STABILITY AND BIFURCATION ANALYSIS OF ACTIVATOR-INHIBITOR REACTION DIFFUSION SYSTEM

A THESIS SUBMITTED TO THE DEPARTMENT OF MATHEMATICS JIMMA UNIVERSITY IN PARTIAL FULFILLMENT FOR THE REQUIREMENTS OF THE DEGREE OF MASTERS OF SCIENCE (M.Sc) IN MATHEMATICS

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***Abstrac*t**

In this thesis, stability and bifurcation analysis of activator-inhibitor reaction diffusion system was considered. The system was analyzed into two parts. The first part is without diffusion. Without diffusion, the system was linearized using Jacobean matrix about equilibrium point. The local stability condition of the equilibrium point was proved by using Routh Hurwitz stability criteria. Hopf bifurcation condition without diffusion was determined by the help of Hopf bifurcation theorem in planar system. The second part is with diffusion. With diffusion, stability conditions are proved by using Routh Hurwitz stability criteria. Diffusive instability condition was also set down. The system undergoes Hopf bifurcation with diffusion provided that specific condition is satisfied. Finally, in order to verify the applicability of the result two numerical examples were solved and MATLAB simulation was implemented to support the findings of the study.

Key words: Diffusion, Local stability, Turing instability, Routh Hurwitz stability criteria, Hopf bifurcation.

Contents

CHAPTER ONE

1. INTRODUCTION

1.1 Background of the Study

Differential equations have a remarkable potential to predict the world around us. They are used in a wide variety of disciplines such as Biology, Economics, Physics, Chemistry and Engineering. Mathematical modeling is the application of mathematics to describe real world problems and investigate important questions that arise from it. Mathematical models by partial differential equation include derivatives of at least two independent variables and hence we can describe the dynamical behavior of our problem of interest in terms of two or more variables at the same time. Mathematical model and theoretical Biology is an interdisciplinary scientific research field with a range of applications. Theoretical Biology focuses more on the development of theoretical principles for Biology while Mathematical Biology focuses on the use of Mathematical tools to study Biological systems (Israel, 1988).

Nowadays Mathematics and Biology have a synergistic relationship. There are several types of Biological problems which can be treated as Mathematical modeling; modeling on activator-inhibitor reaction diffusion system one of it (Barnes, 2010). Mathematical model is a powerful tool for understanding biologically observed phenomena which cannot be understood by verbal reasoning alone (Alder, 2001).

A reaction-diffusion system exhibits diffusion-driven instability, sometimes called instability, if the homogeneous steady state is stable to small perturbations in the absence of diffusion but unstable to small spatial perturbations when diffusion is present. Reaction-diffusion system is often used to model the development of Biological systems, most prominently in the study of Biological pattern formation. In 1972, Alens and Heins proposed that the instabilities in reacting and diffusing systems as a result of interaction of two biochemical substances with different diffusion rates can only occur if one of the substances (activator) influences the production of itself and that of the other one, while the other substance (inhibitor) inhibits the production. All such models that exhibit this property are called Activator-Inhibitor model.

Activator-Inhibitor Reaction-diffusion system is a nonlinear spatiotemporal model given by the following equation (Murray, 2002).

$$\frac{\partial u}{\partial t} = a - bu + \frac{u^2}{v} + d_1 \frac{\partial^2 u}{\partial x^2}$$
$$\frac{\partial v}{\partial t} = u^2 - v + d_2 \frac{\partial^2 v}{\partial x^2} \quad , \qquad (1.1)$$

Where the variables u represents activator, v represents inhibitor, and the constants a represent the rate at which activator is supplied, b represents the rate at which the activator degrades and d_1 and d_2 are rate of diffusion for activator and inhibitor respectively. Reaction-diffusion equations have been studied extensively as mathematical models of systems with reactions and diffusion across a wide range of applications including nerve cell signaling, animal coat patterns, population dispersal, and chemical waves (Sandip and Chakrabartinin, 1999**)**. The concept of diffusion-driven instability also known as Turing instability named after the mathematician (Turing, 1952) which leads to patterns that are stationary in time and periodic in space. For a two-component reaction-diffusion system, a key requirement for diffusion-driven instability is the concept of long-range inhibition and short-range activation (Gierer, 1972).

Stability theory plays a central role in system Engineering especially, in the field of control systems and automation, with regard to both dynamics and control. Turing was the first to present stability analysis which shows that the reaction and diffusion of chemicals can give rise to spatial structure and to suggest that this in turn could be a key event
in the formation of Biological pattern (Turing, 1952). Bifurcation theory is the Mathematical study of changes in the qualitative or topological structure of a given dynamical systems. Bifurcation occurs when a small change made to the parameter values (the bifurcation parameters) of a system causes a sudden 'qualitative' or topological change in its behavior (Blanchard, 2006).

Bifurcations are important scientifically-they provide models of transitions and instabilities as some control parameters are varied. In scientific fields as diverse as fluid Mechanics, Electronics, Chemistry and Theoretical Ecology, there is the application of what is referred to as bifurcation analysis ; the analysis of a system of differential equations under parameter variation.

A local bifurcation is in which an equilibrium point of a dynamical system loses stability as a pair of complex conjugate eigenvalues of linearized system crosses the imaginary axis of the complex plane (Strogatz, 1994).

In 2019, Makwata *et al*. investigated stability and bifurcation analysis of Fisher Mathematical model with allee Effects and they obtained the three different equilibrium solutions as one being stable and with two being saddles. Pijush *et al*. (2018) investigated the stability and bifurcation analysis of three-species food chain model with fear and they concluded that Chaotic dynamics can be controlled by the fear factors. In 2017, Yang *et al*. described Chamostat model which involve control strategy with threshold window are analyzed. They investigated the qualitative analysis such as existence and stability of equilibrium points of the system and it is proved that Pseudo-equilibrium cannot coexist. In 2016, Tee and Salleh investigated Hopf bifurcation of a nonlinear modified Lorenz system using normal form theory. In 1999, Sandip and Chakrabartinon investigate linear Bifurcation Analysis of Reaction-Diffusion Activator-Inhibitor system and determine the nature of the bifurcation point of the system and state solution of critical point.

However, stability and bifurcation analysis of mathematical model represented by Eq. (1.1) is not yet well investigated in the existing literature. Therefore, the central goal of this study is to analyze spatiotemporal model of activator-inhibitor reaction diffusion system represented by

Eq. (1.1).

1.2 Statement of the Problem

Nonlinear Mathematical models of real-world phenomena that are formulated in terms of spatio temporal model as in Eq. (1.1) are not easy to directly solve for their solution and hence it is necessary to use qualitative approaches, such as stability and bifurcation analysis, to investigate their solution behaviors. As a result, this study focuses on the following points.

- ❖ Local stability conditions of the system without diffusion,
- ❖ Turing instability conditions of the system,
- ❖ Hopf bifurcation conditions of the system (1.1) without diffusion,
- ❖ Hopf bifurcation conditions of diffusion driven system given by Eq. (1.1).

1.3 Objective of the Study

1.3.1 General Objective of the Study

The main objective of this study is stability and bifurcation analysis of activator-inhibitor reaction diffusion system given by Eq. (1.1).

1.3.2 Specific Objectives of the Study

The specific objectives of the study are:

- ❖ To determine local stability conditions of the system without diffusion.
- ❖ To determine Turing instability conditions of the system.
- ❖ To determine Hopf bifurcation conditions of the system (1.1) without diffusion.
- ❖ To determine Hopf bifurcation conditions of diffusion driven system given by Eq. (1.1).

1.4 Significance of the Study

This study helps biologists the way to well regulate systems involving reaction diffusion one being activator and the other being inhibitor under the influence of chemical or enzyme.

1.5 Delimitation of the study

This study is delimited to discussing the local stability and hopf bifurcation analysis of activator-inhibitor reaction-diffusion system given by Eq. (1.1).

CHAPTER TWO

2. LITERATURE REVIEW

Mathematical Biology is a fast-growing, well recognized, and the most exciting modern application of Mathematics. The increasing use of Mathematics in Biology is inevitable as Biology becomes more quantitative. Mathematical Biology is the application of Mathematical modeling to solve problems in Biology. It is one of the fastest growing research areas in Mathematics and is contributing significantly to our understanding of the Biological world (Murray, 2002). Biochemical reactions are continually taking place in all living organisms and most of them involve proteins called enzymes, which act as remarkably efficient catalysts. Enzymes react selectively on definite compounds called substrates. Enzymes are important in regulating Biological processes, for example, as activators or inhibitors in a reaction. To understand their role we have to study enzyme kinetics which is mainly the study of rates of reactions the temporal behavior of the various reactants and the conditions which influence them.

Reaction and diffusion are mathematical models which correspond to several physical phenomena. The most common is the change in space and time of the concentration of one or more chemical substances, local chemical reaction in which the substances are transformed into each other and diffusion which causes the substances to spread out over a surface in space. Reaction–diffusion systems are naturally applied in chemistry. However, the system can also describe dynamical processes of non-chemical nature. The Activator-Inhibitor model originally arose in studies of pattern-formation in biology. Mathematically, reaction–diffusion system takes the form of semi-linear parabolic partial differential equation.

Reaction-diffusion systems have been playing a significant role in different fields of science such as chemical reactions, electronic devices, combustion processes, neuron structures, population of organisms etc. (Murray, 2002). In 2015, Chakrabarthi and Debanjana investigated activator-inhibitor model of a dynamical system and application to an oscillating chemical reaction system. In 2018, Minhui Zhu investigated activator-inhibitor model for seashell pattern and many features of seashell patterns and their molecular bases remains unexplained.

In 2019, Yanqiu and Juncheng studied dynamics and patterns of an activator-inhibitor model with cubic polynomial source and they illustrate that weakly linear coupling in the activator-inhibitor model can caused synchronous and also ant-phase patterns. In 2011, Bonni Jean investigated a nonlinear stability analysis of vegetative Turing pattern formation for an interaction-diffusion plant-surface water model system in an arid flat environment.

In 1974, Gierer and Meinhardt have shown that stable inhomogeneous patterns can be formed if the auto catalytic production of the activator is short-ranged, while the formation of the inhibitor is long-ranged. In other words, the self-enhancing process involving the activator is chiefly local, whereas the inhibitor should have a long-range behavior characterized by rapid spreading, producing activator removal at long distances. The role of the diffusion range is thus of central importance in the dynamics of activator–inhibitor systems. Bifurcation theory is the Mathematical study of changes in the qualitative or topological structure of a given dynamical systems. Bifurcation occurs when a small change made to the parameter values (the bifurcation parameters) of a system causes a sudden 'qualitative' or topological change in its behavior (Blanchard, 2006).

In 2019, Makwata *et al*. investigated stability and bifurcation analysis of Fisher Mathematical model with allee Effects and they obtained the three different equilibrium solutions as one being stable and with two being saddles. Pijush *et al*. (2018) investigated the stability and bifurcation analysis of three-species food chain model with fear and they concluded that Chaotic dynamics can be controlled by the fear factors. In 2017, Yang *et al*. described Chamostat model which involve control strategy with threshold window are analyzed. They investigated the qualitative analysis such as existence and stability of equilibrium points of the system and it is proved that Pseudo-equilibrium cannot coexist. In 2016, Tee and Salleh investigated Hopf bifurcation of a nonlinear modified Lorenz system using normal form theory. In 1999, Sandip and Chakrabartinon investigate linear Bifurcation Analysis of Reaction-Diffusion Activator-Inhibitor system and determine the nature of the bifurcation point of the system and state solution of critical point.

CHAPTER THREE

3. METHODOLOGY

3.1 Study Period

This study was conducted from September, 2019 to October, 2021.

3.2 Study Design

The study employs mixed design (analytical and experimental approaches).

3.3 Source of Information

The sources of information for the study were journals, published article and related information from internet.

3.4 Mathematical Procedure

This study was conducted based on the following procedures

1. Determining the equilibrium point of the system without diffusion;
2. Linearizing the system without diffusion about positive equilibrium point;
3. Determining the local stability conditions of the system without diffusion;
4. Determining Hopf bifurcation conditions of the system without diffusion;
5. Linearizing diffusion driven system about positive equilibrium point;
6. Determining Turing instability conditions of the system;
7. Determining Hopf bifurcation conditions of diffusion driven system;
8. Verifying the result using numerical simulation.

CHAPTER FOUR

4. RESULTS AND DISCUSSION

4.1 Preliminaries

Definition 4.1 : Asymptotically stable(the world's leading publisher of open access Books)

Consider the two dimensional system

$$x' = f(x, y)$$
$$y' = g(x, y)$$

Suppose that $\left(x^*, y^*\right)$ is an equilibrium point. That is

$$f(x^*, y^*) = 0 \ \ and \ g\left(x^*, y^*\right) = 0$$

Then an equilibrium point $\left(x^*, y^*\right)$of the system is stable if all the eigenvalues of the Jacobean matrix J evaluated at $\left(x^*, y^*\right)$ have negative real part.

An equilibrium point is unstable if at least one eigenvalues of the Jacobean matrix J has positive real parts.

Definition4.2: Routh Hurwitz Stability Criterion (Katsuhiko, 1997)

The local stability of the equilibrium points is applying the Routh's stability criterion for the given the characteristic polynomial of the form

$$a_0\lambda^n + a_1\lambda^{n-1} + \ldots a_n = 0 \quad ,$$

Where $a_0 \neq 0$ and $a_n > 0$

Then the Routh Hurwitz table is given as follows:

$$
\begin{array}{c|ccccc}
\lambda^{n} & a_0 & a_2 & a_4 & a_6 & \cdots \\
\lambda^{n-1} & a_1 & a_3 & a_5 & a_7 & \cdots \\
\lambda^{n-2} & b_1 & b_2 & b_3 & b_4 & \cdots \\
\lambda^{n-3} & c_1 & c_2 & c_3 & c_4 & \cdots \\
\lambda^{n-4} & d_1 & d_2 & d_3 & d_4 & \cdots \\
\vdots & \vdots & \vdots & \vdots & & \\
\lambda^{2} & e_1 & e_2 & & & \\
\lambda^{1} & f_1 & & & & \\
\lambda^{0} & g_0 & & & &
\end{array},
$$

Where the coefficients of the parameter

$$b_1 = \frac{a_1 a_2 - a_0 a_3}{a_1}$$

$$b_2 = \frac{a_1 a_4 - a_0 a_5}{a_1}$$

$$b_3 = \frac{a_1 a_6 - a_0 a_7}{a_1}$$

$$c_1 = \frac{b_1 a_3 - a_1 b_2}{b_1}$$

$$c_2 = \frac{b_1 a_5 - a_1 b_3}{b_1}$$

$$c_3 = \frac{b_1 a_7 - a_1 b_4}{b_1}$$

$$d_1 = \frac{c_1 b_2 - b_1 c_2}{c_1}$$

$$d_2 = \frac{c_1 b_3 - b_1 c_3}{c_1}$$

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Respectively.

The equilibrium point is stable if there is no sign change in the first column and the equilibrium point is unstable if there is sign change in the first column of the Routh table above (Khalil,2003).

Particularly, for quadratic equation of the form $a_0\lambda^2 + a_1\lambda + a_2 = 0$, we have

$$\begin{array}{c|cc} \lambda^2 & a_0 & a_2 \\ \lambda & a_1 & \\ \lambda^0 & a_2 & \end{array}$$

If the first column is positive, that is $a_0 > 0, a_1 > 0, a_2 > 0$, then the system is asymptotically stable.

Hopf Bifurcation Theorem (Roberto, 2011)

Consider the planar system

$$\frac{dx}{dt} = f_\mu(x, y),$$
$$\frac{dy}{dt} = g_\mu(x, y),$$

Where μ is a parameter. Suppose it has a fixed point, which without loss of generality we may assume to be located at $(x, y) = (0,0)$. Let the eigenvalues of the linearized system about the fixed point be given by $\lambda(\mu), \bar{\lambda}(\mu) = \alpha(\mu) \pm i\beta(\mu)$. Suppose further that for a certain value of μ (which we may assume to be 0) the following conditions are satisfied:

1. $\alpha(0) = 0$, $\beta(0) = \omega \neq 0$ where $\text{sgn}(\omega) = \text{sgn}[(\partial g\mu / \partial x) | \mu = 0(0,0)]$

(Non-hyperbolicity condition: conjugate pair of imaginary eigenvalues)

2. $\frac{d\alpha(\mu)}{d\mu}|_{\mu=0} = d \neq 0$ (transversality condition: the eigenvalues cross the imaginary axis with non-zero speed

3. $a \neq 0$, where $a = \frac{1}{16}\left(f_{xxx} + f_{xyy} + g_{xxy} + g_{yyy}\right) + \frac{1}{16\omega}\left(f_{xy}\left(f_{xx} + f_{yy}\right) - g_{xy}\left(g_{xx} + g_{yy}\right) - f_{xx}g_{xx} + f_{yy}g_{yy}\right)$,

with $f_{xy} = \frac{\partial^2 f_\mu}{\partial x \partial y}\Big|_{\mu=0}(0,0)$, etc. (genericity condition)

Then a unique curve of periodic solutions bifurcates from the origin into the region $\mu > 0$ if ad < 0 or $\mu < 0$ if ad > 0. The origin is a stable fixed point for $\mu > 0$ (resp. $\mu < 0$) and an unstable fixed point for $\mu < 0$ (resp. $\mu > 0$) if $d < 0$ (resp. $d > 0$) whilst the periodic solutions are stable (resp. unstable) if the origin is unstable (resp. stable) on the side of $\mu = 0$ where the periodic solutions exist.

4.2. Equilibrium point

The mathematical model under consideration from equation (1.1)

$$\frac{\partial u}{\partial t} = a - bu + \frac{u^2}{v} + d_1 \frac{\partial^2 u}{\partial x^2}$$
$$\frac{\partial v}{\partial t} = u^2 - v + d_2 \frac{\partial^2 v}{\partial x^2}$$

Equilibrium point without Diffusion is obtained by taking $d_1 = d_2 = 0$

$$\frac{du}{dt} = a - bu + \frac{u^2}{v} \tag{4.1}$$
$$\frac{dv}{dt} = u^2 - v \tag{4.2}$$

To find Equilibrium point equate equations (4.1) and (4.2) with zero

$$a - bu + \frac{u^2}{v} = 0 \tag{4.3}$$
$$u^2 - v = 0 \tag{4.4}$$

From equation (4.4)

$$v = u^2 \tag{4.5}$$

Substituting equation (4.5) into equation (4.3)

$$a - bu + 1 = 0$$

$$u = \frac{a+1}{b}$$

$$v = \frac{(a+1)^2}{b^2}$$

The Equilibrium point is:

$$E = \left(\frac{a+1}{b}, \frac{(a+1)^2}{b^2} \right)$$

4.3. Local stability Analysis without Diffusion

$$f_1 = a - bu + \frac{u^2}{v}$$

Let $\quad f_2 = u^2 - v$

In order to calculate Jacobean matrix J differentiate f_1 and f_2 with respect to u and v.

$$J = \begin{pmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \end{pmatrix}$$

$$J = \begin{pmatrix} -b + \frac{2u}{v} & \frac{-u^2}{v^2} \\ 2u & -1 \end{pmatrix}$$

The Jacobean matrix evaluated at equilibrium point for the system (1.1) is given by

$$J = \begin{pmatrix} -b + \frac{2u}{v} & -\frac{u^2}{v^2} \\ 2u & -1 \end{pmatrix}_{E=\left(\frac{(a+1)}{b}, \frac{(a+1)^2}{b^2} \right)}$$

$$J = \begin{pmatrix} \frac{b - ab}{a+1} & \frac{-b^2}{(a+1)^2} \\ 2\frac{(a+1)}{b} & -1 \end{pmatrix}$$

The characteristics Equation of Jacobean matrix is given by

$$|J-\lambda I|=0$$

$$\begin{vmatrix} \dfrac{b-ab-a\lambda-\lambda}{a+1} & \dfrac{-b^2}{(a+1)^2} \\ \dfrac{2(a+1)}{b} & -1-\lambda \end{vmatrix}=0$$

$$\lambda^2+\left(\frac{a+1+b(a-1)}{a+1}\right)\lambda+b\frac{(a+1)}{a+1}=0$$

$$\lambda^2+\left(\frac{a-b+ab+1}{a+1}\right)\lambda+b=0$$

The Routh Hurwitz table is given by

$$\begin{array}{c|cc} \lambda^2 & 1 & b \\ \lambda^1 & \dfrac{1+a+ab-b}{a+1} & 0 \\ \lambda^0 & b & \end{array}$$

For the first column to be positive,

$$\frac{1+a+ab-b}{a+1}>0$$

Case I: if $a \geq 1,$ then $1+a+ab-b>0$

Therefore the equilibrium point is asymptotically stable

Case II: if $0<a<1,$ then

$$1+a+ab-b>0 \qquad 4.6$$

Therefore, for $0<a<1$ the equilibrium point is asymptotically stable if condition (4.6) is satisfied

4.4. Hopf Bifurcation without Diffusion

$$\frac{du}{dt} = a - bu + \frac{u^2}{v}$$
$$\frac{dv}{dt} = u^2 - v$$

The equilibrium point is

$$E = \left(\frac{a+1}{b}, \frac{(a+1)^2}{b^2} \right)$$

Applying the coordinate transformation about equilibrium point

$$u_1 = u - \left(\frac{a+1}{b} \right) \Rightarrow u = u_1 + \left(\frac{a+1}{b} \right)$$
$$v_1 = v - \left(\frac{a+1}{b} \right)^2 \Rightarrow v = v_1 + \frac{(a+1)^2}{b^2}$$

For the sake of simplicity, let us use u_1 and v_1 as u and v respectively

$$\frac{du}{dt} = a - b\left(u + \frac{a+1}{b} \right) + \frac{\left(u + \frac{(a+1)}{b} \right)^2}{v + \frac{(a+1)^2}{b^2}} \tag{4.7}$$
$$\frac{dv}{dt} = \left(u + \frac{(a+1)}{b} \right)^2 - \left(v + \frac{(a+1)^2}{b^2} \right) \tag{4.8}$$

To find equilibrium point equate equation (4.7) and (4.8) with zero

$$a-b\left(u+\frac{a+1}{b}\right)+\frac{\left(u+\frac{(a+1)}{b}\right)^2}{v+\frac{(a+1)^2}{b^2}}=0 \tag{4.9}$$

$$\left(u+\frac{(a+1)}{b}\right)^2-\left(v+\frac{(a+1)^2}{b^2}\right)=0 \tag{4.10}$$

From equation (4.10)

$$v+\frac{(a+1)^2}{b^2}=\left(u+\frac{(a+1)}{b}\right)^2 \tag{4.11}$$

Substituting Equation (4.11) into equation (4.9)

$$a-b\left(u+\frac{(a+1)}{b}\right)+1=0$$

$$-b\left(u+\frac{(a+1)}{b}\right)=-(a+1)$$

$$u+\frac{a+1}{b}=\frac{a+1}{b}$$

$$u=0$$

$$v=\left(u+\frac{(a+1)}{b}\right)^2-\frac{(a+1)^2}{b^2}$$

$$v=\left(\frac{(a+1)}{b}\right)^2-\frac{(a+1)^2}{b^2}$$

$$v=0$$

The equilibrium point about coordinate transformation is $\mathrm{E}=(0,\ 0)$

The Jacobean matrix about equilibrium point is

Let $f = a - b\left(u + \frac{a+1}{b}\right) + \frac{\left(u + \frac{(a+1)}{b}\right)^2}{v + \frac{(a+1)^2}{b^2}}$

$$g = \left(u + \frac{(a+1)}{b^2}\right)^2 - \left(v + \frac{(a+1)^2}{b^2}\right)$$

$$J = \begin{pmatrix} \frac{df}{du} & \frac{df}{dv} \\ \frac{dg}{du} & \frac{dg}{dv} \end{pmatrix}_{E=(0,0)} \Rightarrow J = \begin{pmatrix} \frac{b-ab}{a+1} & \frac{-b^2}{(a+1)^2} \\ \frac{2(a+1)}{b} & -1 \end{pmatrix}$$

The characteristics equation

$$\lambda^2 + \left(\frac{a+1+b(a-1)}{a+1}\right)\lambda + b\frac{(a+1)}{a+1} = 0$$

$$\lambda^2 + \left(\frac{1+a+ab-b}{a+1}\right)\lambda + b = 0$$

The eigenvalues of the Jacobean matrix is

$$\lambda = -\frac{\frac{(1+a+ab-b)}{a+1} \pm \sqrt{\left(\frac{1+a+ab-b}{a+1}\right)^2 - 4b}}{2}$$

$$\lambda = \frac{-(1+a+ab-b)}{2(a+1)} \pm \frac{\sqrt{(1+a+ab-b)^2 - 4b(a+1)^2}}{2(a+1)}$$

$$\lambda = \frac{-(1+a+ab-b)}{2(a+1)} \pm \frac{i\sqrt{4b(a+1)^2 - (1+a+ab-b)^2}}{2(a+1)}$$

$$\alpha(a,b) = \operatorname{Re}\lambda(a,b) = \frac{-(1+a+ab-b)}{2(a+1)}$$

$$\beta(a,b) = \operatorname{Im}\lambda(a,b) = \frac{\sqrt{4b(a+1)^2 - (1+a+ab-b)^2}}{2(a+1)}$$

1. Non hyperbolicity condition

$$\alpha(a,b) = 0$$

$$\frac{-(1+a+ab-b)}{2(a+1)} = 0$$

$$b(a-1) = -(a+1)$$

$$b = -\frac{(a+1)}{a-1}, \textit{provided that} \quad a \neq 1$$

$$\beta(a,b) = \frac{\sqrt{4b(a+1)^2 - (1+a+ab-b)^2}}{2(a+1)} = \omega \neq 0, \textit{because} \operatorname{sgn}(\omega) = \operatorname{sgn}[(\frac{\partial g}{\partial u})|_{E=(0,0}]$$

Since $\frac{dg}{du} = \frac{2(a+1)}{b} \neq 0$

2. **Transversality condition**

$$\frac{d\alpha(a,b)}{db} = d \neq 0$$

$$\frac{d(-(1+a+ab-b))}{db} = -\frac{(a-1)}{2(a+1)} = \frac{1-a}{2(a+1)}$$

$$d = \frac{1-a}{2(a+1)}, d \neq 0, \text{becuase } \ a \neq 1$$

3. **Genericity condition**

Let $$f = a - b\left(u + \frac{a+1}{b}\right) + \frac{\left(u + \frac{(a+1)}{b}\right)^2}{v + \frac{(a+1)^2}{b^2}}$$

$$g = \left(u + \frac{(a+1)}{b^2}\right)^2 - \left(v + \frac{(a+1)^2}{b^2}\right)$$

$k \neq 0$, where

$$k = \frac{1}{16}\left(f_{uuu} + f_{uvv} + g_{uuv} + g_{vvv}\right) + \frac{1}{16\omega}\left(f_{uv}\left(f_{uu} + f_{vv}\right) - g_{uv}\left(g_{uu} + g_{vv}\right) - f_{uu}g_{uu} + f_{vv}g_{vv}\right)$$

With $f_{uv} = \left(\frac{\partial^2 fb}{\partial u \partial v}\right)_{b=(0,0)}$ *etc.*

$$k = \frac{4b^5}{16(a+1)^5} + \frac{1}{16\omega}\left(\frac{-8b^9}{(a+1)^9} - \frac{4b^2}{(a+1)^2}\right)$$

$$k = \frac{b^5}{4(a+1)^5} - \frac{4b^2}{16(a+1)^2\omega}\left(\frac{2b^7}{(a+1)^7} + 1\right)$$

$$k = \frac{b^5}{4(a+1)^5} - \frac{b^2}{4(a+1)^2\omega}\left(\frac{2b^7}{(a+1)^7} + 1\right)$$

$$k = \frac{b^5}{4(a+1)^5} - \frac{b^2}{4(a+1)^2 \frac{\sqrt{4b(a+1)^2 - (1+a+ab-b)^2}}{2(a+1)}}\left(\frac{2b^7}{(a+1)^7} + 1\right)$$

$$k = \frac{b^5}{4(a+1)^5} - \frac{b^2}{2(a+1)\sqrt{4b(a+1)^2 - (1+a+ab-b)^2}}\left(\frac{2b^7}{(a+1)^7} + 1\right)$$

4.5. Turing Instability

The linearized form of the system (1.1) with diffusion can be written as:-

$$\frac{\partial u}{\partial t} = a_{11}u + a_{12}v + d_1 \frac{\partial^2 u}{\partial x^2} \tag{4.12}$$

$$\frac{\partial v}{\partial t} = a_{21}u + a_{22}v + d_2 \frac{\partial^2 v}{\partial x^2} \tag{4.13}$$

To obtain characteristics equation

Let $u = A_1 e^{\lambda t}\cos(qx) \quad and \quad v = A_2 e^{\lambda t}\cos(qx)$

Substituting u and v into equations (4.12) and (4.13)

$$\frac{\partial u}{\partial t} = a_{11}u + a_{12}v + d_1 \frac{\partial^2 u}{\partial x^2}$$

$$\begin{aligned}
&A_1\lambda e^{\lambda t}\cos(qx) = a_{11}(A_1 e^{\lambda t}\cos(qx)) + a_{12}(A_2 e^{\lambda t}\cos(qx)) - d_1 q^2 A_1 e^{\lambda t}\cos(qx)\\
&A_1\lambda = a_{11}A_1 + a_{12}A_2 - d_1 q^2 A_1\\
&(\lambda + d_1 q^2 - a_{11})A_1 - a_{12}A_2 = 0
\end{aligned}$$

$$\frac{\partial v}{\partial t} = a_{21}u + a_{22}v + d_2 \frac{\partial^2 v}{\partial x^2}$$

$$\begin{aligned}
&A_2\lambda e^{\lambda t}\cos(qx) = a_{21}(A_1 e^{\lambda t}\cos(qx)) + a_{22}(A_2 e^{\lambda t}\cos(qx)) - d_2 q^2 A_2 e^{\lambda t}\cos(qx)\\
&A_2\lambda = a_{21}A_1 + a_{22}A_2 - d_2 q^2 A_2\\
&-a_{21}A_1 + (\lambda + d_2 q^2 - a_{22})A_2 = 0
\end{aligned}$$

$$\begin{aligned}
&(\lambda + d_1 q^2 - a_{11})A_1 - a_{12}A_2 = 0\\
&-a_{21}A_1 + (\lambda + d_2 q^2 - a_{22})A_2 = 0
\end{aligned} \tag{4.14}$$

For the system of equation (4.14) to have non-trivial solution, the determinant of the coefficient matrix must be zero.

$$\begin{vmatrix} \lambda + d_1 q^2 - a_{11} & -a_{12} \\ -a_{21} & \lambda + d_2 q^2 - a_{22} \end{vmatrix} = 0 \quad ,$$

where

$$a_{11} = \left(\frac{df_1}{du}\right)_{(u,v)} = \frac{b-ab}{a+1} \text{ , } a_{12} = \left(\frac{df_1}{du}\right)_{(u,v)} = -\frac{b^2}{(a+1)^2}$$
$$a_{21} = \left(\frac{df_2}{dv}\right)_{(u,v)} = \frac{2(a+1)}{b} \text{ , } a_{22} = \left(\frac{df_2}{dv}\right)_{(u,v)} = -1$$

$$\begin{vmatrix} \lambda + d_1 q^2 - \dfrac{b-ab}{a+1} & \dfrac{b^2}{(a+1)^2} \\ -2\dfrac{(a+1)}{b} & \lambda + d_2 q^2 + 1 \end{vmatrix} = 0$$

$$\lambda^2 - \left(\frac{b-ab}{a+1} - d_1 q^2 - d_2 q^2 - 1\right)\lambda + \left(d_1 q^2 - \frac{b-ab}{a+1}\right)\left(d_2 q^2 + 1\right) + \frac{2b}{a+1} = 0$$
$$\lambda^2 - \left(\frac{b-ab}{a+1} - q^2\left(d_1 + d_2\right) - 1\right)\lambda + \left(d_1 q^2 - \frac{b-ab}{a+1}\right)\left(d_2 q^2 + 1\right) + \frac{2b}{a+1} = 0 \qquad (4.15)$$

The Routh Hurwitz table is given by

$$\begin{array}{c|cc} \lambda^2 & 1 & \left(d_1 q^2 - \dfrac{(b-ab)}{a+1}\right)\left(d_2 q^2 + 1\right) + \dfrac{2b}{a+1} \\ \lambda & -\left(\dfrac{b-ab}{a+1} - d_1 q^2 - d_2 q^2 - 1\right) & 0 \\ \lambda^0 & \left(d_1 q^2 - \dfrac{(b-ab)}{a+1}\right)\left(d_2 q^2 + 1\right) + \dfrac{2b}{a+1} & \end{array}$$

The system (1.1) is stable with diffusion if the following conditions are satisfied

$$\frac{b-ab}{a+1} - q^2\left(d_1 + d_2\right) - 1 < 0 \qquad (4.16)$$
$$\left(d_1 q^2 - \frac{(b-ab)}{a+1}\right)\left(d_2 q^2 + 1\right) + \frac{2b}{a+1} > 0 \qquad (4.17)$$

The diffusive instability will occur if

$$\left(d_1q^2-\frac{(b-ab)}{a+1}\right)\left(d_2q^2+1\right)+\frac{2b}{a+1}<0 \tag{4.18}$$

4.6. Hopf Bifurcation with Diffusion

Suppose the characteristics equation (4.15) has a simple pair of pure imaginary eigenvalues $\lambda=\pm i\omega,\omega>0$

Substitute $\lambda=\pm i\omega$ into the characteristics equation (4.15)

$$\lambda^2-\left(\frac{b-ab}{a+1}-d_1q^2-d_2q^2-1\right)\lambda+\left(d_1q^2-\frac{b-ab}{a+1}\right)\left(d_2q^2+1\right)+\frac{2b}{a+1}=0$$

$$(\omega i)^2-\left(\frac{b-ab}{a+1}-d_1q^2-d_2q^2-1\right)\omega i+\left(d_1q^2-\frac{b-ab}{a+1}\right)\left(d_2q^2+1\right)+\frac{2b}{a+1}=0$$

$$-\omega^2-\left(\frac{b-ab}{a+1}-d_1q^2-d_2q^2-1\right)\omega i+\left(d_1q^2-\frac{b-ab}{a+1}\right)\left(d_2q^2+1\right)+\frac{2b}{a+1}=0$$

Equating real parts and imaginary parts with zero

$$-\left(\frac{b-ab}{a+1}-d_1q^2-d_2q^2-1\right)\omega=0 \tag{4.19}$$

$$-\omega^2+\left(d_1q^2-\frac{b-ab}{a+1}\right)\left(d_2q^2+1\right)+\frac{2b}{a+1}=0 \tag{4.20}$$

From Equation (4.19) since $\omega>0$,then

$$-\left(\frac{b-ab}{a+1}-d_1q^2-d_2q^2-1\right)=0$$

$\dfrac{b-ab}{a+1}-d_1q^2-d_2q^2-1=0$, solving for b

$$b = \left(\frac{1+a}{1-a}\right)\left[d_1q^2 + d_2q^2 + 1\right]$$

Substituting the value of b into characteristics equation and solving for λ

$$\lambda^2 - \left(\frac{b-ab}{a+1} - d_1q^2 - d_2q^2 - 1\right)\lambda + \left(d_1q^2 - \frac{b-ab}{a+1}\right)\left(d_2q^2 + 1\right) + \frac{2b}{a+1} = 0$$

$$\lambda^2 - \left(\frac{(1-a)b}{a+1} - d_1q^2 - d_2q^2 - 1\right)\lambda + \left(d_1q^2 - \frac{(1-a)b}{a+1}\right)\left(d_2q^2 + 1\right) + \frac{2}{a+1}b = 0$$

$$\lambda^2 - \left(\frac{(1-a)}{a+1}\left(\frac{1+a}{1-a}\right)\left[d_1q^2 + d_2q^2 + 1\right] - d_1q^2 - d_2q^2 - 1\right)\lambda +$$
$$\left(d_1q^2 - \frac{(1-a)}{a+1}\left(\frac{1+a}{1-a}\right)\left[d_1q^2 + d_2q^2 + 1\right]\right)\left(d_2q^2 + 1\right) + \frac{2}{a+1}\left(\frac{1+a}{1-a}\right)\left[d_1q^2 + d_2q^2 + 1\right] = 0$$

$$\lambda^2 - \left(\left[d_1q^2 + d_2q^2 + 1\right] - d_1q^2 - d_2q^2\right)\lambda + \left(d_1q^2 - \left[d_1q^2 + d_2q^2 + 1\right]\right)\left(d_2q^2 + 1\right) + \frac{2}{1-a}\left[d_1q^2 + d_2q^2 + 1\right] = 0$$

$$\lambda^2 - \left(d_2q^2 + 1\right)\left(d_2q^2 + 1\right) + \frac{2}{1-a}\left[d_1q^2 + d_2q^2 + 1\right] = 0$$

$$\lambda^2 = \left(d_2q^2 + 1\right)^2 - \frac{2}{1-a}\left[d_1q^2 + d_2q^2 + 1\right]$$

$$\lambda = \pm\sqrt{\left(d_2q^2 + 1\right)^2 - \frac{2}{1-a}\left[d_1q^2 + d_2q^2 + 1\right]}$$

$$\lambda = \pm\sqrt{\frac{(1-a)\left(d_2q^2 + 1\right)^2 - 2\left(d_1q^2 + d_2q^2 + 1\right)}{1-a}}$$

$$\lambda_{1,2} = \sqrt{\frac{-1\left(2\left(d_1q^2 + d_2q^2 + 1\right) - (1-a)\left(d_2q^2 + 1\right)^2\right)}{1-a}}$$

$$\lambda_{1,2} = \pm i\sqrt{\frac{2\left(d_1q^2 + d_2q^2 + 1\right) - (1-a)\left(d_2q^2 + 1\right)^2}{1-a}}$$

$$\omega = \sqrt{\frac{2\left(d_1q^2 + d_2q^2 + 1\right) - (1-a)\left(d_2q^2 + 1\right)^2}{1-a}}$$

$$\frac{2\left(d_1q^2 + d_2q^2 + 1\right) - (1-a)\left(d_2q^2 + 1\right)^2}{1-a} > 0 \qquad (4.21)$$

$\omega > 0$, if condition (4.21) is satisfied.

To find $\frac{d\lambda}{db}$ differentiate characteristics equation both side with respect to b

$$\lambda^2 - \left(\frac{b-ab}{a+1} - d_1q^2 - d_2q^2 - 1\right)\lambda + \left(d_1q^2 - \frac{b-ab}{a+1}\right)\left(d_2q^2 + 1\right) + \frac{2b}{a+1} = 0$$

$$2\lambda\frac{d\lambda}{db} - \left(\frac{b-ab}{a+1} - d_1q^2 - d_2q^2 - 1\right)\frac{d\lambda}{db} - \frac{(1-a)}{a+1}\lambda - \frac{(1-a)}{a+1}\left(d_2q^2 + 1\right) + \frac{2}{a+1} = 0$$

$$\frac{d\lambda}{db}\left(2\lambda - \left(\frac{b-ab}{a+1} - d_1q^2 - d_2q^2 - 1\right)\right) - \frac{(1-a)}{a+1}\lambda - \frac{(1-a)}{a+1}\left(d_2q^2 + 1\right) + \frac{2}{a+1} = 0$$

$$\frac{d\lambda}{db}\left(2\lambda - \left(\frac{b-ab}{a+1} - d_1q^2 - d_2q^2 - 1\right)\right) = \frac{(1-a)}{a+1}\lambda + \frac{(1-a)}{a+1}\left(d_2q^2 + 1\right) - \frac{2}{a+1}$$

$$\frac{d\lambda}{db} = \frac{(1-a)\lambda + (1-a)\left(d_2q^2 + 1\right) - 2}{(a+1)\left(2\lambda - \left(\frac{b-ab}{a+1} - d_1q^2 - d_2q^2 - 1\right)\right)}$$

$$\frac{d\lambda}{db} = \frac{(1-a)\lambda + (1-a)\left(d_2q^2 + 1\right) - 2}{2(a+1)\lambda + (a+1)\left(d_1q^2 + d_2q^2 + 1\right) + ab - b}$$

$$\frac{db}{d\lambda}=\frac{2\left(a+1\right)\lambda+\left(a+1\right)\left(d_1q^2+d_2q^2+1\right)+ab-b}{\left(1-a\right)\lambda+\left(1-a\right)\left(d_2q^2+1\right)-2}$$

$$\left(\frac{d\lambda}{db}\right)^{-1}=\frac{2(a+1)\lambda+(a+1)(d_1q^2+d_2q^2+1)+ab-b}{\left(1-a\right)\lambda+\left(1-a\right)\left(d_2q^2+1\right)-2}$$

$$\left(\frac{d\lambda}{db}\right)_{\lambda=\omega i}^{-1}=\frac{(a+1)(d_1q^2+d_2q^2+1)+ab-b+2(a+1)\omega i}{\left(1-a\right)\left(d_2q^2+1\right)-2+\left(1-a\right)\omega i}$$

$$\left(\frac{d\lambda}{db}\right)_{\lambda=\omega i}^{-1}=\frac{\left((a+1)(d_1q^2+d_2q^2+1)+ab-b\right)+2(a+1)\omega i}{\left(\left(1-a\right)\left(d_2q^2+1\right)-2\right)+\left(1-a\right)\omega i}\times\frac{\left(\left(1-a\right)\left(d_2q^2+1\right)-2\right)-\left(1-a\right)\omega i}{\left(\left(1-a\right)\left(d_2q^2+1\right)-2\right)-\left(1-a\right)\omega i}$$

$$\left(\frac{d\lambda}{db}\right)_{\lambda=\omega i}^{-1}=\frac{\begin{array}{c}\left((a+1)(d_1q^2+d_2q^2+1)+ab-b\right)\left(\left(1-a\right)\left(d_2q^2+1\right)-2\right)-\left(1-a\right)\left((a+1)(d_1q^2+d_2q^2+1)+ab-b\right)\omega i\\+2(a+1)\left(\left(1-a\right)\left(d_2q^2+1\right)-2\right)\omega i+2(a+1)\left(1-a\right)\omega^2\end{array}}{\left(\left(1-a\right)\left(d_2q^2+1\right)-2\right)^2+\left(1-a\right)^2\omega^2}$$

$$\left(\frac{d\lambda}{db}\right)_{\lambda=\omega i}^{-1}=\frac{\left[2(a+1)\left(\left(1-a\right)\left(d_2q^2+1\right)-2\right)-\left(1-a\right)\left((a+1)(d_1q^2+d_2q^2+1)+ab-b\right)\right]\omega i}{\left(\left(1-a\right)\left(d_2q^2+1\right)-2\right)^2+\left(1-a\right)^2\omega^2}+\frac{\left((a+1)(d_1q^2+d_2q^2+1)+ab-b\right)\left(\left(1-a\right)\left(d_2q^2+1\right)-2\right)+2(a+1)\left(1-a\right)\omega^2}{\left(\left(1-a\right)\left(d_2q^2+1\right)-2\right)^2+\left(1-a\right)^2\omega^2}$$

$$\mathrm{Re}\left(\frac{d\lambda}{db}\right)^{-1}=\frac{\left((a+1)(d_1q^2+d_2q^2+1)+ab-b\right)\left(\left(1-a\right)\left(d_2q^2+1\right)-2\right)+2(a+1)\left(1-a\right)\omega^2}{\left(\left(1-a\right)\left(d_2q^2+1\right)-2\right)^2+\left(1-a\right)^2\omega^2}$$

$$\mathrm{Re}\left(\frac{d\lambda}{db}\right)^{-1}=\frac{\left((a+1)(d_1q^2+d_2q^2+1)-\left(1-a\right)\left(\frac{(a+1)}{1-a}\left(d_1q^2+d_2q^2+1\right)\right)\right)\left(\left(1-a\right)\left(d_2q^2+1\right)-2\right)+2(a+1)\left(1-a\right)\omega^2}{\left(\left(1-a\right)\left(d_2q^2+1\right)-2\right)^2+\left(1-a\right)^2\omega^2}$$

$$= \frac{2(a+1)(1-a)\omega^2}{\left((1-a)\left(d_2q^2+1\right)-2\right)^2+(1-a)^2\omega^2} \neq 0, \text{since} \quad a \neq 1$$

Condition (2) of Hopf bifurcation is also satisfied.

As a result, the system (1.1) undergoes Hopf bifurcation at $b = \frac{(a+1)}{1-a}\left(d_1q^2+d_2q^2+1\right)$, when condition (4.21) is satisfied.

4.7. Numerical Examples

Example 1. Consider parameters value $a = \frac{1}{10}$ and $b = \frac{3}{5}$ then the system (1.1) are reduced to:

$$\frac{\partial u}{\partial t} = \frac{1}{10} - \frac{3}{5}u + \frac{u^2}{v} + d_1\frac{\partial^2 u}{\partial x^2} \tag{4.22}$$

$$\frac{\partial v}{\partial t} = u^2 - v + d_2\frac{\partial^2 v}{\partial x^2} \tag{4.23}$$

Equilibrium point without diffusion when $\left(d_1 = d_2 = 0\right)$

$$\frac{1}{10} - \frac{3}{5}u + \frac{u^2}{v} = 0 \tag{4.24}$$

$$u^2 - v = 0 \tag{4.25}$$

From equation (4.25)

$$v = u^2 \tag{4.26}$$

Substitute equation (4.26) into equation (4.24)

$$\frac{1}{10} - \frac{3}{5}u + 1 = 0$$

$$u = \frac{11}{6} \quad and \quad v = \frac{121}{36}$$

Equilibrium point is $E = \left(\frac{11}{6}, \frac{121}{36}\right)$

Local stability analysis without diffusion

Let $f_1 = \frac{1}{10} - \frac{3}{5}u + \frac{u^2}{v}$ and $f_2 = u^2 - v$

$$J = \begin{pmatrix} \frac{df_1}{du} & \frac{df_1}{dv} \\ \frac{df_2}{du} & \frac{df_2}{dv} \end{pmatrix} = \begin{pmatrix} -\frac{3}{5} + \frac{2u}{v} & \frac{-u^2}{v^2} \\ 2u & -1 \end{pmatrix} = \begin{pmatrix} \frac{27}{55} & -\frac{36}{121} \\ \frac{11}{3} & -1 \end{pmatrix}$$

The characteristics equation of the system (1.1) is given by

$$|J - \lambda I| = 0$$

$$\begin{vmatrix} -\frac{3}{5} + \frac{2u}{v} - \lambda & \frac{-u^2}{v^2} \\ 2u & -1-\lambda \end{vmatrix} = 0$$

$$\lambda^2 + \frac{28}{55}\lambda + \frac{3}{5} = 0$$

The Routh Hurwitz table is given by

λ^2	1	$\frac{3}{5}$
λ^1	$\frac{28}{55}$	0
λ^0	$\frac{3}{5}$	

Since all first columns are positive the system is asymptotically stable.

Hopf bifurcation without diffusion

In order to have Hopf bifurcation,

$$b = -\frac{(a+1)}{a-1} = \frac{11}{9}$$

The system under consideration is reduced to

$$\frac{\partial u}{\partial t} = \frac{1}{10} - \frac{11}{9}u + \frac{u^2}{v} \tag{4.27}$$

$$\frac{\partial v}{\partial t} = u^2 - v \tag{4.28}$$

The Equilibrium point is $E = \left(\frac{9}{10}, \frac{81}{100}\right)$

Applying the coordinate transformation about equilibrium points $E = (0,0)$ and the corresponding eigenvalues are $\lambda = \pm i\sqrt{\frac{11}{9}}$

This indicates that real part is zero and imaginary part $= \sqrt{\frac{11}{9}}$ hence, **Non hyperbolicity** condition is satisfied.

Transversality condition

$$d = \frac{1-a}{2(a+1)} = \frac{9}{22} \neq 0$$

The second condition of Hopf bifurcation theorem is also satisfied.

Genericity condition $k \neq 0,$ where

$$k = \frac{b^5}{4(a+1)^5} - \frac{b^2}{2(a+1)\sqrt{4b(a+1)^2 - (1+a+ab-b)^2}}\left(\frac{2b^7}{(a+1)^7} + 1\right)$$

Using $a = \frac{1}{10}$ *and* $b = \frac{11}{9}$

$$k \approx -2.33003...$$

This means, $d > 0$ and $kd < 0,$ hence the origin stable in the region $b < \frac{11}{9}$ and the origin is un stable in the region $b > \frac{11}{9}$. More precisely, there is Hopf bifurcation at $b = \frac{11}{9}$

Turing instability

The linearized form of the system with diffusion can be written as

$$\frac{\partial u}{\partial t} = \frac{27}{55}u - \frac{36}{121}v + d_1 \frac{\partial^2 u}{\partial x^2} \tag{4.29}$$

$$\frac{\partial v}{\partial t} = \frac{11}{3}u - v + d_2 \frac{\partial^2 v}{\partial x^2} \tag{4.30}$$

To obtain the characteristics equation

Let $u = A_1 e^{\lambda t} \cos(qx)$ and $v = A_2 e^{\lambda t} \cos(qx)$

Let $d_1 = d_2 = q = 1$ and equation (4.30) and (4.31) is leads to

$$\left(\lambda + \frac{28}{55}\right)A_1 + \frac{36}{121}A_2 = 0$$

$$-\frac{11}{3}A_1 + (\lambda + 2)A_2 = 0 \tag{4.31}$$

For the system of equation (4.31) to have non trivial solution the determinant of the coefficient matrix must be zero.

$$\begin{vmatrix} \lambda + \frac{28}{55} & \frac{36}{121} \\ -\frac{11}{3} & \lambda + 2 \end{vmatrix} = 0$$

$$\lambda^2 + \frac{138}{55}\lambda + \frac{116}{55} = 0$$

The Routh table is given by

$$
\begin{array}{c|cc}
\lambda^2 & 1 & \frac{116}{55} \\
\lambda & \frac{138}{55} & 0 \\
\lambda^0 & \frac{116}{55} &
\end{array}
$$

Since all first column has the same sign, then the equilibrium point is asymptotically stable by Routh Hurwitz criteria.

MATLAB Simulation

The following diagrams indicate MATLAB simulation and phase portrait that shows stability of the equilibrium point.

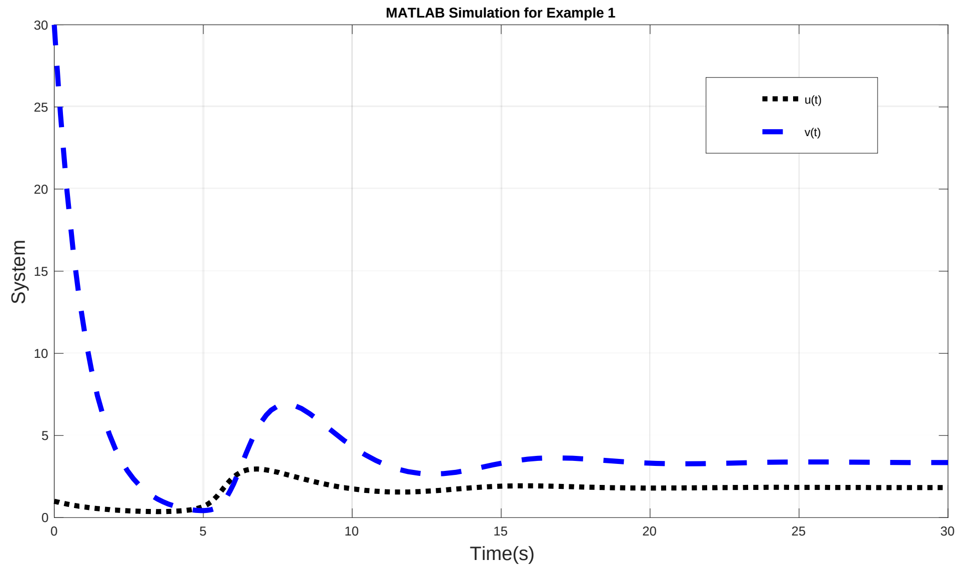

Figure 1: This graph indicates that the system converges to the equilibrium point, which in turn indicates the stability of the equilibrium points.

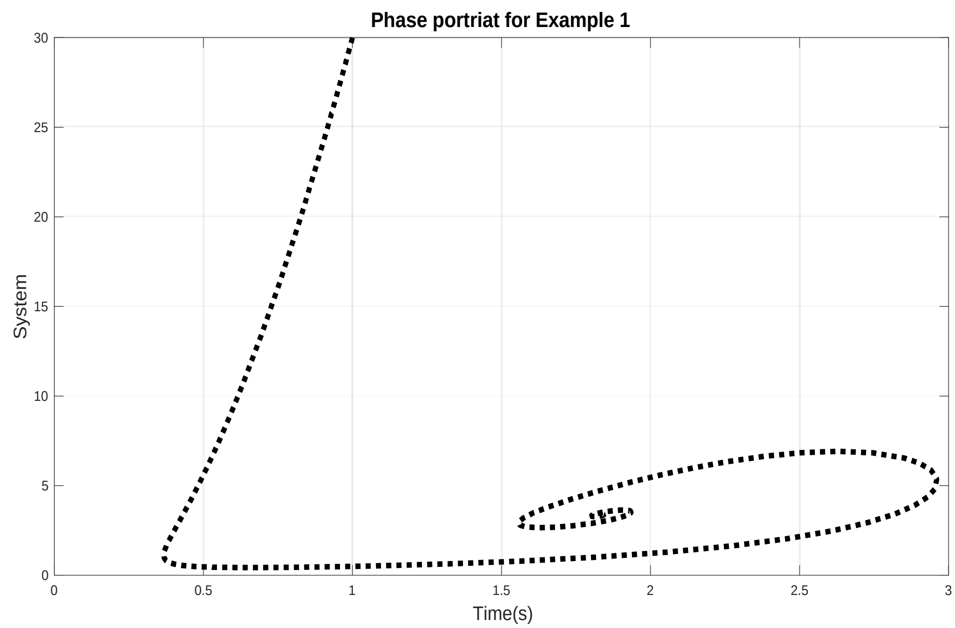

Figure 2: The phase portrait of the system which indicates that the system trajectories are attracted toward equilibrium point.

Hopf bifurcation with diffusion

In order to have Hopf Bifurcation

$$b=\frac{a+1}{1-a}\left(d_1q^2+d_2q^2+1\right)$$
$$b=\frac{11}{3}$$

Substituting the values of b into characteristics equation (4.15) and solving for λ

$$\lambda^2-\left(\frac{b-ab}{a+1}-d_1q^2-d_2q^2-1\right)\lambda+\left(d_1q^2-\frac{b-ab}{a+1}\right)\left(d_2q^2+1\right)+\frac{2b}{a+1}=0$$

$$\lambda^2+\frac{5}{3}=0$$
$$\lambda=\pm\sqrt{-\frac{5}{3}}$$
$$\lambda=\pm i\omega$$

$$\omega=\sqrt{\frac{5}{3}}>0$$

Hence one of the Hopf bifurcation condition is satisfied.

$$\operatorname{Re}\left(\frac{d\lambda}{db}\right)^{-1}=\frac{2(a+1)\left(1-a\right)\omega^2}{\left(\left(1-a\right)\left(d_2q^2+1\right)-2\right)^2+\left(1-a\right)^2\omega^2}\neq 0$$

Using $a=\frac{1}{10},\omega^2=\frac{5}{3},d_2=1$ and $q=1$

$$\operatorname{Re}\left(\frac{d\lambda}{db}\right)^{-1}=\frac{275}{157}\neq 0$$

As a result the second Hopf bifurcation condition is also satisfied.

Example2. Consider parameters value $a = \frac{1}{2}$ and $b = \frac{1}{2}$ then the system (1.1) are reduced to:

$$\frac{\partial u}{\partial t} = \frac{1}{2} - \frac{1}{2}u + \frac{u^2}{v} + d_1 \frac{\partial^2 u}{\partial x^2} \tag{4.32}$$

$$\frac{\partial v}{\partial t} = u^2 - v + d_2 \frac{\partial^2 v}{\partial x^2} \tag{4.33}$$

Equilibrium point without diffusion when $(d_1 = d_2 = 0)$

$$\frac{1}{2} - \frac{2}{2}u + \frac{u^2}{v} = 0 \tag{4.34}$$

$$u^2 - v = 0 \tag{4.35}$$

From equation (4.35)

$$v = u^2 \tag{4.36}$$

Substitute equation (4.36) into equation (4.34)

$$\frac{1}{2} - \frac{1}{2}u + 1 = 0$$

$$u = 3 \;\; and \;\; v = 9$$

Equilibrium point is $E = (3,9)$

Local stability Analysis without diffusion

Let $f_1 = \frac{1}{2} - \frac{1}{2}u + \frac{u^2}{v}$ and $f_2 = u^2 - v$

$$J = \begin{pmatrix} \frac{df_1}{du} & \frac{df_1}{dv} \\ \frac{df_2}{du} & \frac{df_2}{dv} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} + \frac{2u}{v} & \frac{-u^2}{v^2} \\ 2u & -1 \end{pmatrix} = \begin{pmatrix} \frac{1}{6} & -\frac{1}{9} \\ 6 & -1 \end{pmatrix}$$

The characteristics equation of the system (1.1) is given by

$$\left|J-\lambda I\right|=0$$

$$\begin{vmatrix} -\frac{1}{2}+\frac{2u}{v}-\lambda & \frac{-u^2}{v^2} \\ 2u & -1-\lambda \end{vmatrix}=0$$

$$\lambda^2+\frac{5}{6}\lambda+\frac{1}{2}=0$$

The Routh Hurwitz table is given by

$$\begin{array}{c|cc} \lambda^2 & 1 & \frac{1}{2} \\ \lambda^1 & \frac{5}{6} & 0 \\ \lambda^0 & \frac{1}{2} & \end{array}$$

Since all first column have the same sign the system is asymptotically stable.

Hopf bifurcation without diffusion

In order to have Hopf bifurcation

$$b=-\frac{(a+1)}{a-1}=3$$

The system under consideration is reduced to

$$\frac{\partial u}{\partial t}=\frac{1}{2}-3u+\frac{u^2}{v} \tag{4.37}$$

$$\frac{\partial v}{\partial t}=u^2-v \tag{4.38}$$

The Equilibrium point is $E=\left(\frac{1}{2},\frac{1}{4}\right)$

Applying the coordinate transformation about equilibrium points $E=(0,0)$ and the corresponding eigenvalues are $\lambda=\pm i\sqrt{3}$

This indicates that real part is zero and imaginary part $=\sqrt{3}$ hence, **Non hyperbolicity** condition is satisfied.

Transversality condition

$$d=\frac{1-a}{2(a+1)}=\frac{1}{6}\neq 0$$

Hence transversality condition is also satisfied.

Genericity condition

$k\neq 0,$ where

$$k=\frac{b^5}{4(a+1)^5}-\frac{b^2}{2(a+1)\sqrt{4b(a+1)^2-(1+a+ab-b)^2}}\left(\frac{2b^7}{(a+1)^7}+1\right)$$

Using $a=\frac{1}{2}$ and $b=3$

$$k\approx -140.3790...$$

This means, $d>0$ and $kd<0,$ hence the origin stable in the region $b<3$ and the origin is un stable in the region $b>3$. More precisely, there is Hopf bifurcation at $b=3$ the Hopf bifurcation applied.

Turing instability

The linearized form of the system with diffusion can be written as

$$\frac{\partial u}{\partial t}=\frac{1}{6}u-\frac{1}{9}v+d_1\frac{\partial^2 u}{\partial x^2} \tag{4.39}$$

$$\frac{\partial v}{\partial t}=6u-v+d_2\frac{\partial^2 v}{\partial x^2} \tag{4.40}$$

To obtain the characteristics equation

Let $u = A_1 e^{\lambda t}\cos(qx)$ and $v = A_2 e^{\lambda t}\cos(qx)$

Let $d_1 = d_2 = q = 1$ and equation (4.39) and (4.40) is leads to

$$\begin{aligned}&\left(\lambda + \frac{5}{6}\right)A_1 + \frac{1}{9}A_2 = 0\\ &-6A_1 + (\lambda + 2)A_2 = 0\end{aligned} \qquad (4.41)$$

For the system of equation (4.41) to have non trivial solution the determinant of the coefficient matrix must be zero.

$$\begin{vmatrix}\lambda + \frac{5}{6} & \frac{1}{9}\\ -6 & \lambda + 2\end{vmatrix} = 0$$

$$\lambda^2 + \frac{17}{6}\lambda + \frac{7}{3} = 0$$

The Routh table is given by

$$\begin{array}{c|cc}\lambda^2 & 1 & \frac{7}{3}\\ \lambda & \frac{17}{6} & 0\\ \lambda^0 & \frac{7}{3} & \end{array}$$

Since all first column are the same sign then the equilibrium point is locally asymptotically stable by Routh Hurwitz criteria.

Hopf bifurcation with diffusion

In order to have Hopf bifurcation

$$b = \frac{a+1}{1-a}\left(d_1 q^2 + d_2 q^2 + 1\right)$$
$$b = 9$$

Substitute the values of b in to characteristics equation (4.15) and solve for λ

$$\lambda^2 - \left(\frac{b-ab}{a+1} - d_1 q^2 - d_2 q^2 - 1 \right) \lambda + \left(d_1 q^2 - \frac{b-ab}{a+1} \right) \left(d_2 q^2 + 1 \right) + \frac{2b}{a+1} = 0$$

$$\lambda^2 + 6 = 0$$

$$\lambda = \pm\sqrt{-6}$$

$$\lambda = \pm i\sqrt{6}$$

Since $\lambda = \pm i\omega$ then

$$\omega = \sqrt{6} > 0$$

Hence one of the Hopf bifurcation condition is satisfied.

$$\mathrm{Re}\left(\frac{d\lambda}{db} \right)^{-1} = \frac{2(a+1)(1-a)\omega^2}{\left((1-a)\left(d_2 q^2 + 1 \right) - 2 \right)^2 + (1-a)^2 \omega^2} \neq 0$$

Using $a = \frac{1}{2}, \omega^2 = 6, d_2 = 1, \text{and } q = 1$

$$\mathrm{Re}\left(\frac{d\lambda}{db} \right)^{-1} = \frac{54}{19} \neq 0$$

As a result the second condition of Hopf bifurcation also satisfied.

CHAPTER FIVE

CONCLUSION AND FUTURE SCOPE

5.1. Conclusions

In this study, stability and bifurcation analysis of activator-inhibitor reaction diffusion system was considered. At the first glance, some preliminary part was discussed for the successful accomplishment of the finding of the study. The findings of the study revealed that, the system was analyzed into two parts. The first part is without diffusion. Without diffusion, the critical point of the system was determined. The system was linearized using Jacobean matrix about equilibrium point. The local stability condition of the critical point was proved by using Routh Hurwitz stability criteria. Hopf bifurcation condition without diffusion was determined by the help of Hopf bifurcation theorem in planar system. Furthermore, stability conditions with diffusion are proved by using Routh Hurwitz stability criteria. Diffusive instability condition was also set down. The system undergoes Hopf bifurcation with diffusion provided that specific condition is satisfied. Finally, in order to verify the applicability of the result two numerical examples were solved and MATLAB simulation was implemented to support the findings of the study.

5.2. Future Scope

One can investigate stability and bifurcation analysis of activator-inhibitor Reaction-diffusion system by considering time delay effect. Furthermore, direction and stability of Hopf bifurcation of the system is another area of future work. Moreover, qualitative analysis with regard to limit cycle, periodic solution and chaotic behavior are further area of future work.

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