RADIATION FROM FALLING PARTICLES ONTO COMPACT OBJECTS

By

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SUBMITTED IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE OF
MASTER OF SCIENCE IN PHYSICS
(ASTROPHYSICS)
AT
JIMMA UNIVERSITY
COLLEGE OF NATURAL SCIENCES
JIMMA,ETHIOPIA
DECEMBER 2021

JIMMA UNIVERSITY

PHYSICS

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JIMMA UNIVERSITY

Date: **December 2021**

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Title: **RADIATION FROM FALLING PARTICLES ONTO COMPACT OBJECTS**

Department: **Physics**

Degree: **MSc.**
Convocation: **February**
Year: **2022**

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To those who have lost their lives to survive the others

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Abstract

In this thesis we studied the radiation of particles infalling into compact objects where General Relativity Field Equations (GRFEs) were used. In particular the Schwarzschild and Kerr black holes were considered. The radiation is considered to occur at the horizons due to the strong gravity of the compacts. The calculated total energy radiated and the time of fall into the black holes depend both on the black hole and the falling particles dynamical parameters specification. For mathematical analysis the latest Mathematica 11.3 was used. The results are in good agreement with the works of the others. Specially, the falling of particles into Schwarzschild black hole fits with that of the results of Zerilli (1970).

Key words: GRFEs, Schwarzschild metric, Kerr metric, COs, event horizon, electromagnetic radiation, gravitational radiation.

Acknowledgements

Primarily I would thank God for being able to complete this thesis. Then I would like to express my special thanks of gratitude to my Astrophysics Instructor and advisor **Dr. Tolu Biressa** for his able guide and support in completing my work. Then my thanks go to my families (specially, my mother), who has been such an enormous support to me throughout all the years that it took to come up with this work. Last but not the least I would like to thank my companions who have helped me a lot.

Chapter 1

Introduction

1.1 Background and Literature Review

In astronomy, the term compact objects (compact stars) is used to refer collectively to white dwarfs, neutron stars, other exotic dense stars and black holes. Their compactness gives them many extreme properties which make them relevant for the high energy astrophysics, emission of X-rays and Gamma-rays [1]. When a star has exhausted all its fuel, the gas pressure of the hot interior can no longer support the weight of the star and the star dies by collapsing to a denser state, then a compact star is born [2].

A low-mass star, such as our Sun, will end up as a white dwarf, in which the degeneracy pressure of the electron gas balances the gravity of the object. For a more massive star, the formed compact object can be more massive than around $1.4M_{\odot}$, the so-called Chandrasekhar limit, in which the degeneracy pressure of the electron gas cannot resist the gravity, as pointed out by Chandrasekhar. In this case, the compact object has to further contract to become a neutron star, in which most of the free electrons are pushed into protons to form neutrons as suggested by Zwicky and

Landau. Then as Oppenheimer and others noted, if the neutron star is too massive, for example, more than around $3M_{\odot}$, the internal pressure in the object also cannot resist the gravity and the object must undergo catastrophic collapse and form a black hole [3].

The term black hole was first introduced by an American physicist J.A. Wheeler because everything, including the light wave and any electro-magnetic wave, that went into its zone was not able to get out and consequently it was appeared as black. In the 18th century, Laplace and Michell hypothesized for the first time that there exist an astronomical body with a massive mass that was able to cause an escape velocity greater than the speed of light in vacuum for which no light was able to resist the strongest gravitational force generated by this celestial body to escape from it. However, this hypothesis was not supported by the wave theory of light rather by the corpuscular theory of light. On this account, the concept of black hole was abandoned at that time. But some months after the publication of General theory of relativity by Einstein in 1916, the black hole was again become famous because the theory of general relativity predicted that a sufficiently huge and compact mass can deform space-time to form a black hole [4]. However, nowadays the question of how black holes modify space and time around them is still open [5].

The process of capture of masses by black holes investigated, in the hypotesis that the infalling mass M_2 is smaller than a black hole of mass M_1, with the technics of first-order perturbations of the background geometry. Einsteins perturbed equations split into two separated sets of equations that can be reduced to two second-order inhomogeneous wave equations with real (and different) potential barriers and different source terms describing the radial part of axial and polar perturbations (Regge and Wheeler, 1957, Zerilli, 1970). The source term for these equations comes out from the stress-energy tensor of the mass M_2 which is assumed to fall along a geodesic of the unperturbed Schwarzschild spacetime. A particle can fall following a radial or a spiralling trajectory, and an analysis of the source term shows that for polar perturbations it is different from zero in both cases and for axial perturbations it is non-null only in the second case. It means that axial perturbations can be excited only if the particle has an initial angular momentum.

The case of a particle falling onto a star was no more studied before. However, researchers believed that one way of identifying a collapsed star or black hole in space is by observing the electromagnetic radiation spectrum emitted by interstellar gas accreting onto the object. Knowing from theoretical considerations the characteristic frequency spectrum of the emitted radiation, one can presumably determine from observations whether or not a black hole has indeed been located. Spherically symmetric, steady-state accretion onto stars for simple polytropic gases has been examined by Bondi (1952) in the nonrelativistic limit. Michel (1972) considered the general-relativistic version of the same problem and applied his analysis to the accretion of matter onto condensed objects. The possibility that gas accreting onto a black hole might be an important source of radiant energy was first suggested by Zeldovich (1964) and Salpeter (1964). Shvartsman (1971) employed nonrelativistic approximations for both the fluid dynamics and radiative processes to estimate the total energy radiated by a fully ionized gas accreting onto a black hole [6].

Later on, interaction of radiation and matter in the context of black hole astrophysics was investigated as early as 1974 by Wickramasinghe, when he studied the radiation pressure driven mass loss from the outermost region of an accretion disc. Icke (1980) studied the effect of radiative acceleration of gas flow above a Sakura-Sunyaev Keplerian disc (1973). However, the effect of radiation drag on the gas flow was ignored [7].

One of indirect evidence for the existence of BHs is with the radiative efficiency when matter falls toward a central compact object. A matter in a gravitational potential well must continue to fall inward, either through the event horizon of a BH or hitting the surface of a compact object not enclosed by an event horizon but with a radius either larger or smaller than the event horizon of the given mass (called a compact star or naked compact object, respectively). No further radiation is produced after the matter falls through the event horizon of the BH; thus, the majority of the kinetic energy of the infalling matter is carried into the BH. On the other hand, surface emission will be produced when matter hits the surface of the compact star or naked compact object, because it is not a BH. Therefore, the radiative efficiencies for these different scenarios are significantly different [3].

Black holes are most often detected by the radiation produced when they gravitationally pull in surrounding gas. The efficiency with which the hot gas radiates its thermal energy strongly influences the geometry and dynamics of the accretion flow. Both radiatively efficient thin disks and radiatively inefficient thick disks are observed. When the accreting gas gets close to the central black hole, the radiation it produces becomes sensitive to the spin of the hole and the presence of an event horizon. Analysis of the luminosities and spectra of accreting black holes has yielded tantalizing evidence for both rotating holes and event horizons [8].

Studying the X-ray spectrum of an accreting black hole reveals X-rays of different energies emitted by different processes. It is possible to see both the thermal X-rays emitted from the surface of the accretion disc and the X-rays emitted from high energy particles in the hot corona around the black hole. In addition, the X-rays emitted from the corona can be reflected from the accretion disc. Reflection from the accretion disc imprints a number of atomic features on the spectrum we observe, not least emission lines. When atoms are excited, their electrons emit light at very specific energies. It is why different metal compounds glow different colors when heated [9].

The amount of energy emitted by a star in the form of electromagnetic radiation can be inferred directly from the temperature of certain layers of gas in the stars atmosphere (called the effective temperature of the star) and from the stars size. When gas, dust or other kinds of matter fall towards a compact object (such as a black hole or a neutron star), a disk of infalling matter forms around the central object (accretion disk). The energy that matter gains in its fall is transformed into heat energy of the disk matter. Consequently, accretion disks are extremely hot. Their thermal radiation they emit is an important tool for indirect observation of neutron stars and black hole. Within the disk, matter spirals around and coming closer and closer to the central object until at last it falls onto its surface (or in the case of a black hole, through its event horizon). Accretion disks emit large amounts of energy in the form of electromagnetic radiation. Most of that energy is radiated away at very high frequencies, in the form of X-rays.

If the central body is a black hole, matter can fall directly towards the black holes horizon and into the black hole, never to be seen again. In both cases, matter takes a straight plunge. But this is by no means the only possibility in fact, it was not ruled out rather than the exception . Usually, matter will be in motion even before it is close enough for the central object to exert a significant pull. Unless this motion is directed exactly towards the central object a special case, and thus very rare there will be a component of sideways motion, and if that component is large enough, the falling matter will not hit the central object, but go past it [10].

A number of investigators have studied the gravitational radiation given off when a particle falls into a black-hole. Zel'dovich and Novikov (1964a), using linearized general relativity, calculated that an energy of $0.01M_2c^2(M_2/M_1)$ is radiated as gravitational radiation when a small mass M_2 falls straight into a black-hole of mass M_1. They also calculate that up to 5.7 percent of the incoming particle's rest mass can be radiated as gravitational radiation if the particle orbits the black-hole and slowly spirals inward. Orbiting particles were also considered by Peters and Mathews (1963). Zerilli (1970), used the full nonlinear Einstein theory, calculates $0.0016M_2c^2(M_2/M_1)$

instead of the above for zero angular momentum fall. However, Bardeen (1971) considered an extreme Kerr rotating black-hole and found that a particle in an antirotating orbit radiates 3.8 percent of its rest mass while a particle in a corotating orbit radiates 42.3 percent of its rest mass. Zel'dovich and Novikov have also considered that the thermal electromagnetic radiation given off as gas funneled into a black-hole is compressed and heated to a very high temperature. The gas as a whole emits the radiation and not individual particles [11].

This was a plausible extension to the models for X-ray and radio emission from matter falling into (much lighter) stellar-mass black hole candidates, such as the ones observed by the group led by Riccardo Giacconi, the 2002 Nobel Laureate. The gravitational pull must come from an extremely massive object, or else they would exceed the Eddington limiting luminosity, $1.3 \times 10^{31}(M_1/M_\odot)$W (where $M_\odot$ is the mass of the sun), at which point the radiation pressure would overcome gravity, rendering instabilities, which would blow the object apart [12].

1.2 Statement of problem

An accelerated charged particle interacting with a neutral Schwarzschild black-hole, the electromagnetic radiation was computed when particle infalling with zero angular momentum and in the case of spiraling black hole. The gravitational radiation due to a point-like particle falling into a black hole (BH) is a classic problem in General Relativity have computed. It has considered as an application of BH perturbation theory to study the effect of strong gravity on environment. In the case of the gravitational radiation from a particle falling into a Kerr BH, the motion of a particle is derived from the the Kerr metric. However, its radiation is calculated using flat space linearized theory of gravitation. In addition to this, in the case of inspiralling particle, the power radiated observed at infinity is not fully derived classically. The problem of gravitational radiation waves by bodies moving in the field of a collapsing star have also studied. Unfortunately, such considerations can only be valid for bodies which move at distances large compared to the Schwarzschild radius of the central body [13]. Then, in subsequent studies, many-particles in fall radiation have treated in various ways. By now, at least the general features of the radiated waveform, spectra and energies are computed and understood. However, still an extension of the study is an active research area to complement.

Research questions

- What does strong gravity spacetime geometry do on in falling particles into black hole?
- How particles in falling onto black holes be affected at the horizons of Schwarzschild and Kerr BH?
- What amount of energy is being radiated from in falling particles into gravitational BH?
- How fast does a particle orbit during in spiralling into Schwarzschild black hole?

1.3 Objectives

1.3.1 General objective

To study the radiation from falling particles onto compact object.

1.3.2 Specific objectives

- To study the effect of strong gravity in falling particles into black hole.
- To derive equation of motion of particles falling into Schwarzschild and Kerr black hole.
- To derive energy radiated from in falling particles into Schwarzschild and Kerr black hole.
- To derive time of infalling particle and spirals into Schwarzschild black hole.

1.4 Significance of the study

The significance of this study will be:

- Contribute enlightenment in the field study of Astrophysics.
- To provide interesting in scientific input for the progress of research in Astronomy and space science.
- To provide clear mathematical procedures with their physical explanations for other researchers who wish to carry out related investigations.

1.5 Methodology

General Theory of Relativity field equations are used to derive equations of motion of freely falling and in spiralling particle into Schwarzschild and Kerr black hole. Subsequently, electromagnetic radiation equations for charged particle and gravitational radiation in case of uncharged one in falling into Schwarzschild is also used. For Kerr black hole, the power radiated energy of co-rotating particle from infinity with and without angular momentum after infalling is derived. To derive the electromagnetic radiation equation, the semi-classical electrodynamics accelerated charged system is used where curvature effect is included by the metric of the spacetime. For reasonably nearby environment to the central gravitating system, the Schwarzschild and Kerr geometries are considered in our case. Finally, the analytically obtained equations will be used to generate some numerical values using the latest Mathematics software (Mathematica 11.3) to compare with others.

Chapter 2

Introduction to General Relativity Theory

Einstein's General Theory of Relativity (GTR) is a powerful theory to describe compact stars mathematically. It is a theory that describes gravity as curved spacetime. Einstein presented a description of gravity in the sense that the relative acceleration of particles is not viewed as a consequence of gravitational forces, but results from the curvature of the spacetime in which the particles are moving. Consequently, black holes are an extreme form of curved spacetime containing a curvature singularity that swallows matter and light. Later on he developed Equivalence Principle and Metricity. Einstein's principle of equivalence has played an important role in the development of gravitation theory. The Einstein equivalence principle (EEP) has played an important role in gravitational theory, for it is possible to argue convincingly that if EEP is valid, then gravitation must be a curved spacetime phenomenon, then the effects of gravity must be equivalent to the effects of living in a curved spacetime [1].

The modern solution of general theory of relativity that would suggest a black hole was found by Karl Schwarzschild in 1916, although its interpretation as a region of space from which nothing can escape including light was first introduced by David Finkelstein in 1958. However, the problem with the Schwarzschild metric is that it describes the geometry as measured by observers at rest. It is now realized that once inside the Schwarzschild radius, there can be no observers at rest: everything plunges inevitably to the central singularity [14].

On 13 January 1916, after Einstein completed his theory of general relativity on 18 November 1915, Karl Schwarzschild published a solution to Einsteins field equations, describing the curved space-time around a spherically symmetric, non-rotating, mass of black hole. Nevertheless, after two years later(1918) the Austrian physicists Lense and Thirring studied the equations of the General Relativity Theory (GRT) and concluded that a spinning black hole will drag time and space around, an effect since called the Lense-Thirring effect. The effect predicts that the rotation axis of the disk should precede around the rotation axis of the black hole. In 2016 (almost a century after the prediction) there is a claim that the effect has been detected [15].

However, Schwarzschilds theories were predicted by Einstein and then borne out mathematically in 1939 by American astrophysicists Robert Oppenheimer and Hartland Snyder. The Schwarzschild solution is the simplest non-trivial solution to Einsteins equation and it is well suited to introduce some important concepts as the event horizon, geodesics. The Schwarzschild solution describes the simplest category of the black holes, i.e. with only mass and neither rotation nor charge [12].

Most of the current tests of General Relativity are performed in the weak gravitational fields present in our solar system. The consequences of General Relativity (GR) of the "no-hair" theorem, for which black holes can be fully characterized by their mass, angular momentum and charge was understated. Since no charge on astrophysical black holes, the spacetime that surrounds a black hole can be nearly exactly described by the Kerr metric. The only way to test this theorem is to investigate the spacetime very close to the hole. Fortunately, the X-ray emission of accreting black holes carries information about the inner region of the accretion disk, within a few gravitational radii ($R_g = GM_1/c^2$) from the hole, encoded in the fast variability of its spectrum [5].

2.1 The geodesic equation

The freely falling inertial coordinate frame in which the effects of gravity are locally absent is denoted by, ξ^α. In this frame, the equation of motion for a particle is

$$\frac{d^2\xi^\alpha}{d\tau^2} = 0 \tag{2.1.1}$$

with

$$c^2 d\tau^2 = -\eta_{\alpha\beta} d\xi^\alpha d\xi^\beta \tag{2.1.2}$$

being the invariant time interval. Next, we write this equation in any other set of coordinates we like, and call them x^μ . Our inertial coordinates ξ^α will be some function or other of the x^μ so

$$0 = \frac{d^2\xi^\alpha}{d\tau^2} \tag{2.1.3}$$

$$= \frac{d}{d\tau}\left(\frac{\partial\xi^\alpha}{\partial x^\mu}\frac{dx^\mu}{d\tau}\right) \tag{2.1.4}$$

where we have used the chain rule to express $d\xi^\alpha/d\tau$ in terms of $dx^\mu/d\tau$. Carrying out the differentiation,

$$0 = \frac{\partial \xi^\alpha}{\partial x^\mu}\frac{d^2 x^\mu}{d\tau^2} + \frac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} \tag{2.1.5}$$

where now the chain rule has been used on $\partial\xi^\alpha/\partial x^\mu$. This may not look very promising. But if we multiply this equation by $\partial x^\lambda/\partial\xi^\alpha$, and remember to sum over α now, then the chain rule in the form

$$\frac{\partial x^\lambda}{\partial \xi^\alpha}\frac{\partial \xi^\alpha}{\partial x^\mu} = \delta^\lambda_\mu \tag{2.1.6}$$

Our equation becomes

$$\frac{d^2 x^\lambda}{d\tau^2} + \Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0 \tag{2.1.7}$$

where

$$\Gamma^\lambda_{\mu\nu} = \frac{\partial x^\lambda}{\partial \xi^\alpha}\frac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\nu} \tag{2.1.8}$$

is known as the affine connection, and is a quantity of central importance in the study of Riemannian geometry and relativity theory in particular. We should be able to prove, using the chain rule of partial derivatives, an identity for the second derivatives of ξ^α a that we will use shortly:

$$\frac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\nu} = \frac{\partial \xi^\alpha}{\partial x^\lambda}\Gamma^\lambda_{\mu\nu} \tag{2.1.9}$$

In our locally inertial coordinates, the invariant spacetime interval is

$$c^2 d\tau^2 = -\eta_{\alpha\beta} d\xi^\alpha d\xi^\beta \tag{2.1.10}$$

so that in any other coordinates, $d\xi^\alpha = (\partial\xi^\alpha/dx^\mu)dx^\nu$ and

$$c^2 d\tau^2 = -\eta_{\alpha\beta}\frac{\partial \xi^\alpha}{\partial x^\mu}\frac{\partial \xi^\beta}{\partial x^\nu} dx^\mu dx^\nu \equiv -g_{\mu\nu} dx^\mu dx^\nu \tag{2.1.11}$$

where

$$- g_{\mu\nu} = \eta_{\alpha\beta}\frac{\partial\xi^\alpha}{\partial x^\mu}\frac{\partial\xi^\beta}{\partial x^\nu}dx^\mu dx^\nu \tag{2.1.12}$$

is known as the metric tensor. The metric tensor embodies the information of how coordinate differentials combine to form the invariant interval of our spacetime, and once we know $g_{\mu\nu}$, we know everything, including the affine connections $\Gamma^\lambda_{\mu\nu}$.

2.2 The Newtonian limit

For a slowly moving mass (of course means relative to c, the speed of light) in a weak gravitational field ($GM_1/rc^2 \ll 1$). Since $cdt \gg |d\mathbf{x}|$, the geodesic equation greatly simplifies:

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{00}\left(\frac{cdt}{d\tau}\right)^2 = 0 \tag{2.2.1}$$

Now

$$\Gamma^\mu_{00} = \frac{1}{2}g^{\mu\nu}\left(\frac{\partial g_{0\nu}}{\partial(cdt)} + \frac{\partial g_{0\nu}}{\partial(cdt)} - \frac{\partial g_{00}}{\partial x^\nu}\right) \tag{2.2.2}$$

In the Newtonian limit, the largest of the g derivatives is the spatial gradient, hence

$$\Gamma^\mu_{00} \simeq -\frac{1}{2}g^{\mu\nu}\frac{\partial g_{00}}{\partial x^\nu} \tag{2.2.3}$$

Since the gravitational field is weak, $g_{\alpha\beta}$ differs very little from the Minkoswki value:

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} \tag{2.2.4}$$

$h_{\alpha\beta} \ll 1$ and the $\mu = 0$ geodesic equation is

$$\frac{d^2t}{d\tau^2} + \frac{1}{2}\frac{\partial h_{00}}{\partial t}\left(\frac{dt}{d\tau}\right)^2 = 0 \tag{2.2.5}$$

The second term is zero for a static field, so that the spatial components of the geodesic equation become

$$\frac{d^2\mathbf{r}}{dt^2} - \frac{c^2}{2}\nabla h_{00} \quad = \quad 0 \tag{2.2.6}$$

This equation rewritten as

$$\frac{d^2\mathbf{r}}{dt^2} + \nabla\Phi \quad = \quad 0, \tag{2.2.7}$$

where Φ is being the classical gravitational potential. The two views are consistent if

$$h_{00} \simeq -\frac{2\Phi}{c^2} \simeq -\frac{2GM_1}{c^2 r} \tag{2.2.8}$$

We get

$$g_{00} \simeq -\left(1 - \frac{2GM_1}{c^2 r}\right) \tag{2.2.9}$$

2.3 The relationship between the metric tensor and affine connection

Because of their reliance of the local freely falling inertial coordinates ξ^α, the $g_{\mu\nu}$ and $\Gamma^\lambda_{\mu\nu}$ quantities are difficult to use in their present formulation. Fortunately, there is a direct relationship between $\Gamma^\lambda_{\mu\nu}$ and the first derivatives of $g_{\mu\nu}$ with the ξ^α altogether. Differentiate equation 2.1.12 with respect to $\partial/\partial x^\lambda$

$$\frac{\partial g_{\mu\nu}}{\partial x^\lambda} \quad = \quad \eta_{\alpha\beta}\frac{\partial^2\xi^\alpha}{\partial x^\lambda \partial x^\mu}\frac{\partial\xi^\beta}{\partial x^\nu} + \eta_{\alpha\beta}\frac{\partial\xi^\alpha}{\partial x^\mu}\frac{\partial^2\xi^\beta}{\partial x^\lambda \partial x^\nu} \tag{2.3.1}$$

Now we use equation 2.1.9 for the second derivatives of ξ:

$$\frac{\partial g_{\mu\nu}}{\partial x^\lambda} \quad = \quad \eta_{\alpha\beta}\frac{\partial\xi^\alpha}{\partial x^\rho}\frac{\partial\xi^\beta}{\partial x^\nu}\Gamma^\rho_{\lambda\mu} + \eta_{\alpha\beta}\frac{\partial\xi^\alpha}{\partial x^\mu}\frac{\partial\xi^\beta}{\partial x^\rho}\Gamma^\rho_{\lambda\nu} \tag{2.3.2}$$

All remaining ξ derivatives may be absorbed as part of the metric tensor, leading to

$$\frac{\partial g_{\mu\nu}}{\partial x^\lambda} = g_{\rho\nu}\Gamma^\rho_{\lambda\mu} + g_{\mu\rho}\Gamma^\rho_{\lambda\nu} \tag{2.3.3}$$

By adding $\partial g_{\lambda\nu}/\partial x^\mu$ to the above, then subtracting it with indices μ and ν reversed.

$$\begin{aligned}\frac{\partial g_{\mu\nu}}{\partial x^\lambda} + \frac{\partial g_{\lambda\nu}}{\partial x^\mu} - \frac{\partial g_{\lambda\mu}}{\partial x^\nu} &= g_{\rho\nu}\Gamma^\rho_{\lambda\mu} + g_{\mu\rho}\Gamma^\rho_{\lambda\nu} + g_{\rho\nu}\Gamma^\rho_{\mu\lambda} + g_{\rho\lambda}\Gamma^\rho_{\mu\nu} - g_{\rho\mu}\Gamma^\rho_{\nu\lambda} \\ &\quad - g_{\rho\lambda}\Gamma^\rho_{\nu\mu} \end{aligned} \tag{2.3.4}$$

Remembering that Γ is symmetric in its bottom indices, only the $g_{\rho\nu}$ terms survive, leaving

$$\frac{\partial g_{\mu\nu}}{\partial x^\lambda} + \frac{\partial g_{\lambda\nu}}{\partial x^\mu} - \frac{\partial g_{\lambda\mu}}{\partial x^\nu} = 2g_{\rho\nu}\Gamma^\rho_{\mu\lambda} \tag{2.3.5}$$

Finally, we multiply by the inverse matrix $g^{\nu\sigma}$, defined by

$$g^{\nu\sigma}g_{\rho\nu} = \delta^\sigma_\rho \tag{2.3.6}$$

then it yields

$$\Gamma^\sigma_{\mu\lambda} = \frac{g^{\nu\sigma}}{2}\left(\frac{\partial g_{\mu\nu}}{\partial x^\lambda} + \frac{\partial g_{\lambda\nu}}{\partial x^\mu} - \frac{\partial g_{\lambda\mu}}{\partial x^\nu}\right) \tag{2.3.7}$$

2.4 The Einstein Field Equations

Einsteins field equations are the relativistic generalization of Newtons law of gravitation. It also connects the curvature of spacetime (Einstein tensor) and the properties of an object that curves spacetime (Energy-Momentum tensor). Newtons gravitational law tells how mass generates gravitational force, while Einstein's field equations tell how matter and energy curves space-time. In the general theory of relativity the Einstein Field Equations (EFE) in vacuum space-time is;

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu} \tag{2.4.1}$$

For non-vacuum space-time, EFEs take the form;

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu} \tag{2.4.2}$$

where Λ is cosmological constant, $R_{\mu\nu}$ is Ricci tensor, R is Curvature scalar, G is Newton's constant, $T_{\mu\nu}$ is energy-momentum tensor, $g_{\mu\nu}$ general metric tensor [16].

2.5 The Schwarzschild Solution

Now we determine the form of the metric tensor $g_{\mu\nu}$ for the spacetime surrounding a point mass M_1 by solving the equation $R_{\mu\nu} = 0$, subject to the appropriate boundary conditions. Because the spacetime is static and spherically symmetric, we expect the invariant line element to take the form

$$d\tau^2 = Bdt^2 - Adr^2 - Cd\Omega^2 \tag{2.5.1}$$

where $d\Omega$ is the solid angle, $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ and A, B, and C are all functions of the radial variable. We may choose our coordinates so that C is defined to be r^2. A and B will then be some unknown functions of r to be determined. Our metric is now in standard form:

$$d\tau^2 = B(r)dt^2 - A(r)dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{2.5.2}$$

We may now read the components of $g_{\mu\nu}$:

$$g_{tt} = -B(r) \tag{2.5.3}$$

$$g_{rr} = A(r) \tag{2.5.4}$$

$$g_{\theta\theta} = r^2 \tag{2.5.5}$$

$$g_{\phi\phi} = r^2 \sin^2\theta \tag{2.5.6}$$

and its inverse $g^{\mu\nu}$,

$$g^{tt} = -B^{-1}(r) \tag{2.5.7}$$

$$g^{rr} = A^{-1}(r) \tag{2.5.8}$$

$$g^{\theta\theta} = r^{-2} \tag{2.5.9}$$

$$g^{\phi\phi} = r^{-2}(\sin\theta)^{-2} \tag{2.5.10}$$

The determinant of $g_{\mu\nu}$ is $-g$, where

$$g = r^4 AB \sin^2\theta \tag{2.5.11}$$

The affine connection for a diagonal metric tensor will be of the form

$$\begin{aligned} \Gamma^a_{ab} &= \Gamma^a_{ba} \\ &= \frac{1}{2g_{aa}} \frac{\partial g_{aa}}{\partial x^b} \end{aligned}$$

no sum on a, with $a = b$, then

$$\Gamma^a_{bb} = -\frac{1}{2g_{aa}} \frac{\partial g_{bb}}{\partial x^a} \tag{2.5.12}$$

The nonvanishing components are:

$$
\begin{aligned}
\Gamma^{t}_{tr} &= \Gamma^{t}_{rt} = \frac{B'}{2B} \\
\Gamma^{r}_{tt} &= \frac{B'}{2A} \\
\Gamma^{r}_{rr} &= \frac{A'}{2A} \\
\Gamma^{r}_{\theta\theta} &= -\frac{r}{A} \\
\Gamma^{r}_{\phi\phi} &= -\frac{r\sin^2\theta}{A} \\
\Gamma^{\theta}_{r\theta} &= \Gamma^{\theta}_{\theta r} = \frac{1}{r} \\
\Gamma^{\theta}_{\phi\phi} &= -\sin\theta\cos\theta \\
\Gamma^{\phi}_{\phi r} &= \Gamma^{\phi}_{r\phi} = \frac{1}{r} \\
\Gamma^{\phi}_{\phi\theta} &= \Gamma^{\phi}_{\theta\phi} = \cot\theta
\end{aligned}
\tag{2.5.13}
$$

where

$$
\begin{aligned}
A' &= \frac{dA}{dr} \\
B' &= \frac{dB}{dr}
\end{aligned}
$$

Next, the Ricci Tensor is given as:

$$
\begin{aligned}
R_{\mu\kappa} &= R^{\lambda}_{\mu\lambda\kappa} \\
&= \frac{\partial\Gamma^{\lambda}_{\mu\lambda}}{\partial x^{\kappa}} - \frac{\partial\Gamma^{\lambda}_{\mu\kappa}}{\partial x^{\lambda}} + \Gamma^{\eta}_{\mu\lambda}\Gamma^{\lambda}_{\kappa\eta} - \Gamma^{\eta}_{\mu\kappa}\Gamma^{\lambda}_{\lambda\eta}
\end{aligned}
\tag{2.5.14}
$$

where by definition

$$
\Gamma^{\lambda}_{\lambda\mu} = \frac{g^{\lambda\rho}}{2}\left(\frac{\partial}{\partial x^{\mu}}g_{\rho\lambda} + \frac{\partial}{\partial x^{\lambda}}g_{\rho\mu} - \frac{\partial}{\partial x^{\rho}}g_{\lambda\mu}\right) \tag{2.5.15}
$$

$g^{\lambda\rho}$ is symmetric in its indices, whereas the last two g derivatives are antisymmetric in the same indices, so they disappeared. We are left with $\Gamma^{\lambda}_{\lambda\mu} = \frac{g^{\lambda\rho}}{2}\frac{\partial}{\partial x^{\mu}}g_{\rho\lambda}$, in which

$\lambda = \rho$ for nonvanishing entries, and $g^{\lambda\rho}$ is the reciprocal of $g_{\lambda\rho}$.

Hence

$$\begin{aligned}\Gamma^{\lambda}_{\lambda\mu} &= \frac{g^{\lambda\rho}}{2}\frac{\partial}{\partial x^{\mu}}g_{\rho\lambda} \\ &= \frac{1}{2}\frac{\partial}{\partial x^{\mu}}\ln|g| \end{aligned} \tag{2.5.16}$$

Then Ricci Tensor becomes

$$R_{\mu\kappa} = \frac{1}{2}\frac{\partial^2 \ln g}{\partial x^{\kappa}\partial x^{\mu}} - \frac{\partial \Gamma^{\lambda}_{\mu\kappa}}{\partial x^{\lambda}} + \Gamma^{\eta}_{\mu\lambda}\Gamma^{\lambda}_{\kappa\eta} - \frac{\Gamma^{\eta}_{\mu\kappa}}{2}\frac{\partial \ln g}{\partial x^{\eta}} \tag{2.5.17}$$

By using 2.5.11 and 2.5.17, the R_{tt} tensor of static fields becomes

$$\begin{aligned}R_{tt} &= -\frac{\partial \Gamma^{r}_{tt}}{\partial r} + \Gamma^{\eta}_{t\lambda}\Gamma^{\lambda}_{t\eta} + \Gamma^{\eta}_{tt}\Gamma^{t}_{t\eta} - \frac{\Gamma^{\eta}_{\mu\kappa}}{2}\frac{\partial \ln g}{\partial x^{\eta}} \\ &= -\frac{\partial}{\partial r}\left(\frac{B'}{2A}\right) + \Gamma^{t}_{tr}\Gamma^{r}_{tt} + \Gamma^{r}_{tt}\Gamma^{t}_{tr} - \frac{\Gamma^{r}_{tt}}{2}\frac{\partial}{\partial r}\ln(r^4 AB\sin^2\theta) \\ &= -\left(\frac{B''}{2A}\right) + \frac{B'A'}{2A^2} + \frac{B'^2}{4AB} - \frac{B'}{4A}\left(\frac{A'}{A} + \frac{B'}{B} + \frac{4}{r}\right) \\ &= -\frac{B''}{2A} + \frac{B'}{4A}\left(\frac{B'}{B} + \frac{A'}{A}\right) - \frac{B'}{rA} \end{aligned} \tag{2.5.18}$$

Next, R_{rr}:

$$\begin{aligned}R_{rr} &= \frac{1}{2}\frac{\partial^2 \ln g}{\partial r^2} - \frac{\Gamma^{r}_{rr}}{\partial r} + \Gamma^{\eta}_{r\lambda}\Gamma^{\lambda}_{r\eta} - \frac{\Gamma^{r}_{rr}}{2}\frac{\partial \ln g}{\partial r} \\ &= \frac{1}{2}\frac{\partial}{\partial r}\left[\frac{\partial}{\partial r}\ln(r^4 AB\sin^2\theta)\right] - \frac{\partial}{\partial r}\frac{A'}{2A} + \Gamma^{\eta}_{r\lambda}\Gamma^{\lambda}_{r\eta} - \frac{A'}{2A}\left[\frac{\partial}{\partial r}\ln(r^4 AB\sin^2\theta)\right] \\ &= \frac{1}{2}\frac{\partial}{\partial r}\left(\frac{A'}{A} + \frac{B'}{B} + \frac{4}{r}\right) - \frac{\partial}{\partial r}\left(\frac{A'}{2A}\right) + \Gamma^{\eta}_{r\lambda}\Gamma^{\lambda}_{r\eta} + \frac{A'}{4A}\left(\frac{A'}{A} + \frac{B'}{B} + \frac{4}{r}\right) \\ &= \frac{B''}{2B} - \frac{1}{2}\left(\frac{B'}{B}\right)^2 - \frac{2}{r^2} + \left(\Gamma^{t}_{rt}\right)^2 + \left(\Gamma^{r}_{rr}\right)^2 + \left(\Gamma^{\theta}_{r\theta}\right)^2 + \left(\Gamma^{\phi}_{r\phi}\right) - \frac{1}{4}\left(\frac{A'}{A}\right)^2 - \frac{A'B'}{4AB} - \frac{A'}{rA} \\ &= \frac{B''}{2B} - \frac{1}{2}\left(\frac{B'}{B}\right)^2 - \frac{2}{r^2} + \frac{B'^2}{4B^2} + \frac{A'^2}{4A^2} + \frac{1}{r^2} + \frac{1}{r^2} - \frac{1}{4}\left(\frac{A'^2}{A}\right) - \frac{A'B'}{4AB} - \frac{A'}{rA} \\ &= \frac{B''}{2B} - \frac{1}{4}\frac{B'}{B}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{A'}{rA} \end{aligned} \tag{2.5.19}$$

$$
\begin{aligned}
R_{\theta\theta} &= \frac{\partial\Gamma^{\lambda}_{\theta\lambda}}{\partial\theta} - \frac{\partial\Gamma^{\lambda}_{\theta\theta}}{\partial x^{\lambda}} + \Gamma^{\eta}_{\theta\lambda}\Gamma^{\lambda}_{\theta\eta} - \Gamma^{\eta}_{\theta\theta}\Gamma^{\lambda}_{\lambda\eta} \\
&= \frac{1}{2}\frac{\partial^2 \ln g}{\partial\theta^2} - \frac{\partial\Gamma^{r}_{\theta\theta}}{\partial r} + \Gamma^{\eta}_{\theta\lambda}\Gamma^{\lambda}_{\theta\eta} - \Gamma^{r}_{\theta\theta}\Gamma^{\lambda}_{\lambda r} \\
&= \frac{d(\cot\theta)}{d\theta} + \frac{d}{dr}\left(\frac{r}{A}\right) + \Gamma^{\eta}_{\theta\lambda}\Gamma^{\lambda}_{\theta\eta} + \frac{r}{2A}\frac{\partial \ln g}{\partial r} \\
&= -\frac{1}{\sin^2\theta} + \frac{1}{A} - \frac{rA'}{A^2} + \Gamma^{r}_{\theta\lambda}\Gamma^{\lambda}_{\theta r} + \Gamma^{\theta}_{\theta\lambda}\Gamma^{\lambda}_{\theta\theta} + \Gamma^{\phi}_{\theta\lambda}\Gamma^{\lambda}_{\theta\phi} + \frac{r}{2A}\left(\frac{A'}{A} + \frac{B'}{B} + \frac{4}{r}\right) \\
&= -\frac{1}{\sin^2\theta} + \frac{3}{A} - \frac{rA'}{2A^2} + \Gamma^{r}_{\theta\theta}\Gamma^{\theta}_{\theta r} + \Gamma^{\theta}_{\theta r}\Gamma^{r}_{\theta\theta} + \left(\Gamma^{\phi}_{\theta\phi}\right)^2 + \frac{rB'}{2AB} \\
&= -\frac{1}{\sin^2\theta} + \frac{3}{A} - \frac{rA'}{2A^2} - \frac{2}{A} + \cot^2\theta + \frac{rB'}{2AB} \\
&= -\csc^2\theta + \cot^2\theta + \frac{3}{A} - \frac{rA'}{2A^2} - \frac{2}{A} + \frac{rB'}{2AB} \qquad (2.5.20)
\end{aligned}
$$

But from trigonometric identities we have ($-\csc^2\theta + \cot^2\theta = -1$), then equation 2.5.20 becomes

$$
R_{\theta\theta} = -1 + \frac{1}{A} + \frac{r}{2A}\left(-\frac{A'}{A} + \frac{B'}{B}\right) \qquad (2.5.21)
$$

$R_{\phi\phi}$ is the last nonvanishing Ricci component. The first term in equation 2.5.14 vanishes, since nothing in the metric depends on ϕ. Then,

$$
\begin{aligned}
R_{\phi\phi} &= -\frac{\partial\Gamma^{\lambda}_{\phi\phi}}{\partial x^{\lambda}} + \Gamma^{\eta}_{\phi\lambda}\Gamma^{\lambda}_{\phi\eta} - \frac{\Gamma^{\eta}_{\phi\phi}}{2}\frac{\partial \ln|g|}{\partial x^{\eta}} \\
&= -\frac{\Gamma^{r}_{\phi\phi}}{\partial r} - \frac{\Gamma^{\theta}_{\phi\phi}}{\partial\theta} + \Gamma^{r}_{\phi\lambda}\Gamma^{\lambda}_{\phi r} + \Gamma^{\theta}_{\phi\lambda}\Gamma^{\lambda}_{\phi\theta} + \Gamma^{\phi}_{\phi\lambda}\Gamma^{\lambda}_{\phi\phi} - \frac{1}{2}\Gamma^{r}_{\phi\phi}\frac{\partial \ln|g|}{\partial\theta} \\
&= \frac{\partial}{\partial r}\left(\frac{r\sin^2\theta}{A}\right) + \frac{\partial}{\partial\theta}(\sin\theta\cos\theta) + \Gamma^{r}_{\phi\phi}\Gamma^{\phi}_{\phi r} + \Gamma^{\theta}_{\phi\phi}\Gamma^{\phi}_{\phi\theta} + \Gamma^{\phi}_{\phi r}\Gamma^{r}_{\phi\phi} + \Gamma^{\phi}_{\phi\theta}\Gamma^{\theta}_{\phi\phi} \\
&\quad + \frac{1}{2}\sin\theta\cos\theta\frac{\partial \ln\sin^2\theta}{\partial\theta} + \frac{1}{2}\left(\frac{r\sin^2\theta}{A}\right)\left(\frac{A'}{A} + \frac{B'}{B} + \frac{4}{r}\right) \\
&= \frac{\sin^2\theta}{A} - \frac{rA'\sin^2\theta}{A^2} + \cos^2\theta - \sin^2\theta - \frac{\sin^2\theta}{A} - \cos^2\theta - \frac{\sin^2\theta}{A} - \cos^2\theta + \cos^2\theta \\
&\quad + \frac{r\sin^2\theta}{2A}\left(\frac{A'}{A} + \frac{B'}{B} + \frac{4}{r}\right) \\
&= \sin^2\theta\left[\frac{r}{2A}\left(-\frac{A'}{A} + \frac{B'}{B}\right) + \frac{1}{A} - 1\right] \\
&= \sin^2\theta R_{\theta\theta} \qquad (2.5.22)
\end{aligned}
$$

To solve the equations $R_{\mu\nu} = 0$, we have only A and B and all others components then vanish identically. R_{rr} and R_{tt}, both of which must separately vanish, so

$$\frac{R_{rr}}{A} + \frac{R_{tt}}{B} = -\frac{1}{rA}\left(\frac{A'}{A} + \frac{B'}{B}\right) = 0 \tag{2.5.23}$$

Hence we find

$$AB = constant = 1 \tag{2.5.24}$$

Furthermore, we impose on A and B the boundary condition that for $r \to \infty$ the metric tensor must approach Minkowski tensor in spherical coordinates, that is

$$\lim_{r\to\infty} A = \lim_{r\to\infty} B = 1 \tag{2.5.25}$$

From 2.5.24 and 2.5.25, we have

$$A \;\; = \;\; \frac{1}{B} \tag{2.5.26}$$

Plugging this result into the expression for $R_{\theta\theta}$, we obtain

$$R_{\theta\theta} = 0 \quad \Rightarrow \quad -1 + B'r + B(r) = 0 \tag{2.5.27}$$

Then we get

$$\frac{d}{dr}(rB) = rB' + B = 1$$

The solution is

$$rB = r + C \quad \Rightarrow \quad B = 1 + \frac{C}{r} \tag{2.5.28}$$

To fix C, we recall that at great distances from a central mass M_1, the component $g_{tt} = -B$ must approach $-1-2\Phi$, where Φ is Newtonian potential $(-GM_1/r)$. Hence $C = -2GM_1$ and we have

$$B \;\; = \;\; 1 - \frac{2GM_1}{c^2\, r}, \quad A = \left(1 - \frac{2GM_1}{c^2\, r}\right)^{-1} \tag{2.5.29}$$

Finally, the Schwarzschild Metric for the spacetime around a point mass is

$$c^2 d\tau^2 = ds^2 = \left(1 - \frac{2GM_1}{c^2\, r}\right) c^2 dt^2 - \left(1 - \frac{2GM_1}{c^2\, r}\right)^{-1} dr^2 - r^2 d\Omega^2 \qquad (2.5.30)$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$, t is the time coordinate, r is the radial coordinate, θ and ϕ are the usual angles for polar coordinates, and τ is the proper time.

2.5.1 The Schwarzschild Radius

At the end of the 18th century Laplace showed that a sufficiently massive body would prevent the escape of light from its surface. According to classical mechanics, the escape velocity from a body of radius R and mass M_1 is given by;

$$v_e = \sqrt{\frac{2GM_1}{R}} \qquad (2.5.31)$$

This is greater than the speed of light if the radius is smaller than the critical radius, then the Schwarzschild radius is given by

$$R_{Sch} = 2\left(\frac{GM_1}{c^2}\right) \qquad (2.5.32)$$

where

R_{Sch} = Schwarzschild radius.

G = Gravitational constant.

M_1 = Mass of Black Hole.

c = Speed of light.

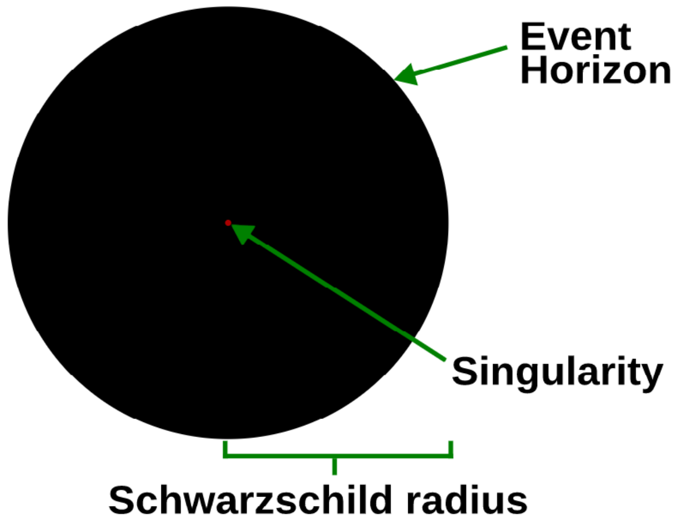

Figure 2.1: Showing Schwarzschild radius (Schwarzschild radius Wikidata; 2020)

This critical radius is now called the Schwarzschild radius and is the radius of the event horizon of non-rotating black holes. The term Black Hole is often attributed to John A. Wheeler, who wrote according to Einsteins general theory of relativity, as mass is added to a degenerate star a sudden collapse will take place and the intense gravitational field of the star will close in on itself. Such a star then forms a black hole in the universe.

The Schwarzschild geometry is the geometry of spacetime outside a spherical star. It is determined by one parameter, the mass M_1 [16]. The Schwarzschild geometry is asymptotically flat, because the metric tends to the Minkowski metric in polar coordinates at large radius as $r \to \infty$.

High-energy astrophysics required to study of the processes that occur within stars, black holes and supernovae. These processes monitored by measuring the high-energy electromagnetic radiation and particles that they emit including x-rays, ultra-violet light and gamma rays. These sources are remnants of stellar explosions, such as neutron stars, white dwarfs and black holes. Due to their extraordinary properties, compact objects form unique laboratories for understanding the physics of extreme environments. In addition, many questions about their origin, composition and evolution are still open today, more than 50 years after they were first discovered. The main study focus in the High Energy Astrophysics group Tbingen was X-ray binaries. These binary systems consist of a star and a compact object orbiting one another. In certain configurations and evolutionary states, the gravitational interactions of both objects leads to matter transfer from the outer envelope of the star onto the compact object. When this happens, the kinetic energy of the in-falling matter is converted into heat once it reaches the surface and X-rays can be observed. This physical process is called accretion [1].

2.6 The Kerr Solution

Kerr metric is the second exact solution of Einstein field equation, which can be used to describe space-time geometry in the vacuum area near a rotational, axialsymmetric heavenly body (Kerr, 1963). It is a generalized form of Schwarzschild metric. Kerr metric in Boyer-Lindquist coordinate system can be expressed in [17]

$$\begin{aligned} ds^2 &= \left(1 - \frac{2M_1 r}{\rho^2}\right) dt^2 + \frac{4M_1\, r\, a \sin^2\theta}{\rho^2} dt d\phi - \frac{\rho^2}{\Delta} dr^2 - \rho^2 d\theta^2 \\ &\quad - \left(r^2 + a^2 + \frac{2M_1\, r\, a^2 \sin^2\theta}{\rho^2}\right) \sin^2\theta d\phi^2 \end{aligned} \tag{2.6.1}$$

where

$$\rho^2 = r^2 + a^2 \cos^2\theta \tag{2.6.2}$$

$$\Delta = r^2 - 2M_1 r + a^2 \tag{2.6.3}$$

$$a = \frac{J}{M_1} \tag{2.6.4}$$

Here, M_1 is the mass of the black hole, and a is its angular momentum per unit mass, ϕ is the angle around the axis of symmetry, t is the time coordinate. Far from the black hole, $r \gg GM_1$; a, the metric reduces to flat Minkowski spacetime, with (t, r, θ, π) the usual coordinates, with $\theta \in [0; \pi]$ and $\phi \in [0; 2\pi)$.

By examining the components of metric tensor $g_{\mu\nu}$ in equation 2.6.1 , one can obtain:

$$\begin{aligned} g_{00} &= 1 - \frac{2M_1 r}{\rho^2} \\ g_{11} &= -\frac{\rho^2}{\Delta} \\ g_{22} &= -\rho^2 \\ g_{03} = g_{30} &= \frac{2M_1\, r\, a \sin^2\theta}{\rho^2} \\ g_{33} &= -\left(r^2 + a^2 + \frac{2M_1\, r\, a^2 \sin^2\theta}{\rho^2}\right)\sin^2\theta \end{aligned} \tag{2.6.5}$$

Both $g_{03}(g_{t\phi})$ and $g_{30}(g_{\phi t})$ off-diagonal terms in Kerr metric are not present in Schwarzschild metric, apparently due to rotation. If the rotation parameter $a = 0$, these two terms vanish. $g_{00}g_{11} = g_{tt}g_{rr} = -1$ in Schwarzschild metric, but not in Kerr metric. When spin parameter $a = 0$, Kerr metric turns into Schwarzschild metric and therefore is a generalized form of Schwarzschild metric.

Chapter 3

Radiation From Particles Falling Into Black-Holes

Once the solutions of the Einstein field equations (EFE) are found, we can study the geodesic equation of the free falling particles onto compact objects (COs). Since particles freely falling from infinity into event horizon of black hole, some amount of energy liberated due to strong gravitational pull of the black hole. This energy occurred in the form of radiation, travels through space at the speed of light. It has an electric field and magnetic field associated with it, and has wave-like properties. In our case the energy emitted in the form of electromagnetic radiation when a charged particle falls into a neutral Schwarzschild black-hole with zero angular momentum and gravitational radiation for uncharged particle of mass M_2 is calculated. The gravitational radiation due to a pointlike particle falling into a black hole (BH) is a classic problem in general relativity [18].

Another preview in this chapter is the study of Kerr black holes, exhibit frame-dragging effect. This effect predicts that objects coming close to a rotating central-mass will be entrained to participate in its rotation. Following Kerr metric in Boyer-Lindquist, the power radiated energy of particle falls onto neutral rotating black hole is calculated.

3.1 Free Falling of Charged Particles with Zero Angular Momentum

The total power passing out through a spherical surface at radius r is the integral of the Poynting vector [19] and given by

$$\mathbf{P}(\mathbf{r}) = \oint \mathbf{S} \cdot d\mathbf{a} = \frac{1}{\mu_0} \oint (\mathbf{E} \times \mathbf{B}) \cdot d\mathbf{a} \tag{3.1.1}$$

For an accelerated point charge q in straight-line motion, the angular distribution of the radiation through out solid angle ($d\Omega = \sin\theta d\theta d\phi$) is given by

$$\frac{d\mathbf{P}}{d\Omega} = \frac{\mu_0 q^2 \mathbf{a}^2}{16\pi^2 c} \frac{\sin^2\theta}{(1-\beta\cos\theta)^5} \tag{3.1.2}$$

where ($\mu_0 = 1/c^2\epsilon_0$, $\beta = \mathbf{v}/c$). Using equation 3.1.2, we can now calculate the electromagnetic radiation emitted by a charged particle falling radially into a black-hole. Then the energy given off per unit time is given by

$$-\frac{dE(\theta)}{dt} = \frac{\dot{\mathbf{v}}^2}{c^3} \frac{e^2 z^2}{16\pi^2\epsilon_0} \frac{\sin^2\theta}{(1-\beta\cos\theta)^5} \tag{3.1.3}$$

where ze is the charge, $\beta = \mathbf{v}/c$, θ is the angle between the line of sight and the particles velocity $\mathbf{v}$, and a dot denotes differentiation with respect to time. Since we are not interested in the angular distribution, we integrate over θ to get

$$-\frac{dE}{dt} = \frac{\dot{\mathbf{v}}^2}{c^3}\frac{e^2}{6\pi\epsilon_0}\left\{\frac{1}{(1-\beta^2)^3}\right\} \tag{3.1.4}$$

Through terms of order β^3 and for the velocities of interest to us, we can approximate this by

$$-\frac{dE}{dt} = \frac{\dot{\mathbf{v}}^2}{c^3}\frac{e^2}{6\pi\epsilon_0}\left(1+3\beta^2\right) \tag{3.1.5}$$

This gives the energy emitted by the accelerating charge, where retardation effects have been taken into account. The total energy received at infinity, neglecting the radiation lost down the black-hole would be calculated. To get this we must first correct equation 3.1.5 for redshift. If a clock at infinity records the passive of a time Δt_∞, then one near the black hole will record a time

$$\begin{aligned}\Delta t_0 &= \Delta t_\infty\left(1-\frac{2GM_1}{r_0c^2}\right)^{\frac{1}{2}} \\ &= \Delta t_\infty\left(1-\frac{R_{Sch}}{r_0}\right)^{\frac{1}{2}}\end{aligned} \tag{3.1.6}$$

Thus objects near R_{Sch} appear to outside observers to slow down, until at R_{Sch} they become entirely frozen in time.

The frequency of a photon of frequency ν_0 emitted at a radius r_0 around a black hole and received at infinity will have frequency

$$\begin{aligned}\nu_\infty &= \nu_0\left(1-\frac{2GM_1}{r_0c^2}\right)^{\frac{1}{2}} \\ &= \nu_0\left(1-\frac{R_{Sch}}{r_0}\right)^{\frac{1}{2}}\end{aligned} \tag{3.1.7}$$

Note that this formula has the property that, as $r_0 \to R_{Sch}$, the observed frequency $\nu_\infty \to 0$. Thus, light emitted near the event horizon becomes more and more redshifted, until finally at the event horizon it becomes infinitely redshifted and can no longer be observed by the outside world. The light coming out of the clock near the black hole will be redshifted, so its frequency will diminish as seen from an observer at infinity. This means that fewer wave crests pass the detector on that clock than pass the detector at infinity in the same amount of time. When the clock near the black hole is pulled back up, it will have recorded fewer ticks than the clock at infinity. Therefore, time must slow down near the black hole.

Since the particle is in falling, then the radial part from equation 2.5.30 being calculated as

$$c^2 d\tau^2 = \left(1 - \frac{R_{Sch}}{r}\right) c^2 dt^2 - \left(1 - \frac{R_{Sch}}{r}\right)^{-1} dr^2 \tag{3.1.8}$$

For an object at fixed r, then the spacing of ticks of the clock in proper time is given by

$$d\tau_r = \left(1 - \frac{R_{Sch}}{r}\right)^{\frac{1}{2}} dt \tag{3.1.9}$$

Thus for an observer at infinity (as $r \rightarrow \infty$, $\left(1 - \frac{R_{Sch}}{r}\right)^{\frac{1}{2}} \rightarrow 1$), then

$$d\tau_\infty = dt \tag{3.1.10}$$

Thus the frequency of a photon observed at infinity ν_∞ is related to the frequency ν_r emitted at a distance r from a black-hole by

$$\begin{aligned} \frac{\nu_\infty}{\nu_r} &= \frac{d\tau_r}{d\tau_\infty} \\ &= \left(1 - \frac{2GM_1}{c^2 r}\right)^{\frac{1}{2}} \frac{dt}{dt} \\ &= \left(1 - \frac{R_{Sch}}{r}\right)^{\frac{1}{2}} \end{aligned} \tag{3.1.11}$$

where G is the gravitational constant, R_{Sch} is Schwarzschild radius. Thus the energy received at infinity is related to the energy emitted near a black-hole by

$$dE_{received} = dE_{emitted}\left(1-\frac{2M_1}{r}\right)^{\frac{1}{2}} \tag{3.1.12}$$

and we have

$$-\frac{dE_{received}}{dt} = \frac{\dot{v}^2}{c^3}\frac{e^2}{6\pi\epsilon_0}\left(1+3\beta^2\right)\left(1-\frac{2M_1}{r}\right)^{\frac{1}{2}} \tag{3.1.13}$$

The redshift correction is very important near the Schwarzschild radius $\mathbf{r} = 2M_1$. To integrate equation 3.1.13 we need $\mathbf{v}$ and $\dot{\mathbf{v}}$ for a particle falling in a Schwarzschild field. We are considering the problem of a small mass $M_2 \ll M_1$ so that the falling particle does not perturb the metric. To find the geodesics of the Schwarzschild metric, we use variational principle along a curvature by the parameter τ. We have

$$s = \int \mathcal{L}\, d\tau = extremum \tag{3.1.14}$$

So that equation 2.5.30 rewritten as

$$\begin{aligned}
1 = \frac{ds^2}{d\tau^2} &= \left(1-\frac{2M_1}{r}\right)\frac{dt^2}{d\tau^2} - \left(1-\frac{2M_1}{r}\right)\frac{dr^2}{d\tau^2} - r^2\frac{d\theta^2}{d\tau^2} - r^2\sin^2\theta\frac{d\phi^2}{d\tau^2} \\
\frac{ds}{d\tau} &= \left[\left(1-\frac{2M_1}{r}\right)\dot{t}^2 - \left(1-\frac{2M_1}{r}\right)\dot{r}^2 - r^2\dot{\theta}^2 - r^2\sin^2\theta\dot{\phi}^2\right]^{\frac{1}{2}} \\
\int ds &= \int\left[\left(1-\frac{2M_1}{r}\right)\dot{t}^2 - \left(1-\frac{2M_1}{r}\right)\dot{r}^2 - r^2\dot{\theta}^2 - r^2\sin^2\theta\dot{\phi}^2\right]^{\frac{1}{2}} d\tau \\
s &= \int\left[\left(1-\frac{2M_1}{r}\right)\dot{t}^2 - \left(1-\frac{2M_1}{r}\right)\dot{r}^2 - r^2\dot{\theta}^2 - r^2\sin^2\theta\dot{\phi}^2\right]^{\frac{1}{2}} d\tau \qquad (3.1.15)
\end{aligned}$$

From 3.1.14 and 3.1.15, the Euler-Lagrange equation of the Schwarzschild metric becomes

$$\mathcal{L} = 1 = \left[\left(1 - \frac{2M_1}{r}\right)c^2\dot{t}^2 - \frac{\dot{r}^2}{1-\frac{2M_1}{r}} - r^2\dot{\theta}^2 - r^2\sin^2\theta\dot{\phi}^2\right]^{\frac{1}{2}} \tag{3.1.16}$$

The Euler-Lagrange equation for t is then

$$\frac{d}{d\tau}\frac{\partial\mathcal{L}}{\partial\dot{t}} - \frac{\partial\mathcal{L}}{\partial t} = 0 \tag{3.1.17}$$

which since $\frac{\partial\mathcal{L}}{\partial t} = 0$,

$$\frac{\partial\mathcal{L}}{\partial\dot{t}} = \frac{1}{2}\left[\left(1 - \frac{2M_1}{r}\right)c^2\dot{t}^2 - \frac{\dot{r}^2}{1-\frac{2M_1}{r}} - r^2\dot{\theta}^2 - r^2\sin^2\theta\dot{\phi}^2\right]^{-\frac{1}{2}}\left(1 - \frac{2M_1}{r}\right)2\dot{t} \tag{3.1.18}$$

where, ($\dot{t} = \frac{dt}{d\tau}$, $\dot{\theta} = \frac{d\theta}{d\tau}$, $\dot{\phi} = \frac{d\phi}{d\tau}$)

so that eq. 3.1.18 reduced to

$$c\dot{t}\left(1 - \frac{2M_1}{r}\right) = constant. \tag{3.1.19}$$

This implies that there is a conserved quantity we will call energy per unit mass:

$$\frac{\partial\mathcal{L}}{\partial\dot{t}} = \frac{E}{M_2} \tag{3.1.20}$$

We have

$$c\dot{t}\left(1 - \frac{2M_1}{r}\right) = \frac{E}{M_2} \tag{3.1.21}$$

For $r \to \infty$, the Schwarzschild metric goes to the Minkowski metric and for the Minkowski metric $\dot{t} = E/M_2$ for (c=1).

Next we find the ϕ equation:

$$\frac{d}{d\tau}\frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \frac{\partial\mathcal{L}}{\partial\phi} = 0 \tag{3.1.22}$$

so again we have a conserved quantity $p_\phi = \partial\mathcal{L}/\partial\dot{\phi} = -L/m$, where we will call this conserved quantity the angular momentum per unit mass. The metric does not depend explicitly on the angle ϕ, we get that result here. Hence we have

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial\dot{\phi}} &= \frac{1}{2}\left[\left(1-\frac{2M_1}{r}\right)c^2\dot{t}^2 - \frac{\dot{r}^2}{1-\frac{2M_1}{r}} - r^2\dot{\theta}^2 - r^2\sin^2\theta\dot{\phi}^2\right]^{-\frac{1}{2}} \\ &\quad\times\left(-r^2\sin^2\theta 2\dot{\phi}\right) \\ &= -r^2\sin^2\theta\dot{\phi} \qquad (3.1.23)\end{aligned}$$

Thus our ϕ equation reads

$$\frac{L}{M_2} = r^2\sin^2\theta\dot{\phi} \qquad (3.1.24)$$

So it makes sense that we called the constant of motion L/M_2.

Next, we consider the θ equation. Here we find that

$$\frac{d}{d\tau}\frac{\partial\mathcal{L}}{\partial\dot{\theta}} = \frac{\partial\mathcal{L}}{\partial\theta} \neq 0 \qquad (3.1.25)$$

thus we do not have a conserved quantity for this equation. We find

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial\theta} &= \frac{1}{2}\left[\left(1-\frac{2M_1}{r}\right)c^2\dot{t}^2 - \frac{\dot{r}^2}{1-\frac{2M_1}{r}} - r^2\dot{\theta}^2 - r^2\sin^2\theta\dot{\phi}\right]^{-\frac{1}{2}} \\ &\quad\times\left(-r^2 2\sin\theta\cos\theta\dot{\phi}^2\right) \\ &= -r^2\dot{\phi}^2\sin\theta\cos\theta \qquad (3.1.26)\end{aligned}$$

and

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial\dot{\theta}} &= \frac{1}{2}\left[\left(1-\frac{2M_1}{r}\right)c^2\dot{t}^2 - \frac{\dot{r}^2}{1-\frac{2M_1}{r}} - r^2\dot{\theta}^2 - r^2\sin^2\theta\dot{\phi}\right]^{-\frac{1}{2}}\left(-r^2 2\dot{\theta}\right) \\ &= -r^2\dot{\theta} \qquad (3.1.27)\end{aligned}$$

Thus our θ equation reads:

$$\frac{d}{dr}\left(r^2\dot{\theta}\right) = r^2\dot{\phi}^2\sin\theta\cos\theta \tag{3.1.28}$$

In our case, the whole squared equations of motion from 3.1.16 becomes

$$1 = \left(1-\frac{2GM_1}{r}\right)\dot{t}^2 - \frac{\dot{r}^2}{(1-\frac{2GM_1}{r})} - r^2\dot{\theta}^2 - r^2\sin^2\theta\dot{\phi}^2 \tag{3.1.29}$$

Since our object and metric are spherically symmetric we can simplify things by only considering motion in the equatorial plane ($\theta = \pi/2$, $\dot{\theta} = 0$). In this case then, equation 3.1.16 gives:

$$1 = \left(1-\frac{2GM_1}{r}\right)\dot{t}^2 - \frac{\dot{r}^2}{(1-\frac{2GM_1}{r})} - r^2\dot{\phi}^2 \tag{3.1.30}$$

Now eliminate variables other than r, using the constants of motion we have from the above equations:

$L/M_2 = r^2\dot{\phi}$ and $E/M_2 = (1-\frac{2GM_1}{r})\dot{t}^2$, to get:

$$1 = \frac{E^2}{M_2^2(1-\frac{2GM_1}{r})} - \frac{\dot{r}^2}{(1-\frac{2GM_1}{r})} - \frac{L^2}{M_2^2 r^2} \tag{3.1.31}$$

We can write this by solving for $M_2\dot{r}^2$,

$$M_2\dot{r}^2 = \frac{E^2}{M_2} - \left(M_2 + \frac{L^2}{M_2 r^2}\right)(1-\frac{2GM_1}{r}), \tag{3.1.32}$$

then this equation rewritten as:

$$M_2\left(\frac{dr}{d\tau}\right)^2 = \frac{E^2}{M_2 c^2} - \left(1-\frac{2GM_1}{rc^2}\right)\left(M_2 c^2 + \frac{L^2}{M_2 r^2}\right) \tag{3.1.33}$$

This geodesic equation shows that things fall due to the spacetime curvature of the metric as a step in proper time $d\tau$ forces a step dr in the r direction. Suppose the angular momentum $L = 0$, which we might expect for radial infall towards a spherical mass. Our equation then is:

$$M_2 \left(\frac{dr}{d\tau}\right)^2 = \frac{E^2}{M_2 c^2} - \left(1 - \frac{2GM_1}{rc^2}\right) M_2 c^2 = 0 \tag{3.1.34}$$

Next we consider a case where we start at rest far from the object so that at proper time $\tau = 0$, $M_2 \left(\frac{dr}{d\tau}\right)^2 = 0$, and $r \to \infty$. Our equation becomes:

$$\frac{E^2}{M_2 c^2} - M_2 c^2 + \lim_{r\to\infty} \frac{2GM_1 M_2}{r} \quad = \quad 0, \tag{3.1.35}$$

or $E^2/M_2c^2 = M_2c^2$, or simply $E = M_2c^2$. So at $\tau = 0$ the total energy is just $E = M_2c^2$. In Newtonian mechanics the energy at infinity is usually defined as $E = 0$. Also since energy E is conserved along geodesics we know that $E = M_2$ always. This E is not the Newtonian energy; it is the conserved quantity, which is better than the sum of $\frac{1}{2}M_2v^2 + V$. If we would have started with some velocity at $r \to \infty$ then $E > M_2c^2$ but it still would have been conserved.

At later times during this radial infall from rest, our equation becomes:

$$M_2 \left(\frac{dr}{d\tau}\right)^2 = \frac{E^2}{M_2 c^2} - M_2 c^2 + \frac{2GM_1}{rc^2} M_2 c^2 = \frac{2GM_1 M_2}{r} \tag{3.1.36}$$

That is simply

$$\frac{1}{2} M_2 \left(\frac{dr}{d\tau}\right)^2 - \frac{GM_1 M_2}{r} \quad = \quad 0, \tag{3.1.37}$$

which looks remarkably like the Newtonian case $\frac{1}{2}M_2v^2 - GM_1M_2/r = 0$. But this is not the same equation.

If we consider the case where the particle has zero velocity at infinity and falls from there, we find that the constant in equation 3.1.19 is equal to 1. Since our particle radially infalling ($d\theta = d\phi = 0$), then dividing eq. 2.5.30 by dt^2

$$\left(\frac{ds}{dt}\right)^2 = \left(1 - \frac{2M_1}{r}\right)c^2 - \left(1 - \frac{2M_1}{r}\right)^{-1}\frac{dr^2}{dt^2} \tag{3.1.38}$$

But equation 3.1.19 rewritten as

$$\begin{aligned} c\left(\frac{dt}{ds}\right)\left(1 - \frac{2M_1}{r}\right) &= 1 \\ \Rightarrow \frac{ds}{dt} &= c\left(1 - \frac{2M_1}{r}\right) \end{aligned} \tag{3.1.39}$$

Substituting equation 3.1.39 into 3.1.38 gives

$$c^2\left(1 - \frac{2M_1}{r}\right)^2 = \left(1 - \frac{2M_1}{r}\right)c^2 - \left(1 - \frac{2M_1}{r}\right)^{-1}\frac{dr^2}{dt^2} \tag{3.1.40}$$

By collecting like terms we get

$$\begin{aligned} c^2\left(1 - \frac{2M_1}{r}\right)\left[1 - \frac{2M_1}{r} - 1\right] &= -\left(1 - \frac{2M_1}{r}\right)^{-1}\frac{dr^2}{dt^2} \\ -\left(\frac{dr}{dt}\right)^2 &= c^2\left(1 - \frac{2M_1}{r}\right)^2\left(\frac{2M_1}{r}\right) \\ \mathbf{v} = \frac{dr}{dt} &= -c\left(1 - \frac{2M_1}{r}\right)\sqrt{\frac{2M_1}{r}} \end{aligned} \tag{3.1.41}$$

Differentiating this equation with respect to t then gives the acceleration.

$$\begin{aligned} \dot{\mathbf{v}} &= \frac{d^2r}{dt^2} \\ &= -\frac{M_1c^2}{r^2}\left[1 - \frac{2M_1}{r} - \frac{3(dr/dt)^2}{1 - \frac{2M_1}{r}}\right] \\ &= -\frac{M_1c^2}{r^2}\left(1 - \frac{2M_1}{r}\right)\left(1 - \frac{6M_1}{r}\right) \\ &= -\frac{M_1c^2}{r^2}\left(1 - \frac{8M_1}{r} + \frac{12M_1^2}{r^2}\right) \end{aligned} \tag{3.1.42}$$

If $\frac{M_1}{r} \ll 1$, equations 3.1.41 and 3.1.42 reduce to the Newtonian values. We note from equation 3.1.41 that the maximum velocity is $\beta_{max} = 0.385$ and that it occurs at

$r = 6M_1$. If we had taken the velocity to be zero at a radius r_α instead of at infinity, the constant in equation 3.1.19 would be $\left(1 - \frac{2M_1}{r_\alpha}\right)^{1/2}$, then equation 3.1.41 becomes

$$\frac{d\mathbf{r}}{dt} = -\left[1 - \frac{2M_1}{r}\right]\left[1 - \frac{\left(1 - \frac{2M_1}{r}\right)}{\left(1 - \frac{2M_1}{r_\alpha}\right)}\right]^{\frac{1}{2}} c \qquad (3.1.43)$$

Now multiplying equation 3.1.13 through by dt/dr and putting in our expressions for $\mathbf{v}$ and $\dot{\mathbf{v}}$ gives

$$-\frac{dE_{obs}}{d\mathbf{r}} = -\frac{e^2 z^2 c}{6\pi\epsilon_o 16 M_1}\alpha^4 (1-3\alpha)^2 (1-\alpha)^{3/2}\left[1 + 3\alpha(1-\alpha)^2\right]\sqrt{\frac{1}{\alpha}} \qquad (3.1.44)$$

where $\alpha = 2M_1/r$. Integrating this equation from $r = \infty$ to some radius $r = r_{min}$ by changing variables to α, then gives

$$E_{observed} = M_e c^2 \left(\frac{r_e}{M_1}\right)\left(\frac{I}{12}\right) z^2 \qquad (3.1.45)$$

where r_e is the classical electron radius and M_e is the electron rest mass ($M_e c^2 r_e = e^2/4\pi\epsilon_o$). The I in equation 3.1.45 is given by

$$\begin{aligned} I = {} & 0.017166\left[\frac{\pi}{2} - \arcsin(1-2x)\right] + 0.034332(2x-1)(x-x^2)^{1/2} + (x-x^2)^{3/2} \\ & \times\left[-0.0915528 + 0.290137x - 0.811272x^2 - 3.568081x^3\right. \\ & \left. +10.834821x^4 - 9.5625x^5 + 3x^6\right] \end{aligned} \qquad (3.1.46)$$

where $x = 2M_1/r_{min}$. Since the particle is falling onto the Schwarzschild radius, then $r_{min} = 2M_1$ and equation 3.1.46 reduces to $I = 0.017166\pi$. Finally, our equation 3.1.45 becomes

$$E_{observed} = 0.0044932\ M_e c^2 \left(\frac{r_0}{M_1}\right) z^2 \qquad (3.1.47)$$

This is the result for the electromagnetic radiation which escapes to infinity if a particle of charge ze falls into a black-hole from $r = \infty$ to $r = 2M_1$ with zero angular momentum. The mass of the falling particle M_2 does not enter. It is interesting that if the redshift and β^2 corrections are not made in equation 3.1.13 the above coefficient is increased to 0.008. If higher order terms are neglected in v and $\dot{v}$, the coefficient is further increased to 0.033.

3.2 Particle Spiralling Black-Hole

Using equation 3.1.31, the orbital energy of a particle spiraling a black-hole is

$$\begin{aligned} E^2 &= \left[\frac{\dot{r}^2}{(1-\frac{2M_1}{r})} + (1+\frac{L^2}{M_2^2 r^2})\right] M_2^2 (1-\frac{2M_1}{r}) \\ &= \left[\dot{r}^2 + (1-\frac{2M_1}{r})(1+\frac{L^2}{M_2^2 r^2})\right] M_2^2 \\ E &= M_2 \left[\dot{r}^2 + (1-\frac{2M_1}{r})(1+\frac{L^2}{M_2^2 r^2})\right]^{1/2} \qquad (3.2.1) \end{aligned}$$

For the case of particle starts at rest far from the black hole, then the proper time $\tau = 0$; $M_2(\frac{dr}{d\tau})^2 = 0$, thus we get the energy of spiralling particle as

$$E(r) = M_2 c^2 \left(1 + \frac{L^2}{M_2^2 c^2 r^2}\right)^{\frac{1}{2}} \left(1 - \frac{2M_1}{r}\right)^{\frac{1}{2}} \qquad (3.2.2)$$

where L is the angular momentum of the orbiting particle, $M_1 = GM_1/c^2$, M_2 and M_1 is mass of spiraling particle and black hole respectively. The particle will orbit stably where equation 3.2.2 goes through a minimum as a function of r. This radius is given by differentiating 3.2.2 as

$$
\begin{aligned}
\frac{d}{dr}\left[M_2\left(1+\frac{L^2}{M_2^2r^2}\right)^{\frac{1}{2}}\left(1-\frac{2M_1}{r}\right)^{\frac{1}{2}}\right] &= 0\\
-\frac{L^2\left(1-\frac{2M_1}{r}\right)^{\frac{1}{2}}}{r^3M_2\left(1+\frac{L^2}{M_2^2r^2}\right)}+\frac{M_1M_2\left(1+\frac{L^2}{M_2^2r^2}\right)^{\frac{1}{2}}}{\left(1-\frac{2M_1}{r}\right)^{\frac{1}{2}}r^2} &= 0\\
L^2r^2(1-\frac{2M_1}{r})-M_1M_2^2r^3(1+\frac{L^2}{M_2^2r^2}) &= 0\\
L^2r^2-2\frac{L^2r^2M_1}{r}-M_1M_2^2r^3-\frac{M_1M_2^2L^2r^3}{M_2^2r^2} &= 0\\
M_1M_2^2r^2-L^2r+3M_1L^2 &= 0 \qquad (3.2.3)
\end{aligned}
$$

which is in the form of quadratic equation. Now we can find the circular radius of spiraling massive particle from this equation, then it yields

$$
\begin{aligned}
r &= \frac{L^2\pm\sqrt{L^4-12M_1^2M_2^2L^2}}{2M_1M_2^2}\\
&= \frac{L^2}{2M_1M_2^2}\pm\sqrt{\frac{L^4}{4M_1^2M_2^4}-\frac{3L^2}{M_2^2}} \qquad (3.2.4)
\end{aligned}
$$

This equation can be rewritten as

$$
\begin{aligned}
r &= \left(a^2+\sqrt{a^4-3a^2}\right)2M_1\\
&= 2M_1a(a+\sqrt{a^2-3}) \qquad (3.2.5)
\end{aligned}
$$

where $a = L/2M_1M_2$. So as we see from 3.2.5, the radius is real only if $a \geq \sqrt{3}$. An orbit with $a = \sqrt{3}$ has the minimum possible angular momentum. In this optimum case a particle will spiral down to $r = 6M_1$ before it falls into the black-hole without further radiation of energy. Particle in this material will gradually lose energy because of friction in the disc and so its value of E will decrease. As a result r will decrease: the particle will gradually spiral in to smaller and smaller r. Eventually the particle will reach the innermost stable circular orbit (ISCO), which has $E = M_2c^2\sqrt{\frac{8}{9}}$, after which it falls rapidly into the hole. The energy that the particle loses as it moves towards the ISCO leaves the disc as radiation, $M_2c^2(1-\sqrt{\frac{8}{9}})$ is radiated away. This is 5.7% of the rest mass of the orbiting particle.

A charged and uncharged particle with the same mass will radiate away the same total energy as they spiral inward. This is in contradistinction to the straight fall case. The difference now is that charged and uncharged particles spiral inward at different rates. It is of interest then to calculate the time required for a charged and uncharged particle to spiral in from a given distance away from the black-hole.

In order to calculate the spiral time, we shall use equation 3.2.2 with the value $a = \sqrt{3}$. This is the most interesting case since it gives the most energy out. The orbital energy of the particle is then

$$E(r) = M_2c^2\left(1+\frac{12M_1^2}{r^2}\right)^{1/2}\left(1-\frac{2M_1}{r}\right)^{1/2} \tag{3.2.6}$$

We can now differentiate this with respect to r to get

$$\begin{aligned}\frac{dE}{dr} &= -\frac{12M_1^2M_2c^2(1-\frac{2M_1}{r})^{1/2}}{(1+\frac{12M_1^2}{r^2})^{1/2}r^3} + \frac{M_1M_2(1+\frac{12M_1^2}{r^2})^{1/2}}{(1-\frac{2M_1}{r})^{1/2}r^2} \\ &= -3M_2c^2\alpha^3\frac{(1-\alpha)^{1/2}}{(1+3\alpha^2)^{1/2}} + \frac{M_2c^2}{4M_1}\alpha^2\frac{(1+3\alpha^2)^{1/2}}{(1-\alpha)^{1/2}}\end{aligned} \tag{3.2.7}$$

where $\alpha = 2M_1/r$. Equation 3.2.7 holds for both charged and uncharged particles. To calculate the spiral time, we need expressions for dE/dt for the gravitational and electromagnetic cases separately. These expressions for dE/dt can be combined with equation (20) to give dr/dt which in turn can be integrated over r to give the spiral time Δt. This dr/dt is the radial velocity as the particle spirals inward and not the orbital velocity. From Jackson (1999), for quasi-circular motion we have for electromagnetic radiation

$$-\left(\frac{dE}{dt}\right)_{e.m} = \left(\frac{e^2 z^2 \dot{v}^2}{6\pi\epsilon_0 c^3}\right)\gamma^4 \tag{3.2.8}$$

where $\gamma = 1/\sqrt{1-\frac{v^2}{c^2}}$. Approximating the β dependence, equation 3.2.8 can be written as

$$-\left(\frac{dE}{dt}\right)_{e.m} = \left(\frac{e^2 z^2 \dot{v}^2}{6\pi\epsilon_0 c^3}\right)(1+2\beta^2_{orbital}) \tag{3.2.9}$$

The angular dependence has been integrated out. β is the orbital velocity of the particle now. We can get this easily from the angular momentum. Since

$$L = \frac{M_2\, c\, r\, \beta_{orbital}}{\sqrt{1-\beta^2_{orbital}}}$$

and since $a = \sqrt{3}$, we have

$$\beta^2_{orbital} = \frac{12M_1^2}{(r^2+12M_1^2)} \tag{3.2.10}$$

The radial acceleration is the same as for the straight fall case and is given by equation 3.1.42. Putting equations 3.1.42 and 3.2.10 into equation 3.2.9 then gives

$$\left(-\frac{dE}{dt}\right)_{e.m} = \frac{z^2 M_e\, c^3\, r_0}{24M_1}\alpha^4(1-3\alpha)^2(1-\alpha)^2\frac{(1+9\alpha^2)}{(1+3\alpha^2)} \tag{3.2.11}$$

Because we will combine equation 3.2.7 with equation 3.2.11 and cancel out the dE in order to obtain dr/dt, we will work to second order in α only. This will introduce very little error in Δt since the main contribution to Δt comes from large radii and since $\alpha_{max} = 1/3$ for the smallest radius of interest. From equation 3.2.11 we then have

$$\left(\frac{dt}{dE}\right)_{e.m} = -\left(\frac{12M_1^2}{z^2 M_e c^3 r_0}\right)\frac{1}{\alpha^4}(1+8\alpha+36\alpha^2) \tag{3.2.12}$$

To second order in α we can write equation 3.2.7 as

$$\frac{dE}{dr} = \frac{M_2 c^2}{4M_1}\alpha^2(1 - \frac{11}{2}\alpha + \frac{39}{8}\alpha^2) \tag{3.2.13}$$

Multiplying equations 3.2.12 and 3.2.13 gives

$$\begin{aligned}
\left(\frac{dt}{dr}\right)_{e.m} &= -\frac{6M_1M_2}{z^2 M_e\, c\, r_0}\frac{1}{\alpha^2}(1+8\alpha+36\alpha^2)(1-\frac{11}{2}\alpha+\frac{39}{8}\alpha^2) \\
&= -\frac{6M_1M_2}{z^2 M_e\, c\, r_0}\frac{1}{\alpha^2}\left(1-\frac{11}{2}\alpha+\frac{39}{8}\alpha^2+8\alpha-44\alpha^2+36\alpha^2\right) \\
dt_{e.m} &= -\frac{6M_1M_2}{z^2 M_e\, c\, r_0}\frac{1}{\alpha^2}\left(1+\frac{5}{2}\alpha-\frac{25}{8}\alpha^2\right)dr \qquad (3.2.14)
\end{aligned}$$

But we have $dr = -\frac{2M_1}{\alpha^2}d\alpha$, then we can integrate equation 3.2.14 over r by way of a variable change to α gives

$$\begin{aligned}
\Delta t_{e.m} &= \frac{12M_1^2M_2}{z^2 M_e\, c\, r_0}\int_{r_{min}}^{r_{max}}\left(\frac{1}{\alpha^4}+\frac{5}{2\alpha^3}-\frac{25}{8\alpha^2}\right)d\alpha \\
&= \frac{4M_1^2M_2}{z^2 M_e\, c\, r_0}\int_{r_{min}}^{r_{max}}\left(\frac{3}{\alpha^4}+\frac{15}{2\alpha^3}-\frac{75}{8\alpha^2}\right)d\alpha \\
&= \frac{4M_1^2M_2}{z^2 M_e\, c\, r_0}\left[A^3\left(1+\frac{15}{4}\frac{1}{A}-\frac{75}{8}\frac{1}{A^2}\right)-B^3\left(1+\frac{15}{4}\frac{1}{B}-\frac{75}{8}\frac{1}{B^2}\right)\right]
\end{aligned} \tag{3.2.15}$$

where $A = r_{max}/2M_1$, $B = r_{min}/2M_1$ and r_{max} and r_{min} are the beginning and ending radii of the spiraling orbit. The gravitational radiation from an uncharged particle in a circular orbit [20] is

$$\left(\frac{dt}{dE}\right)_{grav} = -\frac{5G}{c^5}\left(\frac{M_1}{M_2}\right)^2\frac{1}{\alpha^5} \tag{3.2.16}$$

using the linearized gravitational field equations and $M_2 \ll M_1$. Combining equation 3.2.13 and 3.2.16 and integrating as above then gives

$$\Delta t_{grav} = \frac{5}{8}\left(\frac{G}{c^3}\right)M_2\left(\frac{M_1}{M_2}\right)^2\left[A^4\left(1-\frac{22}{3}\frac{1}{A}+\frac{39}{4}\frac{1}{A^2}\right)-B^4\left(1-\frac{22}{3}\frac{1}{B}+\frac{39}{4}\frac{1}{B^2}\right)\right] \tag{3.2.17}$$

From the equations 3.2.15 and 3.2.17, we have the results of coordinate time for the observer. By taking the mass of the black-hole, M_1 equal to one solar mass, we can compare our results for $\Delta t_{e.m}$ and Δt_{grav}. Let us consider the very end of the spiral process where events happen most rapidly and choose $A = 10$ and $B = 3$ (the minimum possible value). If the orbiting particle is a system of N electrons, we find from equations 3.2.15 and 3.2.17 that $\Delta t_{e.m} = 0.394 \times 10^9/N$ years and $\Delta t_{grav} = 0.79 \times 10^{51}/N$ years. Therefore, we have that $\Delta t_{grav}/\Delta t_{e.m} = 2 \times 10^{42}$ so that charged particles spiral in much faster. Δt_{grav} is so prohibitively long that the particles essentially never spiral in.

3.3 Geodesics Equations of Radially Infalling Particles

In this section we consider the simple spacetime trajectory of a test particle moving radially with respect to a spherical mass. A test particle is sufficiently small in comparison with the gravitating spherical mass so that the particles contribution to the overall gravitational field is negligible. Since we are considering the situation where no other external forces are present we expect that these particles will travel on timelike equations governed by the geodesic equations.

The particle dropped from a radius $r_0 \gg r_s$ (where r_s is the Schwarzschild radius of the black hole) and fall radially inward starting from rest [21], then the initial conditions of this particle will be:

$$
\begin{aligned}
x_0^\alpha &= (0, r_0, \frac{\pi}{2}, 0) \\
u_0^\alpha &= (u_0^t, 0, 0, 0) \qquad (3.3.1)
\end{aligned}
$$

The value of u_0^t found from

$$
g_{\mu\nu} u_0^\mu u_0^\nu = -1
$$

then u_0^t becomes

$$
u_0^t = \frac{1}{\sqrt{1 - \frac{2M_1}{r_0}}} \qquad (3.3.2)
$$

Using the Christoffel symbols from 2.5.13 into the geodesic equation of 2.1.7, we get

$$
\begin{aligned}
\frac{d^2x^t}{d\tau^2} &= -\frac{2M_1}{r^2(1-\frac{2M_1}{r})} u^t u^r \\
\frac{d^2x^r}{d\tau^2} &= -\frac{M_1}{r^2}\left(1 - \frac{2M_1}{r}\right)(u^t)^2 + \frac{M_1}{r^2(1-\frac{2M_1}{r})}(u^r)^2 - (1 - \frac{2M_1}{r})(u^\theta)^2 - (1 - \frac{2M_1}{r}) r \sin^2\theta (u^\phi)^2 \\
\frac{d^2x^\theta}{d\tau^2} &= -\frac{2}{r} u^r u^\theta + \sin\theta\cos\theta (u^\phi)^2 \\
\frac{d^2x^\phi}{d\tau^2} &= -\frac{2}{r} u^r u^\phi - \frac{\cos\theta}{\sin\theta} u^\phi u^\phi \qquad (3.3.3)
\end{aligned}
$$

By writing $\frac{d^2x^t}{d\tau^2} = \frac{du^t}{d\tau}$ and $u^r = \frac{dr}{d\tau}$, then integrating the first of the geodesic equations using the initial conditions becomes:

$$
\begin{aligned}
\frac{du^t}{d\tau} &= -\frac{2M_1}{r^2(1-\frac{2M_1}{r})} u^t \frac{dr}{d\tau} \\
\int_{u_0^t}^{u^t} \frac{du^t}{u^t} &= -\int_{r_0}^{r} \frac{2M_1}{r^2(1-\frac{2M_1}{r})} dr \\
\ln\left(\frac{u^t}{u_0^t}\right) &= \ln\left(\frac{1-\frac{2M_1}{r_0}}{1-\frac{2M_1}{r}}\right) \\
u^t &= u_0^t \left(\frac{1-\frac{2M_1}{r_0}}{1-\frac{2M_1}{r}}\right) \qquad (3.3.4)
\end{aligned}
$$

Since we are considering the particle is expected to fall radially only, then u^θ and u^ϕ are zero. Therefore $\frac{d^2x^\theta}{d\tau^2}$ and $\frac{d^2x^\phi}{d\tau^2}$ are constant.

The radial component of the geodesic equation can be found by 4-velocity and inserting the values for u^t, u^θ and u^ϕ

$$\begin{aligned}
-1 &= -(1-\frac{2M_1}{r})(u^t)^2 + \frac{(u^r)^2}{(1-\frac{2M_1}{r})} + r^2u^\theta + r^2\sin^2\theta(u^\phi)^2 \\
&= \frac{1-\frac{2M_1}{r_0}}{1-\frac{2M_1}{r}} + \frac{(u^r)^2}{1-\frac{2M_1}{r}} \\
\frac{(u^r)^2}{1-\frac{2M_1}{r}} &= \frac{1-\frac{2M_1}{r_0}}{1-\frac{2M_1}{r}} - 1 \\
u^r &= \sqrt{(1-\frac{2M_1}{r_0}) - (1-\frac{2M_1}{r})} \\
&= -\sqrt{\frac{2M_1}{r} - \frac{2M_1}{r_0}} \qquad (3.3.5)
\end{aligned}$$

This yields the left components of the 4-velocity,

$$\begin{aligned}
u^t &= u_0^t\left(\frac{1-\frac{2M_1}{r_0}}{1-\frac{2M_1}{r}}\right) \\
&= \frac{1}{\sqrt{1-\frac{2M_1}{r_0}}}\left(\frac{1-\frac{2M_1}{r_0}}{1-\frac{2M_1}{r}}\right) \\
\frac{dt}{d\tau} &= \frac{\sqrt{1-\frac{2M_1}{r_0}}}{1-\frac{2M_1}{r}} \qquad (3.3.6) \\
u^\theta &= 0 \\
u^\phi &= 0 \qquad (3.3.7)
\end{aligned}$$

The radial equation 3.3.5 can be integrated to find the proper time for the particle to start at r_0 and fall to the horizon at $r = 2M_1$. We get

$$\frac{dr}{d\tau} = -\sqrt{\frac{2M_1}{r} - \frac{2M_1}{r_0}} \tag{3.3.8}$$

$$\tau = -\int_{r_0}^{r} \frac{dr}{\sqrt{\frac{2M_1}{r} - \frac{2M_1}{r_0}}}$$

$$= r_0 \left(\sqrt{\frac{r_0}{2M_1}} \left(\frac{\pi}{2} - \sin^{-1} \sqrt{\frac{2M_1}{r_0}} \right) + \sqrt{1 - \frac{2M_1}{r_0}} \right) \tag{3.3.9}$$

When the initial radius is much larger than $2M_1$ ($r_0 \gg 2M_1$), then the proper time that it would take for the particle to fall from r_0 to $2M_1$ would be approximately

$$c\tau = r_0 \left(\sqrt{\frac{r_0}{2M_1}} \left(\frac{\pi}{2} - \sin^{-1} \sqrt{\frac{2M_1}{r_0}} \right) + \sqrt{1 - \frac{2M_1}{r_0}} \right)$$

$$\tau = \frac{r_0}{c} \left(\sqrt{\frac{r_0}{2M_1}} \frac{\pi}{2} + 1 \right)$$

$$\approx \frac{\pi}{2c} \sqrt{\frac{r_0^3 c^2}{2GM_1}}$$

$$= \sqrt{\frac{\pi^2 r_0^3}{8GM_1}} \tag{3.3.10}$$

Using this, a value for r_0 can be substituted and the proper time elapsed for the particle to fall into the event horizon of the black hole can be found. At $r_0 = 8M_1$, then we have

$$\tau = 8M_1 \left(\sqrt{\frac{8M_1}{2M_1}} \left(\frac{\pi}{2} - \sin^{-1} \sqrt{\frac{2M_1}{8M_1}} \right) + \sqrt{1 - \frac{2M_1}{8M_1}} \right)$$

$$= 8M_1 \left(2(\frac{\pi}{2} - \sin^{-1}(\frac{1}{2})) + \sqrt{\frac{3}{4}} \right)$$

$$= 8M_1 \left(\frac{2\pi}{3} + \frac{\sqrt{3}}{2} \right)$$

$$\approx 9.83M_1 \tag{3.3.11}$$

where $M_1 = \frac{GM_1}{c^2}$. From equation 3.3.6 and 3.3.8 we consider the relation of finite amount of proper time to coordinate time, then

$$\begin{aligned}
\frac{dt}{dr} &= \frac{(\frac{dt}{d\tau})}{(\frac{dr}{d\tau})} \\
&= -\frac{\frac{\sqrt{1-\frac{2M_1}{r_0}}}{1-\frac{2M_1}{r}}}{\sqrt{\frac{2M_1}{r}-\frac{2M_1}{r_0}}} \\
dt &= -\frac{\sqrt{1-\frac{2M_1}{r_0}}}{(1-\frac{2M_1}{r})\sqrt{\frac{2M_1}{r}-\frac{2M_1}{r_0}}}dr \\
t &= -\int \frac{\sqrt{1-\frac{2M_1}{r_0}}}{(1-\frac{2M_1}{r})\sqrt{\frac{2M_1}{r}-\frac{2M_1}{r_0}}}dr \\
&= -\int_{r_0}^{r} \left(1-\frac{2M_1}{r}\right)^{-1} \sqrt{\frac{r}{2M_1}}dr \\
&= -\frac{2}{3}\left(\frac{1}{2M_1}\right)^{\frac{1}{2}} \left(r^{\frac{3}{2}} - r_0^{\frac{3}{2}} + 6M_1 r^{\frac{1}{2}} - 6M_1 r_0^{\frac{1}{2}}\right) \\
&\quad + 2M_1 \ln \left[\frac{\left(r^{\frac{1}{2}}+(2M_1)^{\frac{1}{2}}\right)\left(r_0^{\frac{1}{2}}-(2M_1)^{\frac{1}{2}}\right)}{\left(r_0^{\frac{1}{2}}+(2M_1)^{\frac{1}{2}}\right)\left(r^{\frac{1}{2}}-(2M_1)^{\frac{1}{2}}\right)}\right] \qquad (3.3.12)
\end{aligned}$$

3.4 Rotating black hole

3.4.1 The Kerr metric

The Kerr metric or Kerr geometry describes the geometry of empty spacetime around a rotating uncharged axially-symmetric black hole with a quasispherical event horizon. The Kerr metric is the second exact solution of the Einstein field equations of general relativity; these equations are highly non-linear, which makes exact solutions very difficult to find. The Kerr metric is a generalization to a rotating body of the Schwarzschild metric, discovered by Karl Schwarzschild in 1915. The corresponding solution for a charged, spherical, non-rotating body, the Reissner Nordstrm metric, was discovered soon afterwards (1916-1918). However, the exact solution for an uncharged, rotating black-hole, the Kerr metric, remained unsolved until 1963, when it was discovered by Roy Kerr [22].

According to the Kerr metric, a rotating body should exhibit frame-dragging, a distinctive prediction of general relativity. The first measurement of this frame dragging effect was done in 2011 by the Gravity Probe B experiment. This effect predicts that objects coming close to a rotating mass will be entrained to participate in its rotation, not because of any applied force or torque that can be felt, but rather because of the swirling curvature of spacetime itself associated with rotating bodies. In the case of a rotating black hole, at close enough distances, all objects even light must rotate with the black hole; the region where this holds is called the ergosphere. Rotating black holes have surfaces where the metric seems to have apparent singularities; the size and shape of these surfaces depends on the black hole's mass and angular momentum. The outer surface encloses the ergosphere and has a shape similar to a flattened sphere. The inner surface marks the event horizon; objects passing into the interior of this horizon can never again communicate with the world outside that horizon. The LIGO experiment that first detected gravitational waves, announced in 2016, provided the first direct observation of a pair of Kerr black holes [23].

When $a = 0$, the Kerr metric reduces to the Schwarzschild metric. The event horizon is situated at the point where the sign of the dr term changes, i.e. at $\Delta = 0$. Solving $\Delta = 0$ for r now gives two horizons,

$$\begin{aligned} 0 &= r^2 - 2M_1 r + a^2 \\ r = r_\pm &= M_1 \pm \sqrt{M_1^2 - a^2} \end{aligned} \tag{3.4.1}$$

The outer horizon, r_+, is the event horizon while the inner one, r_-, is called a Cauchy horizon. We see that $a < M$, else a black hole can not exist. A black hole with $a = M_1$ is called a maximally rotating black hole. When $a < M_1$ the event horizon r_+ is smaller than the Schwarzschild radius, and if $a = 0$ then $r_+ = r_s$. The horizon of a rotating black hole is thus smaller than a stationary one. Another surface of rotating black hole is ergosphere (outer surface). Its solution is given at $g_{tt} = 0$

$$\begin{aligned} g_{tt} &= -\frac{\Delta - a^2 \sin^2\theta}{\rho^2} \\ 0 &= \frac{\Delta - a^2 \sin^2\theta}{\rho^2} \\ 0 &= r^2 - 2M_1 r + a^2 - a^2 \sin^2\theta \\ r = r_\pm &= M_1 \pm \sqrt{M_1^2 - a^2\cos^2\theta} \end{aligned} \tag{3.4.2}$$

3.4.2 Geodesics in the equatorial plane

The general equations for non-null and null geodesics in the Kerr geometry are much less persuasive than in the Schwarzschild case, and particle trajectories exhibit complicated behavior. For example, in general the trajectory of a massive particle or photon is not constrained to lie in a plane. This is a direct consequence of the fact that the spacetime is not spherically symmetric and so, in general, the angular momentum of a test particle is not conserved. Since the Kerr geometry is stationary and axisymmetric, the conserved quantities along particle trajectories are p_t and p_ϕ. The component of angular momentum along the rotation axis is conserved. Since the motion of the particle is in equatorial plane, then we have $\theta = \pi/2$. The Kerr metric from 2.6.1 rewritten as

$$ds^2 = \left(1 - \frac{2M_1}{r}\right) dt^2 + \frac{4M_1\, a}{r} dt d\phi - \frac{r^2}{\Delta} dr^2 - \left(r^2 + a^2 + \frac{2M_1\, a^2}{r}\right) d\phi^2 \tag{3.4.3}$$

From this equation we can immediately write down the corresponding Lagrangian $\mathcal{L} = g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu$. In our case, for a massive particle we shall take the particle to have unit rest mass and for a photon we shall choose an appropriate affine parameter along the null geodesic such that, in both cases, $p^\mu = \dot{x}^\mu$. We may obtain the geodesic equations by writing down the appropriate Euler-Lagrange equations. However, we use the fact that p_t and p_ϕ are conserved along geodesics (since the metric does not depend explicitly on t and ϕ). The Lagrangian equation of 3.4.3 becomes

$$\mathcal{L} = \left(1 - \frac{2M_1}{r}\right) \dot{t}^2 + \frac{4M_1\, a}{r} \dot{t}\dot{\phi} - \frac{r^2}{\Delta} \dot{r}^2 - \left(r^2 + a^2 + \frac{2M_1\, a^2}{r}\right) \dot{\phi}^2 \tag{3.4.4}$$

Overdots denote differentiation with respect to an affine parameter (λ). By using the Euler-Lagrange equation

$$\frac{d}{d\lambda}\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} - \frac{\partial \mathcal{L}}{\partial x^\mu} = 0 \tag{3.4.5}$$

From this expression we obtain for the momenta

$$\begin{aligned} p_t &= g_{tt}\dot{t} + g_{t\phi}\dot{\phi} \\ &= \left(1 - \frac{2M_1}{r}\right)\dot{t} + \frac{4M_1\, a}{r}\dot{\phi} \\ &= E \end{aligned} \tag{3.4.6}$$

$$p_r = \frac{r^2}{\Delta}\dot{r} \tag{3.4.7}$$

$$\begin{aligned} p_\phi &= g_{\phi t}\dot{t} + g_{\phi\phi}\dot{\phi} \\ &= \frac{2M_1\, a}{r}\dot{t} - \left(r^2 + a^2 + \frac{2M_1\, a^2}{r}\right)\dot{\phi} \\ &= -L \end{aligned} \tag{3.4.8}$$

The corresponding Hamiltonian is given by

$$\begin{aligned} \mathcal{H} &= p_t\dot{t} + p_r\dot{r} + p_\phi\dot{\phi} - \mathcal{L} \\ &= \frac{1}{2}\left(1 - \frac{2M_1}{r}\right)\dot{t}^2 + \frac{2M_1\, a}{r}\dot{t}\dot{\phi} - \frac{r^2}{2\Delta}\dot{r}^2 - \frac{1}{2}\left[r^2 + a^2 + \frac{2a^2M_1}{r}\right]\dot{\phi}^2 \end{aligned} \tag{3.4.9}$$

Since $\mathcal{H}$ is independent of t, we find

$$2\mathcal{H} = \left[\left(1 - \frac{2M_1}{r}\right)\dot{t} + \frac{2M_1\, a}{r}\dot{\phi}\right]\dot{t} - \frac{r^2}{\Delta}\dot{r}^2 - \left[\left(r^2 + a^2 + \frac{2a^2M_1}{r}\right)\dot{\phi} + \frac{2M_1\, a}{r}\dot{t}\right]\dot{\phi} \tag{3.4.10}$$

This can be written as

$$2\mathcal{H} = E\dot{t} - L\dot{\phi} - \frac{r^2}{\Delta}\dot{r}^2 = \delta_1 \tag{3.4.11}$$

where δ_1 is an integration constant, which can be chosen as $\delta_1 = -1$ for time-like geodesics and $\delta_1 = 0$ for null geodesics. With these expressions we can solve for the velocity components in terms of the conserved quantities E and L

$$\dot{t} = \frac{1}{\Delta}\left[\left(r^2 + a^2 + \frac{2M_1\, a^2}{r}\right)E - \frac{2M_1\, a}{r}L\right] \tag{3.4.12}$$

$$\dot{\phi} = \frac{1}{\Delta}\left[\frac{2M_1 a}{r}E + \left(1 - \frac{2M_1}{r}\right)L\right] \tag{3.4.13}$$

When we insert this into the equation 3.4.11 , we obtain the radial equation

$$r^3\dot{r}^2 = r^3E^2 + 2M_1(aE - L)^2 + r(a^2E^2 - L^2) - r\Delta \tag{3.4.14}$$

Both constants E and L may be obtained by considering the limit as $r \to \infty$, we can rewrite the energy equation 3.4.14 in the form

$$\frac{1}{2}\dot{r}^2 + V_{eff}(r; E, L) = \frac{1}{2}(E^2 - 1) \tag{3.4.15}$$

where we have identified the effective potential per unit mass as

$$V_{eff}(r; E, L) = -\frac{M_1}{r} + \frac{E^2 - a^2(E^2 - 1)}{2r^2} - \frac{M_1(L - aE)^2}{r^3} \tag{3.4.16}$$

This equation reduces to the Schwarzschild result in the limit $a \to 0$. When $a \neq 0$, however an effective potential depends on the energy E of the particle (as well as the usual dependence on the angular momentum L). Nevertheless, by differentiating 3.4.15 with respect to τ, one finds that the radial acceleration of a particle is still given by $\ddot{r} = -dV_{eff}/dr$. An incoming particle will fall into the black hole only if the parameters L and E defining its trajectory are such that the maximum value of $V_{eff}(r; E, L)$ exceeds $1/2(E^2 - 1)$.

3.4.3 Equatorial motion of massive particles with zero angular momentum

For a particle falling into a Kerr black hole whose angular momentum about the black hole is zero, we have $L = 0$. Therefore, equation 3.4.14 becomes

$$r^3\dot{r}^2 = E^2(r^3 + 2M_1a^2 + ra^2) - r\Delta \tag{3.4.17}$$

From this equation we can find the energy of infalling particle as

$$E = \pm\sqrt{\frac{r^3\dot{r}^2 + r\Delta}{r^3 + 2M_1a^2 + ra^2}} \tag{3.4.18}$$

We will also consider the limit in which the particle starts at rest from infinity, in which case $E = 1$. In this case the particle will initially be moving radially. Using these values of L and E, the geodesic equations become

$$\dot{t} = \frac{1}{\Delta}\left(r^2 + a^2 + \frac{2M_1a^2}{r}\right) \tag{3.4.19}$$

$$\dot{\phi} = \frac{2M_1a}{\Delta} \tag{3.4.20}$$

$$\dot{r}^2 = \frac{2M_1}{r}\left(1 + \frac{a^2}{r^2}\right) \tag{3.4.21}$$

From these expression, we see that both $\dot{t}$ and $\dot{\phi}$ are infinite at the horizons (when $\Delta = 0$), the singular behaviours of the t and ϕ coordinates cancel in the expression for $\dot{r}^2$. The above equations may in turn be used to obtain expressions relating differentials of the coordinates along the particle trajectory. In particular, we find that

$$\frac{dr}{dt} = \frac{\dot{r}}{\dot{t}} = -\Delta\left[\frac{2M_1}{r}\left(1 + \frac{a^2}{r^2}\right)\right]^{\frac{1}{2}}\left(r^2 + a^2 + \frac{2M_1a^2}{r}\right)^{-1} \tag{3.4.22}$$

$$\frac{d\phi}{dt} = \frac{\dot{\phi}}{\dot{t}} = \frac{2M_1 a}{r}\left(r^2 + a^2 + \frac{2M_1 a^2}{r}\right)^{-1} \tag{3.4.23}$$

$$\frac{d\phi}{dr} = \frac{\dot{\phi}}{\dot{r}} = -\frac{2M_1 a}{r\Delta}\left[\frac{2M_1}{r}\left(1 + \frac{a^2}{r^2}\right)\right]^{-\frac{1}{2}} \tag{3.4.24}$$

Now differentiating equation 3.4.18 with respect to r for the energy of particle with zero angular momentum infalling onto Kerr black hole as

$$\frac{dE}{dr} = \frac{M_1 a^4 + M_1 r^4 + a^2 r(r^2\dot{r}^2 - 4M_1^2 + M_1 r(2 + 3\dot{r}^2))}{(r^3 + 2M_1 a^2 + a^2 r)^{\frac{3}{2}}\,(r^3\dot{r}^2 + r^3 - 2M_1 r^2 + a^2)^{\frac{1}{2}}} \tag{3.4.25}$$

Multiplying this equation by 3.4.22, then the power radiated energy of particle is given as

$$\begin{aligned} -\frac{dE}{dt} &= \frac{M_1 a^4 + M_1 r^4 + a^2 r(r^2\dot{r}^2 - 4M_1^2 + M_1 r(2 + 3\dot{r}^2))}{(r^3 + 2M_1 a^2 + a^2 r)^{\frac{3}{2}}\,(r^3\dot{r}^2 + r^3 - 2M_1 r^2 + a^2)^{\frac{1}{2}}} \\ &\times \left\{\Delta\left[\frac{2M_1}{r}\left(1 + \frac{a^2}{r^2}\right)\right]^{\frac{1}{2}}\left(r^2 + a^2 + \frac{2M_1 a^2}{r}\right)^{-1}\right\} \end{aligned} \tag{3.4.26}$$

where $\Delta = r^2 - 2M_1 r + a^2$

Using 3.4.21 into 3.4.26, then we get

$$-\frac{dE}{dt} = \frac{\left(M_1 a^4 + M_1 r^4 + a^2 r\left(2M_1^2 + 4M_1 r + \frac{2M_1 a^2}{r} + \frac{6M_1^2 a^2}{r^2}\right)\right)}{(r^3 + 2M_1 a^2 + a^2 r)^{\frac{3}{2}}\,(r^3 + 2M_1 a^2 + a^2)^{\frac{1}{2}}}\,\frac{\Delta\left(\frac{2M_1}{r}\left(1 + \frac{a^2}{r^2}\right)\right)^{\frac{1}{2}}}{\left(r^2 + a^2 + \frac{2M_1 a^2}{r}\right)} \tag{3.4.27}$$

An off-diagonal term of Kerr metric is given as

$$g_{t\phi} = g_{\phi t} = -a\frac{2M_1 r\sin^2\theta}{\rho^2} \tag{3.4.28}$$

Where Kerr metric is independent of ϕ and t. p^ϕ and p^t will then be conserved. And we have:

$$p^\phi = M_2\frac{d\phi}{d\tau} \tag{3.4.29}$$

$$p^t = M_2 \frac{dt}{d\tau} \tag{3.4.30}$$

and thus with angular velocity

$$\omega(r,t) = \frac{d\phi}{dt} = \frac{p^\phi}{p^t} \tag{3.4.31}$$

For the sake of particle with zero angular momentum at spatial infinity, then $p_\phi = 0$ and using the equations 3.4.23, 3.4.29 and 3.4.30

$$\omega(r,t) = \frac{2M_1 a}{r^3 + a^2 r + 2M_1 a^2} \tag{3.4.32}$$

As of gravitational redshift expressed in Schwarzschild black hole, we can also do for Kerr black hole. We first derive the gravitational time dilation to obtain the gravitational redshift. It is given for a clock that is stationary (motionless) in a gravitational field ($dr = d\theta = d\phi = 0$) [24]. Hence the Kerr metric in Boyer-Lindquist coordinate system reduced to

$$\begin{aligned} ds^2 &= g_{tt} c^2 dt^2 \\ &= \left(1 - \frac{2GM_1 r}{r^2 + a^2\cos^2\theta}\right) dt^2 \end{aligned} \tag{3.4.33}$$

and we have

$$d\tau = \sqrt{g_{tt}}\, dt \tag{3.4.34}$$

where $G = c = 1$. It is not possible to determine dt at any specific point in the gravitational field because all devices are affected in exactly the same way. Thus, two points in the gravitational field are required to determine the influence of g_{tt}. By considering two points in the gravitational field, point A, where electromagnetic radiation of a specific frequency is emitted and point B, where it is received. We have from equation 3.4.34

$$d\tau_A = \sqrt{g_{tt}(r_A)}\, dt_A = \sqrt{g_{tt}(r_{em})}\, dt_{em} \tag{3.4.35}$$

$$d\tau_B = \sqrt{g_{tt}(r_B)}\, dt_B = \sqrt{g_{tt}(r_{re})}\, dt_{re} \tag{3.4.36}$$

As the time intervals, dt_A and dt_B , we assign the time between adjacent crests of electromagnetic radiation. This means they are equal to the reciprocal of the period or frequency, ν_A and ν_B , of the electromagnetic radiation. That is:

$$d\tau_A = d\tau_{emitted} = \frac{1}{\nu_{emitted}} \tag{3.4.37}$$

$$d\tau_B = d\tau_{received} = \frac{1}{\nu_{received}} \tag{3.4.38}$$

From equations 3.4.35, 3.4.36, 3.4.37 and 3.4.38, we get

$$\frac{\nu_{received}}{\nu_{emitted}} = \left(\frac{g_{tt}(r_{em})}{g_{tt}(r_{re})}\right)^{1/2} \tag{3.4.39}$$

Plugging equation 3.4.33 into 3.4.39

$$\frac{\nu_{received}}{\nu_{emitted}} = \left(\frac{1 - \frac{2M_1\, r_{em}}{r_{re}^2 + a^2\cos^2\theta}}{1 - \frac{2M_1\, r_{re}}{r_{re}^2 + a^2\cos^2\theta}}\right)^{1/2} \tag{3.4.40}$$

For the receiver at infinity ($r_{received} \gg r_{emitted}$),

$$\left(1 - \frac{2M_1\, r_{re}}{r_{re}^2 + a^2\cos^2\theta}\right)^{1/2} \to 1,$$

then equation 3.4.40 becomes

$$\frac{\nu_{received}}{\nu_{emitted}} = \left(1 - \frac{2M_1 r_{em}}{r_{em}^2 + a^2\cos^2\theta}\right)^{1/2} \tag{3.4.41}$$

The energy received at infinity related to the energy emitted near a Kerr black-hole is given as

$$dE_{received} = dE_{emitted}\left(1 - \frac{2M_1\, r_{em}}{r_{em}^2 + a^2\cos^2\theta}\right)^{1/2} \tag{3.4.42}$$

The energy per unit time of this equation is given by

$$\frac{dE_{received}}{dt} = \frac{dE_{emitted}}{dt}\left(1 - \frac{2M_1 r_{em}}{r_{em}^2 + a^2\cos^2\theta}\right)^{1/2} \tag{3.4.43}$$

Plugging 3.4.27 into 3.4.43, then we have

$$-\frac{dE_{re}}{dt} = \frac{\left(M_1a^4 + M_1r^4 + a^2r\left(2M_1^2 + 4M_1r + \frac{2M_1a^2}{r} + \frac{6M_1^2a^2}{r^2}\right)\right)}{(r^3 + 2M_1a^2 + a^2r)^{\frac{3}{2}}\,(r^3 + 2M_1a^2 + a^2)^{\frac{1}{2}}}\frac{\Delta\left(\frac{2M_1}{r}\left(1 + \frac{a^2}{r^2}\right)\right)^{\frac{1}{2}}}{\left(r^2 + a^2 + \frac{2M_1a^2}{r}\right)} \times\left(1 - \frac{2M_1 r_{em}}{r_{em}^2 + a^2\cos^2\theta}\right)^{1/2} \tag{3.4.44}$$

3.4.4 Equatorial circular motion of massive particles

For circular motion, we require that $\dot{r} = 0$ and, for the particle to remain in a circular orbit, that the radial acceleration $\ddot{r}$ must also vanish. For time-like geodesics we may write this in order of decreasing powers in r as

$$r^3\dot{r}^2 = r^3E^2 - r\Delta - r(L^2 - a^2E^2) + 2M_1(aE - L)^2 \tag{3.4.45}$$

E is now the total energy per unit mass of a particle (or star) and L the angular momentum per unit mass. This radial equation can directly be obtained from the conservation laws and the normalization of the four momentum, $p^2 = -M_2^2$. The radial equation reduces to the Schwarzschild form in the case $a = 0$.

$$\left(\frac{dr}{d\tau}\right)^2 = E^2 - V_s^2 \tag{3.4.46}$$

$$\frac{d\phi}{d\tau} = \frac{L}{r^2}, \tag{3.4.47}$$

with the effective potential defined as

$$V_s^2 = \left(1 - \frac{2M_1}{r}\right)\left(1 + \frac{L^2}{r^2}\right) \tag{3.4.48}$$

However, in the case of Kerr, we cannot transform to a simple effective potential. For this reason, we introduce, as in the Newtonian case, the variable $u = 1/r$ and write the radial equation as

$$u^{-3}\dot{u}^2 = 2M_1(L - aE)^2u^3 - (L^2 - a^2E^2)u^2 - (a^2u^2 - 2M_1u + 1) + E^2 \tag{3.4.49}$$

Since particle is in the circular orbits, $\dot{u} = 0$, for given values of E and L. For circular orbits, the above cubic polynomial will have a double root. This is easily calculated to be the case for

$$2M_1l^2u^3 - (l^2 + 2alE)u^2 - (a^2u^2 - 2M_1u + 1) + E^2 = 0 \tag{3.4.50}$$

and

$$3M_1l^2u^2 - (l^2 + 2alE)u - (a^2u - M_1) = 0 \tag{3.4.51}$$

where we have introduced the reduced angular momentum

$$l = L - aE \tag{3.4.52}$$

These two equations can be combined to give

$$E^2 \;=\; 1 - M_1 u + M_1 l^2 u^3 \tag{3.4.53}$$

and

$$2alEu \;=\; l^2(3M_1 u - 1)u - (a^2 u - M_1) \tag{3.4.54}$$

We can eliminate the energy E from these two equations and combine them into a quadratic equation for l^2

$$\begin{aligned} u^2 \left[(3M_1 u - 1)^2 - 4a^2 M_1 u^3\right] l^4 & \\ & - 2u\left[(3M_1 u - 1)(a^2 u - M_1) - 2a^2 u(M_1 u - 1)\right] l^2 \\ & + (a^2 u - M_1)^2 = 0 \end{aligned} \tag{3.4.55}$$

The discriminant of this equation is

$$D = 4a^2 M_1 u^3 D_\mu^2, \qquad D_\mu = a^2 u^2 - 2M_1 u + 1 \tag{3.4.56}$$

we get

$$R_\pm \;=\; 1 - 3M_1 u \pm 2a\sqrt{M_1 u^3} \tag{3.4.57}$$

and the identity

$$(3M_1 u - 1)^2 - 4a^2 M_1 u^3 \;=\; R_+ R_- \tag{3.4.58}$$

We find

$$l_\pm \;=\; -\frac{a\sqrt{u} \pm M_1}{\sqrt{uR_\mp}} \tag{3.4.59}$$

Inserting this solution into the energy equation 3.4.53, we find

$$E \;=\; \frac{1 - 2M_2 u \mp a\sqrt{M_1 u^3}}{\sqrt{R_\mp}} \tag{3.4.60}$$

and the value of L to be associated with this value for E is

$$\begin{aligned} L &= l + aE \\ &= \mp \frac{\sqrt{M_1(a^2u^2 + 1 \pm 2a\sqrt{M_1u^3})}}{\sqrt{uR_\mp}} \end{aligned} \tag{3.4.61}$$

Finally, the energy per unit mass of circular orbits is given as

$$E = \frac{r^2 - 2M_1r \mp a\sqrt{M_1r}}{r\sqrt{r^2 - 3M_1r \mp 2a\sqrt{M_1r}}} \tag{3.4.62}$$

and the specific angular momentum as

$$L = \mp \frac{\sqrt{M_1r(r^2 - 2a\sqrt{M_1r + a^2})}}{r\sqrt{r^2 - 3M_1r \mp 2a\sqrt{M_1r}}} \tag{3.4.63}$$

where the plus sign correspond to co-rotating orbits and the minus to counter-rotating ones. We can now differentiate equation 3.4.62 for co-rotating with respect to r as

$$\frac{dE}{dr} = \frac{M_1^2\left(8aM_1r - 3a^2\sqrt{M_1r} + r\sqrt{M_1r}(r - 6M_1)\right)}{2(M_1r)^{3/2}(r^2 - 3M_1r + 2a\sqrt{M_1r})^{3/2}} \tag{3.4.64}$$

Multiplying this equation with 3.4.22, then it yields

$$-\frac{dE}{dt} = \frac{M_1^2\left(8aM_1r - 3a^2\sqrt{M_1r} + r\sqrt{M_1r}(r - 6M_1)\right)}{2(M_1r)^{3/2}(r^2 - 3M_1r + 2a\sqrt{M_1r})^{3/2}} \frac{\Delta\left(\frac{2M_1}{r}\left(1 + \frac{a^2}{r^2}\right)\right)^{\frac{1}{2}}}{\left(r^2 + a^2 + \frac{2M_1a^2}{r}\right)} \tag{3.4.65}$$

By using the r-component of the Euler-Lagrangian equation, the circular geodesic of the particles spiralling Kerr spacetime is $\dot{r} = \ddot{r} = 0$. Thus equation 3.4.5 rewritten as

$$\frac{d}{d\lambda}\frac{\partial \mathcal{L}}{\partial \dot{r}} = \frac{\partial \mathcal{L}}{\partial r} \tag{3.4.66}$$

Being $g_{r\mu} = 0$ if $\mu \neq r$, then we have

$$\frac{d}{d\lambda}(g_{rr}\dot{r}) = \frac{1}{2}g_{\mu\nu,r}\dot{x}^{\mu}\dot{x}^{\nu} \tag{3.4.67}$$

This equation reduced to

$$g_{tt,r}\dot{t}^2 + 2g_{t\phi,r}\dot{t}\dot{\phi} + g_{\phi\phi,r}\dot{\phi}^2 = 0 \tag{3.4.68}$$

The angular velocity is

$$\omega = \frac{\dot{\phi}}{\dot{t}} = \frac{u^{\phi}}{u^{t}}, \tag{3.4.69}$$

Dividing both sides of 3.4.68 by $\dot{t}^2$, then we get

$$g_{\phi\phi,r}\omega^2 + 2g_{t\phi,r}\omega + g_{tt,r} = 0 \tag{3.4.70}$$

where we have on the equatorial plane,

$$g_{tt} = -\left(1 - \frac{2M_1}{r}\right) \tag{3.4.71}$$

$$g_{t\phi} = -\frac{2M_1 a}{r} \tag{3.4.72}$$

$$g_{\phi\phi} = r^2 + a^2 + \frac{2M_1 a^2}{r^2}, \tag{3.4.73}$$

then

$$2\left(r - \frac{M_1 a^2}{r^2}\right)\omega^2 + \frac{4M_1 a}{r^2}\omega^2 - \frac{2M_1}{r^2} = 0 \tag{3.4.74}$$

This quadratic equation can be rewritten in the form of,

$$\left(r^3 - M_1 a^2\right)\omega^2 + 2M_1 a\omega - M_1 = 0 \tag{3.4.75}$$

has discriminant

$$M_1^2 a^2 + M_1(r^3 - M_1 a^2) = M_1 r^3, \tag{3.4.76}$$

then we get

$$\omega_{\pm} = \frac{-M_1 a \pm \sqrt{M_1 r^3}}{r^3 - M_1 a^2}$$
$$= \pm\frac{\sqrt{M_1}}{r^{3/2} \pm a\sqrt{M_1}} \quad (3.4.77)$$

This is the relation between angular velocity and radius of circular orbits, and reduces, in Schwarzschild limit $a = 0$, to

$$\omega_{\pm} = \pm\left(\frac{M_1}{r^3}\right)^{1/2}, \quad (3.4.78)$$

Four velocity of a stationary point on the surface can be written as,

$$u^{\mu} = (u^t, 0, 0, u^{\phi}) \quad (3.4.79)$$

Using equation 3.4.69 in above equation 3.4.79

$$u^{\mu} = (u^t,\ 0,\ 0,\ \omega u^{\phi}) \quad (3.4.80)$$

Through the normalization condition of the four velocity given by [25]

$$u_{\mu}u^{\nu} = g_{\mu\nu}u^{\mu}u^{\nu} = 1 \quad (3.4.81)$$

$$g_{tt}u^t u^t + 2g_{t\phi}u^t u^{\phi} + g_{\phi\phi}u^{\phi}u^{\phi} = 1 \quad (3.4.82)$$

Using equation 3.4.80 in to 3.4.82, we obtain the time-like component of the four velocities in terms of the metric components of Kerr field and angular velocity of rotation (ω)

$$
\begin{aligned}
g_{tt} + 2g_{t\phi}\omega + g_{\phi\phi}\left(\frac{u^{\phi}}{u^{t}}\right)^2 &= \left(\frac{1}{u^t}\right)^2 \\
\left(g_{tt} + 2g_{t\phi}\omega + g_{\phi\phi}\omega^2\right)^{1/2} &= \frac{1}{u^t} \\
u^t &= \frac{1}{\left(g_{tt} + 2g_{t\phi}\omega + g_{\phi\phi}\omega^2\right)^{1/2}} \qquad (3.4.83)
\end{aligned}
$$

Any observer measures the frequency ν of a photon following null geodesic $x^{\mu}(\lambda)$ can be calculated by the expression given by

$$
\begin{aligned}
\nu &= u^{\mu}\frac{dx^{\mu}}{dx_{\lambda}} \\
&= u^{\mu}g_{\mu\nu}\frac{dx^{\nu}}{dx_{\lambda}} \qquad (3.4.84)
\end{aligned}
$$

If photon is emitted at $r = \theta =$ constant, $dr = d\theta = 0$, then using 3.4.84 the frequency ν can be expressed as

$$
\nu = u^t(g_{tt}\dot{t} + g_{t\phi}\dot{\phi}) + u^{\phi}(g_{t\phi}\dot{t} + g_{\phi\phi}\dot{\phi}) \qquad (3.4.85)
$$

Using equation 3.4.6 and 3.4.8 in above equation 3.4.85, we can write

$$
\nu = u^t(-E) + u^{\phi}(L) = u^t(-E + \omega L) \qquad (3.4.86)
$$

Using equation 3.4.83 in above equation 3.4.86, we can write the expression of frequency observed as

$$
\nu = \frac{(-E + \omega L)}{\left(g_{tt} + 2g_{t\phi}\omega + g_{\phi\phi}\omega^2\right)^{1/2}} \qquad (3.4.87)
$$

In general relativity gravitational redshift (z) is defined as

$$
\frac{1}{z+1} = \frac{\nu_{received}}{\nu_{emitted}} = \left(g_{tt} + 2g_{t\phi}\omega + g_{\phi\phi}\omega^2\right)^{1/2} \qquad (3.4.88)
$$

Now we can write the energy received at infinity related to the energy emitted near a rotating black-hole as

$$dE_{received} = dE_{emitted}\left(g_{tt} + 2g_{t\phi}\omega + g_{\phi\phi}\omega^2\right)^{1/2} \tag{3.4.89}$$

Using equations 3.4.65 and 3.4.89, then the power radiated energy observed at infinity is

$$-\frac{dE_{obs}}{dt} = \frac{M_1^2\left(8aM_1r - 3a^2\sqrt{M_1r} + r\sqrt{M_1r}(r-6M_1)\right)}{2(M_1r)^{3/2}(r^2 - 3M_1r + 2a\sqrt{M_1r})^{3/2}}\frac{\Delta\left(\frac{2M_1}{r}\left(1+\frac{a^2}{r^2}\right)\right)^{\frac{1}{2}}}{\left(r^2 + a^2 + \frac{2M_1a^2}{r}\right)} \times\left(g_{tt} + 2g_{t\phi}\omega + g_{\phi\phi}\omega^2\right)^{1/2} \tag{3.4.90}$$

Chapter 4

Result and Discussion

Using General Theory of Relativity (GTR) we have developed the equations of motion of freely in falling particle into Schwarzschild and Kerr black hole. Since equations of motion free falling particles were computed, the equation of energy emitted in the form of electromagnetic radiation in the case of charged particle infalling with zero angular momentum and gravitational radiation for uncharged particle into black hole is derived for both Schwarzschild and Kerr case. The time of infalling particles into Schwarzschild black hole is derived and for the orbit case we have calculated the time required for both a charged and an uncharged particle to spiral into the black-hole. The results for the particle falling with zero angular momentum and for the particle orbiting the black-hole is calculated in turn and compared with the corresponding results for gravitational radiation.

4.1 Equations of motion of particle freely infalling into Schwarzschild spacetime

We have considered a radially falling particle in a Schwarzschild spacetime and the way to calculate the equations of motion. The Schwarzschild spacetime is the geometry of the vacuum spacetime outside a spherical star. It is determined by one parameter, the mass M_1, and has the line element

$$ds^2 = \left(1 - \frac{2M_1}{r}\right) dt^2 - \left(1 - \frac{2M_1}{r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 \quad (4.1.1)$$

For Schwarzschild metric, we have the Lagrangian equation as

$$\mathcal{L} = \frac{1}{2}\left[\left(1 - \frac{2M_1}{r}\right)\dot{t}^2 - \frac{\dot{r}^2}{1 - \frac{2M_1}{r}} - r^2\dot{\theta}^2 - r^2\sin^2\theta\dot{\phi}^2\right] \quad (4.1.2)$$

The corresponding canonical momenta are

$$p_t = \frac{\partial\mathcal{L}}{\partial\dot{t}} = \left(1 - \frac{2M_1}{r}\right)\dot{t} = \frac{E}{M_2} \quad (4.1.3)$$

$$p_r = -\frac{\partial\mathcal{L}}{\partial\dot{r}} = \left(1 - \frac{2M_1}{r}\right)^{-1}\dot{r} \quad (4.1.4)$$

$$p_\theta = -\frac{\partial\mathcal{L}}{\partial\dot{\theta}} = r^2\dot{\theta} \quad (4.1.5)$$

$$p_\phi = -\frac{\partial\mathcal{L}}{\partial\dot{\phi}} = r^2\sin^2\theta\dot{\phi} = \frac{L}{M_2} \quad (4.1.6)$$

For equatorial plane ($\theta = \pi/2$) 4.1.2 becomes

$$1 = \left(1 - \frac{2GM_1}{r}\right)\dot{t}^2 - \frac{\dot{r}^2}{(1 - \frac{2GM_1}{r})} - r^2\dot{\phi}^2 \quad (4.1.7)$$

Since the particle is radially infalling, the the equation of motion 2.5.30 becomes

$$\left(\frac{ds}{dt}\right)^2 = \left(1 - \frac{2M_1}{r}\right)c^2 - \left(1 - \frac{2M_1}{r}\right)^{-1}\frac{dr^2}{dt^2} \quad (4.1.8)$$

then the velocity of infalling particle is given by

$$\mathbf{v} = \frac{dr}{dt} = -c\left(1 - \frac{2M_1}{r}\right)\sqrt{\frac{2M_1}{r}} \tag{4.1.9}$$

Differentiating this equation with respect to t then gives the acceleration,

$$\frac{d^2r}{dt^2} = -\frac{M_1c^2}{r^2}\left(1 - \frac{8M_1}{r} + \frac{12M_1^2}{r^2}\right) \tag{4.1.10}$$

4.2 Amount of energy radiated during infalling of particles onto black hole

4.2.1 The Schwarzschild black hole

For an accelerated point charge q in straight-line motion, the angular distribution of the radiation through out solid angle ($d\Omega = \sin\theta d\theta d\phi$) is given by

$$\frac{d\mathbf{P}}{d\Omega} = \frac{\mu_0 q^2 \mathbf{a}^2}{16\pi^2 c}\frac{\sin^2\theta}{(1-\beta\cos\theta)^5} \tag{4.2.1}$$

where ($\mu_0 = 1/c^2\epsilon_0$, $\beta = \mathbf{v}/c$). The energy given off per unit time by a charged particle falling radially into a black-hole is given by

$$-\frac{dE(\theta)}{dt} = \frac{\dot{\mathbf{v}}^2}{c^3}\frac{e^2z^2}{16\pi^2\epsilon_0}\frac{\sin^2\theta}{(1-\beta\cos\theta)^5} \tag{4.2.2}$$

By integrating this equation over θ to get

$$-\frac{dE}{dt} = \frac{\dot{\mathbf{v}}^2}{c^3}\frac{e^2}{6\pi\epsilon_0}\left\{\frac{1}{(1-\beta^2)^3}\right\} \tag{4.2.3}$$

Thus the energy emitted by the accelerating charge is given by

$$-\frac{dE}{dt} = \frac{\dot{\mathbf{v}}^2}{c^3}\frac{e^2}{6\pi\epsilon_0}\left(1+3\beta^2\right) \tag{4.2.4}$$

We would like to calculate the total energy received at infinity, if a clock at infinity records the passive of a time Δt_∞, then one near the black hole will record a time

$$\Delta t_r = \Delta t_\infty \left(1 - \frac{2GM_1}{rc^2}\right)^{\frac{1}{2}} \tag{4.2.5}$$

The frequency of a photon of frequency ν_r emitted at a radius r around a black hole and received at infinity will have frequency

$$\nu_\infty = \nu_r \left(1 - \frac{2GM_1}{rc^2}\right)^{\frac{1}{2}} \tag{4.2.6}$$

We observe from this equation that as $r \to R_{Sch}$, the observed frequency $\nu_\infty \to 0$. Thus, light emitted near the event horizon becomes more and more redshifted, until finally at the event horizon it becomes infinitely redshifted and can no longer be observed by the outside world.

Since the particle is in falling, then the radial part from equation 2.5.30 being calculated as

$$c^2 d\tau^2 = \left(1 - \frac{R_{Sch}}{r}\right) c^2 dt^2 - \left(1 - \frac{R_{Sch}}{r}\right)^{-1} dr^2 \tag{4.2.7}$$

For an observer at infinity (as $r \to \infty$, $\left(1 - 1 - \frac{2GM_1}{rc^2}\right)^{\frac{1}{2}} \to 1$), then

$$d\tau_\infty = dt \tag{4.2.8}$$

Thus the energy received at infinity is related to the energy emitted near a black-hole by

$$dE_{received} = dE_{emitted} \left(1 - \frac{2M_1}{r}\right)^{\frac{1}{2}} \tag{4.2.9}$$

Combining equation 4.2.4 and 4.2.9, then we get

$$-\frac{dE_{received}}{dt} = \frac{\dot{\mathbf{v}}^2}{c^3} \frac{e^2}{6\pi\epsilon_0} \left(1 + 3\beta^2\right) \left(1 - \frac{2M_1}{r}\right)^{\frac{1}{2}} \tag{4.2.10}$$

By using equation of motion of infalling particle onto Schwarzschild black hole, the values of $\mathbf{v}$ and $\dot{\mathbf{v}}$ were calculated.

Finally, the result of electromagnetic radiation which escapes to infinity for a particle of charge ze falls into a black-hole from $r = \infty$ to $r = 2M_1$ with zero angular momentum is given as

$$E_{e.m} \;=\; 0.0044932\, m_e c^2 \left(\frac{r_e}{M_1}\right) z^2 \qquad (4.2.11)$$

where

$m_e = \frac{M_e G}{c^2}$, $M_e = 9.11 \times 10^{-31} kg$ (mass of electron), $G = 6.67384 \times 10^{-11} \frac{m^3}{kg\, s^2}$ (gravitational constant), $r_e = 2.8 \times 10^{-15} m$ (radius of electron), M_1 is mass of black hole and $c = 3 \times 10^8 \frac{m}{s}$ (speed of light.) The mass of the falling particle M_2 does not enter.

When an uncharged particle of mass M_2 emits gravitational radiation in zero angular momentum fall [13], then an amount of radiated energy,

$$\begin{aligned} E_{grav} &= \frac{1}{625}\left(\frac{M_2}{M_1}\right) M_2\, c^2 \\ &= 0.0016\left(\frac{M_2}{M_1}\right) M_2\, c^2 \qquad (4.2.12) \end{aligned}$$

where M_1 is mass of black hole.

By taking M_2 to a system composed of N electrons, then the ratio of electromagnetic radiation to gravitational one is

$$\begin{aligned} \frac{E_{e.m}}{E_{grav}} &= 2.8 \left(\frac{r_e}{m_e}\right) \\ &= 2.8 \left(\frac{2.8 \times 10^{-15} m \;\times\; 9 \times 10^{16} m^2/s^2}{9.11 \times 10^{-31} kg \;\times\; 6.67 \times 10^{-11} m^3\; kg^{-1} s^{-2}}\right) \\ &= 1.16 \times 10^{43} \qquad (4.2.13) \end{aligned}$$

We get far more electromagnetic radiation in this case. The ratio in equation 4.2.13 will, of course, become smaller as the mass M_2 increases and the charge decreases.

For the case of particle starts at rest far from the black hole, then the energy of spiralling particle and effective potential per unit rest-mass of particle are given as

$$E(r) = M_2 c^2 \left(1 + \frac{L^2}{M_2^2 c^2 r^2}\right)^{\frac{1}{2}} \left(1 - \frac{2M_1}{r}\right)^{\frac{1}{2}} \tag{4.2.14}$$

$$V(r) = \left(1 + \frac{L^2}{r^2}\right) \left(1 - \frac{2M_1}{r}\right) \tag{4.2.15}$$

respectively.

The particle will orbit stably where equation 4.2.14 goes through a minimum as a function of r, which is written as

$$\begin{aligned} r &= \left(a^2 + \sqrt{a^4 - 3a^2}\right) 2M_1 \\ &= 2M_1 a(a + \sqrt{a^2 - 3}) \end{aligned} \tag{4.2.16}$$

where $a = L/2M_1M_2$. By following this equation, the radius is real only if $a \geq \sqrt{3}$. From this value, we have two cases for angular momentum (L) for the particle.

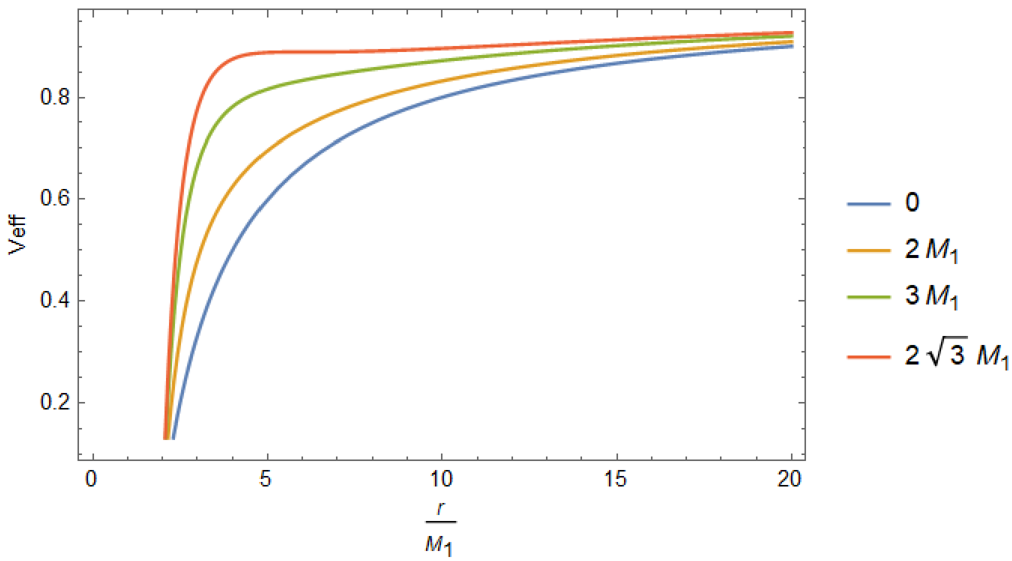

Figure 4.1: Effective potential as a function of radius for various values of the angular momentum $L < 2\sqrt{3}M_1$. The abscissa is a dimensionless radius in units of gravitational radii, and GM_1/c is the natural unit for the specific angular momentum L of a particle.

Case I: As illustrated in figure 4.1; if $L < 2\sqrt{3}M_1$, then there are no turning points. For these values of angular momentum no more type of finite motion is possible, then a test body will inevitably fall in the black hole whatever values of E it may have. The effective potential is a monotonically increasing function of r.

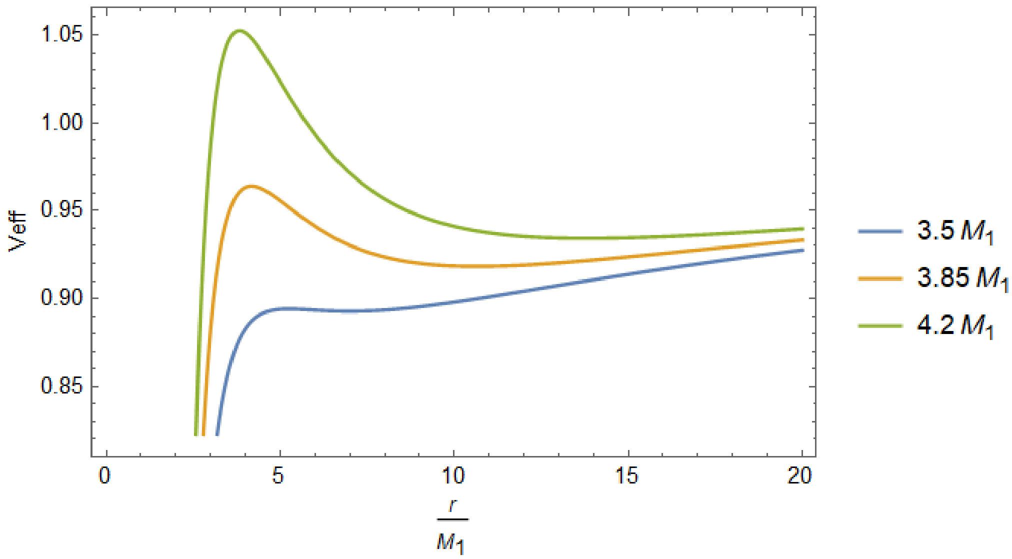

Figure 4.2: Effective potential as a function of radius for various values of the angular momentum $L > 2\sqrt{3}M_1$. The abscissa is a dimensionless radius in units of gravitational radii, and GM_1/c is the natural unit for the specific angular momentum L of a particle. The relativistic effective potential attains a maximum for $L > 2\sqrt{3}M_1$ and then vanishes at the Schwarzschild radius $2M_1$

Case II: Figure 4.2 shows that if $L > 2\sqrt{3}M_1$, then there are two turning points. The effective potential has two extrema: maximum and minimum, at the radii of which unstable and stable circular motion are possible correspondingly. In the case when L equals to the boundary value $2\sqrt{3}M_1$, two extrema of effective potential (EP) merge into one inflection point. This boundary value of L defines parameters of the last stable orbit which is also called the innermost stable circular orbit (ISCO), i. e. the boundary orbit on which the finite motion is still possible [26]. For Schwarzschild BH case, a minimum of the ISCO is $r = 6M_1$.

The existence of an innermost stable orbit has some interesting astrophysical consequences. Gas in an accretion disc around a massive compact central body settles into circular orbits around the compact object. However, the gas slowly loses angular momentum because of turbulent viscosity. As the gas loses angular momentum it moves slowly inwards, losing gravitational potential energy and heating up. Eventually it has lost enough angular momentum that it can no longer follow a stable circular orbit, and so it spirals rapidly inwards onto the central object. Therefore, particle will spiral down to $r = 6M_1$ before it falls into the black-hole without further radiation of energy. The energy per unit rest-mass of a particle at this lowest orbit is given by

$$\begin{aligned} E^2 &= \frac{(r-2M_1)^2}{r(r-3M_1)} \\ E &= \frac{r-2M_1}{\sqrt{r(r-3M_1)}} \\ &= \frac{6M_1-2M_1}{\sqrt{6M_1(6M_1-3M_1)}} \\ &= \sqrt{\frac{8}{9}} \\ E &= 0.9428 \end{aligned} \tag{4.2.17}$$

Therefore, the amount of energy of a particle radiated away at $r = 6M_1$ is

$$\begin{aligned} E_{emitted} &= M_2\, c^2 \left(1 - \sqrt{\frac{8}{9}}\right) \\ &= 0.0572\; M_2\, c^2 \end{aligned} \tag{4.2.18}$$

which is 5.7% of the rest mass of the orbiting particle.

The time of spiralling charged particle into Schwarzschild BH written as

$$\Delta t_{e.m} = \frac{4M_1^2 M_2}{z^2 M_e\, c\, r_0}\left[A^3\left(1+\frac{15}{4}\frac{1}{A}-\frac{75}{8}\frac{1}{A^2}\right)-B^3\left(1+\frac{15}{4}\frac{1}{B}-\frac{75}{8}\frac{1}{B^2}\right)\right] \tag{4.2.19}$$

and time of uncharged one becomes,

$$\Delta t_{grav} = \frac{5}{8}\left(\frac{G}{c^3}\right)M_2\left(\frac{M_1}{M_2}\right)^2\left[A^4\left(1-\frac{22}{3}\frac{1}{A}+\frac{39}{4}\frac{1}{A^2}\right)-B^4\left(1-\frac{22}{3}\frac{1}{B}+\frac{39}{4}\frac{1}{B^2}\right)\right] \tag{4.2.20}$$

where $A = r_{max}/2M_1$, $B = r_{min}/2M_1$ and r_{max} and r_{min} are the beginning and ending radii of the spiraling orbit for both cases.

From these results, we observed that the energy radiated when a particle orbits a black-hole is much greater than when it falls with zero angular momentum and is the same for charged and uncharged particles. However, the charged particle spirals in much faster than the uncharged one.

4.2.2 The Kerr black hole

In the case of Kerr metric, the energy per unit mass of circular orbits of particle is given as

$$E = \frac{r^2 - 2M_1 r \mp a\sqrt{M_1 r}}{r\sqrt{r^2 - 3M_1 r \mp 2a\sqrt{M_1 r}}} \tag{4.2.21}$$

and the specific angular momentum as

$$L = \mp\frac{\sqrt{M_1 r(r^2 - 2a\sqrt{M_1 r} + a^2)}}{r\sqrt{r^2 - 3M_1 r \mp 2a\sqrt{M_1 r}}} \tag{4.2.22}$$

For the neutral rotating BH, the innermost stable circular orbit of particle with co-rotating is $r = M_1$, then 4.2.21 becomes

$$E = \frac{1}{\sqrt{3}} \tag{4.2.23}$$

This gives the maximum energy per unit mass which a stable circular orbit can have in a Kerr geometry with $a^2 \leq M_1^2$. Hence the amount of energy radiated in this case is

$$\begin{aligned} E &= M_2\, c^2 (1 - \frac{1}{\sqrt{3}}) \\ &= 0.4227\, M_2\, c^2 \end{aligned} \tag{4.2.24}$$

Therefore, a particle in a co-rotating orbit with Kerr BH radiates 42.3% of its rest mass. This high gravitational energy is the reason why black holes can so efficiently transform accretion streams into radiation. Now, we can compare the results from 4.2.11, 4.2.12 and 4.2.18 that the energy radiated away of a particle in co-rotating with Kerr spacetime is more than that of Schwarzschild in both cases (radially freely falling and spiralling).

The energy received at infinity related to the energy emitted near a Kerr black-hole is given as

$$dE_{received} = dE_{emitted} \left(1 - \frac{2M_1 r_{em}}{r_{em}^2 + a^2 \cos^2\theta}\right)^{1/2} \tag{4.2.25}$$

For $a = 0$, this equation reduced to 4.2.9 of the Schwarzschild case. The power radiation of a particle falling into a Kerr black hole whose angular momentum about the black hole is zero, we have $L = 0$ is given by

$$-\frac{dE_{re}}{dt} = \frac{\left(M_1 a^4 + M_1 r^4 + a^2 r\left(2M_1^2 + 4M_1 r + \frac{2M_1 a^2}{r} + \frac{6M_1^2 a^2}{r^2}\right)\right)}{(r^3 + 2M_1 a^2 + a^2 r)^{\frac{3}{2}}\,(r^3 + 2M_1 a^2 + a^2)^{\frac{1}{2}}} \frac{\Delta\left(\frac{2M_1}{r}\left(1 + \frac{a^2}{r^2}\right)\right)^{\frac{1}{2}}}{\left(r^2 + a^2 + \frac{2M_1 a^2}{r}\right)} \times \left(1 - \frac{2M_1 r_{em}}{r_{em}^2 + a^2\cos^2\theta}\right)^{1/2} \tag{4.2.26}$$

From 3.4.70 to 3.4.77, then the angular velocity of particle spiralling Kerr space-time is

$$\omega = \pm\frac{\sqrt{M_1}}{r^{3/2} \pm a\sqrt{M_1}} \tag{4.2.27}$$

The energy received at infinity related to the energy emitted near a rotating black-hole is given as

$$dE_{received} = dE_{emitted}\left(g_{tt} + 2g_{t\phi}\omega + g_{\phi\phi}\omega^2\right)^{1/2} \tag{4.2.28}$$

where

$$g_{tt} = -\left(1 - \frac{2M_1}{r}\right) \tag{4.2.29}$$

$$g_{t\phi} = -\frac{2M_1 a}{r} \tag{4.2.30}$$

$$g_{\phi\phi} = r^2 + a^2 + \frac{2M_1 a^2}{r^2} \tag{4.2.31}$$

Finally, the power radiated observed at infinity is given as

$$-\frac{dE_{obs}}{dt} = \frac{M_1^2\left(8aM_1 r - 3a^2\sqrt{M_1 r} + r\sqrt{M_1 r}(r - 6M_1)\right)}{2(M_1 r)^{3/2}(r^2 - 3M_1 r + 2a\sqrt{M_1 r})^{3/2}} \frac{\Delta\left(\frac{2M_1}{r}\left(1 + \frac{a^2}{r^2}\right)\right)^{\frac{1}{2}}}{\left(r^2 + a^2 + \frac{2M_1 a^2}{r}\right)} \times \left(g_{tt} + 2g_{t\phi}\omega + g_{\phi\phi}\omega^2\right)^{1/2} \tag{4.2.32}$$

Chapter 5

Summary and Conclusion

To summarize briefly, the Schwarzschild geometry is the geometry of the vacuum spacetime outside a spherical star. It is determined by one parameter, the mass M. The Kerr black hole is axially symmetric but not spherically symmetric (that is rotationally symmetric about one axis only, which is the angular-momentum axis), and is characterized by two parameters, mass (M) and angular-momentum (J). In the Schwarzschild solution, the horizon was the place where $g_{tt} = 0$ and $g_{rr} = \infty$. In the Kerr solution, the ergosphere occurs at $g_{tt} = 0$ and the horizon is at $g_{rr} = \infty$, i.e. where $\Delta = 0$. Then we have seen that the energy emitted when a charged particle such as an electron falls straight into a Schwarzschild black-hole is much more than the energy emitted when an uncharged particle falls in. The energy radiated when a particle orbits a black-hole is much greater than when it falls with zero angular momentum and is the same for charged and uncharged particles. Concerning to the time of inspiralling, a charged particle spirals in much faster than the uncharged one. In fact, for real bodies Δt_{grav} is so prohibitively long that the energy theoretically available is never emitted.

A particle freely infalling straight in from infinity is dragged just by the influence of gravity so that it acquires an angular velocity in the same sense as rotating black hole (Kerr BH). However, a particle slowly loses angular momentum, then moves slowly inwards and losing gravitational potential energy, then heating up. Eventually it has lost enough angular momentum that it can no longer follow a stable circular orbit and so it spirals rapidly inwards onto the central compact object.

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