} \frac{|g'(z)|^q}{(1+\phi'(z))^q} e^{q\phi(\psi(z))-q\phi(z)} dm(z) < \infty. \tag{4.0.5}$$

If $\sup_{w\in\mathcal{C}} \frac{|g'(w)|(\Delta\phi(w))^{\frac{q-p}{pq}}}{1+\phi'(w)} e^{\phi(\psi(w))-\phi(w)} < \infty$, then using the fact that $\tau(w) \simeq \tau(z)$ for $z \in D(w, \delta\tau(w))$, we have

$$\begin{aligned} &\sup_{w\in\mathbb{C}} \frac{1}{\tau(w)^{\frac{2q}{p}}} \int_{D(w,\delta\tau(w))} \frac{|g'(z)|^q}{(1+\phi'(z))^q} e^{q\phi(\psi(z))-q\phi(z)} dm(z) \\ &\lesssim \sup_{w\in\mathbb{C}} \frac{1}{\tau(w)^{\frac{2q}{p}}} \int_{D(w,\delta\tau(w))} \frac{1}{(\tau(z))^{\frac{2(p-q)}{p}}} dm(z) < \infty. \end{aligned}$$

On the other hand if (4.0.5) holds, then using the local estimate in (1.1.1) and the fact that $1+\phi'(z)\simeq 1+\phi'(w)$ for $z\in D(w,\delta\tau(w))$, we have

$$\frac{|g'(w)|^q(\Delta\phi(w))^{\frac{q-p}{p}}}{(1+\phi'(w))^q}e^{q\phi(\psi(w))-q\phi(w)}\lesssim \frac{\tau(w)^{\frac{-2q}{p}}}{(1+\phi'(w))^q}\int_{D(w,\delta\tau(w))}|g'(z)|^q e^{q\phi(\psi(z))-q\phi(z)}dm(z)$$
$$\simeq \frac{1}{\tau(w)^{\frac{2q}{p}}}\int_{D(w,\delta\tau(w))}\frac{|g'(z)|^q}{(1+\phi'(z))^q}e^{q\phi(\psi(z))-q\phi(z)}dm(z)<\infty.$$

For the compactness part, from Theorem 1 of (Constantin and Peláez, 2015), the embedding operator $I_d:\mathcal{F}^p_\phi\to L^q(\mu_{(q,\psi)})$ is compact if and only if

$$\lim_{|w|\to\infty}\frac{1}{\tau(w)^{\frac{2q}{p}}}\int_{D(w,\delta\tau(w))}\frac{|g'(z)|^q}{(1+\phi'(z))^q}e^{q\phi(\psi(z))-q\phi(z)}dm(z)=0.$$

Proceeding as above this holds if and only if

$$\lim_{|w|\to\infty}\frac{|g'(w)|(\Delta\phi(w))^{\frac{q-p}{pq}}}{1+\phi'(w)}e^{\phi(\psi(w))-\phi(w)}=0.$$

For the case $p\le q=\infty$ we note that, the forward implication of (i) and (ii) follows from the estimates in (4.0.2), (4.0.3) and (4.0.4). Thus, we prove the backward implications. First for $p<\infty$, if $\Delta\phi(z))^{\frac{1}{p}}M_{(g,\psi,\phi)}(z)$ is bounded, then using the local estimate in (1.1.1) and nonconstant linearity of ψ (Lemma 4.0.1),

$$\begin{aligned}\|V_{(g,\psi)}f\|_{\mathcal{F}^\infty_\phi}&\simeq \sup_{z\in\mathbb{C}}\frac{|f(\psi(z))||g'(z)|}{1+\phi'(z)}e^{-\phi(z)}\\ &\lesssim \sup_{z\in\mathbb{C}}\frac{|g'(z)|}{1+\phi'(z)}e^{\phi(\psi(z))-\phi(z)}\left(\frac{1}{\sigma^2\tau(\psi(z))^2}\int_{D(\psi(z),\sigma\tau(z))}|f(\psi(\zeta))|^p e^{-p\phi(\psi(\zeta))}dm(\zeta)\right)^{\frac{1}{p}}\\ &\lesssim \sup_{z\in\mathbb{C}}\frac{|g'(z)|\|f\|_{\mathcal{F}^p_\phi}}{(1+\phi'(z))\tau(\psi(z))^{\frac{2}{p}}}e^{\phi(\psi(z))-\phi(z)}\simeq \|f\|_{\mathcal{F}^p_\phi}\sup_{z\in\mathbb{C}}\tau(\psi(z))^{\frac{-2}{p}}M_{(g,\psi,\phi)}(z)\\ &\lesssim \|f\|_{\mathcal{F}^p_\phi}\sup_{z\in\mathbb{C}}\tau(z)^{\frac{-2}{p}}M_{(g,\psi,\phi)}(z)\simeq \|f\|_{\mathcal{F}^p_\phi}\sup_{z\in\mathbb{C}}\Delta\phi(z)^{\frac{1}{p}}M_{(g,\psi,\phi)}(z).\end{aligned}\tag{4.0.6}$$

The last estimate is by $\frac{\tau(\psi(z))^{\frac{-2}{p}}}{\tau(z)^{\frac{-2}{p}}}\lesssim 1$, which is by an assumption on τ and linearity of $\psi(z)=az+b$ with $|a|\le 1$ ($|az+b|\lesssim|z|$). Therefore, from (4.0.6) we have $V_{(g,\psi)}$ is bounded. Similarly, for $p=\infty$ if $\sup_{z\in\mathbb{C}}M_{(g,\psi,\phi)}(z)<\infty$, then using the estimate in (1.1.2) and Lemma 4.0.1 we obtain,

$$
\begin{aligned}
\|V_{(g,\psi)}f\|_{\mathcal{F}_\phi^\infty} &\simeq \sup_{z\in\mathbb{C}} \frac{|f(\psi(z))||g'(z)|}{1+\phi'(z)}e^{-\phi(z)} \\
&\leq \Big(\sup_{z\in\mathbb{C}} \frac{|g'(z)|}{1+\phi'(z)}e^{\phi(\psi(z))-\phi(z)}\Big)\Big(\sup_{z\in\mathbb{C}}|f(\psi(z))|e^{-\phi(\psi(z))}\Big) \\
&\simeq \Big(\sup_{z\in\mathbb{C}} M_{(g,\psi,\phi)}(z)\Big)\|f\|_{\mathcal{F}_\phi^\infty}
\end{aligned}
$$

from which it follows that, $V_{(g,\psi)} : \mathcal{F}_\phi^\infty \to \mathcal{F}_\phi^\infty$ is bounded.

For the compactness part, we let h_n be arbitrary bounded sequence in $\mathcal{F}_\phi^p$ converging to 0 uniformly on a compact subsets of $\mathbb{C}$ as $n \to \infty$. Then for $r > 0$ and $p < \infty$, using the Littlewood-Paley type estimate (1.1.2), the local estimate (1.1.1) and Lemma 4.0.1,

$$
\begin{aligned}
\|V_{(g,\psi)}h_n\|_{\mathcal{F}_\phi^\infty} &\simeq \sup_{z\in\mathbb{C}} \frac{|h_n(\psi(z))||g'(z)|}{1+\phi'(z)}e^{-\phi(z)} = \Big(\sup_{|z|\leq r} + \sup_{|z|>r}\Big)\frac{|h_n(\psi(z))||g'(z)|}{1+\phi'(z)}e^{-\phi(z)} \\
&\lesssim \sup_{|z|\leq r} ||h_n(\psi(z)| + \|h_n\|_{\mathcal{F}_\phi^p}\big(\sup_{|z|>r} \frac{|g'(z)|}{\tau(\psi(z))^{\frac{2}{p}}(1+\phi'(z))}e^{\phi(\psi(z))-\phi(z)}\big) \\
&\lesssim \sup_{|z|\leq r} ||h_n(\psi(z)| + \sup_{|z|>r} \frac{|g'(w)|\Delta\phi(\psi(z))^{\frac{1}{p}}}{1+\phi'(z)}e^{\phi(\psi(z))-\phi(z)} \\
&\lesssim \sup_{|z|\leq r} ||h_n(\psi(z)| + \sup_{|z|>r} \Delta\phi(z)^{\frac{1}{p}} M_{(g,\psi,\phi)}(z).
\end{aligned}
$$

Taking limit as $n \to \infty$, we obtain

$$
\lim_{n\to\infty} \|V_{(g,\psi)}h_n\|_{\mathcal{F}_\phi^\infty} \lesssim \sup_{|z|>r} \Delta\phi(z)^{\frac{1}{p}} M_{(g,\psi,\phi)}(z). \tag{4.0.7}
$$

Since $\sup_{|z|>r} \Delta\phi(z)^{\frac{1}{p}} M_{(g,\psi,\phi)}(z) \to 0$ as $r \to \infty$, letting $r \to \infty$ in (4.0.7) gives

$$
\lim_{n\to\infty} \|V_{(g,\psi)}h_n\|_{\mathcal{F}_\phi^\infty} = 0
$$

and therefore $V_{(g,\psi)}$ is compact. Similarly, if $p = \infty$, then for $r > 0$,

$$
\begin{aligned}
\|V_{(g,\psi)}h_n\|_{\mathcal{F}_\phi^\infty} &\simeq \sup_{z\in\mathbb{C}} \frac{|h_n(\psi(z))||g'(z)|}{1+\phi'(z)}e^{-\phi(z)} \leq \Big(\sup_{|z|\leq r} + \sup_{|z|>r}\Big)\frac{|h_n(\psi(z))||g'(z)|}{1+\phi'(z)}e^{-\phi(z)} \\
&\lesssim \sup_{|z|\leq r} |h_n(\psi(z)| + \|h_n\|_{\mathcal{F}_\phi^\infty}\big(\sup_{|z|>r} \frac{|g'(z)|}{1+\phi'(z)}e^{\phi(\psi(z))-\phi(z)}\big) \\
&\lesssim \sup_{|z|\leq r} |h_n(\psi(z)| + \sup_{|z|>r} \frac{|g'(z)|}{1+\phi'(z)}e^{\phi(\psi(z))-\phi(z)}.
\end{aligned}
$$

By similar argument as in the case $p < \infty$ above, we obtain the conclusion.

□

Theorem 4.0.4. *Let $0 < q < p \leq \infty$ and (g, ψ) a pair of nonconstant entire functions. Then the following are equivalent.*

(i) $V_{(g,\psi)} : \mathcal{F}_\Phi^p \to \mathcal{F}_\Phi^q$ *is compact.*

(ii) $V_{(g,\psi)} : \mathcal{F}_\Phi^p \to \mathcal{F}_\Phi^q$ *is bounded.*

(iii) The function $M_{(g,\psi,\phi)} \in L^r(\mathbb{C}, dm)$*, where* $r = \begin{cases} \frac{pq}{p-q}, & p < \infty \\ q, & p = \infty. \end{cases}$.

Proof. For any entire function f, using the Littlewood-Paley type formula in (1.1.2), we have

$$\|V_{(g,\psi)}f\|_{\mathcal{F}_\phi^q}^q \simeq \int_{\mathbb{C}} \frac{|f(\psi(z))|^q |g'(z)|^q}{(1+\phi'(z))^q} e^{-q\phi(z)} dm(z)$$
$$= \int_{\mathbb{C}} |f(z)|^q d\mu_{(q,\psi)}(z),$$

where $\mu_{(q,\psi)}$ is a pull-back measure given by

$$\mu_{(q,\psi)}(B) = \int_{\psi^{-1}(B)} \frac{|g'(w)|qp}{(1+\phi'(w))^q} e^{-q\phi(w)} dm(w)$$

for every Borel subset B of $\mathbb{C}$. Thus, $V_{(g,\psi)} : \mathcal{F}_\phi^p \to \mathcal{F}_\Phi^q$ is bounded (compact) if and only if the embedding operator $I : \mathcal{F}_\phi^p \to L^q(\mu_{(q,\psi)})$ is bounded (compact), which by Proposition 3.2 of (Mengestie and Seyoum, 2019) for $p = \infty$ and Theorem 1 of (Constantin and Peláez, 2015) for $p < \infty$ holds if and only if the function

$$T(z) := \frac{1}{\tau(z)^2} \int_{D(z,\delta\tau(z))} e^{q\phi(\zeta)} d\mu_{(q,\psi)}(\zeta)$$

belongs to $L^q(\mathbb{C}, dm)$ for $p = \infty$ and $L^{\frac{p}{p-q}}(\mathbb{C}, dm)$ for $p < \infty$, for some $\delta > 0$. But for $p = \infty$,

$$T(z) = \frac{1}{\tau(z)^2} \int_{D(z,\delta\tau(z))} \frac{|g'(\zeta)|^q}{(1+\phi(\zeta))^q} e^{q\phi(\psi(\zeta)) - q\phi(\zeta)} dm(\zeta).$$

Thus, T belongs to $L^q(\mathbb{C}, dm)$ if and only if

$$\int_{\mathbb{C}} \frac{1}{\tau(z)^{2q}} \int_{D(z,\delta\tau(z))} \frac{|g'(\zeta)|^q}{(1+\phi(\zeta))^q} e^{q\phi(\psi(\zeta)) - q\phi(\zeta)} dm(\zeta) dm(z) < \infty. \tag{4.0.8}$$

First, we assume (4.0.8) holds. Using the estimate in (1.1.1) and the fact that $1 + \phi'(z) \simeq 1 + \phi'(w)$ for $z \in D(w, \delta\tau(w))$, we have

$$\begin{aligned}
&\int_{\mathbb{C}} \frac{|g'(z)|^q}{(1+\phi'(z))^q} e^{q\phi(\psi(z))-q\phi(z)} dm(z) \\
&\qquad \lesssim \int_{\mathbb{C}} \frac{1}{\tau(z)^{2q}(1+\phi'(z))^p} \int_{D(z,\delta\tau(z))} |g'(\zeta)|^q e^{q\phi(\psi(\zeta))-q\phi(\zeta)} dm(\zeta) dm(z) \\
&\qquad \lesssim \int_{\mathbb{C}} \frac{1}{\tau(z)^{2q}} \int_{D(z,\delta\tau(z))} \frac{|g'(\zeta)|^q}{(1+\phi'(\zeta))^q} e^{q\phi(\psi(\zeta))-p\phi(\zeta)} dm(\zeta) dm(z) < \infty.
\end{aligned}$$

Therefore, $M_{(g,\psi,\phi)} \in L^q(\mathbb{C}, dm)$. On the other direction, if $M_{(g,\psi,\phi)}(z) \in L^q(\mathbb{C}, dm(z))$, then $M_{(g,\psi,\phi)}$ is bounded and hence ψ is linear (Lemma 4.0.1). Then

$$\begin{aligned}
\|V_{(g,\psi)} f\|_{\mathcal{F}_\phi^q}^q &\simeq \int_{\mathbb{C}} \frac{|f(\psi(z))|^q |g'(z)|^q}{(1+\phi'(z))^q} e^{-q\phi(z)} dm(z) \\
&\lesssim \left(\sup_{z\in\mathbb{C}} |f(\psi(z))|^q e^{-q\phi(\psi(z))} \right) \left(\int_{\mathbb{C}} \frac{|g'(z)|^q}{(1+\phi'(z))^q} e^{q\phi(\psi(z))-q\phi(z)} dm(z) \right) \\
&\lesssim \|f\|_{\mathcal{F}_\phi^\infty}^q.
\end{aligned}$$

Therefore, $V_{(g,\psi)} : \mathcal{F}_\phi^\infty \to \mathcal{F}_\Phi^q$ is bounded and hence (4.0.8) holds. Which concludes the proof.

Similarly, for $p < \infty$, Substituting back $\mu_{(q,\psi)}$, observe that T belongs to $L^{\frac{p}{p-q}}(\mathbb{C}, dm)$ if and only if

$$\int_{\mathbb{C}} \left(\frac{1}{\tau(z)^2} \int_{D(z,\delta\tau(z))} \frac{|g'(\zeta)|^q}{(1+\phi(\zeta))^q} e^{q\phi(\psi(\zeta))-q\phi(\zeta)} dm(\zeta) \right)^{\frac{p}{p-q}} dm(z) < \infty. \tag{4.0.9}$$

Thus, suppose (4.0.9) holds. Using the estimate in (1.1.1) and the fact that $1+\phi'(z) \simeq 1+\phi'(w)$ for $z \in D(w, \delta\tau(w))$, for $r = \frac{pq}{p-q}$ we have

$$\begin{aligned}
&\int_{\mathbb{C}} \frac{|g'(z)|^r}{(1+\phi'(z))^r} e^{r\phi(\psi(z))-r\phi(z)} dm(z) \\
&\lesssim \int_{\mathbb{C}} \left(\frac{1}{\tau(z)^2(1+\phi'(z))^q} \int_{D(z,\delta\tau(z))} |g'(\zeta)|^q e^{q\phi(\psi(\zeta))-q\phi(\zeta)} dm(\zeta) \right)^{\frac{p}{p-q}} dm(z) \\
&\lesssim \int_{\mathbb{C}} \left(\frac{1}{\tau(z)^2} \int_{D(z,\delta\tau(z))} \frac{|g'(\zeta)|^q}{(1+\phi'(\zeta))^q} e^{q\phi(\psi(\zeta))-q\phi(\zeta)} dm(\zeta) \right)^{\frac{p}{p-q}} dm(z) < \infty.
\end{aligned}$$

On the other hand, if $\frac{g'(z)}{1+\phi'(z)} e^{\phi(\psi(z))-\phi(z)} \in L^r(\mathbb{C}, dm(z))$, then applying Hölder's inequality gives

$$\|V_{(g,\psi)}f\|^q_{\mathcal{F}^q_\phi} \simeq \int_{\mathbb{C}} \frac{|f(\psi(z))|^q|g'(z)|^q}{(1+\phi'(z))^q}e^{-q\phi(z)}dm(z)$$
$$\lesssim \Big(\int_{\mathbb{C}}|f(\psi(z))|^pe^{-p\phi(\psi(z))}dm(z)\Big)^{\frac{q}{p}}\Big(\int_{\mathbb{C}}\frac{|g'(z)|^r}{(1+\phi'(z))^r}e^{r\phi(\psi(z))-r\phi(z)}dm(z)\Big)^{\frac{q}{r}}$$
$$\lesssim \Big(\int_{\mathbb{C}}|f(\psi(z))|^pe^{-p\phi(\psi(z))}dm(z)\Big)^{\frac{q}{p}} \simeq \|f\|^q_{\mathcal{F}^p_\phi}.$$

Where the last estimate is by linearity of $\psi(z) = az+b$ with $0 < |a| \le 1$. Therefore, $V_{(g,\psi)}$ is bounded and hence (4.0.9) holds. □

Corollary 4.0.5. *Let $0 < p, q \le \infty$. Then for a nonconstant entire inducing symbol g,*

(i) $V_g : \mathcal{F}^p_\phi \to \mathcal{F}^p_\phi$ is bounded (respectively, compact) if and only if $\frac{|g'(z)|}{1+\phi'(z)}$ is bounded (respectively, $\lim_{|z|\to\infty} \frac{|g'(z)|}{1+\phi'(z)} = 0$).

(ii) if $p < q$, then $V_g : \mathcal{F}^p_\phi \to \mathcal{F}^q_\phi$ is bounded (compact) if and only if $\lim_{|z|\to\infty} \frac{|g'(z)|}{1+\phi'(z)} = 0$.

(iii) if $q < p$, then $V_g : \mathcal{F}^p_\phi \to \mathcal{F}^q_\phi$ is bounded (compact) if and only if

$$\int_{\mathbb{C}} \frac{|g'(z)|^r}{(1+\phi'(z))^r}dm(z) < \infty, \text{ Where } r = \begin{cases} \frac{pq}{p-q}, & p < \infty \\ q, & p = \infty \end{cases}.$$

Example :- Let $0 < p, q \le \infty$ and $\phi(z) = |z|^m, m > 2$.

(i) If $0 < p < q \le \infty$, then $V_g : \mathcal{F}^p_\phi \to \mathcal{F}^q_\phi$ is bounded (or compact) if and only if g is a polynomial degree $\le m$.

(ii) If $p = q$, the $V_g : \mathcal{F}^p_\phi \to \mathcal{F}^q_\phi$ is bounded (or respectively, compact) if and only if g is a polynomial degree $\le m$, (respectively, g is a polynomial of degree $< m$.)

(iii) If $0 < q < p \le \infty$, then $V_g : \mathcal{F}^p_\phi \to \mathcal{F}^q_\phi$ is bounded (or compact) if and only if $q > \begin{cases} \frac{2p}{p+2}, & p < \infty \\ 2, & p = \infty \end{cases}$ and g is a polynomial degree $< m$.

Chapter 5

Conclusion and Future scope

5.1 Conclusion

This thesis includes a number of results, which characterize generalized Volterra-type integral operators acting between generalized Fock spaces. Our results in chapter 4, which is about boundedness and compactness are new and it is simple to apply to study other properties defined whenever the operator is bounded. In addition, our results improve and generalize some of the results that have been obtained for this important class of linear operators. In particular, Theorem 4.0.3 and 4.0.4 generalize the results of (Constantin and Peláez, 2015, Mengestie and Ueki, 2019) from Volterra-type integral to the generalized Volterra-type integral operators, which is stated in Theorem 2.0.5 and 2.0.6. Moreover, our result shows that the operator has wide number of inducing symbols g and ϕ, which induces bounded and compact generalized Volterra-type integral operators on $\mathcal{F}_p^\phi$, when we compare with the results obtained on the classical Fock spaces.

5.2 Future Scope

The study of generalized Volterra-type integral operator is an active area of research, which attracted interest of many researchers to study acting on several functional spaces, including the classical Fock spaces. Continuing in this area of research, any interested researcher can study different other properties of the operator on the space. For instance, one can study properties like compact difference, path-connected and connected components in the space of bounded generalized Volterra-type integral operators, using the results of this thesis.

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