TWO DIMENSIONAL FOURTH ORDER TIME FRACTIONAL PARABOLIC EQUATIONS AND ITS ANALYTICAL SOLUTION BY USING FRACTIONAL REDUCED DIFFERENTIAL TRANSFORM METHOD.

A THESIS SUBMITTED TO THE DEPARTMENT OF MATHEMATICS, COLLEGE OF NATURAL SCIENCES, JIMMA UNIVERSITY IN PARTIAL FULFILLMENT FOR THE REQUIREMENTS OF THE DEGREE OF MASTERS OF SCIENCE IN MATHEMATICS

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Abstract

In this manuscript, the solution of two dimensional fourth order time-fractional non- homogenous parabolic equations is solved by using Fractional Reduced Differential Transform Method. The efficiency of FRDTM is confirmed by considering two test problems of the two dimensional non-homogenous time fractional parabolic equations. The fractional derivative is taken in Caputo sense. Figures were used to compare approximate and exact solution. FRDTM is a very efficient, effective and powerful mathematical tool which provides exact or very close approximate solutions for a wide range of real-world problems arising in engineering and science.

Keywords:-Partial differential equations, Two dimensional fourth order time fractional parabolic equations, Fractional reduced differential transform method.

Acronyms

ADM -Adomain's Decomposition Method

DTM -Differential Transform Method

FRDTM - Fractional Reduced Differential Transform Method

HAM - Homotopy Analysis Method

HEPE - Higher-Order Elastic Parabolic Equation

HPM - Homotopy Perturbation Method

NIM - New Iteration method

PDE - Partial Differential Equation

RDTM - Reduced Differential Transform Method

VIM - Variational Iteration Method

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CHAPTER-ONE

INTRODUCTION

1.1 Background of the Study

Various physical phenomena in sciences and engineering can be explained successfully by developing models using the fractional calculus theory. Fractional differential equations have achieved much more attention because fractional order system response ultimately converges to the integer order equations. Fractional differentiation is used to mathematically modeling real world physical problems such as the earthquake model, the traffic flow model, measurement of viscoelastic material properties, etc. and this has been widespread in the recent years. Before the nineteenth century, no analytical solution method was available for such type of equations even for the linear fractional differential equations. **(**Gorenflo et al, 1999) developed the fractional reduced differential transform method (FRDTM) for the fractional differential equations and showed that FRDTM is the most easily implemented analytical method and gives the exact solution for both the linear and nonlinear differential equations. Recently, (Keskin and Oturanc, 2010) applied a new approach called the fractional reduced differential transform method for linear and non-linear fractional partial differential equations**.**

Many problems in science and engineering fields can also be described by the partial differential equations. A variety of analytical method have been developed to obtain analytic solutions for the problems in the literature. Partial differential equations (PDEs) are widely used to describe complex physical phenomena in different branches of mathematical physics, engineering science and other areas of natural science (Chen, 1999). Among the different methods used to solve PDEs, the Caputo fractional derivative allows the utilization of initial condition involving integer order derivatives, which have clear physical interpretations compared to the Riemann-Liouville fractional derivatives.

The classical Taylor series method is one of the earliest analytic techniques to many problems especially ordinary differential equations. However, since it requires a lot of symbolic calculation for higher derivatives (Wazwaz, 2002).

The reduced differential transform method is an iterative procedure for obtaining Taylor series solution of differential equations. This method reduces the size of computational work and is easily applicable to many physical problems (Muhammad Sohail and Syed Tauseef, 2012).

One dimensional and two dimensional fourth order homogeneous time fractional parabolic equations by using RDTM has been studied previously (Muhammad, 2012). but two dimensional fourth order time fractional non homogenous parabolic equation by FRDTM is not studied in the existing literature. As a result the main objective of this study is to find analytical solutions of two dimensional fourth order time fractional parabolic equations by FRDTM.

In this study, we have used the fractional reduced differential transformed method to solve non-homogeneous two dimensional fourth order time fractional parabolic equations:

$$\frac{\partial^{2\alpha} u}{\partial t^{2\alpha}} + \left(\eta(\mathrm{x},\mathrm{y}) \frac{\partial^4 u}{\partial x^4} + \lambda(\mathrm{x},\mathrm{y}) \frac{\partial^4 u}{\partial y^4} \right) = f(x,y,t), \qquad n-1 < \alpha \le n,\ n \in \mathbb{N}, \tag{1.1}$$

with initial conditions

$$u(x,y,0) = g_1(x,y)$$

$$u_t(x,y,0) = g_2(x,y), \tag{1.2}$$

where u is a function of x, y, t and $f(x,y,t)$ is analytic function $\alpha\ is$ a parameter which describes the order of time fractional derivative, $n-1 < \alpha \le n,\ n \in \mathbb{N}$, $\eta(\mathrm{x},\mathrm{y})$ and $\lambda(\mathrm{x},\mathrm{y})$ are functions of x and y

Remark: The governing Eq. (1.1) reduces to the classical (or ordinary) two dimensional fourth order parabolic partial differential equation. (Muhammad Sohail, Syed Tauseef, Mohyud-Din 2015)

Searching for exact solutions using FRDTM of two dimensional parabolic equations has long been a major concern for mathematicians. These solutions may well describe various phenomena in physics and other fields. The investigation of the exact solution of these equations is interesting and important. The result in FRDTM on two dimensional parabolic equations helps to study the flow of heat conductivity and it is an efficient technique in finding the exact solution.

1.2 Statement of the Problem

The use of fractional differentiation for the mathematical modeling of real world physical problem such as the earthquake, the traffic flow, measurement of viscoelastic material properties, etc. has been widespread in the recent years. It has been explained that in different existing literature fractional derivatives have a wide variety of applications in engineering and science. The solution of two dimensional fourth order time fractional non-homogeneous parabolic equations by applying fractional reduced differential transformed method is not presumably presented in the existing literature.

As a result, this study mainly focuses on

- Applying fractional reduced differential transform method on the governing equation and the initial conditions to obtain exact solutions.
- Justifying the applicability of the method by considering specific examples.

1.3 Objective of the Study

1.3.1 General Objective

The general objective of this study is to investigate analytical solutions of fourth order two dimensional time fractional non-homogeneous parabolic equations subject to the initial conditions.

1.3.2 Specific Objective

The specific objectives of the study are:

- To determine the solution of two dimensional fourth order parabolic time fractional equations by FRDTM
- To prove some properties and theorems related to FRDTM.
- To verify the applicability of the method by specific examples.

1.4. Significance of the Study

This study is considered of vital importance for the following reason:

- It develops the researcher skills on mathematical research.
- Serve as a reference material for scholars who works on this area.
- It familiarizes the student researcher with the scientific communication in mathematics

1.5. Delimitation of the Study

The study is delimited to find analytical solutions of two dimensional non- homogeneous fourth order time fractional linear and non-linear parabolic equations by FRDTM.

CHAPTER –TWO

LITERATURE REVIEW

Fractional Differential Equations arise in almost all areas of physics, applied and engineering sciences. In order to better understand these physical phenomena, as well as further applications in practical scientific research, it is important to find their exact solutions. The investigation of exact solutions to these equations is interesting and important. In the past, many authors had studied the solution of such equations. Recently, several analytical and numerical techniques were successfully applied to deal with differential equations and fractional differential equations. Studies shows that the Adomian's decomposition method (ADM) (Cherruauit, 1993), Homotopy perturbation method (HPM) (He, 2004), Homotopy analysis method (HAM) (Matinfar. M, and Saeidy, 2010) the solutions of the problem and exactly obtained the solution in the form of convergent. Taylor series and variation of parameter method (VPM) (Biazar and Ghazvini, 2009) are successfully applied to obtain the exact solutions of differential equations. Fractional calculus theory is more than 200 years old to be presented in the literature. Several definitions of fractional integrals and derivatives have been proposed but the first major contribution to give proper and most meaningful definition is due to Riemann-Liouville application of fractional calculus to dynamics of particles, fields and media (Luo. et al, 2006).

Reduced differential transform method has been applied to solve many physical problems, in particular it is applied in solving the heat and wave like equations. The method is applied in direct way without using linearization, transformation, discretization or restrictive assumptions.

See the reference (Muhammad Sohail and Syed Tauseef, 2012). *International Journal of Modern Theoretical Physics.*

The Riemann -Liouville derivative has major drawbacks while modeling the real world with fractional differential equations (Arikoglu and Ozkol, 2005). To overcome this discrepancy,(Caputo and Mainardi,1999) proposed a modified fractional differential operator D^{α} in their work on the theory of viscoelasticity. The Caputo fractional derivative allows the utilization of initial and boundary conditions involving integer order derivatives, which have clear physical interpretations.

The reduced differential transform method is an iterative procedure for obtaining Taylor series solution of differential equations. This method reduces the size of computational work and is easily applicable to many physical problems (Wazwaz, 2002).

The solution obtained by the FRDTM is an infinite power series for appropriate initial condition which finds the solution without any discretization, transformation, perturbation, or restrictive conditions. However, computations show that the FRDTM is very easy to implement and needs a small size of computation contrary to HPM, HAM (Matinfar and Saeidy, 2010), the solutions of the problem and exactly obtained the solution in the form of convergent Taylor series. VIM (Noor and Mohyud, 2008) and ADM (Liao, 2003).

The aim of this study is to apply the fractional reduced differential transform method FRDTM to solve time-fractional fourth order parabolic PDEs. By serving the above mentioned literature it is observed that the proposed technique FRDTM is highly suitable for such problems analytically. Higher order elastic parabolic equation HEPE is derived for wave propagation in depth dependent and weakly range dependent fluid/solid media. In this section, analytical methods already developed for the solution of the heat flow equation which are discussed in some detail in order to provide background for the description of the new method, and also in order to develop several equations that should be discussed.

CHAPTER- THREE

METHODOLOGY

3.1 Study Area and Period

This study is conducted on the topic of PDE which deal with how to obtain solutions of the two dimensional fourth order non-homogeneous time fractional parabolic equations by using FRDTM in Jimma University under the Department of Mathematics, College of Natural Science from September 2019 to September 2021 G.C.

3.2 Study Design

The design of the study is analytical.

3.3 Source of Information

The source of information for this study are collected from related documentary materials such as reference books, internets, published and unpublished articles.

3.4 Mathematical Procedures

In order to achieve the objective of the study, the researcher used the following mathematical procedures. These are

1. Transforming the governing problem using the definition and properties of FRDTM.
2. Transforming the initial conditions using the definition of FRDTM.
3. Substituting $U_0(x,y)$ or/and $U_1(x,y)$ into the transformed governing problemto obtain a recursion system for the unknown functions$U_k(x,y)$ where $k = 0,1,2,\cdots,n\ldots$
4. Use the inverse fractional reduced differential transform method to obtain exact solution of two dimensional time fractional fourth order parabolic equations.
5. Mathematica 7software is used to sketch the solution curve.

CHAPTER- FOUR

RESULTS AND DISCUSSION

4.1 Preliminaries

In this section, some supportive ideas related to fractional calculus theory will be revisited, which we are used to complete this study.

4.1.1The Gamma function

Definition 4.1 The gamma function Γ(z) is defined as

$$\Gamma(z)=\int_0^\infty t^{z-1}e^{-t}dt, z\in\mathbb{C} \ and \ \operatorname{Re}(Z)\geq 0 \tag{4.1}$$

Properties of gamma function

1. $\Gamma(z+1)=z\Gamma(z), z\notin\{0,\mathbb{Z}^-\}$
2. $\Gamma(n+1)=n!, n\in\mathbb{Z}^+$
3. $\Gamma(z)\Gamma(z+\frac{1}{2})=2^{1-2z}\sqrt{\pi}\,\Gamma(2z), z\notin\left\{\frac{-1}{2},0,\mathbb{Z}^-\right\}$
4. $\Gamma(1-z)\Gamma(z)=\dfrac{\pi}{\sin z\pi}, z\notin\mathbb{Z}$

4.1.2 Basic definitions and notations of Fractional Calculus theory

Some essential definitions of fractional order integrals and derivatives are presented in this Literature which is given by Riemann-Liouville and Caputo sense**.**

Definition 4.2 The Reimann-Liouville fractional integral of a function f is defined as

$J^\alpha f(x)=\dfrac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} f(t)dt$,where J^α denotes the fractional integral of order α. (4.2)

Reimann-Liouville fractional derivative (denoted D^α) defined so that $D^\alpha J^\alpha f(x)=f(x)$

Definition 4.3 If $m-1<\alpha\leq m, m\epsilon\mathbb{N}$, then the Reimann-Liouville fractional derivative of $f\in C_\mu$ (Carpinteri and Mainaridi , New York 1997**)** is defined as

$$D_x^\alpha f(x)=\frac{1}{\Gamma(m-\alpha)}\frac{d^m}{dx^m}\int_0^x (x-t)^{m-\alpha-1} f(t)dt$$

In particular, if $0 < \alpha \leq 1$, thenthe Reimann-Liouville fractional derivative of $f \in C_\mu$

$$D_x^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x (x-t)^{-\alpha} f(t) dt \tag{4.3}$$

Definition 4.4let J_x^{α} be Riemann-Liouville fractional integral operator and let $f \epsilon C_\mu$ for $\alpha, \beta \geq 0$ *and* $\gamma > 1$, *then* the following are basic properties.

1. $J_x^{\alpha} J_x^{\beta} f(x) = J_x^{\alpha+\beta} f(x) = J_x^{\beta} J_x^{\alpha} f(x)$ (4.4)

2. $J_x^{\alpha} X^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} X^{\alpha+\gamma}, X > 0$

The Riemann-Liouville derivative has certain limitations when someone tries to model some real physical problems. In their work, Caputo & Mainardi proposed a modified fractional differential operator D_x^{α} to the theory of viscoelasticity to overcome the inconsistency of Riemann-Liouville derivative.

Definition 4.5 If $m - 1 < \alpha \leq m, m \epsilon N$ then Caputo fractional derivative of $f \in C_\mu$ (Carpinteri and Mainaridi , New York 1997**)** is defined as

$$D_x^{\alpha} f(x) = J_x^{m-\alpha} D_x^{m} f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} f^{(m)}(t) dt \tag{4.5}$$

In particular, if $0 < \alpha \leq 1$, then Caputo fractional derivative of f is

$$D_x^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-t)^{-\alpha} f'(t) dt$$

The basic properties of the caputo fractional derivative D_x^{α} are as follows:

1. $D_t^{\alpha} D_t^{\beta} f(t) = D_t^{\alpha+\beta} f(t) = D_t^{\beta} D_t^{\alpha} f(t)$

2. $D_t^{\alpha} t^{\gamma} = \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma-\alpha)} t^{\gamma-\alpha}$**,** $t > 0$

Remark: The difference between Reimann-Liouville and Caputo fractional derivative can be expressed as in the following:

i.e, if we consider the $\frac{3}{2}$ derivative of a function $f(x)$, in terms of Reimann - Liouville we have $D^{\frac{3}{2}} f(x) = D^2 J^{\frac{1}{2}} f(x)$ and in terms of Caputo we have $D^{\frac{3}{2}} f(x) = D^{\frac{1}{2}} J^2 f(x)$.

Example: Find a half derivative of a constant C, using Reimann-Liouville and Caputo fractional derivate.

- Using Reimann-Liouville fractional derivate

$$D_x^{\frac{1}{2}} C = \frac{1}{\Gamma\left(1-\frac{1}{2}\right)} \frac{d}{dx} \int_0^x (x-t)^{-\frac{1}{2}} C dt$$

$$= \frac{1}{\Gamma\left(\frac{1}{2}\right)} \frac{d}{dx}\left(2C\sqrt{x}\right)$$

$$= \frac{\sqrt{x}C}{x\sqrt{\pi}}$$

- Using Caputo fractional derivate

$$D_x^{\frac{1}{2}} C = \frac{1}{\Gamma\left(\frac{1}{2}\right)} \int_0^x (x-t)^{-\frac{1}{2}} (C)' dt$$

$$= \frac{1}{\Gamma\left(\frac{1}{2}\right)} \int_0^x (x-t)^{-\frac{1}{2}} (0) dt = \frac{0}{\sqrt{\pi}} = 0$$

Definition 4.6 Real function $f(t)$, t>0 is said to be in the space $C\mu$, $\mu \epsilon R$, if there exists a real number p(>μ) , such that $f(t) = t^p f_1(t) \ \epsilon \ C(0,\infty)$, and it is said to be in the space $C^n\mu$ if and only if $f(n) = \frac{d^n}{dx^n} f(t) \ \epsilon C\mu \quad n \epsilon \mathbb{N}$ (4.6)

.

4.1.3 Fractional Reduced Differential Transform Method (FRDTM)

The reduced differential transform method was first proposed by the Turkish mathematician (Keskin and Oturance, 2009). It has received much attention since it has applied to solve a wide verity of problems by many authors.

In this section, the basic definitions of the reduced differential transform method (RDTM) and differential inverse transform in (Keskin and Oturance, 2009); (Keskin and Oturance, 2010 a); (Keskin and Oturance, 2010 b) are discussed as follows:-

Consider a function of three variables $u(x, y, t)$ and suppose that it can be represented as a product of two single-variable functions, i.e. $u(x, y, t) = f(x, y)g(t)$. Based on the properties of two-dimensional differential transform the function $u(x, y, t)$ can be represented as:

$$U(x,y,t) = \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty} F(i_1,i_2)x^{i_1}y^{i_2}\sum_{j=0}^{\infty} G(j)t^j = \sum_{k=0}^{\infty} u_k(x,y)t^k \tag{4.7}$$

The basic definition and operation of FRDTM as introduced in (Srivastava et al, 2013),

(Tagavvi and Babaei, 2015), (Boss and Miller , 1993) are given bellow:-

Definition 4.7 If $u(x, y, t)$ is analytic and continuously differentiable with respect to space variable **x** and time variable t in the domain of interest, then the t-dimensional spectrum function FRDTM is given by

$$U_k(x,y) = \frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}} u(x,y,t)\right]_{t=t_0} \tag{4.8}$$

is the reduced transformed function of $u(x, y, t)$, where α is a parameter which describes the order of time fractional derivative in a Caputo sense and $U_k(x, y)$ is the transformed function of $u(x, y, t)$, which is denoted by R_D

Definition 4.8 The inverse FRDTM of $U_k(x,y)$ is defined as:

$$\left[U_K(x,y)\right] = u(x,y,t) = \sum_{k=0}^{\infty} U_K(x,y)(t-t_o)^{k\alpha}$$, which is denoted by R_D^{-1} (4.9)

Combining (4.8) and (4.9) we can find that:-

$$u(x,y,t) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k\alpha+1)} \left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}} u(x,y,t) \right]_{t=t_0} (t-t_o)^{k\alpha} \tag{4.10}$$

Taking the fractional reduced differential transformed on both sides of equation $u(x,y,t) = f(x,y,t)$

Therefore, we get $U_0(x,y) = g_1(x,y)$ and $U_t(x,y) = g_2(x,y)$.

and hence using equation (4.10) , the function $u(x,y,t$**)** can be estimated by a finite series as

$u_n(x,y,t) = \sum_{k=0}^{n} U_k(x,y)(t-t_0)^{k\alpha} + R_n(x,y,t)$ where n represents order of estimated solution.

Here the tail function $R_n(x,y,t)$ is negligibly small. In particular, if $t_0 = 0$**,** this equation takes the form.

$$u_n(x,y,t) = \sum_{k=0}^{n} U_k(x,y) t^{k\alpha} \tag{4.11}$$

Finally, the accurate solution is found by taking limit of the function,

$$\lim_{n\to\infty} u_n(x,y,t) = \sum_{k=0}^{\infty} U_k(x,y) t^{k\alpha} = U_0(x,y) + U_1(x,y)t^{\alpha} + U_2(x,y)t^{2\alpha} + U_3(x,y)t^{3\alpha} + \dots\dots\dots \tag{4.12}$$

Table 1 Basic properties of two dimensional fractional reduced differential transform method are given below (Bhaben Ch. Neog, 2015)

No	$Original\ Function$	$Transformed\ Function$
1	$u(x,y,t) = w(x,y,t)$	$U_k(x,y) = \frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}\left(w(x,y,t)\right)\right]_{t=t_o}$
2	$w(x,y,t) = u(x,y,t) \pm v(x,y,t)$	$W_K(x,y) = U_K(x,y) \pm V_K(x,y)$
3	$w(x,y,t) = \beta u(x,y,t)$	$W_K(x,y) = \beta U_k(x,y), where\ \beta\ is\ a\ constant$
4	$w(x,y,t) = x^m y^n t^p$	$W_k(x,y) = x^m y^n \delta(k-p), where$ $\delta(k-p) = \begin{cases} 1, if k=p \\ 0, if k\neq p \end{cases}$
5	$w(x,y,t) = u(x,y,t)v(x,y,t)$	$W_k(x,y) = \sum_{n=0}^{k} U_n(x,y)V_{k-n}(x,y) = \sum_{n=0}^{k} V_n(x,y)U_{k-n}(x,y)$
6	$w(x,y,t) = \frac{\partial^r}{\partial t^r} u(x,y,t)$	$W_K(x,y) = (k+1)(k+2)\dots(k+r)U_{k+N}(x,y)$
7	$w(x,y,t) = \frac{\partial}{\partial x} u(x,y,t)$	$W_K(x,y) = \frac{\partial}{\partial x} U_k(x,y)$
8	$w(x,y,t) = \frac{\partial^{mr}}{\partial x^{mr}} u(x,y,t)$	$W_K(x,y) = \frac{\partial^{mr}}{\partial x^{mr}} U_k(x,y)$
9	$w(x,y,t) = \frac{\partial^{N\alpha}}{\partial t^{N\alpha}} u(x,y,t)$	$W_K(x,y) = \frac{\Gamma[k\alpha+N\alpha+1]}{\Gamma[k\alpha+1]} U_{k+N}(x,y)$
10	$w(x,y,t) = f(x,y)u(x,y,t)$	$W_k(x,y) = f(x,y)U_k(x,y)$
11	$w(x,y,t) = f(x,y)\frac{\partial^{k\alpha}}{\partial t^{k\alpha}} u(x,y,t)$	$W_k(x,y) = f(x,y)\frac{\partial^{k\alpha}}{\partial t^{k\alpha}} U_k(x,y)$

From the important properties depicted in table 1 we present the proof of some properties which may help us to justify our work.

I. If $w(x,y,t)=\beta u(x,y,t)$, then $W_K(x,y)=\beta U_k(x,y)$, where β is a constant

Proof: From equation (4.9), we have

$$W_k(x,y)=\frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}w(x,y,t)\right]_{t=t_0}$$

$$=\frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}\beta u(x,y,t)\right]_{t=t_0}$$

$$=\beta\frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}u(x,y,t)\right]_{t=t_0}$$

$$=\beta U_k(x,y)$$

II. If $w(x,y,t)=x^m y^n t^p$, then $W_k(x,y)=x^m y^n\boldsymbol{\delta}(k-p)$,where

$$\delta(k-p)=\begin{cases}1, & if\ k=p\\ 0, & if\ k\neq p\end{cases}$$

Proof: By definition (4.8),we have

$$W_k(x,y)=\frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}w(x,y,t)\right]_{t=0}$$

$$W_k(x,y)=\frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}\left(x^m y^n t^p\right)\right]_{t=0}$$

$$W_k(x,y)=\frac{x^m y^n}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}\left(t^p\right)\right]_{t=0}=x^m y^n\left[\frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}\left(t^p\right)\right]_{t=0}\right]$$

$W_k(x,y)=x^m y^n\delta(k-p)$,where

$\delta(k-p)=\begin{cases}1, & if\ k=p\\ 0, & if\ k\neq p\end{cases}$ and hence the theorem is proved.

III. If $w(x,y,t) = u(x.y.t)v(x,y,t),\ then\ W_k(x,y) = \sum_{n=0}^{k} U_n(x,y) V_{k-n}(x,y) = \sum_{n=0}^{k} V_n(x,y) U_{k-n}(x,y)$

Proof:

$$W_k(x,y) = \frac{1}{\Gamma(k\alpha+1)} \left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}} u(x,y,t) v(x,y,t) \right]_{t=0}$$
$$W_k(x,y) = \frac{1}{\Gamma(k\alpha+1)} \left[u(x,y,t) \frac{\partial^{k\alpha}}{\partial t^{k\alpha}} v(x,y,t) \right]_{t=0}$$

$$let \frac{\partial^{k\alpha}}{\partial t^{k\alpha}} u(x,y,t) = v(x,y,t)$$
$$W_k(x,y) = RDT(u(x,y,t)v(x,y,t))$$
$$W_k(x,y) = \sum_{n=0}^{k} U_n(x,y) V_{k-n}(x,y)$$

IV. If $w(x,y,t) = \frac{\partial^{mr}}{\partial x^{mr}} u(x,y,t)$, then $W_K(x,y) = \frac{\partial^{mr}}{\partial x^{mr}} U_k(x,y)$.

Proof: From equation (4.9), we have

$$W_K(x,y) = \frac{1}{\Gamma(k\alpha+1)} \left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}} w(x,y,t) \right]_{t=t_o}$$

$$= \frac{1}{\Gamma(k\alpha+1)} \left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}} \left(\frac{\partial^{mr}}{\partial x^{mr}} u(x,y,t) \right) \right]_{t=t_o}$$

$$= \frac{1}{\Gamma(k\alpha+1)} \left[\frac{\partial^{mr}}{\partial x^{mr}} \left(\frac{\partial^{k\alpha}}{\partial t^{k\alpha}} u(x,y,t) \right) \right]_{t=t_o}$$

$$= \frac{\partial^{mr}}{\partial x^{mr}} \left[\frac{1}{\Gamma(k\alpha+1)} \left(\frac{\partial^{k\alpha}}{\partial t^{k\alpha}} u(x,y,t) \right) \right]_{t=t_0} = \frac{\partial^{mr}}{\partial x^{mr}} U_k(x,y)$$

This completes the proof of theorem

V. If $w(x,y,t)=\frac{\partial^{N\alpha}}{\partial t^{N\alpha}}u(x,y,t)$, then $W_K(x,y)=\frac{\Gamma[k\alpha+N\alpha+1]}{\Gamma[k\alpha+1]}U_{k+N}(x,y)$

Proof: let $W_K(x,y)$ and $U_K(x,y)$ be t-dimensional spectrum functions of $w(x,y,t)$ and $u(x,y,t)$ respectively and is analytic and K-time continuous differentiable function with respect to time **t** and **x** in the domain of our interest.

Applying FRDTM to the equation $w(x,y,t)=\frac{\partial^{N\alpha}}{\partial t^{N\alpha}}u(x,y,t)$, we obtain

$$W_K(x,y)=\frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}w(x,y,t)\right]_{t=t_o}$$

$$=\frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}\left(\frac{\partial^{N\alpha}}{\partial t^{N\alpha}}u(x,y,t)\right)\right]_{t=to}$$

$$=\frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha+N\alpha}}{\partial t^{k\alpha+N\alpha}}u(x,y,t)\right]_{t=t_o} \text{ by exponential properties.}$$

$$=\frac{\Gamma(k\alpha+N\alpha+1)}{\Gamma(k\alpha+N\alpha+1)}\frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha+N\alpha}}{\partial t^{k\alpha+N\alpha}}u(x,y,t)\right]_{t=t_o}$$

$$=\frac{\Gamma(k\alpha+N\alpha+1)}{\Gamma(k\alpha+1)}\underbrace{\frac{1}{\Gamma(k\alpha+N\alpha+1)}\left[\frac{\partial^{k\alpha+N\alpha}}{\partial t^{k\alpha+N\alpha}}U(x,y,t)\right]_{t=t_o}}_{U_{k+N}(x,y)}$$

$$W_K(x,y)=\frac{\Gamma(k\alpha+N\alpha+1)}{\Gamma(k\alpha+1)}U_{k+N}(x,y) \text{ and hence,}$$

if $w(x,y,t)=\frac{\partial^{N\alpha}}{\partial t^{N\alpha}}u(x,y,t)$, then $W_K(x,y)=\frac{\Gamma[k\alpha+N\alpha+1]}{\Gamma[k\alpha+1]}U_{k+N}(x,y)$

VI. Suppose that w (x,y,t)= $f(x,y)\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}u(x,y,t), then W_{\mathrm{k}}(x,y) = f(x,y)U_k\ (x,y).$

Proof: By the definition of the FRDTM,

$$W_k(x,y) = \frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}w(x,y,t)}{\partial t^{k\alpha}}\right]_{t=0}$$

$$W_k(x,y) = \frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}f(x,y)u(x,y,t)}{\partial t^{k\alpha}}\right]_{t=0}$$

$$W_k(x,y) = \frac{1}{\Gamma(k\alpha+1)}\,f(x,y)\left[\frac{\partial^{k\alpha}u(x,y,t)}{\partial t^{k\alpha}}\right]_{t=0}$$

$$W_k(x,y) = f(x,y)\frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}u(x,y,t)}{\partial t^{k\alpha}}\right]_{t=0},$$

$$U_k(x,y) = \frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}u(x,y,t)}{\partial t^{k\alpha}}\right]_{t=0}$$

$$W_k(x,y) = f(x,y)U_k(x,y)$$

4.2 Main Results

The fractional Reduced Differential Transform method (FRDTM) introduced recently by (Keskin and Oturance in 2010) is used to solve fractional partial differential equations. Therefore, this study presents the solution of two dimensional fourth order time fractional homogeneous and non-homogeneous parabolic equations by using FRDTM.

4.2.1 Two-dimensional fourth order parabolic Equation.

I: consider two dimensional linear time fractional non-homogeneous fourth order parabolic equation in Caputo sense of the form:

$$\frac{\partial^{2\alpha}u}{\partial t^{2\alpha}}+\left(\eta(\mathrm{x},\mathrm{y})\frac{\partial^4 u}{\partial x^4}+\lambda(\mathrm{x},\mathrm{y})\frac{\partial^4 u}{\partial y^4}\right)=f(x,y,t),\quad n-1<\alpha\le n,\ n\in\mathbb{N}, \tag{4.13}$$

with initial conditions

$$u(x,y,0)=g_1(x,y)$$

$$u_t(x,y,0)=g_2(x,y), \tag{4.14}$$

where u is a function of x,y,t and $f(x,y,t)$is analytic function α is a parameter which describes the order of time fractional derivative, $n-1<\alpha\le n,\ n\in\mathbb{N}$, $\eta(\mathrm{x},\mathrm{y})$ and $\lambda(\mathrm{x},\mathrm{y})$ are functions of x and y

Remark: The governing Eq. (1.1) reduces to the classical (or ordinary) two dimensional fourth order parabolic partial differential equation. (Muhammad Sohail, Syed Tauseef, Mohyud-Din 2015)

Applying properties of FRDTM to equations (4.13**)** into (4.14), we get the following recurrence relation**:**

$$\frac{\Gamma(k\alpha+2\alpha+1)}{\Gamma(k\alpha+1)}U_{k+2}(x,y)+\eta(\mathrm{x},\mathrm{y})\frac{\partial^4}{\partial x^4}U_k(x,y)+\lambda(\mathrm{x},\mathrm{y})\frac{\partial^4}{\partial y^4}U_k(x,y)=F_k\left(x,y\right)$$

Or

$$U_{k+2}(x,y)=\frac{\Gamma(\alpha k+1)}{\Gamma(\alpha k+2\alpha+1)}\left[-\eta(\mathrm{x},\mathrm{y})\frac{\partial^4}{\partial x^4}U_k(x,y)-\lambda(\mathrm{x},\mathrm{y})\frac{\partial^4}{\partial y^4}U_k(x,y)+F_k\left(x,y\right)\right] \tag{4.15}$$

$$U_0(x,y)=g_1(x,y)$$
$$U_1(x,y)=g_2(x,y) \tag{4.16}$$

When k=0, 1, 2, 3,…, n…, where n$\epsilon\mathbb{N}$, then from equations (4.15) and (4.16)we obtain the successive transformed iterations as follows:

$$U_2(x,y)=\frac{1}{\Gamma(2\alpha+1)}\left[-\eta(\mathrm{x,\ y})\frac{\partial^4}{\partial x^4}g_1(x,y)-\lambda(\mathrm{x,\ y})\frac{\partial^4}{\partial y^4}g_1(x,y)+F_1(x,y)\right],$$

$$U_3(x,y)=\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+2\alpha+1)}\left[-\eta(\mathrm{x,\ y})\frac{\partial^4}{\partial x^4}g_2(x,y)-\lambda(\mathrm{x,\ y})\frac{\partial^4}{\partial y^4}g_2(x,y)+F_2(x,y)\right]_, \cdots$$

Therefore, the exact solution of Eq. (4.14) is given by

$$u(x,y,t)=\sum_{k=0}^{\infty}U_k(x,y)t^{k\alpha}=U_0(x,y)+U_1(x,y)t^{\alpha}+U_2(x,y)t^{2\alpha}+\cdots \text{or equivalently}$$

$$u(x,y,t)=g_1(x,y)+g_2(x,y)t^{\alpha}+\frac{1}{\Gamma(2\alpha+1)}\left[-\eta(\mathrm{x,\ y})\frac{\partial^4}{\partial x^4}g_1(x,y)-\lambda(\mathrm{x,\ y})\frac{\partial^4}{\partial y^4}g_1(x,y)+F_1(x,y)\right]t^{2\alpha}+\cdots$$

4.2.2 Supportive examples

Here is an example to illustrate the effectiveness of the given FRDTM to problems of this type,

Example 4.1 Consider the following two dimensional linear fourth-order time fractional parabolic partial differential equation with variable coefficients.

$$\frac{\partial^{2\alpha}u}{\partial t^{2\alpha}}+2\left(\frac{1}{x^2}+\frac{x^4}{6!}\right)\frac{\partial^4 u}{\partial x^4}+2\left(\frac{1}{y^2}+\frac{y^4}{6!}\right)\frac{\partial^4 u}{\partial y^4}=0,\frac{1}{2}<x,y<1,t>0,0<\alpha\le 1 \tag{4.17}$$

with initial conditions-

$$u(x,y,0)=0$$
$$u_t(x,y,0)=2+\frac{x^6}{6!}+\frac{y^6}{6!} \tag{4.18}$$

Applying FRDTM operator to both sides of the equations (4.17) and (4.18), we obtain

$$\frac{\Gamma(\alpha(k+2)+1)}{\Gamma(k\alpha+1)}U_{k+2}(x,y)+2\left(\frac{1}{x^2}+\frac{x^4}{6!}\right)\frac{\partial^4}{\partial x^4}U_k(x,y)+2\left(\frac{1}{y^2}+\frac{y^4}{6!}\right)\frac{\partial^4}{\partial y^4}U_k(x,y)=0, \tag{4.19}$$

$$U_{k+2}(x,y)=-\left(\frac{\Gamma(k\alpha+1)}{\Gamma(k\alpha+2\alpha+1)}\right)\left[2\left(\frac{1}{x^2}+\frac{x^4}{6!}\right)\frac{\partial^4}{\partial x^4}U_k(x,y)+2\left(\frac{1}{y^2}+\frac{y^4}{6!}\right)\frac{\partial^4}{\partial y^4}U_k(x,y)\right] \tag{4.20}$$

From the initial condition Eq. (4.20), we can obtain

$$U_0(x,y)=0$$
$$U_1(x,y)=2+\frac{x^6}{6!}+\frac{y^6}{6!} \tag{4.21}$$

When k=0, then the equations (4.20) and (4.21) gives as

$$U_2(x,y)=-\left(\frac{1}{\Gamma(2\alpha+1)}\right)\left[2\left(\frac{1}{x^2}+\frac{x^4}{6!}\right)\frac{\partial^4}{\partial x^4}U_0(x,y)+2\left(\frac{1}{y^2}+\frac{y^4}{6!}\right)\frac{\partial^4}{\partial y^4}U_0(x,y)\right]=0 \tag{4.22}$$

When k=1, then the equations (4.20) and (4.22) provides as

$$U_3(x,y)=-\left(\frac{\Gamma(\alpha+1)}{\Gamma(3\alpha+1)}\right)\left[2\left(\frac{1}{x^2}+\frac{x^4}{6!}\right)\frac{\partial^4}{\partial x^4}U_1(x,y)+2\left(\frac{1}{y^2}+\frac{y^4}{6!}\right)\frac{\partial^4}{\partial y^4}U_1(x,y)\right]$$

$$U_3(x,y)=-\frac{\Gamma(\alpha+1)}{\Gamma(3\alpha+1)}\left(2(\frac{1}{x^2}+\frac{x^4}{6!})\frac{x^2}{2}+2(\frac{1}{y^2}+\frac{y^4}{6!})\frac{y^2}{2}\right)$$

or

$$U_3(x,y)=-\frac{\Gamma(\alpha+1)}{\Gamma(3\alpha+1)}\left(2+\frac{x^6}{6!}+\frac{y^6}{6!}\right) \tag{4.23}$$

When k=2, then equations (4.20) and (4.23) gives

$$U_4(x,y) = -\left(\frac{\Gamma(2\alpha+1)}{\Gamma(4\alpha+1)}\right)\left[2\left(\frac{1}{x^2}+\frac{x^4}{6!}\right)\frac{\partial^4}{\partial x^4}U_2(x,y)+2\left(\frac{1}{y^2}+\frac{y^4}{6!}\right)\frac{\partial^4}{\partial y^4}U_2(x,y)\right]=0 \quad (4.24)$$

When k=3, then equations (4.20) and (4.24) yields

$$U_5(x,y) = -\left(\frac{\Gamma(3\alpha+1)}{\Gamma(5\alpha+1)}\right)\left[2\left(\frac{1}{x^2}+\frac{x^4}{6!}\right)\frac{\partial^4}{\partial x^4}U_3(x,y)+2\left(\frac{1}{y^2}+\frac{y^4}{6!}\right)\frac{\partial^4}{\partial y^4}U_3(x,y)\right]$$

$$U_5(x,y) = \frac{\Gamma(\alpha+1)}{\Gamma(5\alpha+1)}\left(2+\frac{x^6}{6!}+\frac{y^6}{6!}\right) \quad (4.25)$$

When k=4, then equations (4.20) and (4.25) gives as

$$U_6(x,y) = -\left(\frac{\Gamma(4\alpha+1)}{\Gamma(6\alpha+1)}\right)\left[2\left(\frac{1}{x^2}+\frac{x^4}{6!}\right)\frac{\partial^4}{\partial x^4}U_4(x,y)+2\left(\frac{1}{y^2}+\frac{y^4}{6!}\right)\frac{\partial^4}{\partial y^4}U_4(x,y)\right]$$

$$U_6(x,y) = 0 \quad (4.26)$$

When k=5, then equations(4.20) and (4.26) gives as

$$U_7(x,y) = -\left(\frac{\Gamma(5\alpha+1)}{\Gamma(7\alpha+1)}\right)\left[2\left(\frac{1}{x^2}+\frac{x^4}{6!}\right)\frac{\partial^4}{\partial x^4}U_5(x,y)+2\left(\frac{1}{y^2}+\frac{y^4}{6!}\right)\frac{\partial^4}{\partial y^4}U_5(x,y)\right]$$

$$U_7(x,y) = -\frac{\Gamma(\alpha+1)}{\Gamma(7\alpha+1)}\left(2+\frac{x^6}{6!}+\frac{y^6}{6!}\right) \quad (4.27)$$

When k=6, then the equations (4.20) and (4.27) gives as

$$U_8(x,y) = -\left(\frac{\Gamma(6\alpha+1)}{\Gamma(8\alpha+1)}\right)\left[2\left(\frac{1}{x^2}+\frac{x^4}{6!}\right)\frac{\partial^4}{\partial x^4}U_6(x,y)+2\left(\frac{1}{y^2}+\frac{y^4}{6!}\right)\frac{\partial^4}{\partial y^4}U_6(x,y)\right]$$

$$U_8(x,y) = 0 \quad (4.28)$$

When k=7, then the equations (4.20) and (4.28) gives as

$$U_9(x,y) = -\left(\frac{\Gamma(7\alpha+1)}{\Gamma(9\alpha+1)}\right)\left[2\left(\frac{1}{x^2}+\frac{x^4}{6!}\right)\frac{\partial^4}{\partial x^4}U_7(x,y)+2\left(\frac{1}{y^2}+\frac{y^4}{6!}\right)\frac{\partial^4}{\partial y^4}U_7(x,y)\right]$$

$$U_9(x,y) = \frac{\Gamma(\alpha+1)}{\Gamma(9\alpha+1)}\left(2+\frac{x^6}{6!}+\frac{y^6}{6!}\right) \quad \text{and so on} \quad (4.29)$$

Thus, the Inverse Fractional Reduced Differential Transform $\{U_k(x,y)\}_{k=0}^{\infty}$ and equations (4.21-4.29) gives as the solution of the problem,

$$u(x,y,t)=\sum_{k=0}^{\infty}U_k\left(x,y\right)\left(t\right)^{k\alpha}$$

$$u(x,y,t)=U_o(x,y)t^{0\alpha}+U_1(x,y)t^{\alpha}+U_2(x,y)t^{2\alpha}+U_3(x,y)t^{3\alpha}+U_4(x,y)t^{4\alpha}+U_5(x,y)t^{5\alpha}+........$$

$$u(x,y,t)=(2+\frac{x^6}{6!}+\frac{y^6}{6!})t^{\alpha}-\frac{\Gamma(\alpha+1)}{\Gamma(3\alpha+1)}\left(2+\frac{x^6}{6!}+\frac{y^6}{6!}\right)t^{3\alpha}+\frac{\Gamma(\alpha+1)}{\Gamma(5\alpha+1)}\left(2+\frac{x^6}{6!}+\frac{y^6}{6!}\right)t^{5\alpha}-\frac{\Gamma(\alpha+1)}{\Gamma(7\alpha+1)}\left(2+\frac{x^6}{6!}+\frac{y^6}{6!}\right)t^{7\alpha}+\frac{\Gamma(\alpha+1)}{\Gamma(9\alpha+1)}\left(2+\frac{x^6}{6!}+\frac{y^6}{6!}\right)t^{9\alpha}...$$

$$u(x,y,t)=(2+\frac{x^6}{6!}+\frac{y^6}{6!})\left(t^{\alpha}-\frac{\Gamma(\alpha+1)}{\Gamma(3\alpha+1)}t^{3\alpha}+\frac{\Gamma(\alpha+1)}{\Gamma(5\alpha+1)}t^{5\alpha}-\frac{\Gamma(\alpha+1)}{\Gamma(7\alpha+1)}t^{7\alpha}+\frac{\Gamma(\alpha+1)}{\Gamma(9\alpha+1)}t^{9\alpha}-\right), \quad (4.30)$$

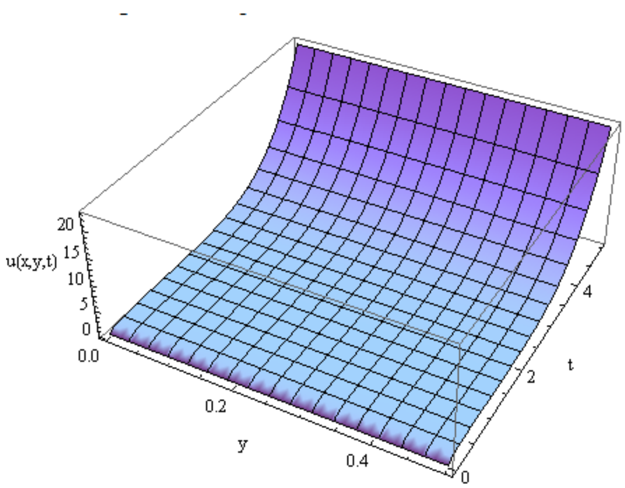
When $\boldsymbol{\alpha}$=1, then from equation (4.30) we obtain

$$u(x,y,t)=(2+\frac{x^6}{6!}+\frac{y^6}{6!})\left(t\ -\frac{\Gamma(2)}{\Gamma(4)}t^3+\frac{\Gamma(2)}{\Gamma(6)}t^5-\frac{\Gamma(2)}{\Gamma(8)}t^7+\frac{\Gamma(2)}{\Gamma(10)}t^9-....\right)$$

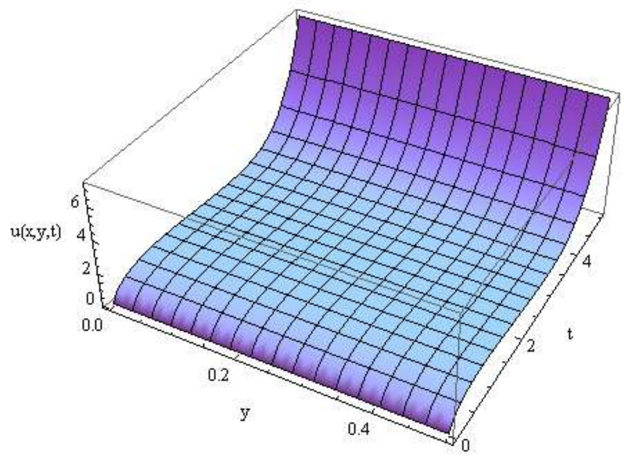
$$u(x,y,t)=(2+\frac{x^6}{6!}+\frac{y^6}{6!})\left(t\ -\frac{t^3}{3!}+\frac{t^5}{5!}-\frac{t^7}{7!}+\frac{t^9}{9!}-....\right)$$

$u(x,y,t)=(2+\frac{x^6}{6!}+\frac{y^6}{6!})\left(\sin t\right)$ Which is the exact solution of the problem and it is the same result obtained by ADM, VIM, HPM and NIM (Al-Jawary, M.A. 2015)

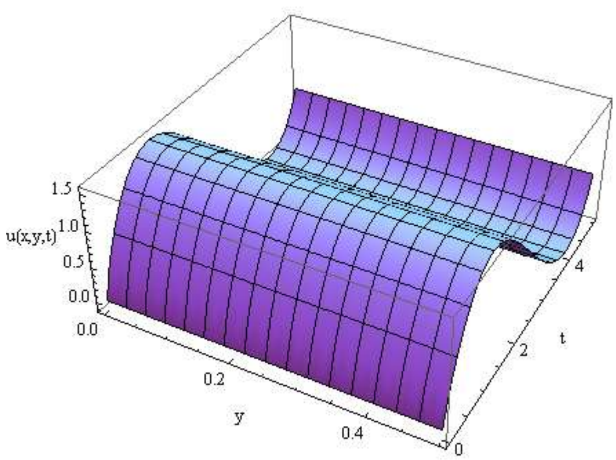
The solution curves of two dimensional time fractional parabolic partial differential equation with variable coefficients given in Example 4.1 above for different values of time fractional order α is depicted below in Figure 4.1.



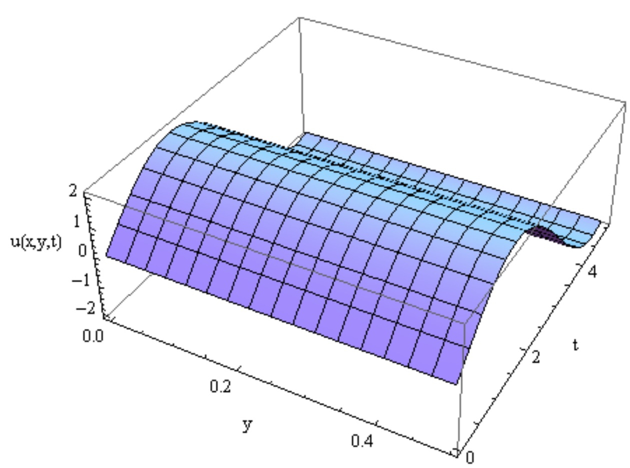
For **α** = 0.4



For **α** = 0.6



For **α** = 0.8



For **α** = 1.00

Figure 4.1: FRDTM solution behaviour of Example 4.1 at x=2 when a) **α**= 0.4, b) **α**=0.6, c)**α**=0.8, d) **α**=1 .

II: consider two dimensional non -linear time fractional non-homogeneous fourth order parabolic equation in Caputo sense of the form.

$$\frac{\partial^{2\alpha}u}{\partial t^{2\alpha}}+\left(\eta(\text{x, y})\ \text{u(x,y)}\ \frac{\partial^4 u}{\partial x^4}+\lambda(\text{x, y})\ \text{u(x,y)}\frac{\partial^4 u}{\partial y^4}\right)= f\left(x,y,t\right),\quad n-1<\alpha\le n,\ n\in\mathbb{N}, \tag{4.31}$$

subject to the initial conditions

$$\begin{aligned} &u(x,y,0)=g_1(x,y)\\ &u_t(x,y,0)=g_2(x,y)\end{aligned} \tag{4.32}$$

where u is a function of x, y, t, and f(x, y, t) is analytic function , α *is* a parameter which describes the order of time fractional derivative . $n-1<\alpha\le n,\ n\in\mathbb{N}$,

η(x, y) and λ(x, y) are functions of x and y

Applying properties of FRDTM to equations (4.31**)** and (4.32), we get the following recurrence relation**:**

$$\frac{\Gamma(k\alpha+2\alpha+1)}{\Gamma(k\alpha+1)}U_{k+2}(x,y)+\left(\eta(\text{x, y})A_k+\lambda(\text{x, y})B_k\right)=F_k\left(x,y\right)$$

$$U_{k+2}(x,y)=\frac{\Gamma(\alpha k+1)}{\Gamma(\alpha k+2\alpha+1)}\left[-\eta(\text{x, y})A_k-\lambda(\text{x, y})\ \text{B}_k+F_k(x,y)\right] \tag{4.33}$$

Where

$$A_k=\sum_{k_1=0}^{k}\text{U}_{k_1}\left(x,y\right)\frac{\partial^4 u}{\partial x^4}U_{k-k_1}(x,y)\ \text{and}\ B_k=\sum_{k_1=0}^{k}\text{U}_{k_1}\left(x,y\right)\frac{\partial^4}{\partial y^4}U_{k-k_1}(x,y)$$

and

$$\begin{aligned} &U_0\left(x,y\right)=g_1(x,y)\\ &U_1(x,y)=g_2(x,y)\end{aligned} \tag{4.34}$$

When k=0, 1, 2, 3,…n …, where nϵN then from equations (4.33) and (4.34),we obtain the successive transformed iterations as follows:

When k=0, 1, 2,… we get

$$A_0 = U_0 \frac{\partial^4}{\partial x^4} U_0 = g_1(x,y) \frac{\partial^4}{\partial x^4} g_1(x,y)$$

$$A_1 = U_1(x,y) \frac{\partial^4}{\partial x^4} U_0(x,y) + U_0(x,y) \frac{\partial^4}{\partial x^4} U_1(x,y) = g_2(x,y) \frac{\partial^4}{\partial x^4} g_1(x,y) + g_1(x,y) \frac{\partial^4}{\partial x^4} g_2(x,y)$$

$$\begin{aligned} A_2 &= U_2(x,y) \frac{\partial^4}{\partial x^4} U_0(x,y) + U_1(x,y) \frac{\partial^4}{\partial x^4} U_1(x,y) + U_0(x,y) \frac{\partial^4}{\partial x^4} U_2(x,y) \\ &= U_2(x,y) \frac{\partial^4}{\partial x^4} g_1(x,y) + g_2(x,y) \frac{\partial^4}{\partial x^4} g_2(x,y) + g_1(x,y) \frac{\partial^4}{\partial x^4} U_2(x,y) \end{aligned}$$

and so on

$$B_0 = U_0 \frac{\partial^4}{\partial y^4} U_0 = g_1(x,y) \frac{\partial^4}{\partial y^4} g_1(x,y)$$

$$B_1 = U_1(x,y) \frac{\partial^4}{\partial y^4} U_0(x,y) + U_0(x,y) \frac{\partial^4}{\partial y^4} U_1(x,y) = g_2(x,y) \frac{\partial^4}{\partial y^4} g_1(x,y) + g_1(x,y) \frac{\partial^4}{\partial y^4} g_2(x,y)$$

$$\begin{aligned} B_2 &= U_2(x,y) \frac{\partial^4}{\partial y^4} U_0(x,y) + U_1(x,y) \frac{\partial^4}{\partial y^4} U_1(x,y) + U_0(x,y) \frac{\partial^4}{\partial y^4} U_2(x,y) \\ &= U_2(x,y) \frac{\partial^4}{\partial y^4} g_1(x,y) + g_2(x,y) \frac{\partial^4}{\partial y^4} g_2(x,y) + g_1(x,y) \frac{\partial^4}{\partial y^4} U_2(x,y) \end{aligned}$$

and so on

Now when k=0, Eq. (4.31) implies

$$U_2(x,y) = \frac{1}{\Gamma(2\alpha+1)} \left[-\eta(\mathrm{x,\ y}) A_0 - \lambda(\mathrm{x,\ y})\ \mathrm{B}_0 + F_0(x,y) \right]$$

$$U_2(x,y) = \frac{1}{\Gamma(2\alpha+1)} \left[-\eta(\mathrm{x,\ y})\ g_1(x,y) \frac{\partial^4}{\partial x^4} g_1(x,y) - \lambda(\mathrm{x,\ y})\ g_1(x,y) \frac{\partial^4}{\partial y^4} g_1(x,y) + F_0(x,y) \right]$$

Now when k=1, Eq. (4.31) implies

$$U_3(x,y) = \frac{\Gamma(\alpha+1)}{\Gamma(3\alpha+1)} \begin{bmatrix} -\eta(\mathrm{x,\ y}) \left(g_2(x,y) \frac{\partial^4}{\partial x^4} g_1(x,y) + g_1(x,y) \frac{\partial^4}{\partial x^4} g_2(x,y) \right) \\ -\lambda(\mathrm{x,\ y}) \left(g_2(x,y) \frac{\partial^4}{\partial y^4} g_1(x,y) + g_1(x,y) \frac{\partial^4}{\partial y^4} g_2(x,y) \right) + F_1(x,y) \end{bmatrix}$$

Therefore, the approximate solution of the Eq. (4.31) is given by

$$u(x,y,t)=\sum_{k=0}^{\infty}U_k(x,y)t^{k\alpha}=U_0(x,y)+U_1(x,y)t^{\alpha}+U_2(x,y)t^{2\alpha}+\cdots$$

$$u(x,y,t)=g_1(x,y)+g_2(x,y)t^{\alpha}+\frac{1}{\Gamma(2\alpha+1)}\left[-\eta(\text{x, y})\,g_1(x,y)\frac{\partial^4}{\partial x^4}g_1(x,y)-\lambda(\text{x, y})\,g_1(x,y)\frac{\partial^4}{\partial y^4}g_1(x,y)+F_0(x,y)\right]t^{2\alpha}+\cdots$$

Example 4.2 Consider the following two dimensional non-linear fourth-order time fractional non-homogeneous parabolic partial differential equation with variable coefficients.

$$\frac{\partial^{2\alpha}u}{\partial t^{2\alpha}}+xu\frac{\partial^4 u}{\partial x^4}+yu\frac{\partial^4 u}{\partial y^4}=2x^4y^4+24t^2x^8y^5+24t^2x^5y^8,\frac{1}{2}<x,y<1,t>o,0<\alpha\le 1 \tag{4.35}$$

with initial conditions

$$u(x,y,0)=0,u_t(x,y,0)=0 \tag{4.36}$$

Applying the related rules in table 1 on equations (4.35) and (4.36),we obtain the following recurrence relations:

$$\frac{\Gamma(\alpha(k+2)+1)}{\Gamma(k\alpha+1)}U_{k+2}(x,y)+xA_k(x,y)+yB_k(x,y)=2x^4y^4\sigma(k)+(24x^8y^5+24x^5y^8)\sigma(k-2)$$

Where, $A_k(x,y)=\sum_{k_1=0}^{k}\text{U}_{k_1}(x,y)\frac{\partial^4}{\partial x^4}U_{k-k_1}(x,y)$ and $B_k(x,y)=\sum_{k_1=0}^{k}\text{U}_{k_1}(x,y)\frac{\partial^4}{\partial y^4}U_{k-k_1}(x,y)$

or equivalently

$$U_{k+2}(x,y)=\frac{\Gamma(k\alpha+1)}{\Gamma(\alpha(k+2)+1)}\left[-xA_k(x,y)-yB_k(x,y)+2x^4y^4\sigma(k)+(24x^8y^5+24x^5y^8)\sigma(k-2)\right] \tag{4.37}$$

and

$$\begin{aligned}U_0(x,y)&=0\\U_1(x,y)&=0\end{aligned} \tag{4.38}$$

When k=0, then the equations (4.37) and (4.38) gives

$$U_2(x,y)=\frac{1}{\Gamma(2\alpha+1)}\left[-xA_0\left(x,y\right)-yB_0\left(x,y\right)+2x^4y^4\right] \tag{4.39}$$

When k=1, then the above equations (4.37) and (4.38) gives

$$U_3(x,y)=\frac{\Gamma\left(\alpha+1\right)}{\Gamma(3\alpha+1)}\left[-xA_1\left(x,y\right)-yB_1\left(x,y\right)\right]$$

$$U_3(x,y)=\frac{\Gamma\left(\alpha+1\right)}{\Gamma(3\alpha+1)}\begin{bmatrix}-x\left(U_1\left(x,y\right)\frac{\partial^4}{\partial x^4}U_0\left(x,y\right)+U_0\left(x,y\right)\frac{\partial^4}{\partial x^4}U_1\left(x,y\right)\right)- \\ y\left(U_1\left(x,y\right)\frac{\partial^4}{\partial y^4}U_0\left(x,y\right)+U_0\left(x,y\right)\frac{\partial^4}{\partial y^4}U_1\left(x,y\right)\right)\end{bmatrix}=0 \tag{4.40}$$

When k=2, then the equation (4.37) and (4.39) gives

$$U_4(x,y)=\frac{\Gamma\left(2\alpha+1\right)}{\Gamma\left(4\alpha+1\right)}\left[24x^8y^5+24x^5y^8-\left(xA_2\left(x,y\right)+yB_2\left(x,y\right)\right)\right]$$

$$U_4(x,y)=\frac{\Gamma\left(2\alpha+1\right)}{\Gamma\left(4\alpha+1\right)}\begin{bmatrix}-x\left(U_2\frac{\partial^4}{\partial x^4}U_0\left(x,y\right)+U_1\left(x,y\right)\frac{\partial^4}{\partial x^4}U_1\left(x,y\right)+U_0\left(x,y\right)\frac{\partial^4}{\partial x^4}U_2\left(x,y\right)\right)- \\ y\left(U_2\frac{\partial^4}{\partial y^4}U_0\left(x,y\right)+U_1\left(x,y\right)\frac{\partial^4}{\partial y^4}U_1\left(x,y\right)+U_0\left(x,y\right)\frac{\partial^4}{\partial y^4}U_2\left(x,y\right)\right)+ \\ (24x^8y^5+24x^5y^8)\end{bmatrix}$$

$$U_4(x,y)=\frac{\Gamma(2\alpha+1)}{\Gamma(4\alpha+1)}\left[24(x^8y^5+x^5y^8)\right] \tag{4.41}$$

When k=3, then the equations (4.37) and (4.39) provides

$$U_5(x,y)=\frac{\Gamma\left(3\alpha+1\right)}{\Gamma(5\alpha+1)}\left[-xA_3\left(x,y\right)-yB_3\left(x,y\right)\right]$$

$$U_5(x,y)=\frac{\Gamma(3\alpha+1)}{\Gamma(5\alpha+1)}\left[\begin{array}{l} -x\left(\begin{array}{l} U_3(x,y)\frac{\partial^4}{\partial x^4}U_0(x,y)+U_2(x,y)\frac{\partial^4}{\partial x^4}U_1(x,y)+ \\ U_1(x,y)\frac{\partial^4}{\partial x^4}U_2(x,y)+U_0(x,y)\frac{\partial^4}{\partial x^4}U_3(x,y) \end{array}\right) \\ -y\left(\begin{array}{l} U_3(x,y)\frac{\partial^4}{\partial y^4}U_0(x,y)+U_2(x,y)\frac{\partial^4}{\partial y^4}U_1(x,y)+ \\ U_1(x,y)\frac{\partial^4}{\partial y^4}U_2(x,y)+U_0(x,y)\frac{\partial^4}{\partial y^4}U_3(x,y) \end{array}\right) \end{array}\right]=0$$

When k=4, then the equations (4.37) and (4.39) gives

$$U_6(x,y)=\frac{\Gamma(4\alpha+1)}{\Gamma(6\alpha+1)}\left[-xA_4(x,y)-yB_4(x,y)\right]$$

$$U_6(x,y)=\frac{\Gamma(4\alpha+1)}{\Gamma(6\alpha+1)}\left[\begin{array}{l} -x\left(\begin{array}{l} U_4(x,y)\frac{\partial^4}{\partial x^4}U_0(x,y)+U_3(x,y)\frac{\partial^4}{\partial x^4}U_1(x,y)+ \\ U_2(x,y)\frac{\partial^4}{\partial x^4}U_2(x,y)+U_1(x,y)\frac{\partial^4}{\partial x^4}U_3(x,y)+U_0(x,y)\frac{\partial^4}{\partial x^4}U_4(x,y) \end{array}\right) \\ -y\left(\begin{array}{l} U_4(x,y)\frac{\partial^4}{\partial y^4}U_0(x,y)+U_3(x,y)\frac{\partial^4}{\partial y^4}U_1(x,y)+ \\ U_2(x,y)\frac{\partial^4}{\partial y^4}U_2(x,y)+U_1(x,y)\frac{\partial^4}{\partial y^4}U_3(x,y)+U_0(x,y)\frac{\partial^4}{\partial y^4}U_4(x,y) \end{array}\right) \end{array}\right]$$

$$U_6(x,y)=\frac{\Gamma(4\alpha+1)}{\Gamma(6\alpha+1)}\left[-\left(\frac{96(x^5y^8+x^8y^5)}{\Gamma(2\alpha+1)\Gamma(2\alpha+1)}\right)\right] \tag{4.42}$$

When k=5, then equations (4.36) and (4.39) implies

$$U_7(x,y)=\frac{\Gamma(5\alpha+1)}{\Gamma(7\alpha+1)}\left[-xU_5(x,y)\frac{\partial^4}{\partial x^4}U_5(x,y)-yU_5(x,y)\frac{\partial^4}{\partial y^4}U_5(x,y)\right] \tag{4.43}$$

$U_7(x,y)=0$,

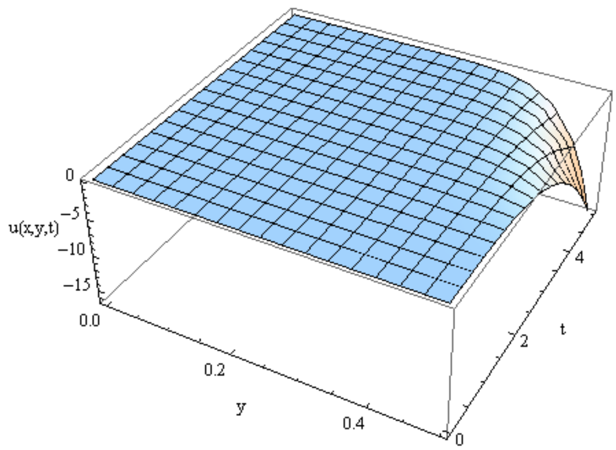
and soon. Thus, the solution of the given PDE is

$$u(x,y,t)=\sum_{k=0}^{\infty}U_k(x,y)(t)^{k\alpha}$$

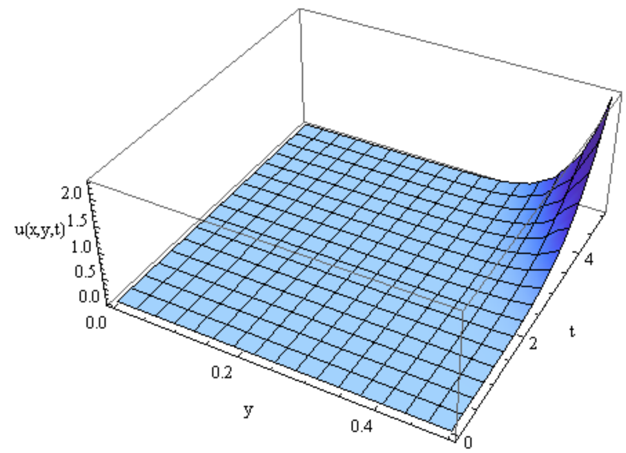
$$\begin{aligned} u(x,y,t)&=U_0(x,y)+U_1(x,y)t^{\alpha}+U_2(x,y)t^{2\alpha} \\ &+U_3(x,y)t^{3\alpha}+U_4(x,y)t^{4\alpha}+U_5(x,y)t^{5\alpha}+\ldots U_n(x,y)t^{n\alpha} \end{aligned} \tag{4.44}$$

when $\alpha = 1$, then equation (4.44) becomes $u(x, y, t) = x^4 y^4 t^2$ which is the required exact solution of the problem and it is the same result obtained by ADM, VIM, HPM (Al-Jawary, M.A. 2015)

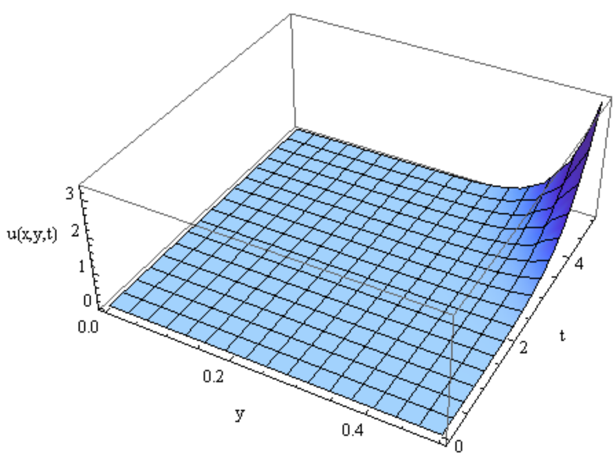
The solution curves of two dimensional time fractional parabolic partial differential equation with variable coefficients given in Examples 4.2 for different values of time fractional order α is depicted below in Figure 4.2.



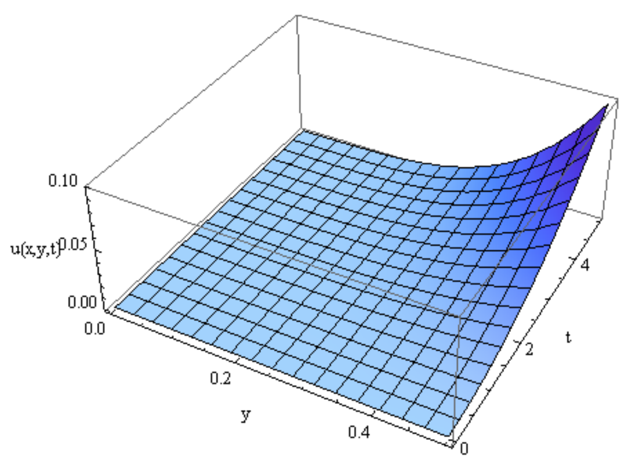
A



B



C



D

Figure 4.2:FRDTM solution behaviour of Example 4.2 at x=0.5 when a) **α**= 0.6, b) **α**=0.75, c)**α**=0.9, d) **α**=1.

4.3 Discussion

As it can be clearly seen from numerical Examples 4.1 and 4.2, the analytical solutions of two dimensional fourth order time fractional parabolic partial differential equations are obtained by using fractional reduced differential transform method.

- The results show that it is easier to make calculations with FRDTM, because it does not include integrals like local fractional VIM, ADM and HAM.
- The proposed solutions are obtained in the form of power series
- FRDTM is very powerful and easy applicable mathematical tool for solve linear and nonlinear PDEs with differentiable terms.
- The validity and efficiency of FRDTM has been confirmed by two test problems. It is found that the obtained solutions are agreed well with the solutions obtained by ADM,VIM,HPM and NIM (Al-Jawary, M.A.2015)
- We compared the results obtained by the proposed method between the approximate solutions and the exact solutions for the fourth order parabolic partial differential equation in two dimension when $\alpha \to 1$.The result shows that when α closure and closure to 1, the approximated solution are closure to the exact solution, and when α is 1, the approximated solution graph fits with the graph of the exact solution.

CHAPTER FIVE

CONCLUSION AND FUTURE WORK

5.1 Conclusion

In this study, the fractional reduced differential transform method namely (FRDTM) is implemented to obtain the exact solution for solving linear and nonlinear two dimensional fourth-order time fractional parabolic partial differential equation with variable coefficient that governs the transverse of a vibrating beam using the initial condition and recursion relations. The FRDTM is simple to understand and easy to implement and does not require any restrictive assumptions as required by some existing techniques. The obtained exact solution of applying the FRDTM is in full agreement with the results obtained with those methods available in the literature such as Adomian decomposition method, Variational iteration method and Homotopy perturbation method. Moreover, by solving some examples the FRDTM appears to be very accurate to employ with reliable results.

5.2 Future work

In this study, we have successfully shown that the fractional reduced differential transform method can be used to solve linear and nonlinear two dimensional fourth-order time fractional parabolic partial differential equation with variable coefficients. Thus, we recommend that the proposed method can be applied to solve different models involving linear and non-linear partial differential equations arising in science and engineering fields. Also we recommend that to obtain the analytic solution of three dimensional fourth-order time fractional parabolic partial differential equation with variable coefficients for future work.

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