A MONOTONE HYBRID FINITE DIFFERENCE METHOD FOR SINGULARLY PERTURBED BURGERS' EQUATION

A THESIS SUBMITTED TO THE DEPARTMENT OF MATHEMATICS, JIMMA UNIVERSITY IN PARTIAL FULFILLMENT FOR THE REQUIREMENTS OF THE DEGREE OF MASTER OF SCIENCE IN MATHEMATICS

(NUMERICAL ANALYSIS)

By

TEKLE MENGESHA OFGEA

Under the supervision of

PROF. GEMECHIS FILE DURESSA

WAKJIRA TOLASSA GOBENA (M.Sc)

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Jimma, Ethiopia

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Approved by the Examiner Board

Chairman

Name: ________________ Sign: ________________ Date: ________________

External Examiner

Name: ________________ Sign: ________________ Date: ________________

Internal Examiner

Name: ________________ Sign: ________________ Date: ________________

Declaration

I, undersigned declare that this thesis entitled ”A monotone hybrid finite difference method for solving singularly perturbed Burgers’ equation ” is my own original work and it has not been submitted for the award of any academic degree or the like in any other institution or university, and that all the sources I have used or quoted have been indicated and acknowledged as complete references.

Student Researcher

Name: ______________________ Sign: ______________________ Date: ______________________

This research has been done under the supervision and approval of

Supervisor

Name: ______________________ Sign: ______________________ Date: ______________________

Co-supervisor

Name: ______________________ Sign: ______________________ Date: ______________________

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Abstract

In this thesis, we deal with a monotone hybrid finite difference method for singularly perturbed Burgers' equation. First, we apply quasilinearization process to tackle the non-linearity in the equation. We constructed a numerical scheme that comprises of an implicit second-order finite difference method to discretize the time derivative on uniform mesh and a monotone hybrid finite difference method to discretize the space derivative with piecewise uniform Shishkin mesh. The method has been shown to be second-order uniformly accurate in the time variable, and in the spatial direction it is first-order parameter uniform convergent in the outer region and almost second-order parameter uniform convergent in the boundary layer region. For small values of the parameter ε, a boundary layer is in the neighborhood of right part of the domain. Accuracy and uniform convergence of the proposed method is demonstrated by numerical experiments.

Chapter 1

Introduction

1.1 Background of the Study

Numerical Analysis is concerned with mathematical derivation of numerical methods, designing an algorithm for the methods, implementation of these algorithms on computers and analysis of the errors associated with the methods all to solve mathematical problems. It does not strive for exactness. Instead, attempts to devise a method which yields an approximation differing from exactness by less than a specified tolerance, or by an amount which has less than a specified probability of exceeding that tolerance. The ultimate aim of the field of numerical analysis is to provide convenient methods for obtaining useful solutions to mathematical problems and for extracting useful information from available solutions which are not expressed in tractable forms. Such problems may each be formulated, for example, in terms of algebraic or transcendental equation, an ordinary or partial differential equation, or in terms of a set of such equations. The wide use of computing methods, combined with the demands of scientific and technical practices, has stimulated the development of numerical methods to a great extent, and in particular, methods for solving differential equations. The efficiency of such methods is governed by their accuracy, simplicity in computing the discrete solution and also their relative insensitivity to parameters in the problem (Shishkin and Shishkina, 2009). At present, numerical methods for solving partial differential equations, in particular, finite difference scheme, are well developed for wide classes of problems (Morton, 1996).

A partial differential equations (PDEs) is an equation that involves two or more independent variables, as unknown function (dependent on those variables), and partial derivatives of the unknown function with respect to the independent variables. In real life, we often encounter many problems which are described by parameter dependent differential equations. The behavior of the solution of these types of differential equation depends on the magnitude of the parameter. Any differential equation in which the highest order derivative is multiplied by a small positive parameter $\varepsilon (0 < \varepsilon << 1)$ is called singular perturbation problem (SPP) and the parameter is known as the perturbation parameter. A boundary layer is small part of the region in which solution changes very rapidly to satisfy the given condition. SPPs arise very frequently in diversified fields of applied mathematics and engineering, for instance fluid mechanics, elasticity, hydrodynamics, quantum mechanics, elasticity, chemical-reaction theory, aerodynamics, plasma dynamics, rarefied-gas dynamics, oceanography, meteorology, modeling of semiconductor devices, diffraction theory and reaction-diffusion processes and many other allied areas (Roos et al., 2008).

Nonlinear phenomena appear in a wide variety of scientific applications such as plasma physics, solid state physics, optical fibers, biology, fluid dynamics and chemical kinetics. The nonlinear partial differential equation is a homogenous quasi-linear parabolic partial differential equation which encounters in the theory of shock waves, mathematical modeling of turbulent fluid and in continuous stochastic processes (Arora and Singh, 2013). Such type of partial differential equation is introduced by Bateman (1915) and he proposes the steady-state solution of the problem. Burger (1939, 1948) used the nonlinear partial differential equation to solve problems in fluid dynamics involving turbulence; later on it is widely referred as Burgers' equation. The distinctive feature of the Burgers' equation is that it contains the non-linear convection term and the diffusion term with viscosity coefficient.

The study of the general properties of the Burgers' equation has attracted attention of scientific community due to its applications in many areas of science and engineering fields such as gas dynamics, heat conduction, elasticity, modelling of traffic flow, modelling of fluid dynamics involving turbulence and fluid flow (Kadalbajoo et al., 2005). The numerical solution of Burgers' equation is an active area of research for the last many years to develop better numerical schemes to approximate its solution. Many researchers have been defined various numerical methods for numerical solutions of Burgers' equation such as Galerkin finite element method by Abdulkadir (2004), quadratic B-spline finite element method by Aksan (2006), modified cubic B-spline differential quadrature method by Arora and Singh (2013), automatic differentiation method by Asai (2010), cubic B-spline method by Dag et al. (2005), fourth-order finite difference method by Hassanien et al. (2005), Haar wavelet quasilinearization approach by Jiwari (2012), weighted average differential quadrature method by Jiwari (2013), a hybrid numerical scheme Jiwari (2014), a numerical method based on Crank-Nicolson by Kadalbajoo and Awasthi (2006), spectral collocation method by Khater et al. (2008), quartic B-spline differential quadrature method by Korkmaz et al. (2011), polynomial based differential quadrature method by Korkmaz and Dag (2011), least-square quadratic B-spline finite element method by Kutulay et al. (2004), an implicit fourth-order compact finite difference scheme by Liao (2008), modified cubic B-splines collocation method by Mittal and Jain (2012), a numerical scheme based on differential quadrature method by Mittal et al. (2013), finite element method by Ozis et al. (2003), non-polynomial spline approach by Ramadan et al. (2007), quartic B-spline collocation method by Saka and Dag (2007), etc. All the above review reports confirm that the numerical solutions for Burgers' equation are with high coefficient of viscosity. Therefore, the main purpose of this study is to propose fitted numerical method for Burgers' equation with very small coefficient of viscosity.

1.2 Statement of the Problem

Recently, research has been conducted on semi-linear and nonlinear types of singularly perturbed problems. For instance, Kabeto and Duressa (2021) developed numerical method for singularly perturbed semilinear parabolic differential difference equations. Very few scholars presented the numerical solution of nonlinear singularly perturbed Burgers' equation. For instance, Kadalbajoo et al. (2005) constructed an implicit upwind difference scheme for solving nonlinear singularly perturbed Burgers' equation whose convergence behavior in the global maximum norm is parameter-uniform. Gupta and Kadalbajoo (2016) developed B-spline collocation with implicit

Euler method on a piecewise uniform mesh of Shishkin type for solving nonlinear singularly perturbed Burgers' equation. Gowrisankar and Natesan (2019) studied singularly perturbed viscous Burgers' equation. Difficulties arise in the numerical solution of nonlinear singularly perturbed Burgers' equation which corresponds to the boundary layers produced in the neighborhood of right part of the spatial domain, by the steeping effect of the non-linear convection term. It is well known facts that, for small values of ε standard numerical methods for solving such problems are unstable and do not give accurate results on uniform mesh. Therefore, it is important to develop efficient numerical methods for solving these problems, whose accuracy does not depend on the value of parameter ε, i.e., methods that are convergent $\varepsilon-$uniformly. However, a fitted mesh numerical method has not been sufficiently developed yet for seeking accurate and efficient numerical solutions of nonlinear singularly perturbed Burgers' equation. Therefore, the main purpose of this study is to develop fitted mesh monotone hybrid numerical method for solving singularly perturbed Burgers' equation. Due to this, the present study attempted to answer the following questions:

- How do we describe fitted mesh monotone hybrid method for the Burgers' equation?
- To what extent the proposed method converges?
- To what extent the proposed method accurate?

1.3 Objectives of the Study

1.3.1 General Objective

The general objective of this study is to formulate monotone hybrid numerical method for solving singularly perturbed Burgers' equation.

1.3.2 Specific Objectives

The specific objectives of the study was:

- To describe fitted mesh monotone hybrid method for singularly perturbed Burgers' equation.
- To establish the uniform convergence of the proposed method.

- To validate the accuracy of the proposed method.

1.4 Significance of the Study

The result obtained from this study may

- Used as a reference material for scholars who works on this area.
- Able the graduate students to acquire research skills and scientific procedures.
- Provide a numerical method for solving nonlinear singularly perturbed Burgers' equations.

1.5 Delimitation of the Study

Singular perturbation problems are perhaps arises in variety of mathematical and physical systems. Though singular perturbation problems are vast topics and has many applications in the real world, this study is delimited to focus on presenting monotone hybrid numerical method for singularly perturbed Burgers' equation of the form:

$$L_\varepsilon u(x,t) \equiv \frac{\partial u}{\partial t} - \varepsilon \frac{\partial^2 u}{\partial x^2} + u\frac{\partial u}{\partial x} = 0, \quad (x,t) \in D = \Omega_x \times \Omega_t = (0,1) \times (0,T], \tag{1.1}$$

with the following initial and boundary conditions, respectively

$$\begin{aligned} &u(x,0) = \phi(x), \quad 0 \le x \le 1, \\ &u(0,t) = 0, \; u(1,t) = 0, \quad 0 \le t \le T, \end{aligned} \tag{1.2}$$

where ε is the diffusion parameter and assume the functions $\phi(x)$ is sufficiently smooth.

Chapter 2

Review of Related Literatures

2.1 Singular Perturbation Theory

Lundwing Prandtl was the first introduce the concept layer in 1904 at the Third International Congruence of Mathematics in Heidelberg Germany. His hypothesis was in the setting of fluid dynamics, fluid adjacent to the boundary sticks to the edge in a thin boundary layer due friction but this friction has no effect to the flow on the interior by (Prandtl, 1904). The term singular perturbation appears to have been first coined (Friedrichs and Wasow, 1946). Wasow continued to the contribute to the area of asymptotic methods over many years and his book " Asymptotic expansion for ordinary differential equation " Vasil'eva (1963), attracted much interest in the area of singular perturbed boundary value problems. In Russia, mainly at Moscow State University, research activity on singular perturbations for ordinary differential equations, originated and developed by Tikhonov (1952) and his students, especially Vasil'eva (1994) continues to be vigorously pursued even today. A brief survey for the historical development of singular perturbation problems is covered in the recent book (O'Malley, 1991) and (Roos et al., 2008). More precisely, a perturbation problem is problem that contains a small parameter ε, called perturbation parameter. If the solution of the problem can be approximated by setting the value of the perturbation parameter equal to zero, then the problem is called regular perturbation problem, otherwise it is called singular perturbation problem. That is, if it is impossible to approximate the solution by asymptotic expansion as the perturbation parameter tends to zero, then the problem is called singular.

Some numerical methods for solving singularly perturbed problems have been studied extensively in the literature. Singularly perturbed differential equations are characterized by the presence of a small parameter multiplying the highest-order derivatives. Such problems arise in many areas of applied mathematics and engineering. Among these are the Navier-Stokes equations of fluid flow at high Reynolds number, mathematical models of liquid crystal materials and chemical reactions, control theory, reaction-diffusion processes, quantum mechanics, and electrical networks. An overview of some existence and uniqueness results and applications of singularly perturbed problems can be found (Roos et al., 2008).

2.2 Singularly Perturbed Burgers' Equation

Kadalbajoo et al. (2005) studied the following singularly perturbed Burgers' equation of the form

$$\begin{cases} L_\varepsilon u(x,t) \equiv \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \varepsilon \frac{\partial^2 u}{\partial x^2}, \quad (x,t) \in D = \Omega_x \times \Omega_t = (0,1) \times (0,T], \\ u(x,0) = f(x), \quad 0 < x < 1, \\ u(0,t) = 0, \;\; u(1,t) = 0, \quad 0 < t \leq T, \end{cases} \tag{2.1}$$

where $0 < \varepsilon << 1$ is the coefficient of kinematic viscosity and the prescribed function $f(x)$ is sufficiently smooth. The authors first semi-discretize the original non-linear Burgers' equation in the temporal direction by backward Euler scheme with the constant time step which produces a set of stationary Burgers' equations. Then using the quasilinearization process, the stationary Burgers' equation obtained from semi-discretization will be linearized. For totally discrete scheme, the authors discretize the set of linear problems resulting from the time semi discretization using the simple upwind finite difference scheme defined on an appropriate piecewise uniform mesh of Shishkin type.

Gupta and Kadalbajoo (2016) studied qualitative analysis and numerical solution of Burgers' equation via B-spline collocation with implicit Euler method on piecewise uniform mesh for the problem of type (1.1)-(1.2). They constructed a numerical scheme that comprises of implicit Euler method to discretize in temporal direction on uniform mesh and a B-spline collocation approach to discretize the spatial variable with piecewise uniform Shishkin mesh. Quasi-linearization process is used to tackle the non-linearity and shown that quasi-linearization process converges quadratically. Asymptotic bounds for the derivatives of the solution are established by decomposing the solution into smooth and singular components. These bounds are applied in convergence analysis of the proposed method on Shishkin mesh. The method has been shown to be first-order convergent in the temporal variable and almost second order accurate in the spatial variable.

Gowrisankar and Natesan (2019) studied problem of type (1.1)-(1.2). They proposed a parameter uniformly convergent numerical method for viscous Burgers' equation. In order to find a numerical approximation to Burgers' equation, they linearize the equation to obtain sequence of linear PDEs. The linear PDEs are solved by a finite difference scheme, which comprises of the backward-difference scheme for the time derivative and upwind finite difference scheme for the spatial derivatives. Layer-adapted nonuniform meshes are invoked at each time level to exhibit layer nature of the solution. The nonuniform meshes are obtained by equidistribution of a positive integrable monitor function, which involves the derivative of the solution. It is shown that the methods converges uniformly with respect to the perturbation parameter. Numerical experiments are carried out to validate the $\varepsilon-$uniform error estimate of $O(N^{-1}+\Delta t)$.

Kadalbajoo and Awasthi (2017) considered singularly perturbed modified Burgers' turbulence model

$$\begin{cases} L_\varepsilon u(x,t) \equiv \frac{\partial u}{\partial t} + kt^{n/2} u \frac{\partial u}{\partial x} = \varepsilon \frac{\partial^2 u}{\partial x^2}, \quad 0<\varepsilon<<1, \ (x,t)\in D = \Omega \times (0,T], \\ u(x,0) = f(x), \quad 0<x<1, \\ u(0,t) = 0 = u(1,t), \quad 0<t\leq T, \end{cases} \tag{2.2}$$

where k is a positive constant, $n=1$ or $n=2$, and $\Omega=(0,1)$. The authors proposed a numerical method which comprises of Euler implicit scheme in time and hybrid scheme in space direction. First, they discretize the continuous problem in temporal direction by Euler implicit method, which yields a set of ordinary differential equations at each time level. The resulting set of differential equations are approximated by a hybrid scheme on Shishkin mesh i.e. upwind in regular region (non-boundary layer region) and central difference in boundary layer regions. The convergence of proposed method has been shown parameter uniform. Some numerical experiments have been carried out to corroborate the theoretical results. As can be seen in the above literature review, very few researchers are tried to find the numerical solution for singularly perturbed nonlinear Burgers' and modified Burgers' equation. As far as the review report is concerned, the numerical solution for singularly perturbed nonlinear Burgers' and modified Burgers' equation is at the initial stage. Therefore, the main purpose of this study is to develop fitted numerical method for solving singularly perturbed nonlinear Burgers' equation.

2.3 Quasilinearzation Technique

The nonlinear partial differential equation is linearized around a nominal solution of the nonlinear partial differential equation which satisfies the boundary conditions. Suppose $u^{(k)}(x)$ is the nominal solution of the nonlinear partial differential equation. The quasilinearization process yields a sequence $< u^{(k)} >$ of linear equations (Bellman and Kalaba, 1965). They developed the quasilinearzation technique which is used to reduce the given nonlinear boundary value problem into the corresponding sequence of linear boundary value problem. The quasilinearzation technique of reducing nonlinear boundary value problem into a sequence of linear boundary value problem involves some steps. First, we linearize the semi-linear ordinary differential equation around a nominal solution, which satisfies the specified boundary conditions. Second, we solve a sequence of boundary value problems in which the solution of the linear boundary value problem satisfies the specified boundary conditions and is taken as the nominal profile for the linear boundary value problem. Quasilinearzation technique is used to linearize the original semi-linear singular perturbation problem into a sequence of linear singular perturbation problems.

2.4 Numerical versus Analytical Methods

Suppose we have a differential equation and we want to find a solution of the differential equation. The best is when we can find out the exact solution using calculus, trigonometry and other techniques. The techniques used for calculating the exact solution are known as analytic methods because we used the analysis to figure it out. Analytical solution is continuous. The exact solution is also referred to as a closed form solution or analytical solution. But this tends to work only for simple differential equations with simple coefficients, but for higher order or non-linear differential equations with complex coefficient, it becomes very difficult to find exact solution. Therefore, we need numerical methods for solving the equations. Numerical methods are commonly used for solving mathematical problems that are formulated in science and engineering where it is difficult or even impossible to obtain exact solutions. Numerical solution is discrete. Numerical methods, on the other hand, can give an approximate solution to an equation.

2.5 Finite Difference Methods

Most problems cannot be solved analytically, henceforth finding good approximation solutions using numerical methods will be very useful. From different classification of numerical methods such as finite difference method, spectral method, finite element method, finite volume method, spline method, finite difference method seems to be the simplest approach for the numerical solution of boundary value problems (Roos et al., 2008). Finite difference methods are widely used by the scientific community and it is always a convenient choice for solving boundary value problems because of their simplicity. In finite difference methods, derivatives appearing in the differential equations are replaced by finite difference approximations obtained by Taylor series expansions at the grid points. This gives a large algebraic system of equations to be solved by Thomas Algorithm in place of the differential equation to give the solution value at the grid points and hence the solution is obtained at grid points. Some of the finite difference methods include forward approximation, backward approximation, central difference approximation.

Chapter 3

Research Methodology

3.1 Study Area and Period

This study was conducted at Jimma University department of Mathematics from September 2020 to June 2021.

3.2 Study Design

This study employed both documentary review and numerical experimental design.

3.3 Source of Informations

The relevant source of information for this study are books and published articles from internet.

3.4 Mathematical Procedures

In order to achieve the aforementioned objectives, the study followed the following steps:

1. Define the problem.

2. Apply the quasilinearization technique to linearize the nonlinear problem.

3. Discretize the temporal interval and replace the differential term with respect to time derivative by second-order finite difference approximation.

4. Discretize the spatial interval using piecewise uniform mesh and formulate the monotone hybrid method.

5. Establish the convergence of the formulated method.

6. Write MATLAB code for the method.

7. Present the numerical illustrations.

Chapter 4

Derivation of the Method, Convergence Analysis and Numerical Results

4.1 Derivation of the Method

In this study, we consider the following singularly perturbed nonlinear one-dimensional Burgers' equation with the initial and boundary conditions, respectively

$$\begin{cases} L_\varepsilon u(x,t) \equiv \dfrac{\partial u}{\partial t} - \varepsilon \dfrac{\partial^2 u}{\partial x^2} + u\dfrac{\partial u}{\partial x} = 0, & (x,t) \in D = (0,1) \times (0,T], \\ \qquad u(x,0) = \phi(x), & 0 \le x \le 1, \\ \qquad u(0,t) = 0, \quad u(1,t) = 0, & 0 \le t \le T, \end{cases} \tag{4.1}$$

where ε is the perturbation parameter and assume the function $\phi(x)$ must be continuous on $[0,1]$ and continuously differentiable on $(0,1)$.

4.1.1 The Quasilinearization Technique

To linearize the nonlinear term in Eq. (4.1), we re-write Eq. (4.1) in the form

$$\begin{cases} F(x,t,u,\dfrac{\partial u}{\partial x}) = \dfrac{\partial u}{\partial t} - \varepsilon \dfrac{\partial^2 u}{\partial x^2}, & (x,t) \in D = (0,1) \times (0,T], \\ u(x,0) = \phi(x), & 0 \le x \le 1, \\ u(0,t) = 0, \quad u(1,t) = 0, & 0 \le t \le T, \end{cases} \tag{4.2}$$

where $F(x,t,u,\frac{\partial u}{\partial x}) = -u\frac{\partial u}{\partial x}$. A reasonable initial approximation $u^{(0)}(x,t)$ for the function $u(x,t)$ is used to linearize the term $F(x,t,u,\frac{\partial u}{\partial x}) = -u\frac{\partial u}{\partial x}$ by applying the quasilinearization technique (Bellman and Kalaba, 1965). For the reasonable initial guess of the form

$$u^{(0)}(x,t) = \phi(x)e^{-ct} = u^{(0)}, \tag{4.3}$$

satisfying the initial and boundary conditions of the problem so that the constant c is taken from the coefficients of the initial condition (Gowrisankar (2019)). Expanding the nonlinear term $F(x,t,u^{(0)},\frac{\partial u^{(0)}}{\partial x})$ around the point $u^{(0)}$, we obtain:

$$\begin{aligned} F\left(x,t,u^{(1)},\frac{\partial u^{(1)}}{\partial x}\right) \cong & F\left(x,t,u^{(0)},\frac{\partial u^{(0)}}{\partial x}\right) + \left(u^{(1)} - u^{(0)}\right)\frac{\partial F}{\partial u}\Bigg|_{(x,t,u^{(0)},\frac{\partial u^{(0)}}{\partial x})} \\ & + \left(\frac{\partial u^{(1)}}{\partial x} - \frac{\partial u^{(0)}}{\partial x}\right)\frac{\partial F}{\partial(\frac{\partial u}{\partial x})}\Bigg|_{(x,t,u^{(0)},\frac{\partial u^{(0)}}{\partial x})} + \cdots \end{aligned} \tag{4.4}$$

Putting Eq. (4.4) into Eq. (4.2), we obtain

$$\begin{aligned} & F\left(x,t,u^{(0)},\frac{\partial u^{(0)}}{\partial x}\right) + \left(u^{(1)} - u^{(0)}\right)\frac{\partial F}{\partial u}\Bigg|_{(x,t,u^{(0)},\frac{\partial u^{(0)}}{\partial x})} \\ & + \left(\frac{\partial u^{(1)}}{\partial x} - \frac{\partial u^{(0)}}{\partial x}\right)\frac{\partial F}{\partial(\frac{\partial u}{\partial x})}\Bigg|_{(x,t,u^{(0)},\frac{\partial u^{(0)}}{\partial x})} = \frac{\partial u^{(1)}}{\partial t} - \varepsilon\frac{\partial^2 u^{(1)}}{\partial x^2} \end{aligned} \tag{4.5}$$

From Eq. (4.5), we get

$$\begin{aligned} & \frac{\partial u^{(1)}}{\partial t} - \varepsilon\frac{\partial^2 u^{(1)}}{\partial x^2} - \frac{\partial F}{\partial(\frac{\partial u}{\partial x})}\Bigg|_{(x,t,u^{(0)},\frac{\partial u^{(0)}}{\partial x})}\frac{\partial u^{(1)}}{\partial x} - \frac{\partial F}{\partial u}\Bigg|_{(x,t,u^{(0)},\frac{\partial u^{(0)}}{\partial x})}u^{(1)} \\ & = F\left(x,t,u^{(0)},\frac{\partial u^{(0)}}{\partial x}\right) - \frac{\partial F}{\partial(\frac{\partial u}{\partial x})}\Bigg|_{(x,t,u^{(0)},\frac{\partial u^{(0)}}{\partial x})}\frac{\partial u^{(0)}}{\partial x} - \frac{\partial F}{\partial u}\Bigg|_{(x,t,u^{(0)},\frac{\partial u^{(0)}}{\partial x})}u^{(0)} \end{aligned} \tag{4.6}$$

At the first iteration, we obtain the following linearized parabolic differential equation

$$\begin{cases} L_\varepsilon u \equiv \left(\dfrac{\partial u}{\partial t} - \varepsilon\dfrac{\partial^2 u}{\partial x^2} + a\dfrac{\partial u}{\partial x} + bu\right)(x,t) = f(x,t), \\ \quad u(x,0) = \phi(x), \qquad 0 \le x \le 1, \\ \quad u(0,t) = 0, \quad u(1,t) = 0, \qquad 0 \le t \le T, \end{cases} \tag{4.7}$$

where $a(x,t) = -\frac{\partial F}{\partial(\frac{\partial u}{\partial x})}\bigg|_{(x,t,u^{(0)},\frac{\partial u^{(0)}}{\partial x})}$, $\qquad b(x,t) = -\frac{\partial F}{\partial u}\bigg|_{(x,t,u^{(0)},\frac{\partial u^{(0)}}{\partial x})}$,
$f(x,t) = F\left(x,t,u^{(0)},\frac{\partial u^{(0)}}{\partial x}\right) - \frac{\partial F}{\partial(\frac{\partial u}{\partial x})}\bigg|_{(x,t,u^{(0)},\frac{\partial u^{(0)}}{\partial x})}\frac{\partial u^{(0)}}{\partial x} - \frac{\partial F}{\partial u}\bigg|_{(x,t,u^{(0)},\frac{\partial u^{(0)}}{\partial x})}u^{(0)}$.
Specifically, we can write $a(x,t) = u^{(0)}$, $b(x,t) = \frac{\partial u^{(0)}}{\partial x}$ and $f(x,t) = u^{(0)}\frac{\partial u^{(0)}}{\partial x}$.

4.1.2 Time discretization

Now, we first discretize the time derivative by means of an implicit second-order finite difference method on a uniform mesh. Let N be a positive integer different from one, we divide the time interval $[0,T]$ with uniform step length Δt. Hence, the interval $[0,T]$ is partitioned into N equal sub-intervals with each nodal points satisfying $0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T$. Thus, the time nodal points are generated by $t_n = n\Delta t$, $\Delta t = \frac{T}{N}$, $n = 0,\cdots,N$, where N denotes the number of mesh intervals in the time direction.
The semi-discretized problem in Eq. (4.7) at the point $(x,t_{n+\frac{1}{2}})$ becomes

$$L_\varepsilon^N U^{n+\frac{1}{2}}(x) \equiv \left(\frac{\partial U}{\partial t} - \varepsilon\frac{\partial^2 U}{\partial x^2} + a\frac{\partial U}{\partial x} + bU\right)(x,t_{n+\frac{1}{2}}) = f(x,t_{n+\frac{1}{2}}), \tag{4.8}$$

Using Taylor series expansion about $(x,t_{n+\frac{1}{2}})$ gives

$$U(x,t_{n+1}) = U(x,t_{n+\frac{1}{2}}) + \frac{\Delta t}{2}\frac{\partial U(x,t_{n+\frac{1}{2}})}{\partial t} + \frac{\Delta t^2}{8}\frac{\partial^2 U(x,t_{n+\frac{1}{2}})}{\partial t^2} + \frac{\Delta t^3}{48}\frac{\partial^3 U(x,t_{n+\frac{1}{2}})}{\partial t^3} + \cdots \tag{4.9}$$

$$U(x,t_n) = U(x,t_{n+\frac{1}{2}}) - \frac{\Delta t}{2}\frac{\partial U(x,t_{n+\frac{1}{2}})}{\partial t} + \frac{\Delta t^2}{8}\frac{\partial^2 U(x,t_{n+\frac{1}{2}})}{\partial t^2} - \frac{\Delta t^3}{48}\frac{\partial^3 U(x,t_{n+\frac{1}{2}})}{\partial t^3} + \cdots \tag{4.10}$$

From Eqs. (4.9) and (4.10), we obtain

$$\frac{\partial U(x,t_{n+\frac{1}{2}})}{\partial t} = \frac{U^{n+1}(x) - U^{n}(x)}{\Delta t} + TE, \tag{4.11}$$

where $TE = -\frac{\Delta t^2}{24}\frac{\partial^3 U(x,t_{n+\frac{1}{2}})}{\partial t^3}$.

From this truncation error, we obtain the following error bound in time semi-discretization

$$\|E\|_\infty \leq C\Delta t^2, \tag{4.12}$$

where C is an arbitrary constant given by $C = \frac{1}{24}|\frac{\partial^3 U(x,t_{n+\frac{1}{2}})}{\partial t^3}|$.

Averaging of all the terms in Eq. (4.8) yields

$$\begin{aligned} &\left(-\varepsilon\frac{\partial^2 U}{\partial x^2} + a\frac{\partial U}{\partial x} + bU - f\right)(x,t_{n+\frac{1}{2}}) \\ &= \frac{1}{2}\left[\left(-\varepsilon\frac{\partial^2 U}{\partial x^2} + a\frac{\partial U}{\partial x} + bU - f\right)(x,t_{n+1}) + \left(-\varepsilon\frac{\partial^2 U}{\partial x^2} + a\frac{\partial U}{\partial x} + bU - f\right)(x,t_n)\right] \end{aligned} \tag{4.13}$$

Putting Eqs. (4.11) and (4.13) into Eq. (4.8) gives

$$\begin{aligned} \frac{U(x,t_{n+1}) - U(x,t_n)}{\Delta t} + \frac{1}{2}\Bigg[&\left(-\varepsilon\frac{\partial^2 U}{\partial x^2} + a\frac{\partial U}{\partial x} + bU - f\right)(x,t_{n+1}) \\ &+ \left(-\varepsilon\frac{\partial^2 U}{\partial x^2} + a\frac{\partial U}{\partial x} + bU - f\right)(x,t_n)\Bigg] = TE. \end{aligned} \tag{4.14}$$

Multiplying both sides Eq. (4.14) by 2 gives the following form

$$\left(-\varepsilon\frac{d^2 U}{dx^2} + a\frac{dU}{dx} + (b + \frac{2}{\Delta t})U\right)(x,t_{n+1}) = Z(x,t_{n+1}), \tag{4.15}$$

subject to the initial-boundary conditions, respectively

$$\begin{cases} U(x,0) = \phi(x), & 0 \leq x \leq 1, \\ U(0,t_{n+1}) = 0, \quad U(1,t_{n+1}) = 0, & 0 \leq t \leq T, \end{cases} \tag{4.16}$$

where $Z(x,t_{n+1}) = f(x,t_{n+1}) + f(x,t_n) + \left(\varepsilon \frac{d^2U}{dx^2} - a\frac{dU}{dx} - (b - \frac{2}{\Delta t})U\right)(x,t_n)$. Assume that the coefficients $a^{n+1}(x)$ and $b^{n+1}(x)$ are sufficiently smooth and bounded functions satisfying the following conditions $a^{n+1}(x) \geq \alpha > 0,\ b^{n+1}(x) \geq \beta > 0$. These conditions ensure that the boundary layer is located at $x = 1$.

4.1.3 Properties of the Semi-discretized Problem

The differential operator in Eq. (4.15)-(4.16) satisfies the following continuous maximum principle.

Lemma 4.1 *Let* $\Psi^{n+1}(x) \in C^2(\bar{\Omega})$ *be a smooth function satisfying* $\Psi^{n+1}(0) \geq 0,\ \Psi^{n+1}(1) \geq 0$. *Then,* $L\Psi^{n+1}(x) \geq 0$, $\forall x$ *implies that* $\Psi^{n+1}(x) \geq 0$, $\forall x$.

Proof: Let x^* be such that $\Psi^{n+1}(x^*) = \min_{x\in\bar{\Omega}} \Psi^{n+1}(x)$ and assume that $\Psi^{n+1}(x^*) < 0$. It is clear that $x^* \notin \{0,1\}$. Therefore, we have $(\Psi^{n+1})_x = 0$ and $(\Psi^{n+1})_{xx} \geq 0$. Then,

$$L\Psi^{n+1}(x^*) = -\varepsilon\Psi^{n+1}_{xx}(x^*) + a(x^*)\Psi^{n+1}_{x}(x^*) + b(x^*)\Psi^{n+1}(x^*) < 0,$$

which contradicts the assumption that $L\Psi^{n+1}(x) \geq 0,\ \forall x$. It follows that $\Psi^{n+1}(x^*) \geq 0$ and thus $\Psi^{n+1}(x) \geq 0$, $\forall x \in \bar{\Omega}$. □

Bounds for the solution of the semi-discretized problem in Eq. (4.15)-(4.16) and its derivatives are established in the following theorem. We need this bounds for the convergence analysis of the fully discrete scheme in next subsection.

Theorem 4.2 *Let* $U^{n+1}(x)$ *be the solution of Eq.* (4.15). *Then, the bounds of the solution and its derivatives satisfy*

$$\left\| \frac{\partial^i U^{n+1}(x)}{\partial x^i} \right\|_{\bar{\Omega}} \leq C\left(1 + \varepsilon^{-i} \exp(\frac{-\alpha(1-x)}{\varepsilon})\right), \quad i = 1,2,3,$$

where the constant C is independent of ε.

Proof: The proof of this theorem is found (Kadalbajoo et al., 2005) and (Kadalbajoo and Gupta, 2010). □

4.1.4 Decomposition of Solution

To obtain the stronger bounds on the solution of problem (4.1), we decompose the solution as the sum $U^{n+1}(x) = v^{n+1}(x) + w^{n+1}(x)$, where $v^{n+1}(x)$ is the solution of the regular component and $w^{n+1}(x)$ is the singular component solution. The bounds of $v^{n+1}(x)$, $w^{n+1}(x)$ and their derivatives are given by the following theorem.

Theorem 4.3 *The regular component $v^{n+1}(x)$ satisfies the bound*

$$\left\| \frac{\partial^i v^{n+1}}{\partial x^i} \right\|_{\bar{\Omega}} \leq C\big(1 + \varepsilon^{2-i}\big), \quad i = 0, 1, 2, 3,$$

and the singular component $w^{n+1}(x)$ satisfies the bound

$$\begin{aligned} \left\| w^{n+1}(x) \right\|_{\bar{\Omega}} &\leq C \exp\Big(\frac{-\alpha(1-x)}{\varepsilon}\Big), \quad \forall x \in \Omega, \\ \left\| \frac{\partial^i w^{n+1}}{\partial x^i} \right\|_{\bar{\Omega}} &\leq C\varepsilon^{-i}, \qquad i = 1, 2, 3, \end{aligned}$$

where C is a constant independent of ε and mesh points.

Proof: The detailed proof of this theorem is established (Kadalbajoo et al., 2005). □

4.1.5 Spatial Discretization

Spatial discretization was made using a hybrid method which is a combination of the upwinding difference scheme in the outer region and the central difference scheme in the inner region based on a piecewise-uniform Shishkin mesh. The construction of a piecewise uniform mesh of Shishkin type was made in such a way that more mesh points are generated in the boundary layer region than outer region. We divide the spatial interval $\bar{\Omega}_x^M = [0, 1]$ into two sub-domains $[0, 1-\sigma]$ and $[1-\sigma, 1]$ by placing a uniform mesh with $M/2$ mesh intervals in each of the sub-domains where σ is the transition point defined as $\sigma = \min\{\frac{1}{2}, \frac{2\varepsilon}{\alpha}\ln(M)\}$, with $M \geq 8$. For $\sigma = 1/2$, the mesh is uniform and for $\sigma = \frac{2\varepsilon}{\alpha}\ln(M)$, the mesh points get condensed at the right side of the domain. For error analysis, we assume that $\sigma = \frac{2\varepsilon}{\alpha}\ln(M)$. The spatial mesh points condensing at the right boundary point $x = 1$ is given by

$$x_m = \begin{cases} \frac{2(1-\sigma)m}{M}, & \text{if } m = 0(1)\frac{M}{2}, \\ 1-\sigma+\frac{2\sigma\left(m-\frac{M}{2}\right)}{M}, & \text{if } m = (\frac{M}{2}+1)(1)M, \end{cases}$$

The spatial mesh widths for $m = 0, 1, 2, \cdots, M$ can be defined as

$$h_m = x_m - x_{m-1} = \begin{cases} \tilde{h}_1 = \frac{2(1-\sigma)}{M}, & \text{if } m = 1(1)\frac{M}{2}, \\ \tilde{h}_2 = \frac{2\sigma}{M}, & \text{if } m = (\frac{M}{2}+1)(1)M, \end{cases}$$

where h_1 and h_2 are the spatial step size in $[0, 1-\sigma]$ and $[1-\sigma, 1]$, respectively. We define the first- and second-order difference operators in space for any discrete function $z(x_m, t_n) \approx z_m^n$

$$D^+ z_m^n = \frac{z_{m+1}^n - z_m^n}{h_{m+1}}, \quad D^0 z_m^n = \frac{z_{m+1}^n - z_{m-1}^n}{\hat{h}_m}, \quad D^- z_m^n = \frac{z_m^n - z_{m-1}^n}{h_m}, \ \ \delta_x^2 z_m^n = \frac{2(D_x^+ z_m^n - D_x^- z_m^n)}{\hat{h}_m}.$$

where $\hat{h}_m = h_m + h_{m+1}$ for $m = 1, 2, \cdots, M-1$. The monotone hybrid difference scheme which is a combination of the upwinding difference scheme on the coarse mesh region and the central difference scheme on the fine mesh region. We discretize Eq. (4.15) by the monotone hybrid difference scheme, where we use the upwind finite difference scheme

$$\begin{aligned} \tilde{L}_{\varepsilon,up}^{M,N} U_m^{n+1} \equiv & -\varepsilon\delta_x^2 U_m^{n+1} + a_m^{n+1} D_x^- U_m^{n+1} + (b_m^{n+1} + \frac{2}{\Delta t}) U_m^{n+1} \\ = & \varepsilon\delta_x^2 U_m^n - a_m^n D_x^- U_m^n - (b_m^n - \frac{2}{\Delta t}) U_m^n + f_m^{n+1} + f_m^n, \quad \text{for } m = 1, \cdots, \frac{M}{2}, \end{aligned} \tag{4.17}$$

on the coarse mesh region and central difference scheme

$$\begin{aligned} \tilde{L}_{\varepsilon,cen}^{M,N} U_m^{n+1} \equiv & -\varepsilon\delta_x^2 U_m^{n+1} + a_m^{n+1} D_x^0 U_m^{n+1} + (b_m^{n+1} + \frac{2}{\Delta t}) U_m^{n+1} \\ = & \varepsilon\delta_x^2 U_m^n - a_m^n D_x^0 U_m^n - (b_m^n - \frac{2}{\Delta t}) U_m^n + f_m^{n+1} + f_m^n, \quad \text{for } m = \frac{M}{2}+1, \cdots, M-1 \end{aligned} \tag{4.18}$$

in the fine mesh region. Substituting the above defined operators into Eqs. (4.17) and (4.18), we have the following on the coarse mesh region for $m = 1, \cdots, \frac{M}{2}$

$$\begin{aligned}\tilde{L}^{M,N}_{\varepsilon,up}U^{n+1}_m \equiv& \frac{-2\varepsilon}{\hat{h}_m}\left(\frac{U^{n+1}_{m+1}-U^{n+1}_m}{h_{m+1}} - \frac{U^{n+1}_m - U^{n+1}_{m-1}}{h_m}\right) + a^{n+1}_m\left(\frac{U^{n+1}_m - U^{n+1}_{m-1}}{h_m}\right) + (b^{n+1}_m + \frac{2}{\Delta t})U^{n+1}_m \\ =& \frac{2\varepsilon}{\hat{h}_m}\left(\frac{U^{n}_{m+1}-U^{n}_m}{h_{m+1}} - \frac{U^{n}_m - U^{n}_{m-1}}{h_m}\right) - a^{n}_m\left(\frac{U^{n}_m - U^{n}_{m-1}}{h_m}\right) - (b^{n}_m - \frac{2}{\Delta t})U^{n}_m + f^{n+1}_m + f^n_m.\end{aligned} \tag{4.19}$$

Again, we have the following on the fine mesh region $m = \frac{M}{2}+1, \cdots, M-1$

$$\begin{aligned}\tilde{L}^{M,N}_{\varepsilon,cen}U^{n+1}_m \equiv& \frac{-2\varepsilon}{\hat{h}_m}\left(\frac{U^{n+1}_{m+1}-U^{n+1}_m}{h_{m+1}} - \frac{U^{n+1}_m - U^{n+1}_{m-1}}{h_m}\right) + a^{n+1}_m\left(\frac{U^{n+1}_m - U^{n+1}_{m-1}}{\hat{h}_m}\right) + (b^{n+1}_m + \frac{2}{\Delta t})U^{n+1}_m \\ =& \frac{2\varepsilon}{\hat{h}_m}\left(\frac{U^{n}_{m+1}-U^{n}_m}{h_{m+1}} - \frac{U^{n}_m - U^{n}_{m-1}}{h_m}\right) - a^{n}_m\left(\frac{U^{n}_m - U^{n}_{m-1}}{\hat{h}_m}\right) - (b^{n}_m - \frac{2}{\Delta t})U^{n}_m + f^{n+1}_m + f^n_m.\end{aligned} \tag{4.20}$$

The totally discrete monotone hybrid numerical scheme now takes the following form

$$\tilde{L}^{M,N}_{\varepsilon}U^{n+1}_m = \begin{cases} U^0_m = \phi(x_m), & m = 0, 1, \cdots, M, \\ \tilde{L}^{M,N}_{\varepsilon,up}U^{n+1}_m = H^n_{up}, & \text{for } 1 \le M \le \frac{M}{2}, \\ \tilde{L}^{M,N}_{\varepsilon,cen}U^{n+1}_m = H^n_{cen}, & \text{for } \frac{M}{2} < m < M. \\ U^{n+1}_0 = 0, \quad U^{n+1}_M = 0, & 0 \le n \le M, \end{cases} \tag{4.21}$$

where the discrete operator $\tilde{L}^{M,N}_{\varepsilon}U^{n+1}_m$ is defined as

$$\begin{cases} \tilde{L}^{M,N}_{\varepsilon,up}U^{n+1}_m \equiv \begin{aligned}&\frac{-2\varepsilon}{\hat{h}_m}\left(\frac{U^{n+1}_{m+1}-U^{n+1}_m}{h_{m+1}} - \frac{U^{n+1}_m - U^{n+1}_{m-1}}{h_m}\right) + a^{n+1}_m\left(\frac{U^{n+1}_m - U^{n+1}_{m-1}}{h_m}\right) + (b^{n+1}_m + \frac{2}{\Delta t})U^{n+1}_m, \\ &\text{for } 1 \le m \le \frac{M}{2},\end{aligned} \\ \tilde{L}^{M,N}_{\varepsilon,cen}U^{n+1}_m \equiv \begin{aligned}&\frac{-2\varepsilon}{\hat{h}_m}\left(\frac{U^{n+1}_{m+1}-U^{n+1}_m}{h_{m+1}} - \frac{U^{n+1}_m - U^{n+1}_{m-1}}{h_m}\right) + a^{n+1}_m\left(\frac{U^{n+1}_m - U^{n+1}_{m-1}}{h_m}\right) + (b^{n+1}_m + \frac{2}{\Delta t})U^{n+1}_m, \\ &\text{for } \frac{M}{2} < m < M.\end{aligned} \end{cases} \tag{4.22}$$

Rearranging Eq. 4.22 in three term recurrence relation form, we obtain the following totally discrete problem

$$\begin{cases} U_m^0 = \phi(x_m), & m = 0,1,\cdots,M, \\ r_m^- U_{m-1}^{n+1} + r_m^c U_m^{n+1} + r_m^+ U_{m+1}^{n+1} = H_{up}^n, & \text{for } 1 \le m \le \frac{M}{2}, \\ s_m^- U_{m-1}^{n+1} + s_m^c U_m^{n+1} + s_m^+ U_{m+1}^{n+1} = H_{cen}^n, & \text{for } \frac{M}{2} < m < M, \\ U_0^{n+1} = 0, \quad U_M^{n+1} = 0, & 0 \le n \le M, \end{cases} \tag{4.23}$$

where the coefficients for $1 \le m \le M/2$ and $M/2+1 \le m \le M-1$ are given by, respectively

$$\begin{aligned} &r_m^- = -\left(\frac{2\varepsilon}{\hat{h}_m h_m} + \frac{a_m^{n+1}}{h_m}\right), \quad r_m^c = \frac{2\varepsilon}{h_m h_{m+1}} + \frac{a_m^{n+1}}{h_m} + b_m^{n+1} + \frac{2}{\Delta t}, \quad r_m^+ = -\left(\frac{2\varepsilon}{\hat{h}_m h_{m+1}}\right), \\ &H_{up}^n = f_m^{n+1} + f_m^n + \frac{2\varepsilon}{\hat{h}_m}\left(\frac{U_{m+1}^n - U_m^n}{h_{m+1}} - \frac{U_m^n - U_{m-1}^n}{h_m}\right) - a_m^n \frac{U_m^n - U_{m-1}^n}{h_m} - (b_m^n - \frac{2}{\Delta t})U_m^n, \end{aligned} \tag{4.24}$$

$$\begin{aligned} &s_m^- = -\left(\frac{2\varepsilon}{\hat{h}_m h_m} + \frac{a_m^{n+1}}{\hat{h}_m}\right), \quad s_m^c = \frac{2\varepsilon}{h_m h_{m+1}} + b_m^{n+1} + \frac{2}{\Delta t}, \quad s_m^+ = -\left(\frac{2\varepsilon}{\hat{h}_m h_{m+1}} - \frac{a_m^{n+1}}{\hat{h}_m}\right), \\ &H_{cen}^n = f_m^{n+1} + f_m^n + \frac{2\varepsilon}{\hat{h}_m}\left(\frac{U_{m+1}^n - U_m^n}{h_{m+1}} - \frac{U_m^n - U_{m-1}^n}{h_m}\right) - a_m^n \frac{U_{m+1}^n - U_{m-1}^n}{\hat{h}_m} - (b_m^n - \frac{2}{\Delta t})U_m^n, \end{aligned} \tag{4.25}$$

The coefficient matrix of the monotone hybrid numerical scheme in (4.23) gives an $(M-1)\times(M-1)$ linear equations which can easily be solved by Thomas Algorithm.

4.2 Convergence Analysis

In this section, we establish the stability and $\varepsilon-$uniform error estimate for the fully discrete scheme by decomposing the approximate solution U_m^n in an analogous manner as that of the continuous solution $U^n(x)$ at nth time step. For the sake of simplicity, we denote the discrete solution U_m^n by $U^{M,N}(x_m,t^n)$ during convergence analysis. In order to attain a monotone discrete operator $\tilde{L}_\varepsilon^{M,N}$, we impose the following mild assumption on the minimum number of mesh points

$$\frac{h_2\|a\|_{\bar{\Omega}}}{2\varepsilon} < 1, \quad \frac{M}{\log M} > 2\frac{\|a\|_{\bar{\Omega}}}{\alpha}. \tag{4.26}$$

The analysis is based on the discrete maximum principle and barrier function technique. We start with stating the following discrete maximum principle.

Lemma 4.4 *Under the assumption in Eq.* (4.26), *the totally discrete scheme in Eq.* (4.21) *satisfies a discrete maximum principle for any mesh function* $Z(x_m,t_n)$ *defined on* $\bar{\Omega}^{M,N}$ *such that if* $Z(x_0,t_n) \geq 0$, $Z(x_m,t_n) \geq 0$ *and* $\tilde{L}_\varepsilon^{M,N} Z(x_m,t_n) \geq 0,\ \forall (x_m,t_n) \in \Omega$, *then* $Z(x_m,t_n) \geq 0$, $(x_m,t_n) \in \bar{\Omega}$.

Proof: Under the assumption in Eq. (4.26), we establish the discrete maximum principle by simply check the following inequalities to show that the associated system matrix is an M-matrix

$$\begin{aligned}
&r_m^- < 0,\ \ r_m^+ < 0,\ \ r_m^- + r_m^c + r_m^+ > 0, \qquad m = 1,2,\cdots,M/2,\\
&s_m^- < 0,\ \ s_m^+ < 0,\ \ s_m^- + s_m^c + s_m^+ > 0, \qquad m = M/2+1,\cdots,M-1.
\end{aligned}$$

From these sign patterns, it is easily seen that the system matrix $\tilde{L}_\varepsilon^{M,N}$ is an irreducible M-matrix and so has a positive inverse. Moreover, discrete system Eq. (4.21) satisfies the desired discrete maximum principle. □

To prove uniform convergence of the proposed scheme, first we construct the following barrier functions for all $n\Delta t \leq T$.

$$\psi_m^n(\alpha) = \begin{cases} \prod_{j=1}^{m} \left(1 + \frac{\alpha h_j}{\varepsilon}\right)^{-1}, & m = 1,\cdots,M, \\ 1, & m = 0. \end{cases} \tag{4.27}$$

Lemma 4.5 *The barrier functions* $\psi_m^n(\alpha)$ *satisfy the following inequalities*

$$\tilde{L}_\varepsilon^{M,N} \psi_m^n(\alpha) \geq \frac{C(\alpha)\Delta t}{\varepsilon + \alpha h_m} \psi_m^n(\alpha), \ \ \textit{for}\ \ m = 1,2,\cdots,M-1, \ \ n\Delta t \leq T.$$

for some positive constant $C(\alpha)$.

Proof: From the definition of barrier functions, we have

$\frac{\psi_m^n(\alpha)-\psi_{m-1}^n(\alpha)}{h_m} = \frac{\alpha\psi_{m-1}^n(\alpha)}{\varepsilon}$, $\frac{\psi_{m+1}^n(\alpha)-\psi_{m-1}^n(\alpha)}{h_m+h_{m+1}} = \frac{\alpha}{\varepsilon}\left(\frac{h_{m+1}\psi_m^n(\alpha)+h_m\psi_{m-1}^n(\alpha)}{h_m+h_{m+1}}\right)$. Now, applying the operator $\tilde{L}_\varepsilon^{M,N}$ on the barrier function $\psi_m^n(\alpha)$ and and doing some simplifications, we obtain

$$\tilde{L}_\varepsilon^{M,N}\psi_m^n(\alpha) = \begin{cases} \left[1+\Delta t\left(\frac{\alpha}{\varepsilon+\alpha h_m}\left(-\frac{2\alpha h_m}{h_m+h_{m+1}}+a_m+b_m\frac{\varepsilon+\alpha h_m}{\alpha}\right)\right)\right]\psi_m^n(\alpha), \;\; m=1,2,\cdots,M/2, \\ \left[1+\Delta t\left(\frac{\alpha}{\varepsilon+\alpha h_m}\left(-\frac{2\alpha h_m}{h_m+h_{m+1}}+a_m+b_m\frac{\varepsilon+\alpha h_m}{\alpha}\right)\right)\right]\psi_m^n(\alpha) \\ +b_m\left(\frac{\alpha^2 h_m h_{m+1}}{\varepsilon^2(h_m+h_{m+1})}\right), \;\; i=M/2+1,\cdots,M-1, \end{cases}$$

from which we deduce the desired result. □

Furthermore, we give the following truncation error bounds for upwind and central difference operators employed in $\tilde{L}_\varepsilon^{M,N}$.

Lemma 4.6 *At the time level t^n, for $u^n(x) \in C^4(\bar{\Omega}_x)$, the local truncation error at spatial discretization stage for the operator $\tilde{L}_\varepsilon^{M,N}$ is given by*

$$\begin{aligned} |\tau_m^{up}| &= |\tilde{L}_\varepsilon^{M,N}(u_m^n) - \tilde{L}_\varepsilon^{M,N}u(x_m,t_n)|, \\ &\leq C\Delta t\left[\varepsilon\int_{x_{m-1}}^{x_{m+1}}\left|\frac{\partial^3 u^n}{\partial s^3}\right|ds + \int_{x_{m-1}}^{x_m}\left|\frac{\partial^2 u^n}{\partial s^2}\right|ds\right], \quad m=1,2,\cdots,M/2, \\ |\tau_m^{cen}| &= |\tilde{L}_\varepsilon^{M,N}(u_m^n) - \tilde{L}_\varepsilon^{M,N}u(x_m,t_n)|, \\ &\leq C\Delta t h_m\left[\varepsilon\int_{x_{m-1}}^{x_{m+1}}\left|\frac{\partial^4 u^n}{\partial s^4}\right|ds + \int_{x_{m-1}}^{x_{m+1}}\left|\frac{\partial^3 u^n}{\partial s^3}\right|ds\right], \quad m=M/2+1,\cdots,M-1, \end{aligned}$$

where C is a positive constant depends on $\|a\|$ and $\|a'\|$.

Proof: By using the valid Taylor series expansion with the integral form of the remainder, or Peano's theorem by Davis (1963), we obtain the desired truncation error estimates. □

To prove the uniform convergence of the proposed scheme, we use the following estimate.

Lemma 4.7 *For each m and $\alpha > 0$, we have*

$$\prod_{j=m+1}^{M}\left(1+\frac{\alpha h_j}{\varepsilon}\right)^{-1} \geq \exp\left(-\alpha\frac{(1-x_m)}{\varepsilon}\right).$$

Lemma 4.8 *For the Shishkin mesh defined above, there exists a constant C, such that*

$$\prod_{j=m+1}^{M}\big(1+\frac{\alpha h_j}{\varepsilon}\big)^{-1} \le \begin{cases} CN^{-2}, & \forall m = 0,1,\cdots,M/2. \\ CN^{-4(1-m/N)}, & \forall m = M/2+1,\cdots,M-1. \end{cases}$$

Proof: For $m = 0,1,\cdots,M/2$, we have

$$\begin{aligned}
\prod_{j=m+1}^{M}\big(1+\frac{\alpha h_j}{\varepsilon}\big)^{-1} &\le \prod_{j=M/2+1}^{M}\big(1+\frac{\alpha h_j}{\varepsilon}\big)^{-1}, \\
&\le \exp\Big(-\alpha(1-x_{M/2})/(\varepsilon+\alpha\tilde{h}_2)\Big), \quad \text{(Lemma 4.1(b) by Kellogg and Tsan (1978)}, \\
&= \exp\big(-\alpha\sigma/(\varepsilon+2\alpha\sigma M^{-1})\big), \\
&= \exp\big(-2\log M/(1+4M^{-1}\log M)\big), \\
&= M^{-2/(1+4M^{-1}\log M)} \\
&\le CM^{-2}.
\end{aligned}$$

The required bound for $m = M/2+1,\cdots,M-1$ also follows using the same argument given in Lemma 4.1(b) (Kellogg and Tsan, 1978). □

To estimate the error in the regular component and singular component separately in the spatial direction at the *nth* time step, we decompose the discrete solution $U^{M,N}(x_m,t_n)$ into a regular and singular part as

$$U^{M,N}(x_m,t_n) = V^{M,N}(x_m,t_n) + W^{M,N}(x_m,t_n), \quad \forall x_m \in \bar{\Omega}_x^M,\ n\Delta t \le T,$$

where the regular component $V^{M,N}(x_m,t_n)$ satisfies the non-homogeneous equation

$$\begin{aligned}
&\tilde{L}_\varepsilon^{M,N} V^{M,N}(x_m,t_n) = H(x_m,t_n), \quad \forall x_m \in \Omega_x^M,\ n\Delta t \le T, \\
&V^{M,N}(x_m,t_n) = v(x_m,t_n),
\end{aligned}$$

and the singular part $W^{M,N}(x_m,t_n)$ is the solution of the problem

$$\tilde{L}_{\varepsilon}^{M,N} W^{M,N}(x_m,t_n) = 0, \quad \forall x_m \in \Omega_x^M, \; n\Delta t \le T,$$
$$W^{M,N}(x_m,t_n) = w(x_m,t_n).$$

Therefore, we have

$$(U^{M,N}-u)(x_m,t_n) = (V^{M,N}-v)(x_m,t_n) + (W^{M,N}-w)(x_m,t_n), \; \forall x_m \in \Omega_x^M, \; n\Delta t \le T.$$

Now, we estimate the error bound in the regular and singular components separately.

Theorem 4.9 *Under the assumption* (4.26)*, the error in the regular component* $V^{M,N}(x_m,t_n)$ *satisfies the following error estimate at the nth time level*

$$|(V^{M,N}-v)(x_m,t_n)| \le \begin{cases} CM^{-1}, & m = 0,1,\cdots,M/2, \quad n\Delta t \le T, \\ CM^{-2}, & m = M/2+1,\cdots,M, \quad n\Delta t \le T. \end{cases}$$

Proof: From Lemma (4.6), the truncation error in the regular component is given by

$$|\tilde{L}_{\varepsilon}^{M,N}(V^{M,N}-v)(x_m,t_n)| \le \begin{cases} CN^{-2}\left[\varepsilon(h_m+h_{m+1})\|\frac{\partial^3 v}{\partial x^3}\|_{\bar{\Omega}_x} + h_m\|\frac{\partial^2 v}{\partial x^2}\|_{\bar{\Omega}_x}\right], \\ m = 1,2,\cdots,M/2, \quad n\Delta t \le T, \\ CN^{-2}\left[h_m(h_m+h_{m+1})\left(\varepsilon\|\frac{\partial^4 v}{\partial x^4}\|_{\bar{\Omega}_x} + \|\frac{\partial^3 v}{\partial x^3}\|_{\bar{\Omega}_x}\right)\right], \\ m = M/2+1,\cdots,M-1, \quad n\Delta t \le T, \end{cases}$$

Note that $h_{m+1}+h_m \le 2M^{-1}$ is always true for both the cases of uniform mesh and piecewise uniform Shishkin mesh. Moreover, using the bounds on the derivatives of v given in Theorem

(4.3), we get

$$|\tilde{L}_\varepsilon^{M,N}(V^{M,N}-v)(x_m,t_n)| \leq \begin{cases} CN^{-2}M^{-1}(\varepsilon+1), & m=1,2,\cdots,M/2, \quad n\Delta t \leq T, \\ CN^{-2}M^{-2}, & m=M/2+1,\cdots,M-1, \quad n\Delta t \leq T. \end{cases}$$

An application of Lemma (4.4) yield the desired bounds. □

Theorem 4.10 *Under the assumption* (4.26), *the error in the singular component* $W^{M,N}(x_m,t_n)$ *satisfies the following error estimate at the nth time level*

$$|(W^{M,N}-w)(x_m,t_n)| \leq \begin{cases} CM^{-2}, & m=0,1,\cdots,M/2, \quad n\Delta t \leq T, \\ CM^{-2}\ln^2 M, & i=M/2+1,\cdots,M, \quad n\Delta t \leq T. \end{cases}$$

Proof: In the sub-interval $[0,1-\sigma]$ with no boundary layer, both $W^{M,N}$ and w are small. After applying triangle inequality, it it is sufficient to find the bounds on w and $W^{M,N}$ separately. Here, we note that

$$\tilde{L}_\varepsilon^{M,N} W^{M,N}(x_m,t_n) = 0, \quad \forall x_m \in \Omega_x^M, \ n\Delta t \leq T,$$
$$|W^{M,N}(x_0,t_n)| = |w(x_0,t_n)| \leq C\exp(-\alpha/\varepsilon) \leq C\prod_{j=1}^{M}\left(1+\frac{\alpha h_j}{\varepsilon}\right)^{-1}, \ n\Delta t \leq T,$$

and

$$|W^{M,N}(x_M,t_n)| = |w(1,t_n)| \leq C, \ n\Delta t \leq T.$$

Furthermore, to obtain the bound on $W^{M,N}$, we consider the following mesh function $\phi_m^n(\alpha)$ for sufficiently large C and $n\Delta t \leq T$,

$$\phi_m^n(\alpha) = C\left[\prod_{j=1}^{M}\left(1+\frac{\alpha h_j}{\varepsilon}\right)^{-1}\right]\psi_m^n(\alpha).$$

Now, using Lemma (4.5), we have

$$\tilde{L}_{\varepsilon}^{M,N}\phi_m^n(\alpha) \geq \frac{C\Delta t}{\varepsilon+\alpha h_i}\prod_{j=m+1}^{M}\left(1+\frac{\alpha h_j}{\varepsilon}\right)^{-1} \geq 0,\ \ m=1,\cdots,M-1,\ \ n\Delta t \leq T,$$

and

$$\phi_0^n(\alpha) = C\prod_{j=1}^{M}\left(1+\frac{\alpha h_j}{\varepsilon}\right)^{-1}, \quad \text{since } \ \psi_0^n(\alpha)=1,$$
$$\phi_M^n(\alpha) = C, \quad n\Delta t \leq T.$$

Therefore, $\phi_m^n(\alpha)$ is a barrier function for $\{W^{M,N}(x_m,t_n)\}$. Discrete maximum principle gives

$$|W^{M,N}(x_m,t_n)| \leq \phi_m^n(\alpha) = C\prod_{j=m+1}^{M}\left(1+\frac{\alpha h_j}{\varepsilon}\right)^{-1},\ \ m=0,\cdots,M,\ \ n\Delta t \leq T. \tag{4.28}$$

Using triangular inequality, we have

$$\begin{aligned}|(W^{M,N}-w)(x_m,t_n)| &\leq |W^{M,N}(x_m,t_n)|+|w(x_m,t_n)|,\\ &\leq C\prod_{j=m+1}^{M}\left(1+\frac{\alpha h_j}{\varepsilon}\right)^{-1}+C\exp\left(-\frac{\alpha(1-x_m)}{\varepsilon}\right),\\ &\leq C\prod_{j=m+1}^{M}\left(1+\frac{\alpha h_j}{\varepsilon}\right)^{-1},\ \ \text{(using Lemma (4.7))}.\end{aligned}$$

In particular, using Lemma (4.8), we get

$$|(W^{M,N}-w)(x_m,t_n)| \leq CM^{-2},\ \ m=0,\cdots,\frac{M}{2},\ \ n\Delta t \leq T. \tag{4.29}$$

On the other hand, we use barrier function technique to estimate $|(W^{M,N}-w)(x_m,t_n)|$ in the fine mesh region $[1-\sigma,1]$. Moreover, truncation error estimate in Lemma (4.6) for $m=M/2+$

$1,\cdots,M-1$ leads to the following estimate

$$\begin{aligned}
|\tilde{L}_{\varepsilon}^{M,N}(W^{M,N}-w)(x_m,t_n) &\le CN^{-2}\tilde{h}_2\left[\varepsilon\int_{x_{m-1}}^{x_{m+1}}\left\|\frac{\partial^4 w}{\partial x^4}\right\|_{\bar{\Omega}_x}dx+\int_{x_{m-1}}^{x_{m+1}}\left\|\frac{\partial^3 w}{\partial x^3}\right\|_{\bar{\Omega}_x}dx\right],\\
&\le CN^{-2}\tilde{h}_2\left[\frac{1}{\varepsilon^3}\int_{x_{m-1}}^{x_{m+1}}\exp\big(-\frac{\alpha(1-x)}{\varepsilon}\big)dx\right],\\
&=\frac{CN^{-2}\tilde{h}_2}{\varepsilon^2\alpha}\left[\exp\left(-\frac{\alpha(1-x_{m+1})}{\varepsilon}\right)-\exp\left(-\frac{\alpha(1-x_{m-1})}{\varepsilon}\right)\right],\\
&=\frac{CN^{-2}\tilde{h}_2}{\varepsilon^2\alpha}\exp\left(-\frac{\alpha(1-x_m)}{\varepsilon}\right)\sinh\left(\frac{\alpha\tilde{h}_2}{\varepsilon}\right),
\end{aligned}$$

Since $\sinh(\xi)\le C\xi$, for $0\le\xi\le 1$, the truncation error estimate reduces to

$$\begin{aligned}
|\tilde{L}_{\varepsilon}^{M,N}(W^{M,N}-w)(x_m,t_n) &\le \frac{CN^{-2}}{\varepsilon}M^{-2}\ln^2 M\exp\left(-\frac{\alpha(1-x_m)}{\varepsilon}\right),\\
&\le \frac{CN^{-2}}{\varepsilon}M^{-2}\ln^2 M\prod_{j=m+1}^{M}\left(1+\frac{\alpha h_j}{\varepsilon}\right)^{-1},\ \text{by Lemma (4.7).}
\end{aligned}\tag{4.30}$$

It is easy to see that for $m=M/2$ in Eq. (4.30), we have

$$|(W^{M,N}-w)(x_{M/2},t_n)|\le CM^{-2}.$$

Furthermore, we have

$$|(W^{M,N}-w)(x_M,t_n)|=0.$$

Using Eq. (4.30), we construct the mesh function

$$\phi_m^n(\alpha)=CM^{-2}\left(1+\ln^2 M\prod_{j=m+1}^{M}\left(1+\frac{\alpha h_j}{\varepsilon}\right)^{-1}\psi_m^n(\alpha)\right),\quad i=M/2,\cdots,M,$$

for sufficiently large value of C. With the help of Lemma (4.5), it is easy to see that

$$\begin{aligned}
&|\tilde{L}_{\varepsilon}^{M,N}\phi_i^n(\alpha)|\ge|\tilde{L}_{\varepsilon}^{M,N}(W^{M,N}-w)(x_m,t_n)|,\ m=M/2+1,\cdots,M-1,\quad n\Delta t\le T,\\
&\phi_{M/2}^n(\alpha)=CM^{-2}\left[1+\ln^2 M\prod_{j=M/2+1}^{M}\left(1+\frac{\alpha h_j}{\varepsilon}\right)^{-1}\right]\ge|(W^{M,N}-w)(x_{M/2},t_n)|,
\end{aligned}$$

and

$$|\phi_M^n(\alpha)| = CM^{-2}\big(1+\ln^2 M\big) \geq |(W^{M,N}-w)(x_M,t_n)|.$$

Therefore, $\phi_m^n(\alpha)$ is a barrier function for $(W^{M,N}-w)(x_m,t_n)$ and consequently by the discrete maximum principle, we have

$$|(W^{M,N}-w)(x_M,t_n)| \leq \phi_m^n(\alpha), \quad m = M/2,\cdots,M, \quad n\Delta t \leq T.$$

Now, Lemma (4.8) gives the estimate

$$|(W^{M,N}-w)(x_m,t_n)| \leq C\max\{M^{-2}, M^{-6+4m/M}\ln^2 M\}, \quad m = M/2,\cdots,M, \quad n\Delta t \leq T. \tag{4.31}$$

Hence, combining the results from the Eq. (4.30) and Eq. (4.31) proves the theorem. □

Theorem 4.11 *Let $U^{M,N}(x_m,t_n)$ be the hybrid finite difference approximation in the spatial direction to the solution $u^n(x) \in C^4(\bar{\Omega}_x)$ of the semi-discrete problem in Eq.* (4.15) *at n^{th} time level. Then, under the assumption in Eq.* (4.26) *following error estimates hold for the proposed scheme in the spatial discretization process at the n^{th} time level*

$$\|(U^{M,N}-u)(x_m,t_n)\|_{\bar{\Omega}_x} \leq \begin{cases} CM^{-1}, & m = 0,1,\cdots,M/2, \quad n\Delta t \leq T, \\ CM^{-2}\ln^2 M, & m = M/2+1,\cdots,M, \quad n\Delta t \leq T. \end{cases}$$

Proof: The proof follows from Theorem (4.9) and Theorem (4.10). □

The above discussions leads us to the following main convergence theorem.

Theorem 4.12 *Let $u(x,t)$ be the continuous solution of problem in Eq.* (4.1)*, $U^n(x)$ be the solution of the semi-discrete problem in Eq.* (4.15) *after the quasilinearization process and the time discretization and $U^{M,N}(x_m,t_n)$ be the solution of the totally discrete problem in Eq.* (4.21)*, then under* *the assumption in Eq.* (4.26) *following error estimates satisfies for the totally discrete scheme*

$$\|(U^{M,N}-u)(x_m,t_n)\|_{\bar{\Omega}^{M,N}} \leq \begin{cases} C\big(M^{-1}+\Delta t^2\big), & m=0,1,\cdots,M/2, \quad n\Delta t \leq T, \\ C\big(M^{-2}\ln^2 M+\Delta t^2\big), & m=M/2+1,\cdots,M, \quad n\Delta t \leq T, \end{cases}$$

where C is a constant independent of ε *and the mesh parameters.*

Proof: The proof follows by combining the estimates given in Eq. (4.11) and Theorem (4.11). □

4.3 Numerical Results

In this section, we do numerical experiments to support the theoretical results.

Example 4.1 *Consider the following singularly perturbed Burger's equation*

$$\begin{cases} \frac{\partial u}{\partial t}-\varepsilon\frac{\partial^2 u}{\partial x^2}+u\frac{\partial u}{\partial x}=0, & (x,t)\in(0,1)\times(0,1], \\ u(x,0)=\sin(\pi x), & x\in[0,1], \\ u(0,t)=0=u(1,t), & t\in[0,1]. \end{cases}$$

Example 4.2 *Consider the following singularly perturbed Burger's equation*

$$\begin{cases} \frac{\partial u}{\partial t}-\varepsilon\frac{\partial^2 u}{\partial x^2}+u\frac{\partial u}{\partial x}=0, & (x,t)\in(0,1)\times(0,1], \\ u(x,0)=x(1-x^2), & x\in[0,1], \\ u(0,t)=0=u(1,t)=0, & t\in[0,1]. \end{cases}$$

Since the exact solutions for the test examples are not available, we use the double mesh principle to estimate the maximum errors. For each ε, the maximum errors and rate of convergence are given

$$e_{\varepsilon}^{M,\Delta t}=\max_{0\leq m\leq M;\, n\in[0,T]}\left|U^{M,\Delta t}(x_m,t_n)-U^{2M,\Delta t/2}(x_m,t_n)\right|, \qquad r_{\varepsilon}^{M,\Delta t}=\frac{\log e_{\varepsilon}^{M,\Delta t}-e_{\varepsilon}^{2M,\Delta t/2}}{\log 2}.$$

where $U^{M,\Delta t}$ denote the numerical solution obtained using the mesh points (x_m, t_n) whereas $U^{2M,\Delta t/2}$ denote the numerical solution obtained when the space mesh interval is multiplied by 2 and the time mesh interval is divided by 2.

Table 4.1: Computed maximum absolute errors for Example (4.1).

$\varepsilon \downarrow$	$M = 32,\ \Delta t = \frac{1}{10}$	$64,\ \frac{1}{20}$	$128,\ \frac{1}{40}$	$256,\ \frac{1}{80}$	$512,\ \frac{1}{160}$
10^{-2}	1.1940e-02	6.2077e-03	3.2758e-03	1.6864e-03	8.5065e-04
10^{-4}	1.4536e-02	7.9277e-03	4.1608e-03	2.1526e-03	1.0981e-03
10^{-6}	1.4601e-02	7.9617e-03	4.1773e-03	2.1610e-03	1.1024e-03
10^{-8}	1.4602e-02	7.9620e-03	4.1775e-03	2.1611e-03	1.1025e-03
10^{-10}	1.4602e-02	7.9620e-03	4.1775e-03	2.1611e-03	1.1025e-03
10^{-12}	1.4602e-02	7.9620e-03	4.1775e-03	2.1611e-03	1.1025e-03

Table 4.2: Comparison of maximum absolute errors and rate of convergence using the proposed method and the method in the literature for Example (4.1).

$\varepsilon \downarrow$	$M = 64,\ \Delta t = \frac{1}{20}$	$128,\ \frac{1}{40}$	$256,\ \frac{1}{80}$	$512,\ \frac{1}{160}$
Our Result				
10^{-2}	6.2077e-03	3.2758e-03	1.6864e-03	8.5065e-04
	0.9222	0.9579	0.9873	-
10^{-4}	7.9277e-03	4.1608e-03	2.1526e-03	1.0981e-03
	0.9300	0.9508	0.9711	-
10^{-6}	7.9617e-03	4.1773e-03	2.1610e-03	1.1024e-03
	0.9305	0.9509	0.9711	-
Gowrisankar and Natesan (2019)				
10^{-2}	1.1303e-1	6.1846e-2	2.7592e-2	1.4072e-2
	0.8700	1.1644	0.9714	-
10^{-4}	2.5946e-1	1.4418e-1	6.8426e-2	3.2474e-2
	0.8477	1.0752	1.0753	-
10^{-6}	2.7194e-1	1.5667e-1	6.9863e-2	3.3902e-2
	0.7955	1.0432	1.0432	-

The calculated maximum absolute errors for Example 4.1 are given in Table 4.1. As demonstrated in Table 4.2, comparison of maximum errors and rate of convergences are given for Example 4.1 with the method in the literature showing that the present method gives accurate results than the existing method. Numerical solution using surface plot for Example 4.1 is depicted in Figure 4.1. Numerical solution in terms of line graph is plotted in Figure 4.2 by showing the effect of perturbation parameter ε. The corresponding log-log plot of the maximum absolute errors are sketched in Figure 4.3 which also reveals the uniform convergence of first-order. Numerical results in Table 4.3 confirms that the present method gives a better numerical results than some existing method in the literature for Example 4.2. Solution profile for Example 4.2 is plotted in Figure 4.4. Numerical solution in terms of line graph for Example 4.2 is sketched in Figure 4.5 and the corresponding maximum point-wise errors using log-log scale is plotted in Figure 4.6. From all the graphs plotted for Example 4.1 in Figures 4.1 and 4.2 and Example 4.2 in Figures 4.4 and 4.5, we conclude that the problem 4.1 has parabolic boundary layer near the boundary $x = 1$.

Table 4.3: Comparison of computed maximum absolute errors for Example (4.2).

$\varepsilon \downarrow$	$M = 32,\ \Delta t = \frac{1}{20}$	$64,\ \frac{1}{40}$	$128,\ \frac{1}{80}$	$256,\ \frac{1}{160}$
Our Result				
2^{-4}	9.8309e-04	5.4418e-04	2.6362e-04	1.4044e-04
2^{-8}	2.3372e-03	1.1961e-03	7.9129e-04	9.4747e-04
2^{-10}	2.4469e-03	1.2578e-03	6.4145e-04	3.2368e-04
2^{-12}	2.4767e-03	1.2753e-03	6.5173e-04	3.2927e-04
2^{-14}	2.4853e-03	1.2806e-03	6.5468e-04	3.3090e-04
2^{-16}	2.4878e-03	1.2820e-03	6.5550e-04	3.3135e-04
2^{-18}	2.4884e-03	1.2824e-03	6.5572e-04	3.3147e-04
Liu et al. (2020)				
2^{-4}	2.2788e-02	1.3365e-02	7.2769e-03	3.8441e-03
2^{-8}	4.4450e-02	2.0109e-02	1.0519e-02	5.9653e-02
2^{-10}	8.3339e-02	6.6120e-02	4.0769e-02	1.9284e-02
2^{-12}	1.8762e-01	8.4106e-02	5.7234e-02	3.8309e-02
2^{-14}	1.9755e-01	1.5347e-01	8.3864e-02	4.6945e-02
2^{-16}	2.1340e-01	1.5016e-01	1.1582e-01	6.1958e-02
2^{-18}	2.8299e-01	1.7036e-01	1.1805e-01	7.4251e-02

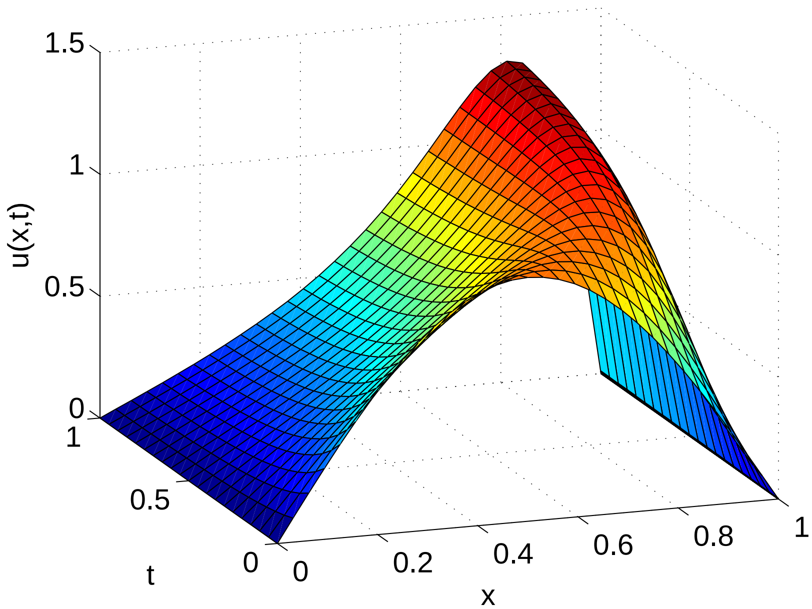

Figure 4.1: Numerical solution using surface plot for Example (4.1) at $M = 64,\ N = 20,\ \varepsilon = 10^{-4}$.

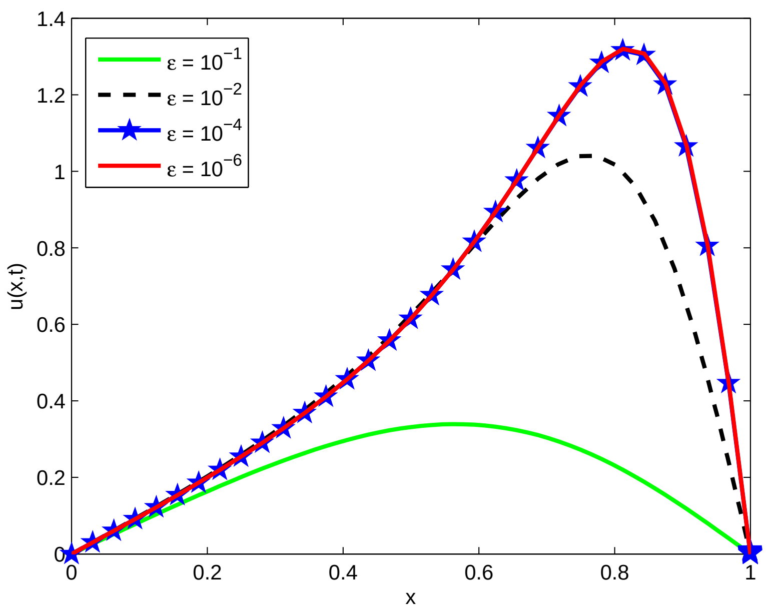

Figure 4.2: Numerical solution in terms of line graph for Example (4.1) at $M = 64,\ N = 20$ and various values of ε.

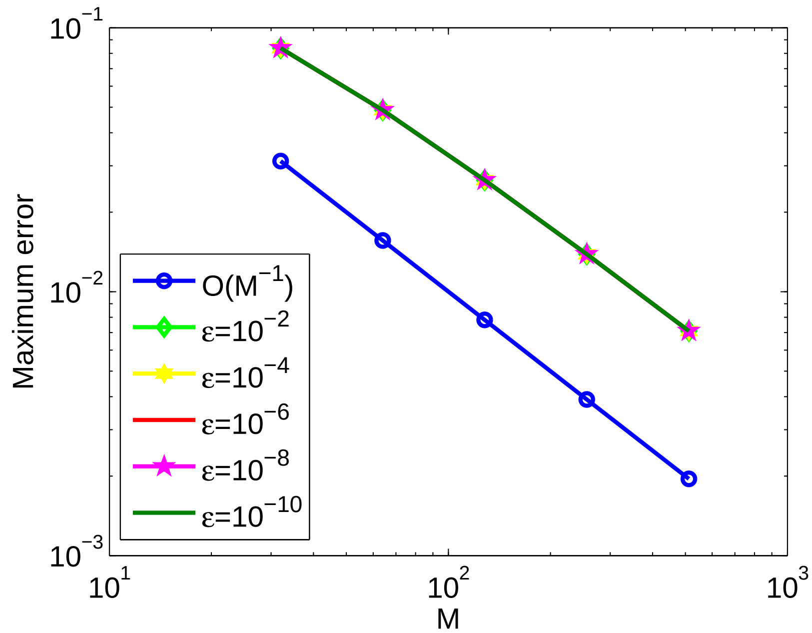

Figure 4.3: Loglog plot of maximum point-wise errors for Example (4.1) using Table (4.1).

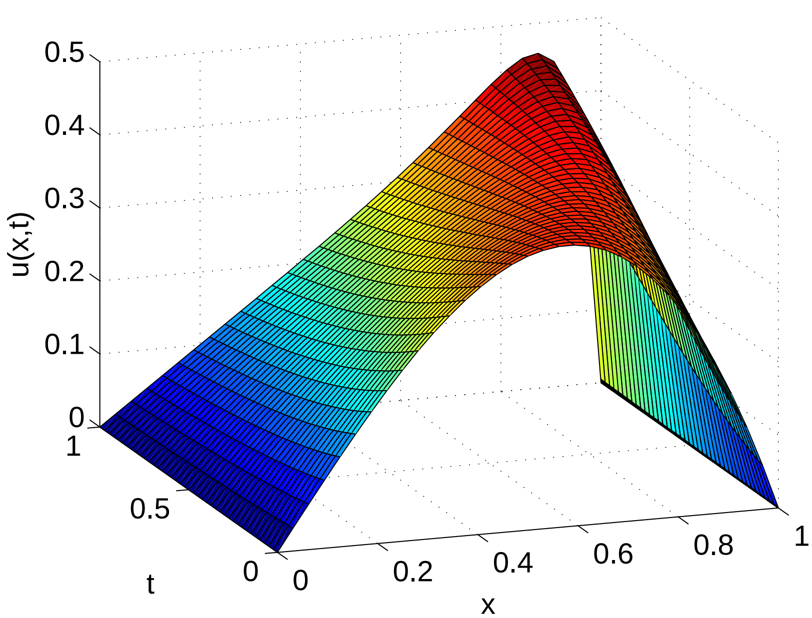

Figure 4.4: Numerical solution using surface plot for Example (4.2) at $M = 64$, $N = 40$, $\varepsilon = 2^{-14}$.

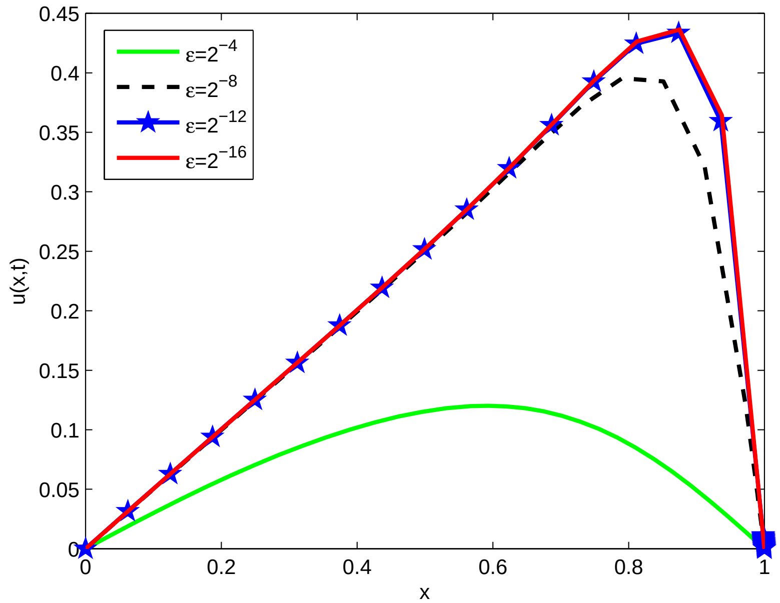

Figure 4.5: Numerical solution in terms of line graph for Example (4.2) at $M = 32$, $N = 20$ and various values of ε.

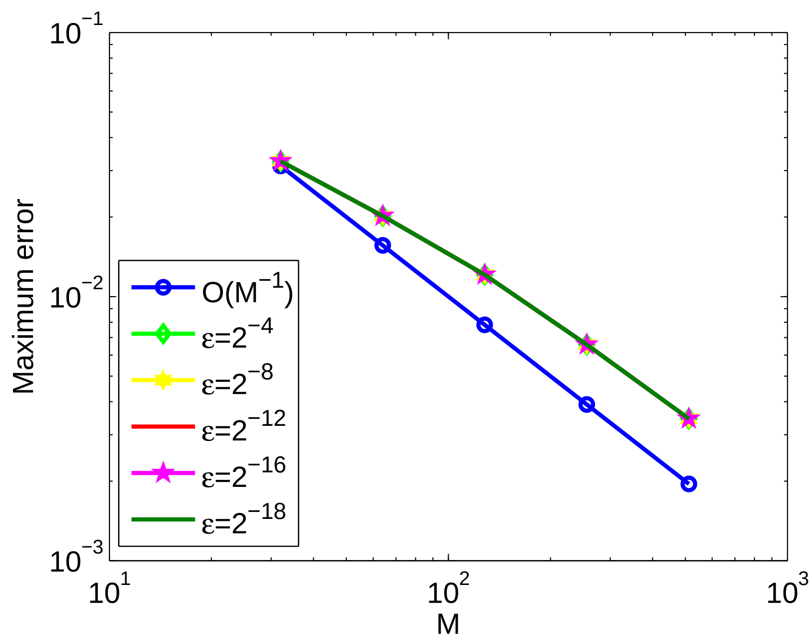

Figure 4.6: Loglog plot of maximum point-wise errors for Example (4.2) using Table (4.3).

Table 4.4: Comparison of rate of convergence for Example (4.2).

$\varepsilon \downarrow$	$M = 32,\ \Delta t = \frac{1}{20}$	$64,\ \frac{1}{40}$	$128,\ \frac{1}{80}$
2^{-4}	0.8532	1.0456	0.9085
2^{-8}	0.9664	0.5961	0.9013
2^{-10}	0.9601	0.9715	0.9868
2^{-12}	0.9576	0.9685	0.9850
2^{-14}	0.9566	0.9680	0.9844
2^{-16}	0.9565	0.9677	0.9842
2^{-18}	0.9564	0.9677	0.9842
Liu et al. (2020)			
2^{-4}	0.7698	0.8771	0.9207
2^{-8}	1.1443	0.9348	0.8183
2^{-10}	0.3339	0.6976	1.0801
2^{-12}	1.1575	0.5553	0.5792
2^{-14}	0.3643	0.8718	0.8371
2^{-16}	0.5076	0.3746	0.9025
2^{-18}	0.7322	0.5292	0.6689

Chapter 5

Conclusion and Future Scope

5.1 Conclusion

In this thesis, a finite difference scheme based on a monotone hybrid finite difference scheme on a piecewise uniform mesh of Shishkin type on spatial direction and an implicit second-order finite difference method on time direction is constructed to solve singularly perturbed Burgers equation. The qualitative aspects of the singularly perturbed Burgers equation have been established. One of the major difficulties occur corresponding to the non-linearity of the Burgers equation, which we resolved by using the layer-adapted mesh of Shishkin type. A brief analysis has been carried out to prove the uniform convergence of the proposed scheme. We show the second-order uniform convergence in the temporal variable and first-order uniform convergence in the coarse region and almost second-order convergence in the boundary layer region for the spatial variable. The numerical results presented in error tables indicate that the proposed scheme is convergent for sufficiently small value of the singular perturbation parameter ε. We observe from the error tables that for a fixed value of ε, point wise errors and maximal nodal errors decrease as the number of mesh points increases. Thus, the present method works nicely for very small values of the singular perturbation parameter $\varepsilon \in (0,1]$ and the numerical results support the theoretical predictions and exhibit good physical behavior.

5.2 Future Scope

The technique presented in this thesis may also be applicable to the construction and study of $\varepsilon-$uniform numerical methods for more complicated non-linear problems.

References

Abdulkadir, D. (2004). A Galerkin finite element approach to Burgers' equation. *Applied Mathematics and Computation, 157,* 331-346.

Aksan, E.N. (2006). Quadratic B-spline finite element method for numerical solution of the Burgers' equation. *Applied Mathematics and Computation, 174,* 884-896.

Arora, G., Singh, B.K. (2013). Numerical solution of Burgers' equation with modified cubic B-spline differential quadrature method. *Applied Mathematics and Computation, 224,* 166-177.

Asai, A. (2010). Numerical solution of the Burgers' equation by automatic differentiation. *Applied Mathematics and Computation, 216*, 2700-2708.

Bateman, H. (1915). Some recent researches on the motion of fluids. *Monthly Weather Rev., 43*, 163–170.

Bellman, R.E., Kalaba. R.E. (1965). *Quasilinearization and nonlinear boundary-value problems.* American Elsevier Publishing Company Inc., New York, 1965.

Burger, J.M. (1939). Mathematical examples illustrating relation occurring in the theory of turbulent fluid motion. *Trans. Roy. Neth. Acad. Sci, Amsterdam*, 17, 1–53.

Burger, J.M. (1948). A mathematical model illustrating the theory of turbulence. *Adv. App. Mech. I*, Academic Press, New York, 171-199.

Dag, I., Irik, D., Saka, B. (2005). A numerical solution of the Burgers' equation using cubic B-splines. *Applied Mathematics and Computation, 163*, 199-211.

Davis, P.J. (1963). *Interpolation and approximations, Blaisdell, Waltham, Mass*.

Friedrichs, K.O., Wasow, W.R. (1946). Singular perturbations of nonlinear oscillations. *Duke Mathematical Journal, 13,* 367-381.

Gowrisankar, S., Natesan, S. (2019). An efficient robust numerical method for singularly perturbed Burgers' equation. *Applied Mathematics and Computation, 346,* 385-394.

Gupta, V., Kadalbajoo, M.K. (2016). Qualitative analysis and numerical solution of Burgers' equation via B-spline collocation with implicit Euler method on piecewise uniform mesh. *Journal of Numerical Mathematics, 24*(2), 73-94.

Hassanien, I.A., Salama, A.A., Hosham, H.A. (2005). Fourth-order finite difference method for solving Burgers' equation. *Applied Mathematics and Computation, 170*, 781-800.

Jiwari, R. (2012). A Haar wavelet quasilinearization approach for numerical simulation of Burgers' equation. *Computational Physics Communication, 183,* 2413-2423.

Jiwari, R., Mittal, R.C., Sharma, K.K. (2013). A numerical scheme based on weighted average differential quadrature method for the numerical solution of Burgers' equation. *Applied Mathematics and Computation, 219,* 6680-6691.

Jiwari, A. (2014). Hybrid numerical scheme for the numerical solution of the Burgers' equation. *Computer Physics Communications,* 1-22.

Kadalbajoo, M.K., Awasthi, A. (2006). A numerical method based on Crank-Nicolson scheme for Burgers' equation. *Applied Mathematics and Computation, 182,* 1430-1442.

Kadalbajoo, M.K., Awasthi, A. (2017). Parameter free hybrid numerical method for solving modified Burgers' equations on a nonuniform mesh. *Asian-European Journal of Mathematics, 10*(2), 1-11.

Gupta, V., Kadalbajoo, M.K. (2010). Numerical approximation of modified Burgers' equation via hybrid finite difference scheme on layer-adaptive mesh. *Neural, Parallel, and Scientific Computations, 18*, 167-194.

Kadalbajoo, M.K., Sharma, K.K., Awasthi, A. (2005). A parameter-uniform implicit difference scheme for solving time-dependent Burgers' equations. *Applied Mathematics and Computation, 170,* 1365-1393.

Kellogg, R.B., Tsan, A. (1978). Analysis of some difference approximations for a singular perturbation problem without turning points. *Mathematical Computations, 32,* 1025-1039.

Khater, A.H., Temsah, R.S., Hassan, M.M. (2008). A Chebyshev spectral collocation method for solving Burgers' type equations. *Journal of Computational and Applied Mathematics, 222,* 333-350.

Korkmaz, A., Murat Aksoy, A., Dag, I. (2011). Quartic B-spline differential quadrature method. *International Journal of Nonlinear Sciences, 11*(4), 403-411.

Korkmaz, A., Dag, I. (2011). Polynomial based differential quadrature method for numerical solution of nonlinear Burgers' equation. *Journal of Franklin Institution, 348*(10), 2863-2875.

Kutulay, S., Esen, A., Dag, I. (2004). Numerical solutions of the Burgers' equation by the least-squares quadratic B-spline finite element method. *Journal of Computational and Applied Mathematics, 167,* 21-33.

Liao, W. (2008). An implicit fourth-order compact finite difference scheme for one-dimensional Burgers' equation. *Applied Mathematics and Computation, 206,* 755-764.

Liu, L.-B., Liang, Y., Zhang, J. (2020). A robust adaptive grid method for singularly perturbed Burger-Huxley equations. *Electronic Research Archive, 28*(4), 1439-1457.

Masho, J.K., Gemechis, F.D. (2021). Robust numerical method for singularly perturbed semilinear parabolic differential difference equations. *Mathematics and Computers in Simulation, 188,* 537-547.

Mittal, R.C., Jiwari, R. (2012). A differential quadrature method for solving Burgers' type equation. *International Journal of Numerical Methods for Heat and Fluid Flow, 22*(7), 880-895.

Mittal, R.C., Jiwari, R., Sharma, K.K. (2013). A numerical scheme based on differential quadrature method to solve time dependent Burgers' equation. *Engineering Computations, 30*(1), 117-131.

Morton, K.W. Numerical Solution of Convection-Diffusion Problems. Chapman & Hall, London, 1996.

Ozis, T., Aksan, E.N., Ozdes, A. (2003). A finite element approach for solution of Burgers' equation. *Applied Mathematics and Computation, 139,* 417-428.

O'Malley, R.E. Jr. *Singular perturbation methods for ordinary differential equations,* Springer, Verlag, 1991.

Prandtl, L. (1904). Uber flussigkeits-bewegung bei kleiner reibung. pages 484-495, III Int. Math. Kongresses Tuebner,Verhandlungen, Leipzig.

Ramadan, M.A., El-Danaf, T.S., Abd Alaal, F.E.I. (2007). Application of the non-polynomial spline approach to the solution of the Burgers' equation. *Open Applied Mathematical Journal,* 15-20.

Roos, H.-G., Stynes, M., Tobiska, L. *Robust numerical methods singularly perturbed differential equations*, Springer, Berlin, 2008.

Saka, B., Dag, I. (2007). Quartic B-spline collocation method to the numerical solutions of the Burgers' equation. *Chaos Solitons and Fractals, 32,* 1125-1137.

Saka, B., Dag, I. (2007). Quartic B-spline collocation method to the numerical solutions of the Burgers' equation. *Chaos Solitons and Fractals, 32,* 1125-1137.

Shishkin, G.I., Shishkina, L.P. *Difference methods for singular perturbation problems*. monographs and surveys in pure and applied mathematic, 2009.

Vasil'eva, A.B. (1963). Asymptotic behavior of solutions to certain problems involving nonlinear ordinary differential equations containing a small parameter multiplying the highest derivatives. *Russian Mathematical Surveys, 18,* 13-84.

Vasil'eva, A.B. (1994). On the development of singular perturbation theory at Moscow State University and Elsewhere. *SIAM Review, 36,* 440-452.

Wasow, W. On boundary layer problems in the theory of ordinary differential equations, PhD thesis, New York University, 1981.