COMMON FIXED POINT RESULTS ON COMPLETE b-METRIC SPACES AND DISLOCATED QUASI b-METRIC SPACES

A RESEARCH SUBMITTED TO THE DEPARTMENT OF MATHEMATICS IN PARTIAL FULFILLMENT FOR THE REQUIREMENTS OF THE DEGREE OF MASTERS OF SCIENCE IN MATHEMATICS

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Declaration

I, the undersigned declare that, this research paper entitled ”Common fixed Point results on complete b-metric spaces and dislocated quasi b-metric Spaces” is my own original work and it has not been submitted for the award of any academic degree or the like in any other institution or university, and that all the sources I have used or quoted have been indicated and acknowledged.
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Abstract

In this thesis, we established common fixed point theorems and proved the existence and uniqueness of common fixed points for a pair of self-maps in the setting of complete b-metric spaces and dislocated quasi b-metric spaces. We employed analytical design and used secondary sources of data such as published articles and related books. Our results provided extension as well as substantial improvements and generalizations of (Faraji and Nourouzi, 2017 ; Sastry and Kumar, 2019). Finally, we provided examples in support of our main findings.

Contents

Chapter 1

Introduction

1.1 Background of the study

Let X be a nonempty set. A map $T : X \to X$ is said to be a self-map of X. An element x in X is called a fixed point of T if $Tx = x$.
Let (X,d) be a metric space. A self-map $T : X \to X$ be a contraction, if there is a real number k in $[0,1)$ such that:

$$d(Tx,Ty) \leq kd(x,y).$$

In this case k is called a contraction constant.
A theory of fixed point is one of the most powerful and popular tools of modern mathematics. It has wide applications in different disciplines such as; economics, chemistry, biology, computer science, engineering, and others. For more details one can refer (Banach, 1922; Nieto and Rodrguez-Lopez, 2005; Beg *et al.* , 2013).
The first most significant result of metric fixed point theory was given by the polish mathematician Stefan Banach, in 1922, which is known as Banach contraction principle. The famous Banach contraction principle states that if (X,d) is a complete metric space and $T : X \to X$ is a contraction, then T has a unique fixed point.
Banach contraction principle is one of the cornerstones in the development of nonlinear analysis. There are a number of extensions and generalizations of Banach contraction theorem by many researchers who have obtained fixed point and common fixed point results in metric spaces, b-metric spaces, dislocated quasi metric spaces,dislocated quasi b-metric spaces, and other spaces. For more results in this line of research we refer (Czerwik, 1993; Zeyada *et al.*, 2006; Klin-Eam and Suanoom, 2015).
In 1968, Kannan gave a fixed point theorem for a self-map $T : X \to X$ which need not be continuous and satisfying the following condition

$$d(Tx,Ty) \leq k[d(x,Tx) + d(y,Ty)]$$

for all $x, y \in X$ where $0 \leq k < \frac{1}{2}$.
Also in 1972, Chatterjea gave the dual of Kannan fixed point theorem as follows:

$$d(Tx, Ty) \leq k[d(x, Ty) + d(y, Tx)]$$

for all $x, y \in X$ where $0 \leq k < \frac{1}{2}$.
The works of Banach, Kannan and Chatterjea are independent.
The concept of quasi-metric spaces was introduced by (Wilson, 1931) as a generalization of metric spaces, and (Hitzler and Seda, 2000) introduced dislocated metric spaces as a generalization of metric spaces, (Zeyada *et al.*, 2005) generalized the result of Hitzler, Seda and Wilson and introduced the concept of dislocated quasi-metric space.
In 1989, Bakhtin introduced b-metric space as a generalization of metric space. Moreover, (Czerwik, 1998) made the results of Bakhtin known more. Finally, many other generalized b-metric spaces such as quasi-b-metric spaces (Shah and Hussain, 2012), b-metric-like spaces (Alghamdi and Hussain, 2013), and quasi-b-metric like spaces (Zhu *et al.*, 2014).
Recently, Faraji and Nourouzi (2017) proved common fixed point theorem which generalized Kannan and Chatterjea fixed point theorems in the setting of b-complete metric spaces.
Very Recently, Sastry and Kumar (2019) established common fixed point theorems in the setting of dislocated quasi b-metric spaces.
Inspired and motivated by the works of (Faraji and Nourouzi, 2017 ; Sastry and Kumar, 2019) the purpose of this research work is to establish common fixed point theorems for a pair of self-maps satisfying contractive conditions in the setting of b-metric spaces and dislocated quasi b-metric spaces and prove the existence and uniqueness of common fixed points. Illustrative examples which support the main results of the study have been provided.

1.2 Statements of the problem

In this study we focused on establishing and proving common fixed point results for a pair of self-maps in the setting of b-metric spaces and dislocated quasi b-metric spaces.

1.3 Objectives of the study

1.3.1 General objective

This study has the following general objective:

- To study common fixed point results for a pair of self-maps defined in the setting of b-metric and dislocated quasi b-metric spaces.

1.3.2 Specific objectives

This study has the following specific objectives:

- To prove the existence of common fixed points for a pair of self-maps defined in the setting of b-metric and dislocated quasi b-metric spaces .
- To prove the uniqueness of common fixed points for a pair of self-maps defined in the setting of b-metric and dislocated quasi b-metric spaces.
- To provide examples in support our main results.

1.4 Significance of the study

The study may have the following importance:

- The outcome of this study may contribute to research activities on study area.
- It may provide basic research skills to the researcher.
- It may help to show existence and uniqueness of solutions for problems involving integral and differential equations.

1.5 Delimitation of the Study

This study was delimited to establish and prove common fixed point results for a pair of self-maps defined on b-metric and dislocated quasi b-metric spaces.

Chapter 2

Review of Related Literature

The applications of fixed point theory are very important in diverse disciplines of mathematics since they can be applied for solving various problems, for instance, equilibrium problems, variation problems, and optimization problems. The Banachs contraction mapping principle is one of the cornerstones in the development of fixed point theory. In particular, this principle is used to demonstrate the existence and uniqueness of a solution of differential equations, integral equations, functional equations, partial differential equations and others. Due to the importance, generalizations of Banachs contraction mapping principle have been investigated heavily by many authors.
Many researchers have obtained fixed point, common fixed point and other fixed point results in metric spaces, cone- metric spaces, partially ordered metric spaces, dislocated b-metric spaces and other spaces.
Fixed point theory is one of the famous and traditional theories in mathematics and has a broad set of applications. In this theory, contraction is one of the main tools to prove the existence and uniqueness of a fixed point. Banachs contraction principle, which gives an answer on the existence and uniqueness of a solution of an operator equation, is the most widely used fixed point theorem in all areas of analysis. This principle is constructive in nature and is one of the most useful tools in the study of nonlinear equations.
Throughout this manuscript, $R^+ = [0,\infty)$ and N is the set of positive integers.

Definition 2.1 *(Czerwik, 1993) Let* X *be a non-empty set and* $k \geq 1$ *be a given real number. A function* $d : X \times X \longrightarrow R^+$ *is a b-metric if and only if for all* $x,y,z \in X$*, the following conditions are satisfied:*
a) $d(x,y) = 0$ *iff* $x = y$;
b) $d(x,y) = d(y,x)$*;*
c) $d(x,z) \leq k[d(x,y) + d(y,z)]$.

The pair (X,d) is called a b-metric space.
Example 2.1.(Dubey *et al.*, 2014). The space L_p $(0 < p < 1)$ of all real functions

$x(t), t \in [0,1]$ such that $\int_0^1 |x(t)|^p dt < \infty$, is b-metric space if we take

$$d(x,y) = [\int_0^1 |x(t) - y(t)|^p dt]^{1/p}$$

for each $x, y \in L_p$.

Definition 2.2 *(Klin-eam and Suanoom, 2015) Let X be a nonempty set and $k \geq 1$ be a real number. Suppose that the mapping*
$d : X \times X \longrightarrow R^+$ satisfies the following conditions:
a) $d(x,y) = d(y,x) = 0$ implies $x = y$ for all $x, y \in X$;
b) $d(x,y) \leq k[d(x,z) + d(z,y)]$ for all $x, y, z \in X$.

The pair (X,d) is called a dislocated quasi-b-metric space (or simply dqb-metric). The number k is called the coefficient of (X,d).

Remark. It is obvious that b-metric spaces, quasi-b-metric spaces are dislocated quasi-b-metric spaces, but the converse is not true.

Example 2.2. (Sastry and Kumar, 2019) Let $X = R^+$ and for $p > 1$, $d : X \times X \to R^+$ defined by
$d(x,y) = |x-y|^p + |x|^p$, for all $x, y \in X$.
Then the pair (X,d) is dqb-metric space with $k = 2^p > 1$. But (X,d) is not dislocated quasi metric spaces.

Definition 2.3 *(Klin-eam and Suanoom, 2015) A sequence $\{x_n\}$ in dqb-metric spaces (X,d) is a dqb-conveges to $x \in X$ if*

$$\lim_{n\to\infty} d(x_n, x) = 0 = \lim_{n\to\infty} d(x, x_n).$$

In this case x is called a dqb-limit of $\{x_n\}$ and write $(x_n \to x)$.

Definition 2.4 *(Klin-eam and Suanoom, 2015) A sequence $\{x_n\}$ in dqb-metric spaces (X,d) is Cauchy sequence if*

$$\lim_{(n,m)\to\infty} d(x_n, x_m) = 0 = \lim_{(n,m)\to\infty} d(x_m, x_n).$$

Definition 2.5 *(Klin-eam and Suanoom, 2015) A dqb-metric spaces (X,d) is complete if every Cauchy sequence in it is dqb-convergent in X.*

Chapter 3

Methodology

3.1 Study area and period

The study was conducted at Jimma University under the department of mathematics from October 2018 G.C to October 2021 G.C.

3.2 Study Design

In this study we followed analytical design method.

3.3 Source of Information

The relevant sources for this research work were books and published articles related to the area of the study.

3.4 Mathematical Procedure of the Study

In this study we followed the procedures stated below:

- Establishing theorems.
- Constructing sequences.
- Showing the constructed sequences are Cauchy sequences.
- Proving the existence of common fixed points.
- Showing uniqueness of common fixed points.
- Providing examples in support of our main results.

Chapter 4

Preliminaries and Main Result

4.1 Preliminaries

Definition 4.1 *(Khan, et al. 1984) A function $\phi : R^+ \to R^+$ is called an altering distance function if the following properties are satisfied:*
a) $\phi(t) = 0$, iff $t = 0$.
b) ϕ is a monotonically non-decreasing function.
c) ϕ is a continuous function.
Φ denote the set of all altering distance functions.

Definition 4.2 *(Jungck and Rhoades, 1998) Let f and g be self-maps of a nonempty set X. If $w = fx = gx$ for some $x \in X$, then x is called a coincidence point of f and g, and w is called point of coincidence f and g.*

Definition 4.3 *(Jungck and Rhoades, 1998) Let f and g be self-maps of a nonempty set X. Then f and g are said to be weakly compatible if they commute at their coincidence point.*

Lemma 4.1.1 (*Abbas and Jungck, 2008*) Let f and g are weakly compatible self-maps of a nonempty set X. If f and g have unique point of coincidence $w = fx = gx$, then w is the unique common fixed point of f and g.

Theorem 4.1.1 *(Faraji and Nourouzi, 2017) Let (X,d) be a complete b-metric space with parameter $k \geq 1$ and T,S be self-mappings on X which satisfy*

$$\begin{aligned} d(Sx,Ty) \quad &\leq \quad a_1 d(x,Sx) + a_2 d(y,Ty) + a_3 d(x,Ty) \\ &\quad + a_4 d(y,Sx) + a_5 d(x,y), \end{aligned}$$

for all $x,y \in X$, where a_1,a_2,a_3,a_4,a_5 are nonnegative real numbers satisfying:
(i) $k^2 a_1 + k^2 a_2 + k^3 a_3 + k^3 a_4 + k^2 a_5 < 1$,
(ii) $a_1 = a_2$ or $a_3 = a_4$.

Then T and S have a unique common fixed point.
Very recently, Sastry and Kumar, (2019) proved the following fixed point theorem in the setting of dqb-complete metric space.

Theorem 4.1.2 *Let (X,d) be a dqb-complete metric space with coefficients $k \geq 1$. Let $f : X \to X$ be self-mapping satisfying:*

$$d(fx, fy) \leq \phi(max\{d(x,y), d(fx,x), d(y, fy)\}),$$

for all $x, y \in X$ where ϕ is altering distance function and $\phi(t) < \frac{t}{k}$ for $t > 0$.Then f has unique fixed point in X.

4.2 Main Result

In this section, we established and proved common fixed point results in the setting of complete b-metric spaces and dislocated quasi-b-metric spaces.

Theorem 4.2.1 *Let (X,d) be a complete b-metric space with $s \geq 1$ and T,S be self-mappings on X which satisfy*

$$\begin{aligned} d(Sx,Ty) \leq\ & a_1 d(x,Sx) + a_2 d(y,Ty) + a_3 d(x,Ty) + a_4 d(y,Sx) \\ & + a_5 d(x,y) + a_6 [d(x,Sx) + d(x,Ty)], \end{aligned} \tag{4.1}$$

for all $x, y \in X$, where a_1, a_2, a_3, a_4, a_5 and a_6 are non-negative real numbers satisfying
(i) $s^2 a_1 + s^2 a_2 + s^3 a_3 + s^3 a_4 + s^2 a_5 + s^2 a_6 + 2s^3 a_6 < 1$,
(ii) $a_1 = a_2$ or $a_3 = a_4$.
Then T and S have a unique common fixed point.
Proof: Let x_0 be any arbitrary point in X. We can choose $x_1 \in X$ such that $x_1 = Sx_0$. Again we can choose $x_2 \in X$ such that $x_2 = Tx_1$ and repeating in the same manner for $x_{2n} \in X$ we can choose $x_{2n+1} \in X$ such that $x_{2n+1} = Sx_{2n}$ and $x_{2n+2} = Tx_{2n+1}, n = 0, 1, 2, \ldots$

By (4.1), we have,

$$
\begin{aligned}
d(x_1,x_2) &= d(Sx_0,Tx_1)\\
&\leq a_1d(x_0,Sx_0)+a_2d(x_1,Tx_1)+a_3d(x_0,Tx_1)+a_4d(x_1,Sx_0)\\
&\quad +a_5d(x_0,x_1)+a_6[d(x_0,Sx_0)+d(x_0,Tx_1)]\\
&= a_1d(x_0,x_1)+a_2d(x_1,x_2)+a_3d(x_0,x_2)+a_4d(x_1,x_1)\\
&\quad +a_5d(x_0,x_1)+a_6[d(x_0,x_1)+d(x_0,x_2)]\\
&\leq a_1d(x_0,x_1)+a_2d(x_1,x_2)+sa_3d(x_0,x_1)+sa_3d(x_1,x_2)+a_5d(x_0,x_1)\\
&\quad +a_6d(x_0,x_1)+sa_6d(x_0,x_1)+sa_6d(x_1,x_2)
\end{aligned}
$$

Therefore,

$$
d(x_1,x_2) \leq \frac{(a_1+sa_3+a_5+a_6+sa_6)}{(1-a_2-sa_3-sa_6)}d(x_0,x_1).
$$

So,

$$
d(x_2,x_3) \leq \frac{(a_2+sa_4+a_5+a_6+sa_6)}{(1-a_1-sa_4-sa_6)}d(x_1,x_2).
$$

By repeating this procedure, we obtain:

$$
d(x_{2n-1},x_{2n}) \leq (r^n)(h^{n-1})d(x_0,x_1), n=1,2,3,\ldots, \tag{4.2}
$$

and

$$
d(x_{2n},x_{2n+1}) \leq (r^n)(h^n)d(x_0,x_1), n=1,2,3,\ldots, \tag{4.3}
$$

where

$$
\frac{(a_1+sa_3+a_5+a_6+sa_6)}{(1-a_2-sa_3-sa_6)} = r \quad and \quad \frac{(a_2+sa_4+a_5+(1+s)a_6}{(1-a_1-sa_4-sa_6)} = h.
$$

Let $m,n \in N$ and $m > n$. Then by (4.2) and (4.3) we have the following

$$\begin{aligned} d(x_{2n},x_{2m}) &\leq sd(x_{2n},x_{2n+1})+s^2d(x_{2n+1},x_{2n+2})+\cdots+s^{2m-2n-1}d(x_{2m-2},x_{2m-1}) \\ &\quad +s^{2m-2n}d(x_{2m-1},x_{2m}) \\ &\leq sr^nh^n\lambda+\cdots+s^{2m-2n-1}r^{m-1}h^{m-1}\lambda+s^{2m-2n}r^mh^{m-1}\lambda \\ &= s\alpha^n\lambda+\cdots+s^{2m-2n-1}\alpha^{m-1}\lambda+s^{2m-2n}r\alpha^{m-1}\lambda \\ &= s\alpha^n\lambda(1+sr)+\cdots+s^{2m-2n-1}\alpha^{m-1}\lambda(1+sr) \\ &= s\alpha^n\lambda(1+sr)[1+s^2\alpha+(s^2\alpha)^2+\cdots+(s^2\alpha)^{m-n-1}], \end{aligned}$$

where $\alpha = rh$ and $\lambda = d(x_0,x_1)$. Since $s^2\alpha < 1$, we get

$$d(x_{2n},x_{2m}) \leq s(1+sr)\lambda\frac{\alpha^n}{1-s^2\alpha}.$$

Therefore $\{x_{2n}\}$ is a Cauchy sequence.
Since X is complete there exists $x \in X$ such $x_{2n} \to x$. Using (4.3), we have:

$$\begin{aligned} d(x,x_{2n+1}) &\leq sd(x,x_{2n})+sd(x_{2n},x_{2n+1}) \\ &\leq sd(x,x_{2n})+\lambda\alpha^n, n=0,1,2,\ldots \end{aligned}$$

So $lim_{n\to\infty}x_{2n+1} = x$ and therefore $lim_{n\to\infty}x_{2n} = x$.
Now, we show that x is the unique fixed point of T and S. Using (4.9) we have

$$\begin{aligned} d(x,Sx) &\leq s[d(x,x_{2n})+d(x_{2n},Sx)] \\ &= sd(x,x_{2n})+sd(Tx_{2n-1},Sx) \\ &\leq sd(x,x_{2n})+sa_1d(x,Sx)+sa_2d(x_{2n-1},Tx_{2n-1})+sa_3d(x,Tx_{2n-1}) \\ &\quad +sa_4d(x_{2n-1},Sx)+sa_5d(x,x_{2n-1})+sa_6[d(x,Sx)+d(x,Tx_{2n-1})] \\ &\leq sd(x,x_{2n})+sa_1d(x,Sx)+sa_2d(x_{2n-1},Tx_{2n-1})+sa_3d(x,Tx_{2n-1}) \\ &\quad +sa_4[s(d(x_{2n-1},x)+d(x,Sx))]+sa_5d(x,x_{2n-1}) \\ &\quad +sa_6d(x,Sx)+sa_6d(x,Tx_{2n-1}) \\ &\leq (sa_1+s^2a_4+sa_6)d(x,Sx). \end{aligned}$$

This implies that $Sx = x$. Again using (4.9) we have

$$\begin{aligned}
d(x,Tx) &\leq s[d(x,x_{2n-1}) + d(x_{2n-1},Tx)] \\
&= sd(x,x_{2n-1}) + sd(Sx_{2n-2},Tx) \\
&\leq sd(x,x_{2n-1}) + sa_1d(x_{2n-2},Sx_{2n-2}) + sa_2d(x,Tx) + sa_3d(x_{2n-2},Tx) \\
&\quad + sa_4d(x,Sx_{2n-2}) + sa_5d(x_{2n-2},x) + sa_6[d(x_{2n-2},Sx_{2n-2}) + d(x_{2n-2},Tx)] \\
&\leq sd(x,x_{2n-1}) + sa_1d(x_{2n-2},Sx_{2n-2}) + sa_2d(x,Tx) + s^2a_3d(x_{2n-2},x) \\
&\quad + s^2a_3d(x,Tx) + sa_4d(x,Sx_{2n-2}) + sa_5d(x_{2n-2},x) \\
&\quad + sa_6d(x_{2n-2},Sx_{2n-2}) + s^2a_6d(x_{2n-2},x) + s^2a_6d(x,Tx) \\
&\leq (sa_2 + s^2a_3 + s^2a_6)d(x,Tx).
\end{aligned}$$

This implies that $Tx = x$. To see the uniqueness of the common fixed point of T and S, assume on the contrary that $Tx = Sx = x$ and $Ty = Sy = y$ but $x \neq y$. By (4.9) we have

$$\begin{aligned}
d(x,y) &= d(Sx,Ty) \\
&\leq a_1d(x,Sx) + a_2d(y,Ty) + a_3d(x,Ty) + a_4d(y,Sx) + a_5d(x,y) \\
&\quad + a_6[d(x,Sx) + d(x,Ty)] \\
&= [a_3 + a_4 + a_5 + a_6]d(x,y).
\end{aligned}$$

Which is a contradiction. Hence, $x = y$ and this implies that the common fixed point of T and S is unique.

Theorem 4.2.2 *Let (X,d) be a dqb-complete metric space. Let $f,g : X \to X$ be self mappings satisfying the inequality*

$$d(fx,fy) \leq \phi[max\{d(gx,gy),d(fx,gx),d(gy,fy)\}], \tag{4.4}$$

for all $x,y \in X$, where ϕ is altering distance function and $\phi(t) < \frac{t}{k}$ for $t > 0$ and $k > 1$. If $f(X) \subseteq g(X)$ and $g(X)$ is dqb-complete subspace of X, then f and g have unique point of coincidence in X. In addition if f and g are weakly compatible, then f and g have unique common fixed point in X.

Proof: Let x_0 be any arbitrary point in X. As $f(X) \subseteq g(X)$ we can choose $x_1 \in X$ such that $fx_0 = gx_1$, and $x_2 \in X$ such that $fx_1 = gx_2, \ldots$ continuing in the same way

for $x_n \in X$, we can choose $x_{n+1} \in X$ such that $fx_n = gx_{n+1}$, where $n = 0, 1, 2, \ldots$
By (4.4), we have:

$$\begin{aligned} d(gx_1, gx_2) &= d(fx_0, fx_1) \\ &\leq \phi[max\{d(gx_0, gx_1), d(fx_0, gx_0), d(gx_1, fx_1)\}] \\ &= \phi[max\{d(gx_0, gx_1), d(gx_1, gx_0), d(gx_1, gx_2)\}] \\ &\leq \phi[max\{d(gx_0, gx_1), d(gx_1, gx_0)\}] = \phi(\alpha_1). \end{aligned}$$

wehre, $\alpha_1 = max\{d(gx_0, gx_1), d(gx_1, gx_0)\}$.
And

$$\begin{aligned} d(gx_2, gx_1) &= d(fx_1, fx_0) \\ &\leq \phi[max\{d(gx_1, gx_0), d(fx_1, gx_1), d(gx_0, fx_0)\}] \\ &= \phi[max\{d(gx_1, gx_0), d(gx_2, gx_1), d(gx_0, gx_1)\}] \\ &\leq \phi[max\{d(gx_0, gx_1), d(gx_1, gx_0)\}] = \phi(\alpha_1). \end{aligned}$$

wehre, $\alpha_1 = max\{d(gx_0, gx_1), d(gx_1, gx_0)\}$.
Similarly,

$$\begin{aligned} d(gx_2, gx_3) &= d(fx_1, fx_2) \\ &\leq \phi[max\{d(gx_1, gx_2), d(fx_1, gx_1), d(gx_2, fx_2)\}] \\ &= \phi[max\{d(gx_1, gx_2), d(gx_2, gx_1), d(gx_2, gx_3)\}] \\ &\leq \phi[max\{d(gx_1, gx_2), d(gx_2, gx_1)] = \phi(\alpha_2). \end{aligned}$$

wehre, $\alpha_2 = max\{d(gx_1, gx_2), d(gx_2, gx_1)\}$.
And

$$\begin{aligned} d(gx_3, gx_2) &= d(fx_2, fx_1) \\ &\leq \phi[max\{d(gx_2, gx_1), d(fx_2, gx_2), d(gx_1, fx_1)\}] \\ &= \phi[max\{d(gx_2, gx_1), d(gx_3, gx_2), d(gx_1, gx_2)\}] \\ &\leq \phi[max\{d(gx_1, gx_2), d(gx_2, gx_1)] = \phi(\alpha_2). \end{aligned}$$

wehre, $\alpha_2 = max\{d(gx_1, gx_2), d(gx_2, gx_1)\}$.
Therefore, $\alpha_2 \leq \phi(\alpha_1)$.
Similarly, $\alpha_3 \leq \phi(\alpha_2)$.
Hence, $\alpha_3 \leq \phi(\alpha_2) \leq \phi(\phi(\alpha_1)) = \phi^2(\alpha_1)$.
In general $\alpha_n \leq \phi^{n-1}(\alpha_1)$ and $\phi^{n-1}(\alpha_1) \to 0$ as $n \to \infty$.
$max\{d(gx_{n-1}, gx_n), d(gx_n, gx_{n-1})\} \to 0$ as $n \to \infty$.
Hence, $d(gx_{n-1}, gx_n) \to 0$ as $n \to \infty$ and $d(gx_n, gx_{n-1}) \to 0$ as $n \to \infty$.
Now, we show $\{gx_n\}$ is a dqb-cauchy sequence.
Consider for $m, n \in N, m > n, m = n + s$.

$$\begin{aligned}
d(gx_{n+1}, gx_n) &= d(fx_n, fx_{n-1}) \\
&\leq \phi[max\{d(gx_n, gx_{n-1}), d(fx_n, gx_n), d(gx_{n-1}, fx_{n-1})\}] \\
&= \phi[max\{d(gx_n, gx_{n-1}), d(gx_{n+1}, gx_n), d(gx_{n-1}, gx_n)\}] \\
&\leq \phi[max\{d(gx_n, gx_{n-1}), d(gx_{n-1}, gx_n)\}].
\end{aligned}$$

And

$$\begin{aligned}
d(gx_n, gx_{n+1}) &= d(fx_{n-1}, fx_n) \\
&\leq \phi[max\{d(gx_{n-1}, gx_n), d(fx_{n-1}, gx_{n-1}), d(gx_n, fx_n)\}] \\
&= \phi[max\{d(gx_{n-1}, gx_n), d(gx_{n+1}, gx_{n-1}), d(gx_n, gx_{n+1})\}] \\
&\leq \phi[max\{d(gx_{n-1}, gx_n), d(gx_n, gx_{n-1})\}].
\end{aligned}$$

Similarly,

$$\begin{aligned}
d(gx_{n+2}, gx_n) &\leq kd(gx_{n+2}, gx_{n+1}) + kd(gx_{n+1}, gx_n) \\
&\leq k\alpha_{n+1} + k\alpha_n \\
&\leq k\phi^n \alpha_1 + k\phi^{n-1}\alpha_1.
\end{aligned}$$

And

$$\begin{aligned}
d(gx_n, gx_{n+2}) &\leq kd(gx_n, gx_{n+1}) + kd(gx_{n+1}, gx_{n+2}) \\
&\leq k\alpha_n + k\alpha_{n+1} \\
&\leq k\phi^{n-1}\alpha_1 + k\phi^n \alpha_1.
\end{aligned}$$

Similarly,

$$\begin{aligned} d(gx_{n+3}, gx_n) &\leq kd(gx_{n+3}, gx_{n+2}) + kd(gx_{n+2}, gx_n) \\ &\leq k\alpha_{n+2} + k[k\phi^n\alpha_1 + k\phi^{n-1}\alpha_1] \\ &< k^2\phi^{n+1}\alpha_1 + k^2\phi^n\alpha_1 + k^2\phi^{n-1}\alpha_1. \end{aligned}$$

And

$$\begin{aligned} d(gx_n, gx_{n+3}) &\leq kd(gx_n, gx_{n+2}) + kd(gx_{n+2}, gx_{n+3}) \\ &\leq k[k\phi^{n-1}\alpha_1 + k\phi^n\alpha_1] + k\alpha_{n+2} \\ &< k^2\phi^{n-1}\alpha_1 + k^2\phi^n\alpha_1 + k^2\phi^{n+1}\alpha_1 \\ &< k^2\frac{\alpha_1}{k^{n-1}} + k^2\frac{\alpha_1}{k^n} + k^2\frac{\alpha_1}{k^{n+1}} \\ &= \alpha_1[\frac{1}{k^{n-3}} + \frac{1}{k^{n-2}} + \frac{1}{k^{n-1}}]. \end{aligned}$$

Again,

$$\begin{aligned} d(gx_{n+4}, gx_n) &\leq kd(gx_{n+4}, gx_{n+3}) + kd(gx_{n+3}, gx_n) \\ &\leq k\alpha_{n+3} + k[k^2\phi^{n+1}\alpha_1 + k^2\phi^n\alpha_1 + k^2\phi^{n-1}\alpha_1] \\ &< k^3\phi^{n+2}\alpha_1 + k^3\phi^{n+1}\alpha_1 + k^3\phi^n\alpha_1 + k^3\phi^{n-1}\alpha_1 \\ &< k^3\frac{\alpha_1}{k^{n+2}} + k^3\frac{\alpha_1}{k^{n+1}} + k^3\frac{\alpha_1}{k^n} + k^3\frac{\alpha_1}{k^{n-1}} \\ &= \alpha_1[\frac{1}{k^{n-1}} + \frac{1}{k^{n-2}} + \frac{1}{k^{n-3}} + \frac{1}{k^{n-4}}]. \end{aligned}$$

And

$$\begin{aligned} d(gx_n, gx_{n+4}) &\leq kd(gx_n, , gx_{n+3}) + kd(gx_{n+3}, gx_{n+4}) \\ &\leq k[k^2\phi^{n-1}\alpha_1 + k^2\phi^n\alpha_1 + k^2\phi^{n+1}\alpha_1] + k\alpha_{n+3} \\ &< k^3\phi^{n-1}\alpha_1 + k^3\phi^n\alpha_1 + k^3\phi^{n+1}\alpha_1 + k^3\phi^{n+2}\alpha_1 \\ &< k^3\frac{\alpha_1}{k^{n-1}} + k^3\frac{\alpha_1}{k^n} + k^3\frac{\alpha_1}{k^{n+1}} + k^3\frac{\alpha_1}{k^{n+2}} \\ &= \alpha_1[\frac{1}{k^{n-4}} + \frac{1}{k^{n-3}} + \frac{1}{k^{n-2}} + \frac{1}{k^{n-1}}]. \end{aligned}$$

Similarly,

$$\begin{aligned}
d(gx_{n+s}, gx_n) &\leq k^{n-s}\phi^{n-1}\alpha_1 + k^{n-s}\phi^n\alpha_1 + \cdots + k^{n-s}\phi^{n+s-2}\alpha_1 \\
&< k^{s-1}[\frac{\alpha_1}{k^{n-1}} + \frac{\alpha_1}{k^n} + \cdots + \frac{\alpha_1}{k^{n+s-2}}] \\
&= \alpha_1[\frac{1}{k^{n-s}} + \frac{1}{k^{n-s+1}} + \cdots + \frac{1}{k^{n-1}}].
\end{aligned}$$

And

$$\begin{aligned}
d(gx_n, gx_{n+s}) &\leq k^{n-s}\phi^{n+s-2}\alpha_1 + k^{n-s}\phi^n\alpha_1 + \cdots + k^{n-s}\phi^{n-1}\alpha_1 \\
&< k^{s-1}[\frac{\alpha_1}{k^{n+s-2}} + \frac{\alpha_1}{k^n} + \cdots + \frac{\alpha_1}{k^{n-1}}] \\
&= \alpha_1[\frac{1}{k^{n-1}} + \frac{1}{k^{n-s+1}} + \cdots + \frac{1}{k^{n-s}}].
\end{aligned}$$

Hence,
$d(gx_n, gx_{n+s}) \to 0$, as $n \to \infty$.
Therefore, $\{gx_n\}$ is a dqb-cauchy sequence.
Now we prove that common fixed point of f and g is unique.
Since, $g(X)$ is dqb-complete, there exists $v \in g(X)$ such that $gx_n \to v$ as $n \to \infty$.
Since $v \in g(X)$, we can find $u \in X$ such that $gu = v$.
Now,

$$\begin{aligned}
d(gx_n, fu) &= d(fx_{n-1}, fu) \\
&\leq \phi[max\{d(gx_{n-1}, gu), d(fx_{n-1}, gx_{n-1}), d(gu, fu)\}] \\
&= \phi[max\{d(gx_{n-1}, gu), d(gu, gx_{n-1}), d(gu, fu)\}].
\end{aligned}$$

As $n \to \infty$, we have

$$\begin{aligned}
d(v, fu) &\leq \phi[max\{d(v,v), d(v,v), d(v, fu)\}] \\
&\leq \phi[max\{d(gu, fu)\}] \\
&= \frac{d(v, fu)}{k}.
\end{aligned}$$

Therefore, the above inequality is possible if

$$d(v, fu) = 0. \tag{4.5}$$

Similarly, we can show that

$$d(fu,v)=0. \tag{4.6}$$

Using (4.5), (4.6) and condition of dislocated quas b-metric we get, $v=fu=gu$ is a point of coincidence of f and g in X.
Assume that $w\in X$ is another coincidence point of f and g in X, such that $fw=gw\neq fu=gu$.

$$\begin{aligned}
d(gu,gw)=d(fu,fw) &\leq \phi[max\{d(gu,gw),d(fu,gu),d(gw,fw)\}]\\
&= \phi[max\{d(gu,gw),d(gu,gu),d(gw,gw)\}]\\
&= \phi[d(gu,gw)]\\
&\leq \frac{d(gu,gw)}{k}.
\end{aligned}$$

Therefore, the above inequality is possible if

$$d(gu,gw)=0. \tag{4.7}$$

Similarly, we can show that

$$d(gw,gu)=0. \tag{4.8}$$

Using (4.7), (4.8) and condition of dislocated quas b-metric we get, $gw=gu$. Which is a contradiction hence, the point of coincidence in X is unique. By Lemma 4.1.1 the self-maps f and g have unique common fixed point in X.
Corollary 4.2.1 Let (X,d) be a complete b-metric space with $s\geq 1$ and T,S be self-mappings on X which satisfy

$$\begin{aligned}
d(Sx,Ty) &\leq a_1d(x,Sx)+a_2d(y,Ty)+a_3d(x,Ty)+a_4d(y,Sx)\\
&\quad +a_5d(x,y),
\end{aligned}$$

for all $x,y\in X$, where a_1,a_2,a_3,a_4,a_5 are non-negative real numbers satisfying
(i) $s^2a_1+s^2a_2+s^3a_3+s^3a_4+s^2a_5<1$,
(ii) $a_1=a_2$ or $a_3=a_4$.
Then T and S have a unique common fixed point.
Proof: The result follows by taking $a_6=0$ in Theorem 4.2.1.
Corollary 4.2.2 Let (X,d) be a dqb-complete metric space with coefficients $k>1$.

Let $f : X \to X$ be self-maps satisfying the inequality

$$d(fx, fy) \leq \phi[max\{d(x,y), d(fx,x), d(y, fy)\}],$$

for all $x, y \in X$, where ϕ is altering distance function and $\phi(t) < \frac{t}{k}$ for $t > 0$. Then f has unique fixed point in X.

Proof: The result follows by taking $g = I(I$ is an identity map in X) in Theorem 4.2.2.

The following example is in support of Theorem 4.2.1

Example 4.1 Let $X = \{4,5,6\}$ and define $d : X \times X \to R^+$ by $d(4,4) = d(5,5) = d(6,6) = 0, d(4,5) = d(5,4) = 1, d(4,6) = d(6,4) = 1/12$ and $d(5,6) = d(6,5) = 5/6$. It is easy to check that (X,d) is a b-metric space with parameter s=4/3.

Define the maps $T, S : X \to X$ by $T4 = T6 = 4$, $T5 = 6$ and $S4 = S5 = S6 = 4$. Let $a_1 = a_2 = a_3 = a_5 = 0$, $a_4 = 1/14$ and $a_6 = 1/7$. Then the conditions of Theorem 4.2.1 are satisfied. The first two conditions of b-metric are clear and the b-triangular inequality is given by

$$1 = d(4,5) \leq 4/3[d(4,6) + d(6,5)] = 11/9,$$

$$1/12 = d(4,6) \leq 4/3[d(4,5) + d(5,6)] = 22/9,$$

$$5/6 = d(5,6) \leq 4/3[d(5,4) + d(4,6)] = 13/9.$$

Therefore (X, d) is a b-metric space.

Now we show that the inequality (4.9) holds:

$$\begin{aligned} d(Sx, Ty) &\leq a_1 d(x, Sx) + a_2 d(y, Ty) + a_3 d(x, Ty) + a_4 d(y, Sx) \\ &\quad + a_5 d(x, y) + a_6 [d(x, Sx) + d(x, Ty)] \\ &= \frac{1}{14} d(y, Sx) + \frac{1}{7}[d(x, Sx) + d(x, Ty)], \end{aligned}$$

using this we consider the following cases:

$$(i) \quad 0 = d(S4, T4) \leq \frac{1}{14} d(4, S4) + \frac{1}{7}[d(4, S4) + d(4, T4)] = 0,$$

$$(ii) \;\; 1/12 = d(S5, T5) \leq \frac{1}{14} d(5, S5) + \frac{1}{7}[d(5, S5) + d(5, T5)] = 1/3,$$

$$(iii)\quad 0 = d(S6,T6) \le \frac{1}{14}d(6,S6) + \frac{1}{7}[d(6,S6)+d(6,T6)] = 5/168,$$

$$(iv)\quad 1/12 = d(S4,T5) \le \frac{1}{14}d(5,S4) + \frac{1}{7}[d(4,S4)+d(4,T5)] = 1/12,$$

$$(v)\quad 0 = d(S5,T4) \le \frac{1}{14}d(4,S5) + \frac{1}{7}[d(5,S5)+d(5,T4)] = 2/7,$$

$$(vi)\quad 0 = d(S4,T6) \le \frac{1}{14}d(6,S4) + \frac{1}{7}[d(4,S4)+d(4,T6)] = 1/168,$$

$$(vii)\quad 0 = d(S6,T4) \le \frac{1}{14}d(4,S6) + \frac{1}{7}[d(6,S6)+d(6,T4)] = 1/12,$$

$$(viii)\ 0 = d(S5,T6) \le \frac{1}{14}d(6,S5) + \frac{1}{7}[d(5,S5)+d(5,T6)] = 49/168,$$

$$(ix)\ 1/12 = d(S6,T5) \le \frac{1}{14}d(5,S6) + \frac{1}{7}[d(6,S6)+d(6,T5)] = 1/12$$

and $s^2a_1 + s^2a_2 + s^3a_3 + s^3a_4 + s^2a_5 + s^2a_6 + 2s^3a_6 = 160/189 < 1$.
Hence, all the conditions of Theorem 4.2.1 are satisfied. Therefore, 4 is a common fixed point of T and S .
The following example is in support of Theorem 4.2.2
Example 4.2 Let $X = [0,2]$, define $d : X \times X \to R^+$ by $d(x,y) = |x-y|^2 + |x|^2$ and $f,g : X \to X$ by

$$f(x) = \begin{cases} 0, & \text{if } 0 \le x \le 1, \\ 1/2, & \text{if } 1 < x \le 2 \end{cases}$$

and

$$g(x) = \begin{cases} x/2, & \text{if } 0 \le x \le 1, \\ 2, & \text{if } 1 < x \le 2. \end{cases}$$

If $\phi(t) = \frac{t}{3}$ for all $t \ge 0$ with $k = 2$, then (X,d) is a dislocated quas b-metric space. But (X,d) is not a quasi b-metric space and not b-metric space. Indeed suppose $x,y,z \in X$.
Now,
(i) $d(x,y) = |x-y|^2 + |x|^2 = |y-x|^2 + |y|^2 = d(y,x) = 0$ implies that $x = y = 0$.

(ii)

$$\begin{aligned} d(x,y) &= |x-y|^2+|x|^2 \\ &\leq [|x-z|+|z-y|]^2+|x|^2 \\ &= |x-z|^2+2|x-z|.|z-y|+|z-y|^2+|x|^2 \\ &\leq |x-z|^2+|x-z|^2+|z-y|^2+|z-y|^2+|x|^2 \\ &\leq 2[|x-z|^2+|z-y|^2]+|x|^2+|z|^2 \\ &\leq 2[d(x,z)+d(z,y)], \end{aligned}$$

where $k=2$ and $x,y,z\in X$. We conclude that (X,d) is a dqb-metric space and $f0=g0=0$. Which implies that 0 is a coincidence point of f and g. Moreover, $g(x)=2\cup[0,1/2]$ and $f(x)=\{0,1/2\}$, hence this implies that $f(X)\subseteq g(X)$. And $f0=fg0=gf0=g0=0$, which implies that f and g are weakly compatible.
To show that the inequality in Theorem (4.2.2) holds true, we consider the following four cases.
Case(i) Suppose $x,y\in[0,1]$
$d(fx,fy)=d(0,0)=|0-0|^2+|0|^2=0,$
$d(gx,gy)=d(x/2,y/2)=|x/2-y/2|+|x/2|^2=(x-y)^2/4+x^2/4,$
$d(fx,gx)=d(0,x/2)=|0-x/2|^2+|0|^2=x^2/4,$
$d(gy,fy)=d(y/2,0)=|y/2-0|^2+|y/2|^2=y^2/2.$
Therefore,

$$\begin{aligned} d(fx,fy) &\leq \phi[max\{d(gx,gy),d(fx,gx),d(gy,fy)\}] \\ 0 &\leq \phi[max\{((x-y)^2+x^2)/4,x^2/4,y^2/2\}]. \end{aligned}$$

Which is true for all $x,y\in[0,1]$.
Case (ii) Suppose $x,y\in(1,2]$
$d(fx,fy)=d(1/2,1/2)=|1/2-1/2|^2+|1/2|^2=0+1/4=1/4,$
$d(gx,gy)=d(2,2)=|2-2|+|2|^2=0+4=4,$
$d(fx.gx)=d(1/2,2)=|1/2-2|+|1/2|^2=9/4+1/4=10/4,$
$d(gy,fy)=d(2,1/2)=|2-1/2|+|2|^2=9/4+4=25/4.$

Therefore,

$$\begin{aligned} d(fx,fy) &\leq \phi[max\{d(gx,gy),d(fx,gx),d(gy,fy)\}] \\ 1/4 &\leq \phi[max\{4,10/4,25/4\}]. \end{aligned}$$

Which is true for all $x,y \in (1,2]$.
Case (iii) Suppose $x \in [0,1]$ and $y \in (1,2]$
$d(fx,fy) = d(0,1/2) = |0-1/2|^2 + |1/2|^2 = 1/4+1/4 = 1/2,$
$d(gx,gy) = d(x(2,)2) = |x/2-2|^2 + |x/2|^2 = ((x-4)^2+x^2)/4,$
$d(fx,gx) = d(0,x/2) = |0-x/2|^2 + |0|^2 = x^2/4,$
$d(gy,fy) = d(2,1/2) = |2-1/2|^2 + |2|^2 = 9/4+4 = 25/4.$
Therefore,

$$\begin{aligned} d(fx,fy) &\leq \phi[max\{d(gx,gy),d(fx,gx),d(gy,fy)\}] \\ 1/2 &\leq \phi[max\{((x-4)^2+x^2)/4,x^2/4,25/4\}]. \end{aligned}$$

Which is true for all $x \in [0,1]$ and $y \in (1,2]$.
Case (iv) Suppose $x \in (1,2]$ and $y \in [0,1]$
$d(fx,fy) = d(1/2,0) = |1/2-0|^2 + |1/2|^2 = 1/4+1/4 = 1/2,$
$d(gx,gy) = d(2,y/2) = |2-y/2|^2 + |2|^2 = ((4-y)^2/4)+4,$
$d(fx,gx) = d(1/2,2) = |1/2-2|^2 + |1/2|^2 = 9/4+1/4 = 10/4,$
$d(gy,fy) = d(y/2,0) = |y/2-0|^2 + (y/2)^2 = y/4^2 + y/4^2 = y^2/2.$
Therefore,

$$\begin{aligned} d(fx,fy) &\leq \phi[max\{d(gx,gy),d(fx,gx),d(gy,fy)\}] \\ 1/2 &\leq \phi[max\{((4-y)^2+16)/4,10/4,y^2/2\}]. \end{aligned}$$

Which is true for all $x \in (1,2]$ and $y \in [0,1]$.
From **Case (i)-(iv)** all the conditions of Theorem 4.2.2 are satisfied and 0 is the unique common fixed point of f and g.

Chapter 5

Conclusion and Future scope

5.1 Conclusion

In this study we established new common fixed point theorems in complete b-metric spaces and dislocated quasi b-metric spaces, and proved the existence and uniqueness of common fixed points .
Our results generalize and extend the work of (Faraji and Nourouzi, 2017) in b-metric spaces and (Sastry and Kumar, 2019) in dislocated quasi b-metric spaces. Finally we have supported the main results of this research work by applicable examples.

5.2 Future scope

There are some published results related to the existence of common fixed point theorems of mappings defined on b-metric spaces and dislocated quasi b-metric spaces. The researcher believes the search for the existence and uniqueness of common fixed points of self-maps satisfying conditions in the setting of b-metric spaces and dislocated quasi b-metric spaces are active areas of study. So, other interested researchers can use this opportunity and conduct their research work in this areas.

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