CONSTRAINING COSMOLOGICAL PARAMETERS AND ESTIMATION OF MASS OF THE UNIVERSE IN THE LAMBDA COLD DARK MATTER MODEL

By

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PHYSICS

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ACRONYMS

GR :	General Relativity
GRT :	General Relativity Theory
CMB :	Cosmic Microwave Background
WMAP:	Wilkinson Microwave Anisotropy Probe
ΛCDM:	Lambda Cold Dark Matter

Abstract

In this thesis we considered the flat ΛCDM cosmological model to constrain the density parameters, the Hubble constant and the age of the universe. We did also estimate the mass of the universe setting some relevant boundary conditions. The analytical derivations were exploited by GR field equations with the expanding coordinates where the cosmic equations were handled with the Friedman-Lemaître equations. Mathematica 11.3 was used to generate numerical data to compare with the observational data of WMAP. To our conclusion, the flat ΛCDM well fits to the observations in the selected constraints. On the other hand, our derived analytical mass well agrees with the result obtained through cosmic dimensional analysis. However, our estimated numerical value is one order higher than obtained earlier which is $m \sim 10^{54}$kg..

Keywords: GR, FLRW-metric, EFE, FLE, Cosmology, Universe parameters, Mass of the universe

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Chapter 1

Introduction

1.1 Background

The universe is the spacetime and their contents and other forms of matter and energy[1]. Mathematical implication by Friedmann and others within Einstein's general relativity tells us that the universe is expanding. The evidence corresponding to the predictions of expansion is that the observed objects in deep space are presifted. This means that these objects are receding from the Earth. This phenomenon obeys a law known as Hubble's law, which Hubble was found empirically in 1929 [2]. However, the law was first derived from general relativity by George Lemaître in 1927 [3]. The simplest cosmological model assumes that the universe is filled with both matter and radiation. Nowadays, it was found that radiation density is low and the universe today is matter dominated. If matter can be treated as a pressureless fluid, then the universe will expand forever (if the spatial geometry is Euclidean or hyperbolic) or eventually recollapse (if the spatial geometry is of a 3-sphere)[4].

The discovery of the accelerating Universe revolutionized 20^{th} century cosmology by indicating the presence of a qualitatively new component in the Universe that dominates the expansion in the last several billion years. The nature of dark energy - the component that causes the accelerated expansion is unknown, and understanding its properties and origin is one of the principal challenges in modern physics. Current measurements are consistent with an interpretation of dark energy as a cosmological constant in general relativity.

The model of the hot expanding universe becomes scientific and philosophical debate to establish reality. Explosive progress in the field of cosmology over the past decade allowed for something that may well have been totally unanticipated in the previous decades: the high-precision determination of several cosmological parameters. Consequently, measuring dark energy properties requires a combination of cosmological probes that are sensitive to both classes of effects to break these parameter and model degeneracies[5][6]. In 1998, published observations of type Ia supernovae by the high-z supernova search team [7], followed in 1999 by the supernova cosmology project [8], suggested that the expansion of the universe is accelerating. The observational evidences indicate that the universe can not be modeled in such a simple way. A hypothesis corresponding to the observations is that the universe consist of some form of dark energy with negative-pressure. The existence of dark energy is needed to reconcile the measured geometry of space by measurements of the cosmic microwave background (CMB) anisotropies and the WMAP satellite. The CMB indicates that the universe is very close to flatness [9][10]. The WMAP seven-year analysis gives an estimate of 72.7% dark energy, 22.7% dark matter and 4.6% ordinary matter. The dust model of cosmology is inconsistent with observation[11].

The mysterious dark energy content of the universe is not well known. There are a number of models proposed to explain it, including models with higher dimensional universe and new principles of physics. Yet, most of the models need to pass observational constraints while others are waiting for advanced tools to test. However, irrespective of the limitations raised on some theoretical background issues, the standard ΛCDM model is successfully passing all observational tests made so far. Further researches on its developments and applications of the ΛCDM model are necessary. Thus, in this thesis we consider the ΛCDM cosmological model to constrain some parameters of the universe where observations are used to predict. We also try to estimate the mass of the universe employing appropriate boundary conditions.

1.2 Statement of the problem

Since the discovery of an accelerated expanding universe [7, 8] scenario, about 70% of the content of the universe is considered with a mysterious dark energy whose origin is yet unknown. There are a number of models proposed to explain it, including models with higher dimensional universe and new principles of physics. Yet, most of the models need to pass observational constraints while others are waiting for advanced tools to test. However, irrespective of the limitations raised on some theoretical background issues, the standard ΛCDM model is successfully passing all observational tests made so far. Further researches on its developments and applications of the ΛCDM model are necessary. Thus, in this thesis we consider the ΛCDM cosmological model to constrain some parameters of the universe where observations are used to predict. We also try to estimate the mass of the universe employing appropriate boundary conditions.

Research questions

1. How does the universe evolve in the ΛCDM model?

2. How and in what way do some of the viable parameters of the universe constrain the ΛCDM model with respect to observation?

3. What is the mass of the universe estimated by ΛCDM model?

1.3 Objectives

1.3.1 General objective

To constrain cosmological parameters and estimate mass of the universe in the ΛCDM model.

1.3.2 Specific objectives

Specific objectives are:

1. To describe evolution of the universe in the ΛCDM model.

2. To address the ways how viable parameters of the universe are being constrained with respect to observation in the ΛCDM model.

3. To estimate mass of the universe with the ΛCDM model.

1.4 Methodology

General Relativity field equations in the presence of positive cosmological constant is used to derive dynamical equations of the universe, where the expanding coordinate system is used. In particular, the Friedmann-Lemaîre equations are exploited to develop relevant equations being addressed in terms of observable dynamic parameters like: cosmic redshift, the Hubble parameter and density parameters. Then, the resulting equations are used to constrain the parameters with respect to observation in ΛCDM model. Furthermore, we estimate the mass of the observable universe where the general relativistic volume element is used to integrate the mass.

The general scheme of the thesis is outlined as: In **chapter two** we introduce the concept universe and cosmological basics to study. In **chapter three** we introduce General Relativity and the conception of modern cosmology. In **chapter four** we present fundamental principles and equations of cosmology in GR framework with positive cosmological constant. In **chapter five** results and discussions follow while our summary and conclusion will be given in **chapter 6**.

Chapter 2

Introduction of basic cosmological concepts of the universe

2.1 Historical overview of the universe

From the get-go in the twentieth century the universe was believed to be static: consistently the equivalent size, neither growing nor contracting. In any case, in 1924 space expert Edwin Hubble utilized a strategy spearheaded by Henrietta Leavitt to quantify distances to remote objects in the sky. Hubble used spectroscopic red-shift data to measure the speeds these objects were travelling then graphed their distance from Earth against their speed. He discovered that the speed at which astronomical objects move apart is proportional to their distance from each other. The relationship Hubble discovered was later used as evidence that the universe is expanding[12]. Today, the consensus among scientists, astronomers and cosmologists is that the universe as we know it was created in a massive explosion that not only created the majority of matter, but the physical laws that govern our ever-expanding cosmos. This is what astronomers and cosmologists today called as the big bang[13].

After the evidence of the expanding universe given by hubble, cosmologists and astronomers rise the question what the universe was like in the past. Then Georges Lemiatre addressed this question and in 1927 he showed that Einstein's attempt to eliminate the prediction of an expanding or contracting space from his general theory of relativity did not work.

Lemaitre reasoned that if space is expanding, then the universe must have been denser in the past than it is now. Using the physics of the general theory of relativity, he realized that there would have been a time in the finite past when the universe would have been infinitely dense and, therefore, he said that our universe must have had a beginning in time. This was contrary to the prevailing opinion among Lemaitre's contemporaries. Most scientists at time considered the universe was infinitely old; that it had always existed and always would exist. Significant philosophical and cosmological breakthrough, it was inadequate in terms of physics. There was little that could be done with the model to see if it could, in fact, evolve into the present universe that would have required a knowledge of nuclear physics that did not exist at the time. There was another reason why scientists were opposing Lemaitre's idea of a universe that had a beginning in time. They stated that Lemaitre's model was inadequate in terms of nuclear physics. George Gamow became interested in Lemaitre's ideas concerning the early universe. However, 1946 Gamow proposed that the high temperatures of the early universe could provide the appropriate conditions for the creation of the chemical elements in their proper ratios. Since the time Lemaitre had first addressed the problem, significant advances in nuclear physics had occurred. In 1948, Gamow and Ralph Alpher developed a model of an early universe consisting of neutrons at a very high temperature[14]. Cosmology is the science about the origin, changes, structure and evolution of the Universe on the large scale, its past, present and future[15][16]. The first era of relativistic cosmology, started in 1917 with the seminal paper by Einstein in which he constructed, at the expense of the introduction of a cosmological constant.

The history of relativistic cosmology can be divided into 6 periods:

- The initial one (1917-1927), during which the first relativistic universe models were derived in the absence of any cosmological observation.

- A period of development (1927-1945), during which the cosmological redshifts were discovered and interpreted in the framework of dynamical Friedmann-Lemaître solutions, whose geometrical and mathematical aspects were investigated in more details.
- A period of consolidation (1945-1965), during which primordial nucleosynthesis of light elements and fossil radiation were predicted.
- A period of acceptation (1965-1980), during which the big bang theory triumphed over the steady state theory.
- A period of enlargement (1980-1998), when high energy physics and quantum effects were introduced for describing the early universe.
- The present period of high precision experimental cosmology, where the fundamental cosmological parameters are now measured with a precision of a few %, and new problematic arise (nature of the dark energy, topology of the universe, new cosmologies in quantum gravity theories, etc.)[17].

2.2 Cosmological parameters

The term "cosmological parameters" is forever increasing in its scope, and nowadays often includes the parameterization of some functions, as well as simple numbers describing properties of the universe. The original usage referred to the parameters describing the global dynamics of the Universe, such as its expansion rate and curvature. Now we wish to know how the matter budget of the universe is built up from its constituents: baryons, photons, neutrinos, dark matter, and dark energy. We also need to describe the nature of perturbations in the universe, through global statistical descriptors such as the matter and radiation power spectra. There may be additional parameters describing the physical state of the universe, such as the ionization fraction as a function of time during the era since recombination. Typical comparisons of cosmological models with observational data now feature between five and ten parameters[18].

2.2.1 Choice of the parameters

Most of the recent work on cosmological parameters has chosen a particular parameter sets, and investigated parameter constraints when faced with different observational data sets. However, the information criteria ask how well different models fit the same data set. A useful division of parameters is into those that are definitely needed to give a reliable fit to the data, which Andrew R Liddle will call the base parameter set, and those that have proved irrelevant, or of marginal significance, in fits to the present data. Cosmological models are typically defined through base parameters mainly through;Hubble parameter,matter density,dark matter density,and dark energy density.

The base parameter set is actually extraordinarily small, and given in Table 2.1.

Ω_m	dark matter density
Ω_r	radiation density
Ω_Λ	Dark energy density
h	Hubble parameter
q	Deceleration Parameter

Table 2.1: Base parameters for a successful of cosmological model

1. Hubble parameter

In 1929 Edwin Hubble published his landmark discovery that distant spiral nebulae are receding from us at speeds proportional to their distances, implying that the Universe is expanding at a constant rate. Recessional velocities were calculated from the doppler shift of spectral lines and distances estimated from luminosity measurements. Despite considerable scatter in the results, Hubble concluded that the rate of expansion was constant, with a value of almost 500 km per second per megaparsec. Hubble's Law can be written as:

$$v = H_0 * d \tag{2.2.1}$$

where v is the radial velocity of the galaxy, d is the galaxy's distance, and H_o is a constant of proportionality that was later coined Hubble's constant. Over the decades, Hubble's constant has been refined by many new and improved probes in the cosmos. Hubble's original diagram is reproduced as2.1.

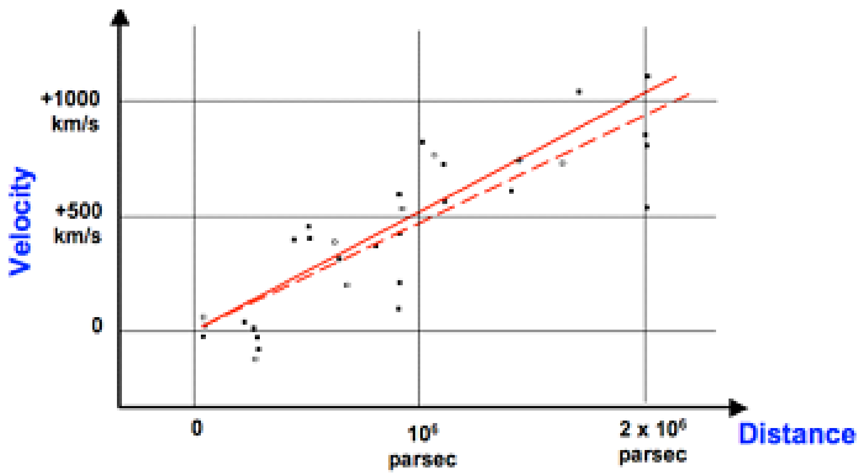

Figure 2.1: A reproduction of Hubble's original diagram. Recessional velocities are plotted against estimated distances.

More recently, the WMAP mission's detailed measurements of the cosmic microwave background radiation give the best current estimate of the Hubble constant as $71 \pm 3.5 kms^{-1} Mpc^{-1}$. [19]

2. Density Parameter

The density parameter is the ratio of the average density of matter and energy in the Universe to the critical density (the density at which the universe would stop expanding only after an infinite time). The density parameter (Ω) is given by:

$$\Omega = \frac{\rho}{\rho_c} \tag{2.2.2}$$

where (ρ) is the actual density of the universe and (ρ_c) the critical density. In other words, it is the sum of a number of different components including both normal and dark matter as well as the dark energy suggested by recent observations. We can therefore write:

$$\Omega = \Omega_m + \Omega_r + \Omega_\Lambda \tag{2.2.3}$$

where Ω_m is the density parameter for matter, Ω_r is the density parameter for radiation and Ω_Λ is the density parameter for dark energy. Current observations suggest that we live in a dark energy dominated universe with Ω_Λ = 0.73, Ω_m = 0.27, and Ω_r = $8.24 * 10^{-5}$. To the accuracy of current cosmological observations, this means that we live in a flat universe(Ω = 1) [14]. By making accurate measurements of the cosmic microwave background fluctuations, WMAP is able to measure the basic parameters of the Big Bang model including the density and composition of the universe. WMAP measures the relative density of baryonic and non-baryonic matter to an accuracy of better than a few percent of the overall density. It is also able to determine some of the properties of the non-baryonic matter: the interactions of the non-baryonic matter with itself, its mass and its interactions with ordinary matter all affect the details of the cosmic microwave background fluctuation spectrum. WMAP determined that the universe is flat, from which it follows that the mean energy density in the universe is equal to the critical density.

WMAP breakdown total density into:

- 4.6% atoms. More than 95% of the energy density in the universe is in a form that has never been directly detected in the laboratory. The actual density of atoms is equivalent to roughly 1 proton per 4 cubic meters.
- 24% cold dark matter. Dark matter is likely to be composed of one or more species of subatomic particles that interact very weakly with ordinary matter
- 71.4% dark energy. The first observational hints of dark energy in the universe date back to the 1980's when astronomers were trying to understand how clusters of galaxies were formed. Their attempts to explain the observed distribution of galaxies were improved if dark energy were present, but the evidence was highly uncertain. In the 1990's, observations of supernova were used to trace the expansion history of the universe (over relatively recent times) and the big surprise was that the expansion appeared to be speeding up, rather than slowing down. There was some concern that the supernova data were being misinterpreted, but the result has held up to this day. In 2003, the first WMAP results came out indicating that the universe was flat and that the dark matter made up only 24% of the density required to produce a flat universe. If 71.4% of the energy density in the universe is in the form of dark energy, which has a gravitationally repulsive effect, it is just the right amount to explain both the flatness of the universe and the observed accelerated expansion. Thus dark energy explains many cosmological observations at once[14].

3. Deceleration Parameter The Hubble parameter H(t) measures the expansion rate at any particular time t for any model obeying the cosmological principle. It does, however, vary with time in a way that depends upon the contents of the universe. One can express this by expanding the cosmic

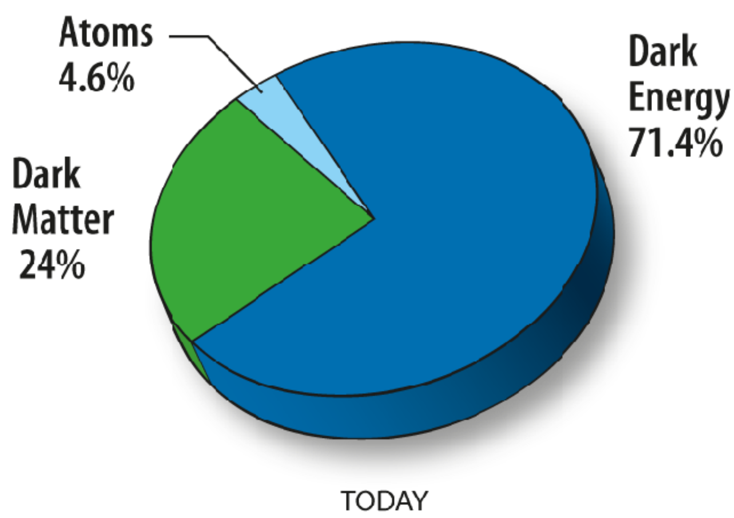

Figure 2.2: Content of the univese

scale factor for times t close to t_0 in a power series:

$$a(t) = a_0[1 + H_0(t - t_0) - \frac{1}{2}q_0 H_0^2(t - t_0)^2 + ...] \tag{2.2.4}$$

where $q_0 = -\frac{\ddot{a}(t_0)a_0}{\dot{a}(t_0)^2}$ is is called the deceleration parameter; the suffix '0', as always, refers to the fact that $q_0 = q(t_0)$. Note that while the Hubble parameter has the dimensions of inverse time, q is actually dimensionless.

The deceleration parameter, q, indicates the rate at which the expansion of universe is slowing due to self-gravitation. It is defined by:

$$q = -\frac{\ddot{a}a}{\dot{a}^2} \tag{2.2.5}$$

where **a** is the scale factor, **a(t)**, of the universe by which all lengths scale, **ȧ** is the first time derivative (rate of change) of **a**, and **ä** is the second time derivative of **a**. Recent observations have suggested that the rate of expansion of the universe is currently accelerating, perhaps due to the effects of dark energy. This yields negative values for the deceleration parameter[18].

2.3 Probes for the expansion of the universe

A lot of independent probes witnessed the accelerated expansion of the universe as a stronghold of modern cosmology. However, cosmologist and astrophysicist proposed that the real physical mechanism driving such accelerated expansion is still unknown; one possible explanation is that it is due to an unknown form of energy, dark energy, which in its simplest embodiment is a cosmological constant, but other possibilities could include a breakdown of general relativity on large scales or an effect of interpreting the observation using a metric which is not correct for our inhomogeneous universe. The ΛCDM model is able to describe the evolution of the universe with a minimal number of cosmological parameters; current data constrain these parameters at the $\%$ level. The universe can be taken as spatially flat, with the dark matter, baryon, and radiation densities requiring to be specified as independent parameters[20]. Several different probes have been used to set constraints on cosmological parameters, and especially on dark energy parameters.

Most of these probes are:

- Cosmic microwave background (CMB)
- Baryonic acoustic oscillation(BAO)
- Supernovae type Ia(SNe)
- Probes of the growth of structure via weak lensing studies and cluster of galaxies abundance

WMAP team gives an overview of combining different probes to obtain a good reference of cosmological parameters[21]. According to the cold dark matter paradigm, dark matter makes up about 27 percent of the universe, but the particles that constitute dark matter are yet to be discovered.

2.3.1 Cosmic microwave background radiation

One of the firm predictions of standard big bang model is the existence of relic radiation from the hot phase the universe has experienced at early times. The cosmic microwave background (CMB) was first serendipitously detected in 1965 by Arno Penzias and Robert Wilson, working on long-distance radio communications at the bell laboratories. This radiation is a relic of the initial hot and dense state of the universe; hence it provided the first compelling evidence for the hot big bang model proposed by George Gamow in 1948. The presence of the radiation at earlier time has never been proven directly[22]. T

2.3.2 Baryon acoustic oscilations

The baryon acoustic oscillations is a phenomenon ocurred at the early times of universe, before the decoupling of matter and radiation, where the perturbation of baryonic matter propagated as a wave. Time of decoupling is placed at years after big-bang. Before that, the universe temperature was around making the photons to be very energetic. Photons interacted with baryons, which, in the cosmology context, refer not only protons but also electrons, via compton scattering. This strong interaction is known as tight coupled limit and it caused that photons and baryons moved together like an unique fluid, the photon-baryon fluid. Because of, the model to describe this behavior is known as fluid approximation. Energetic photons were able to scatter baryons and they avoid that proton and electrons to join into neutral atoms. The previous described conditions finished when the Hubble rate becomes higher than scatter rate. Photons are not energetic enough to scatter baryons, occurring the decoupling of matter and radiation. Protons and electrons to join in neutral atoms and photons follow free ways, being possible to be observed currently in the CMB[23]. The current standard cosmological model, cold dark matter (CDM), assumes that the initial fluctuations in the distribution of matter were seeded by quantum fluctuations pushed to cosmological scales by inflation. Directly after inflation, the universe is radiation-dominated and the baryonic matter is ionized and coupled to radiation through Thomson scattering. The radiation pressure drives sound waves originating from overdensities in the matter distribution. At the time of recombination, the photons decouple from the baryons and shortly after that the sound wave stalls. Through this process, each overdensity of the original density perturbation field has evolved to become a centrally peaked perturbation surrounded by a spherical shell. The radius of these shells is called the sound horizon r_s. Both overdense regions attract baryons and dark matter and will be preferred regions of galaxy formation[24].

2.4 Model of the universe

2.4.1 Einstein static models of universe

A static universe is a cosmological model in which the universe is both spatially infinite, and space is neither expanding nor contracting. Einstein proposed a static model in 1917 as a static solution of his field equations [25]. Modern cosmology began in 1917 with Einstein's cosmological considerations on the general theory of relativity. He applied general relativity to the entire universe. To him it must have been a matter of common sense that we lived in an immutable cosmos; thus, theory had to describe a static universe. However,his original field equations did not give such results. If matter was homogenously distributed and gravitation was the only active force, his universe would collapse. He therefore introduced the famous cosmological term, λ, today usually designed by Λ, so that his fundamental equations took the form

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu} \tag{2.4.1}$$

Λ is Einstein 's constant of gravity. The left hand side describes the geometrical structure of the universe, the right hand side represents the energy-momentum tensor due to the action of matter;

the Λ term acts as a repulsive force. This additional term gave Einstein a static, spherical, spatially closed universe. He emphasised that the known laws of gravitation did not justify the introduction of Λ, its inclusion was motivated by the quest for a static solution of the differential equations[26]. Einstein's universe is constructed on the basis that the universe is static, isotropic and homogeneous. This solution is marked as the birth of modern cosmology. The model is based on the following assumptions:

- The universe is static, i.e., in a proper coordinate system matter is at rest, and the proper pressureP_o and proper density ρ_o are the same everywhere.
- The universe is isotropic, i.e., all the spatial directions are equivalent.
- The universe is homogeneous, i.e., no part of the universe can be distinguished from the other.
- For small values of r the line element takes the form of special relativity of flat space$-$time, since local gravitational field can be neglected for small space$-$time [27].

2.4.2 De Sitter's static models of universe

Willem De Sitter was a Dutch mathematician and astronomer who made major contributions to the field of physical cosmology. He co-authored a paper with Einstein in 1932 in which they discussed the implications of cosmological data for the curvature of the universe. De Sitter also came up with the concept of the de Sitter space, another static solution for Einstein's field equation. The De Sitter universe is the second model of the universe just after the publications of the Einstein's static and closed model. In 1917, Wilhelm De Sitter has developed this model which is a maximally symmetric solution of the Einstein field equation with zero density. The geometry of the de Sitter universe is theoretically more complicated than that of the Einstein universe. The model does not contain matter or radiation. But, it predicts that there is a redshift[28].

2.4.3 Standard cosmological model

The standard model of cosmology, often called lambda cold dark matter (ΛCDM), consists of a spatially flat, homogeneous and isotropic universe on large scales. Initially hot and dense, the universe features four principal energy components: photons (relativistic species), baryonic matter, dark matter, and dark energy in the form of a cosmological constant Λ . The latter dominates the energy content of the universe at late times and is responsible for the current accelerated expansion. Inflation provides a mechanism to seed the structures we see today, which are originated from the hierarchical gravitational collapse of small overdensities generated in the early universe[29]. The past few years has seen the emergence of a "concordant" cosmological model that is consistent both with observational constraints from the background evolution of the universe as well that from the formation of large-scale structures. It is certainly fair to say that the present edifice of the standard cosmological models is robust. A set of foundation and pillars of cosmology have emerged and are each supported by a number of distinct observations. The community is now looking beyond the estimation of parameters of a working standard model of cosmology[30]. Rapid advances in observational cosmology have led to the establishment of a precision cosmological model, with many of the key cosmological parameters determined to one or two significant figure accuracy. Particularly prominent are measurements of cosmic microwave background (CMB) anisotropies, with the highest precision observations being those of the Planck Satellite which supersede the landmark WMAP results. However the most accurate model of the Universe requires consideration of a range of observations, with complementary probes providing consistency checks, lifting parameter degeneracies, and enabling the strongest constraints to be placed. The simple ΛCDM model is based on six parameters: physical baryon density parameter; physical dark matter density parameter; the age of the universe; scalar spectral index; curvature fluctuation amplitude; and reionization optical depth[31].

Chapter 3

Introduction to General Relativity and the conception of modern cosmology

3.1 Introduction to General relativity

In 1905 Einstein introduced special relativity theory, then in 1907 he proposed general relativity theory by including non-inertial reference frames; that is, to include acceleration and gravity[32]. The fundamental physical postulate of GR is that the presence of matter causes curvature in the space time in which it exists. This curvature is taken to be the gravitational field produced by the matter. Einstein′s field equation gives the mathematical description of how the matter and curvature are related. Moreover, once this curvature is given, GR describes how other objects (such as planets and light beams) move in this gravitational field via the geodesic equation. In addition, general relativity states that clocks run slower in strong gravitational fields (or highly accelerated frames), predicting a gravitational redshift. It also predicts the existence of gravitational lensing, gravitational waves, gravitomagnetism, the lense-thirring effect, and relativistic precession of orbiting bodies[33]. General relativity models the physical universe as a 4-dimensional space-time manifold. Albert Einstein used tensor as an essential tool to present his general theory of relativity[28].

3.1.1 Einstein field equations

In a weak static field produced by a non-relativistic mass density ρ, the time component of metric tensor is approximately given by

$$g_{00} = -(1 + 2\phi) \tag{3.1.1}$$

Here ϕ is newtonian potential determined by poisson's equation

$$\nabla^2 \phi = 4\pi G\rho \tag{3.1.2}$$

Furthermore, the energy density for non-relativistic matter is just equal to its mass density

$$T_{00} = \rho \tag{3.1.3}$$

Combaning equation(3.1.1,3.1.2 and 3.1.3

$$\nabla^2 g_{00} \quad = \quad -8\pi G T_{00} \tag{3.1.4}$$

This field equation is only supposed to hold for weak static fields generated by non-relativistic matter. However, 3.1.4 leads us to guess that the weak-field equations for a general distribution T_{ij} of energy and momentum take the form

$$G_{ij} = -8\pi G T_{ij} \tag{3.1.5}$$

Where G_{ij} is linear combination of the metric and its first and second derivatives. It follows then from principle of equivalence that the equations which govern gravitational fields of arbitrary strength must take the form

$$G_{\mu\nu} = -8\pi G T_{\mu\nu} \tag{3.1.6}$$

Where $G_{\mu\nu}$ is a tensor which reduces to G_{ij} for weak fields. The properties need to find $G_{\mu\nu}$ are:

(A) By definition $G_{\mu\nu}$ is a tensor

(B) By assumption, $G_{\mu\nu}$ consists only of term with the total number N=2 of derivative of the metric; that is, $G_{\mu\nu}$ contains only terms that are either linear in the second derivatives or quadratic in the first derivatives of the metric.

(C) Since $T_{\mu\nu}$ is symmetric, so is $G_{\mu\nu}$

(D) Since $T_{\mu\nu}$ is conserved, so is $G_{\mu\nu}$:

$$G^{\mu}_{\nu;\mu} = 0 \tag{3.1.7}$$

(E) For a weak stationary field produced by non-relativistic matter the 00 component of equation 3.1.6 must reduce to 3.1.4, so in this limit

$$G_{00} \simeq \nabla^2 g_{00} \tag{3.1.8}$$

The most general way of constructing a field satisfying (A) and (B) is by contraction of the curvature tensor $R^{\lambda}_{\mu\nu\kappa}$. The anti-symmetry property of $R_{\mu\nu\kappa\lambda}$ shows that there are only two tensors that can be formed by contracting $R_{\mu\nu\kappa\lambda}$; that is, the Ricci tensor $R_{\mu\kappa} \equiv R^{\lambda}_{\mu\lambda\kappa}$, and curvature scalar $R = R^{\mu}_{\mu}$. Hence (A) and (B) require $G_{\mu\nu}$ to take the form:

$$G_{\mu\nu} = C_1 R_{\mu\nu} + C_2 g_{\mu\nu} R \tag{3.1.9}$$

Where C_1 andC_2 are constants.

Using the Bianchi identity gives the covariant divergence of $G_{\mu\nu}$ as

$$G^{\mu}_{\nu;\mu} = (\frac{C_1}{2} + C_2) R_{;\nu} \tag{3.1.10}$$

(D)allows two possibilities:either $C_2 = \frac{-C_1}{2}$ or $R_{;\nu}$ vanishes everywhere. One can reject the second possibility, because 3.1.10 and 3.1.6 give

$$G^{\mu}_{\mu} = (C_1 + 4C_2) R = -8\pi G T^{\mu}_{\mu} \tag{3.1.11}$$

Thus if $R_{;\nu} \equiv \frac{\partial R}{\partial x^\nu}$ vanishes, then so must $\frac{\partial T^\mu_\mu}{\partial x^\nu}$, and this is not the case in the presence of inhomogeneous non-relativistic matter. Then $C_2 = \frac{-C_1}{2}$,so 3.1.8 becomes

$$G_{\mu\nu} = C_1(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R) \tag{3.1.12}$$

Finally, we use the property(E)to fix the constant C_1. A non-$relativistic$ system always has $|T_{ij}| \ll |T_{00}|$, so we are concerned here with a case where $|G_{ij}| \ll |G_{00}|$, or using 3.1.12

$$R_{ij} \simeq \frac{1}{2}g_{ij}R \tag{3.1.13}$$

Furthermore, we deal here with a weak field, so $g_{ij} \simeq \eta_{ij}$. The curvature scalar is therefore given by

$$\begin{aligned} R \simeq R_{kk} - R_{00} \simeq \frac{3}{2}R - R_{00} \\ R \simeq 2R_{00} \end{aligned} \tag{3.1.14}$$

At any point X in an arbitrarily strong gravitational field, we can define a locally inertial coordinate system such that

$$g_{ij}(X) = \eta_{ij} \tag{3.1.15}$$

$$\left(\frac{\partial g_{ij}(x)}{\partial x^\nu}\right)_{x=X} = 0 \tag{3.1.16}$$

Using 3.1.14 and 3.1.15 in 3.1.12 we find

$$G_{00} \simeq 2C_1R_{00} \tag{3.1.17}$$

To calculate R_{00} for a weak field we use the linear part of $R_{\lambda\mu\nu\kappa}$,given as

$$R_{\lambda\mu\nu\kappa} = \frac{1}{2}\left[\frac{\partial^2 g_{\lambda\nu}}{\partial x^\kappa \partial x^\mu} - \frac{\partial^2 g_{\mu\nu}}{\partial x^\kappa \partial x^\lambda} + \frac{\partial^2 g_{\mu\kappa}}{\partial x^\nu \partial x^\lambda}\right] \tag{3.1.18}$$

When the field is static all time derivatives vanish, and the components we need become

$R_{0000} \simeq 0$, $R_{0i0j} \simeq \frac{1}{2}\frac{\partial^2 g_{00}}{\partial x^i \partial x^i}$

Hence 3.1.17 gives

$$G_{00} \simeq 2C_1 \left(R_{i0i0} - R_{0000}\right) \simeq C_1 \nabla^2 g_{00} \tag{3.1.19}$$

Comparing 3.1.19 and 3.1.8 we find that (E)is satisfied if and only if $C_1 = 1$, so equation 3.1.12 becomes

$$G_{\mu\nu} = (R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R) \tag{3.1.20}$$

With equation 3.1.6, this gives the Einstein field equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi G T_{\mu\nu} \tag{3.1.21}$$

Einstein introduced a term $\lambda g_{\mu\nu}$ to these field equations for cosmological reasons and the equations become:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \lambda g_{\mu\nu} = -8\pi G T_{\mu\nu} \tag{3.1.22}$$

For this reason, λ is called the cosmological constant[34].

3.2 The beginning of modern cosmology

Our present understanding of the universe is based upon the successful hot big bang theory, which explains its evolution from the first fraction of a second to our present age.

This theory rests upon four robust pillars.

- A theoretical framework based on general relativity.
- The expansion of the universe.
- The relative abundance of light elements.
- The cosmic microwave background (CMB), the afterglow of the big bang.

Modern cosmology begun with the advent of Einstein's general relativity and the realization that the geometry of spacetime, and thus the general attraction of matter, is determined by the energy content of the universe. Einstein field equations are non-linear, so these non-linear equations are simply too difficult to solve without invoking some symmetries of the problem at hand: the universe itself. Although at small scales the universe looks very inhomogeneous and anisotropic, the universe on large scales is very homogeneous and isotropic. Moreover, the cosmic microwave background, which contains information about the early universe, indicates that the deviations from homogeneity and isotropy were just a few parts per million at the time of photon decoupling. Therefore, we can safely impose those symmetries to the universe at large and determine the corresponding evolution equations. The most general metric satisfying homogeneity and isotropy is the Friedmann-Robertson-Walker (FRW) metric. This metric is characterized by just two quantities: a scale factor a(t), which determines the physical size of the universe, and a constant K, which characterizes the spatial curvature of the universe, Spatially open, flat and closed universes have different three-geometries. Light geodesics on these universes behave differently, and thus could in principle be distinguished observationally.

3.2.1 The matter and energy content of the universe

The most general matter fluid consistent with the assumption of homogeneity and isotropy is a perfect fluid, one in which an observer comoving with the fluid would see the universe around it as isotropic. The energy momentum tensor associated with such a fluid can be written as:

$$T^{\mu\nu} \quad = \quad pg^{\mu\nu}(p+\rho)U^{\mu}U^{\nu} \tag{3.2.1}$$

where p(t) and $\rho(t)$ are the pressure and energy density of the fluid at a given time in the expansion, as measured by this comoving observer, and U^{μ} is the comoving four-velocity, satisfying $U^{\mu}U_{\mu} =$

-1. For such a comoving observer, the matter content looks isotropic,

$$T^{\mu}_{\nu} = \text{diag}(\rho(t), p(t), p(t), p(t)) \tag{3.2.2}$$

The conservation of energy ($T^{\mu}_{\nu};\nu = 0$), a direct consequence of the general covariance of the theory ($G^{\mu}_{\nu};\nu = 0$), can be written in terms of the FRW metric and the perfect fluid tensor 3.2.1 as

$$\dot{\rho} + 3\frac{\dot{a}}{a}(p+\rho) = 0 \tag{3.2.3}$$

In order to find explicit solutions, one has to supplement the conservation equation with an equation of state relating the pressure and the density of the fluid, $p = p(\rho)$. The most relevant fluids in cosmology are barotropic, i.e. fluids whose pressure is linearly proportional to the density, $p = w\rho$, and therefore the speed of sound is constant in those fluids[35].

Chapter 4

Fundamental principles and cosmological equations in GR framework with positive Λ

4.1 The Cosmological Principle

After the introduction of GR scientist were able to study the universe in a more mathematical way than ever before. The study of the evolution of the universe, as well as the properties and the dynamics of it is known today as cosmology. The Einstein field equations describe the dynamics of the universe, but to do that an appropriate form of the energy-momentum tensor is needed, which is connected to the composition of the universe, and the metric, that is related to the Ricci curvature tensor and the Ricci curvature scalar. To construct those object scientists set up axioms. This is named as the cosmological principle. The cosmological principle states that in macroscopic scales the universe can be seen as homogeneous and isotropic[36].

4.2 Spacetime geometry

Geometry is encoded in a metric $g_{ij}(x)$ (with i and j running over the three coordinate directions), or equivalently in a line element $dS^2 = g_{ij}dX_idX_j$, with summation over repeated indices understood.

dS is the proper distance between X and X + dX, meaning that it is the distance measured by a surveyor who uses a coordinate system that is cartesian in a small neighborhood of the point X. Obvious homogeneous isotropic dimensional spaces are:

- Flat space of three-dimensional space with positive definite lengths,with line element

$$dS^2 = dX^2 \tag{4.2.1}$$

- A spherical surface in four-dimensional Euclidean space with some radius a, with line element

$$dS^2 = dX^2 + dz^2, z^2 + X^2 = a^2 \tag{4.2.2}$$

- A hyperspherical surface in four-dimensional pseudo-Euclidean space, with line element

$$dS^2 = dX^2 - dz^2, z^2 - X^2 = a^2 \tag{4.2.3}$$

  where a^2 is an arbitrary positive constant with z instead of time.

By rescale coordinates

$$X^{'} = a * X, z^{'} = a * z \tag{4.2.4}$$

Dropping primes, the line elements in the spherical and hyperspherical cases are

$$dS^2 = a^2 \left[dX^2 \pm dz^2\right], \, z^2 \pm X^2 = a^2 \tag{4.2.5}$$

The differential of the equation $z^2 \pm X^2 = 1$ gives $zdz = \mp X \cdot dX$ so

$$dS^2 = a^2 \left[dX^2 \pm \frac{(X \cdot dX)^2}{1 \mp X^2}\right] \tag{4.2.6}$$

Extending this to the case of Euclidean space by writing it as

$$dS^2 = a^2 \left[dX^2 + k\frac{(X \cdot dX)^2}{1 - kX^2}\right] \tag{4.2.7}$$

Where

$$k = \begin{cases} 1, & \text{spherical;} \\ -1, & \text{hyperspherical;} \\ 0, & \text{Euclidean.} \end{cases} \tag{4.2.8}$$

There is an obvious way to extend this to the geometry of spacetime: just include a term 4.2.7 in the spacetime line element, with **a** now an arbitrary function of time known as the Robertson-Walker scale factor:

$$d\tau^2 = -g_{\mu\nu}(x)dx^\mu dx^\nu = dt^2 - a^2(t)\left[dX^2 + k\frac{(X\cdot dX)^2}{1-kX^2}\right] \tag{4.2.9}$$

The components of the metric in these coordinates are:

$$g_{ij} = a^2(t)\left[\delta_{ij} + k\frac{x^i x^j}{1-kX^2}\right], g_{i0} = 0, g_{00} = -1, \tag{4.2.10}$$

with i and j running over the values 1, 2, and 3, and with $x^0 = t$ the time coordinate in units, with the speed of light equal to unity. Instead of the quasi$-$Cartesian coordinates x^i, we can use spherical polar coordinates, for which

$dx^2 = dr^2 + r^2 d\Omega, d\Omega = d\theta^2 + \sin\theta^2 d\phi^2$ So

$$d\tau^2 = dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\Omega\right] \tag{4.2.11}$$

in which case the metric becomes diagonal.

4.3 The Friedmann-Lemaître-Robertson-Walker metric

The idea of the cosmological principle leads to construct models of the universe in which this principle holds. General relativity is a geometrical theory that regard the universe as a continuous fluid and assign to each fluid element the three spatial coordinates $x^\alpha(\alpha = 1, 2, 3)$. Thus, any point in spacetime can be labeled by the coordinates x^α, corresponding to the fluid element which is passing through the point, and a time parameter which we take to be the proper time t measured by a clock moving with the fluid element. The coordinates x^{α} are called comoving coordinates. The geometrical properties of spacetime are described by a metric. It is possible to choose coordinates r,θ,ϕ,t,for which the metric takes the form given in(4.2.11):

$$d\tau^2 = dt^2 - a^2(t)[\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2] \tag{4.3.1}$$

Where a(t) is unknown function of time ,and k is a constant ,which by suitable choice of units for r can be chosen to have the value +1,-1 or 0.

The metric in equation(4.3.1) is known in cosmology as the Friedmann-Lemaître- Robertson-Walker metric (FLRW) [37]. These metric are:

$$g_{rr} = \frac{a^2(t)}{1-kr^2}, g_{\theta\theta} = a^2(t)r^2, g_{\phi\phi} = a^2 r^2 \sin^2\theta, g_{tt} = -1 \tag{4.3.2}$$

4.4 Friedmann equations

In order to find how the scale factor a(t) evolves, we need to consider the equations of motion, given by the Einstein equation. The Friedmann-Lemaître-Robertson-Walker metric of 4.3.2 has the components

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{a^2(t)}{1-kr^2} & 0 & 0 \\ 0 & 0 & a^2(t)r^2 & 0 \\ 0 & 0 & 0 & a^2 r^2 \sin^2\theta \end{pmatrix} \tag{4.4.1}$$

Calculating the Einstein tensor from this metric gives

$$G_0^0 = -3\frac{\dot{a}^2}{a^2} - 3\frac{k}{a^2} \tag{4.4.2}$$

$$G_j^i = -\left[2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right] \tag{4.4.3}$$

$$G_0^i = 0 \tag{4.4.4}$$

In cosmology, the energy-momentum tensor which is of greatest relevance is that of a perfect fluid:

$$T_{ij} = (p + \rho c^2)U_i U_j - p g_{ij}$$

$$T^{\mu}_{\nu} = \begin{pmatrix} -\rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} \tag{4.4.5}$$

where U_k is the fluid four-velocity, ρ is the energy density and p is the pressure. Homogeneity implies that they only depend on time, $\rho = \rho(t)$, p = p(t). In general, the Einstein equation $G_{\alpha\beta} + \Lambda g_{\alpha\beta} = 8\pi G T_{\alpha\beta}$ is a non-linear system of ten partial differential equations. In the case of the FRW universe, it reduces to two ordinary non-linear differential equations:

$$3\frac{\dot{a}^2}{a^2} + 3\frac{k}{a^2} = 8\pi \mathrm{G}\rho + \Lambda \tag{4.4.6}$$

$$-2\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} - \frac{k}{a^2} = 8\pi \mathrm{Gp} + \Lambda \tag{4.4.7}$$

These are called the Friedmann equations[33].

4.5 Cosmological redshift

The information whether the scale factor a(t) is increasing, decreasing, or constant in the Friedmann-Lemaître-Robertson-Walker metric comes to us from the observation of a shift in the frequencies of spectral lines from distant galaxies as compared with their values observed in terrestrial laboratories. For light world lines (paths through spacetime), $ds^2 = 0$. For a radial trajectory (one with $d\phi = d\theta = 0$) we thus have $c^2 dt^2 = \frac{a^2(t)dr^2}{(1-kr2)}$. Taking the square root, and choosing the sign so that the photon is headed toward the origin ($\frac{dr}{dt} < 0$) we have:

$$cdt = -\frac{a(t)dr}{\sqrt{1-kr^2}} \tag{4.5.1}$$

$$\int_{t_1}^{t_0} \frac{dt}{a(t)} = \int_0^{r_1} \frac{dr}{\sqrt{1-kr^2}} \tag{4.5.2}$$

Taking the differential of this relation, and recalling that the radial coordinate r_1 of co-moving sources is time-independent, we see that the interval δt_1 between departure of subsequent light signals is related to the interval δt_0 between arrivals of these light signals by

$$\frac{\delta t_1}{a(t_1)} = \frac{\delta t_0}{a(t_0)} \tag{4.5.3}$$

The emitted frequency is $\nu_1 = \frac{1}{\delta t_1}$, and the observed frequency is $\nu_0 = \frac{1}{\delta t_0}$, so

$$\frac{\nu_0}{\nu_1} = \frac{a(t_1)}{a(t_0)} \tag{4.5.4}$$

If a(t) is increasing, then this is a redshift, a decrease in frequency by a factor $\frac{a(t_1)}{a(t_0}$), equivalent to an increase in wavelength by a factor conventionally called 1 + z:

$$1 + z = \frac{a(t_0)}{a(t_1)} \tag{4.5.5}$$

Alternatively, if a(t) is decreasing then we have a blueshift.

4.6 Solution of Friedmann equation

For K = 0 we get very simple solutions to Eq. 4.4.6 in the three special cases by using equation of state $p = w\rho$.

4.6.1 Non-relativistic matter

Here pressure is zero. So fluid equation becomes:

$$\dot{\rho} + 3\rho\frac{\dot{a}}{a} = 0 \tag{4.6.1}$$

From this equation ρ can be written as

$$\rho = \rho_0 \left(\frac{a}{a_0}\right)^{-3} \tag{4.6.2}$$

So the solution of Eq. 4.4.6)by using 4.6.2 with K = 0 is

$$a(t) \quad \propto \quad t^{\frac{2}{3}} \tag{4.6.3}$$

This gives $q_0 \equiv -a\frac{\ddot{a}}{\dot{a}^2} = \frac{1}{2}$ and a simple relation between the age of the universe and the Hubble constant

$$t_0 \quad = \quad \frac{2}{3H_0} = 6.5 \star 10^9 h^{-1} yr \tag{4.6.4}$$

Equations 4.6.3 and 4.4.7 show that for k=0, the energy density at time is.

$$\rho = \frac{1}{6\pi G t^2} \tag{4.6.5}$$

This is known as the Einstein-de Sitter model.

4.6.2 Relativistic matter

Here pressure is $\frac{\rho}{3}$. So fluid equation becomes:

$$\dot{\rho} + 4\rho\frac{\dot{a}}{a} \quad = \quad 0 \tag{4.6.6}$$

From this equation ρ can be written as

$$\rho \quad = \quad \rho_0 \left(\frac{a}{a_0}\right)^{-4} \tag{4.6.7}$$

So the solution of Eq. 4.4.6, using 4.6.7 with K = 0 is

$$a(t) \quad \propto \quad t^{\frac{1}{2}} \tag{4.6.8}$$

This gives $q_0 \equiv -a\frac{\ddot{a}}{\dot{a}^2} = 1$ and a simple relation between the age of the universe and the Hubble constant

$$t_0 \quad = \quad \frac{1}{2H_0} \tag{4.6.9}$$

Equations 4.6.8 and 4.4.7 show that for k=0, the energy density at time is.

$$\rho = \frac{3}{32\pi G t^2} \tag{4.6.10}$$

4.6.3 Vacuum energy

In vacuum we have $p = -\rho$. Thus fluid equation becomes

$$\dot{\rho} \quad = \quad 0 \tag{4.6.11}$$

so our energy density is constant.

$$\rho \quad = \quad \beta \tag{4.6.12}$$

where β is some constant Inserting the value of ρ into equation 4.4.6 with k=0, scale factor can be written as the following

$$a(t) \propto e^{\mathrm{Ht}} \tag{4.6.13}$$

where H is the Hubble constant, now really a constant, given by

$$H = \sqrt{\frac{8\pi \mathrm{G}\rho}{3}} \tag{4.6.14}$$

Here $q_0 = -1$, and the age of the universe in this case is infinite.

More generally, for arbitrary K and a mixture of vacuum energy and relativistic and non-relativistic matter, making up fractions Ω_Λ, Ω_M, and Ω_R of the critical energy density, we have

$$\rho = \frac{3H_0^2}{8\pi \mathrm{G}}\left[\Omega_\Lambda + \Omega_M \left(\frac{a_0}{a}\right)^3 + \Omega_R \left(\frac{a_0}{a}\right)^4\right] \tag{4.6.15}$$

where the present energy densities in the vacuum, non-relativistic matter, and relativistic matter (i.e., radiation) are, respectively,

$$\rho_{\mathrm{V0}} = \frac{3H_0^2}{8\pi \mathrm{G}}\Omega_\Lambda, \rho_{\mathrm{M0}} = \frac{3H_0^2}{8\pi \mathrm{G}}\Omega_M, \rho_{\mathrm{R0}} = \frac{3H_0^2}{8\pi \mathrm{G}}\Omega_R \tag{4.6.16}$$

Using this in Equation 4.4.6 gives

$$\mathrm{dt} = \frac{\mathrm{dx}}{H_0 x\sqrt{\Omega_\Lambda + \Omega_k x^{-2} + \Omega_M x^{-3} + \Omega_R x^{-4}}} \tag{4.6.17}$$

$$\text{dt} = \frac{\text{-dz}}{H_0(1+z)\sqrt{\Omega_\Lambda + \Omega_k(1+z)^2 + \Omega_M(1+z)^3 + \Omega_R(1+z)^4}} \tag{4.6.18}$$

where $x \equiv a/a_0 = \frac{1}{(1+z)}$. Therefore, if we define the zero of time as corresponding to an infinite redshift, then the time at which light was emitted that reaches us with redshift z is given by

$$\text{t(z)} \;=\; \int_0^{\frac{1}{1+z}} \frac{\text{dx}}{H_0 x\sqrt{\Omega_\Lambda + \Omega_k x^{-2} + \Omega_M x^{-3} + \Omega_R x^{-4}}} \tag{4.6.19}$$

In a flat Universe,$\rho_m = \rho_{m,0}(\frac{a}{a_0})^{-3}$.

So that we can write the Friedmann equation as:

$$\left(\frac{\dot{a}}{a}\right)^2 \;=\; \frac{8\pi G}{3}\rho_{m,0}a^{-3} + \frac{\Lambda}{3} or \dot{a}^2 = H_0^2\Omega_{m,0}a^{-1} + H_0^2\Omega_{\Lambda,0}a^2 \tag{4.6.20}$$

Thus this becomes

$$\left(\frac{H(z)}{H_0}\right)^2 \;=\; \Omega_{m,0}(1+z)^3 + \Omega_{\Lambda,0} \tag{4.6.21}$$

The right-hand side is often referred to as E(z), so that $H(z) = H_0\,(E(z))^{1/2}$. We can derive a relationship between time t and redshift z by differentiating equation 4.5.5 with respect to z. Thus this becomes

$$da \;=\; -\frac{a_0}{(1+z)^2}dz \tag{4.6.22}$$

Hubble parameter in terms of z can be written as:

$$H(z) = \frac{\dot{a}}{a} = \frac{da}{dz}\frac{dz}{dt}\frac{(1+z)}{a_0} \tag{4.6.23}$$

So that

$$\int_{\text{t1}}^{\text{t2}} dt = \frac{-1}{H_0}\int_{\text{z1}}^{\text{z2}} \frac{1}{(1+z)\,(E(z))^{\frac{1}{2}}}\,dz \tag{4.6.24}$$

The age of the Universe is

$$\int_0^{\text{to}} dt = \frac{1}{H_0}\int_0^{\infty} \frac{1}{(1+z)\,(E(z))^{\frac{1}{2}}}\,dz \tag{4.6.25}$$

4.7 Cosmological distances

The comoving coordinate system relates to proper distance in spaces described by the Friedmann-Lemaître- Robertson-Walker metric. Obviously, however, we cannot measure proper distances to astronomical objects in any direct way. Distant objects are observed only through the light they emit which takes a finite time to travel to us; we cannot therefore make measurements along a surface of constant proper time, but only along the set of light paths traveling to us from the past. One can, however, define operationally other kinds of distance which are, at least in principle, directly measurable. One such distance is the luminosity distance d_L or the angular-diameter distance d_A.

4.7.1 Proper distances

We define a proper distance, as the distance between two events, A and B, in a reference frame for which they occur simultaneously. We must be clearer about the difference between the radial coordinate and the distance. They are equal only when $d\Omega = 0$. The comoving square infinitesimal distance is indeed, from FLRW metric 4.3.1 the following:

$$d\chi^2(t) = \frac{dr^2}{1-kr^2} + r^2 d\Omega^2 \tag{4.7.1}$$

i.e. it has indeed a radial part, but also has a transversal part. So, if χ is the comoving distance between two points, the proper distance at a certain time t is

$$d_p(\chi, t) = a(t)\chi \tag{4.7.2}$$

The comoving distance is a notion of distance which does not include the expansion of the universe and thus does not depend on time. Suppose that $d\Omega = 0$. Then the comoving distance to an object with radial coordinate r is the following:

$$\chi = \int_0^r \frac{dr}{\sqrt{1-kr^2}} \tag{4.7.3}$$

$$\chi = \begin{cases} \frac{1}{k^{1/2}} \sin^{-1}(k^{1/2} r), & \text{k>0;} \\ r, & \text{k=0;} \\ \frac{1}{|k|^{1/2}} \sinh^{-1}(\mid k \mid^{1/2} r), & \text{k<0.} \end{cases} \tag{4.7.4}$$

In a flat universe, the proper distance to an object is just its coordinate distance, $d_p(t) = a(t)r$.Because of $\sin^{-1}(x) > x$, in a closed universe (k $>$ 0) the proper distance to an object is greater than its coordinate distance, while in an open universe (k $<$ 0) the proper distance to an object is less than its coordinate distance because of $\sinh^{-1}(x) < x$. From the FLRW metric, by putting $dS^2 = 0$, we can relate the lookback time with the comoving distance as follows:

$$cdt = a(t)d\chi \tag{4.7.5}$$

This seems quite similar to the proper distance, but careful: the proper distance is defined as $a(t)\chi$ and evidently $a(t)d\chi \neq d(a(t)\chi)$. The lookback time is the photon time of flight and thus it includes cumulatively the expansion of the universe. On the other hand, the proper distance is the distance considered between two simultaneous events and therefore the expansion of the universe is not taken into account cumulatively. By integrating 4.7.5) from t_{em} to t_0 we get the comoving distance from the source to us

$$\chi = \int_{t_{em}}^{t_0} \frac{cdt}{a(t)} = \int_a^1 \frac{cda}{H(a)a(t)^2} \tag{4.7.6}$$

For the dust-dominated case one has $H = H_0/a^{3/2}$ and the comoving distance as a function of the scale factor and of the redshift is:

$$\chi = \frac{c}{H_0} \int_a^1 \frac{da}{\sqrt{a}} = \frac{2c}{H0} \left(1 - \sqrt{a}\right) \tag{4.7.7}$$

$$\chi(z) = \frac{2c}{H_0} \left(1 - \frac{1}{\sqrt{1+z}}\right) \tag{4.7.8}$$

Inserting equation 4.7.8 into 4.7.2 we can write proper distance in terms of redshift as follows

$$d_p = a(t)\frac{2c}{H_0}\left(1 - \frac{1}{\sqrt{1+z}}\right) \tag{4.7.9}$$

This equation is equivalent to

$$d_p = \frac{2ca_0}{H_0}\left(\frac{1}{1+z} - \frac{1}{(1+z)^{\frac{3}{2}}}\right) \tag{4.7.10}$$

4.7.2 Luminosity distance

This is defined in such a way as to preserve the Euclidean inverse-square law for the diminution of light with distance from a point source. Let L denote the power emitted by a source at a point P, which is at a coordinate distance r at time t. Let l be the power received per unit area at time $_{t0}$ by an observer placed at P_0. We then define

$$d_L = \left(\frac{L}{4\pi \mathrm{l}}\right)^{\frac{1}{2}} \tag{4.7.11}$$

The area of a spherical surface centred on P and passing through P_0 at time t_0 is just $4pa_0^2r^2$. The photons emitted by the source arrive at this surface having been redshifted by the expansion of the universe by a factor $\frac{a}{a_0}$. Also, as we have seen, photons emitted by the source in a small interval δt arrive at P_0 in an interval $\delta t_0 = (\frac{a_0}{a})\delta t$ due to a time-dilation effect. We therefore find

$$l = \frac{L}{4\pi \mathrm{a_0}^2 r^2}\left(\frac{a}{a_0}\right)^2 \tag{4.7.12}$$

From which

$$d_L = {a_0}^2\frac{r}{a} \tag{4.7.13}$$

For objects with $z \ll 1$, we can usefully write the relation between luminosity distance and redshift as a power series. Using equation 4.5.5, 2.2.4,and 2.2.5 the redshift is related to the look-back time $t_0 - t$ by

$$z = H_0(t_0 - t) + \frac{1}{2}(q_0 + z)H_0^2(t_0 - t)^2 + ... \tag{4.7.14}$$

This can be inverted, to give the look-back time as a power series in the redshift

$$H_0(t_0 - t) = z - \frac{1}{2}\left(q_0 + z\right) z^2 \tag{4.7.15}$$

The coordinate distance r1 of the luminous object is given by Eq. (4.5.2) as

$$\frac{(t_0 - t_1)}{a\left(t_o\right)} + H_0 \frac{\left(t_0 - t_1\right)^2}{2a\left(t_0\right)} + ... = r_1 + ... \tag{4.7.16}$$

with the dots on the right-hand side denoting terms of third and higher order in r_1. Using Eq. (4.6.5), the solution is

$$r_1 a\left(t_0\right) H_0 = z - \frac{1}{2}\left(1 + q_0\right) z^2 + \ldots \tag{4.7.17}$$

This gives the luminosity distance (4.7.13) as a power series

$$d_L = {H_0}^{-1}\left[z + \frac{1}{2}\left(1 - q_0\right) z^2 + \ldots\right] \tag{4.7.18}$$

4.7.3 Angular-diameter distance

Again, this is constructed in such a way as to preserve a geometrical property of Euclidean space, namely the variation of the angular size of an object with its distance from an observer. Let $d_P(t)$ be the (proper) diameter of a source placed at coordinate r at time t. If the angle subtended by d_P is denoted $\triangle\theta$. In Special Relativity, the invariant interval between two events at coordinates (t, x, y, z) and (t + dt, x + dx, y + dy, z + dz) is defined by

$$dS^2 = c^2 t^2 - \left(dx^2 + dy^2 + dz^2\right) \tag{4.7.19}$$

where dS is invariant under a change of coordinate system and the path of a light ray is given by dS = 0. Equation (4.7.19) implies

$$d_P = ar\Delta\theta \tag{4.7.20}$$

We define d_A to be the distance

$$d_A = \frac{d_P}{\Delta\theta} = ar \tag{4.7.21}$$

Comparison of equation 4.7.21 with equation 4.7.13 shows that the ratio of the luminosity and angulardiameter distances is simply a function of redshift:

$$\frac{d_A}{d_L} = (1+z)^{-2} \tag{4.7.22}$$

Thus, angular diamter distances can be written as[38]:

$$d_A = (1+z)^{-2} {H_0}^{-1} \left[z + \frac{1}{2}\left(1 - q_0\right) z^2 + \ldots \right] \tag{4.7.23}$$

More generally, for arbitrary K and a mixture of vacuum energy and relativistic and non-relativistic matter, making up fractions $\Omega_\Lambda, \Omega_M, and \Omega_R$ of the critical energy density, we have

$$\rho \;\; = \;\; \frac{3H_0^2}{8\pi G}\left[\Omega_\Lambda + \Omega_M \left(\frac{a_0}{a}\right)^3 + \Omega_R \left(\frac{a_0}{a}\right)^4\right] \tag{4.7.24}$$

where the present energy densities in the vacuum, non-relativistic matter, and and relativistic matter (i.e., radiation) are, respectively.

$$\rho_{\mathrm{V0}} = \frac{3{H_0}^2\Omega_\Lambda}{8\pi\mathrm{G}}, \;\; \rho_{\mathrm{M0}} = \frac{3{H_0}^2\Omega_M}{8\pi\mathrm{G}}, \; \rho_{\mathrm{R0}} = \frac{3{H_0}^2\Omega_R}{8\pi\mathrm{G}} \tag{4.7.25}$$

according to Eq. 4.4.6,

$$\Omega_\Lambda + \Omega_M + \Omega_R + \Omega_K \;\; = \;\; 1, \Omega_K = -\frac{K}{a_0^2 H_0^2} \tag{4.7.26}$$

Using this in Eq. 4.4.6 gives

$$\begin{aligned} \mathrm{dt} \;\; &= \;\; \frac{\mathrm{dx}}{H_0 x\sqrt{(\Omega_\Lambda + \Omega_K x^{-2} + \Omega_M x^{-3} + \Omega_R x^{-4})}} \\ &= \;\; \frac{\text{-dz}}{H_0(1+z)\sqrt{(\Omega_\Lambda + \Omega_K (1+z)^2 + \Omega_M (1+z)^3 + \Omega_R (1+z)^4)}} \end{aligned} \tag{4.7.27}$$

where $x \equiv \frac{a}{a_0} = \frac{1}{(1+z)}$. Therefore, if we define the zero of time as corresponding to an infinite redshift, then the time at which light was emitted that reaches us with redshift z is given by:

$$t(z) = \frac{1}{H_0}\int_0^{\frac{1}{1+z}} \frac{\mathrm{dx}}{x\sqrt{\Omega_\Lambda + \Omega_K x^{-2} + \Omega_M x^{-3} + \Omega_R x^{-4}}} \tag{4.7.28}$$

In particular, by setting $z = 0$, one can find the present age of the universe:

$$t_0 = \frac{1}{H_0}\int_0^{1} \frac{\mathrm{dx}}{x\sqrt{\Omega_\Lambda + \Omega_K x^{-2} + \Omega_M x^{-3} + \Omega_R x^{-4}}} \tag{4.7.29}$$

In order to calculate luminosity or angular diameter distances, we also need to know the radial coordinate r of a source that is observed now with redshift z. According to Eqs 4.5.2 and 4.7.28, this is

$$\begin{aligned} r(z) &= S\left[\int_{t(z)}^{t_0} \frac{\mathrm{dt}}{a(t)}\right] \qquad (4.7.30) \\ &= S\left[\frac{1}{a_0 H_0}\int_{\frac{1}{1+z}}^{1} \frac{\mathrm{dx}}{x^2\sqrt{\Omega_\Lambda + \Omega_K x^{-2} + \Omega_M x^{-3} + \Omega_R x^{-4}}}\right] \end{aligned}$$

Where

$$S(y) = \begin{cases} \mathrm{siny}, & \text{K=+1;} \\ \mathrm{y}, & \text{K=0;} \\ \mathrm{sinhy}, & \text{K=-1.} \end{cases}$$

This can be written more conveniently by using Eq. 4.7.26 to express $a_0 H_0$ in terms of Ω_K. We then have a single formula

$$a_0 r(z) = \frac{1}{H_0 {\Omega_K}^{\frac{1}{2}}} \times \sinh\left({\Omega_K}^{\frac{1}{2}}\int_{\frac{1}{1+z}}^{1} \frac{\mathrm{dx}}{x^2\sqrt{\Omega_\Lambda + \Omega_K x^{-2} + \Omega_M x^{-3} + \Omega_R x^{-4}}}\right) \tag{4.7.31}$$

which can be used for any curvature. (Eq. 4.7.29 has a smooth limit for $\Omega_K = 0$, which gives the result for zero curvature. Also, for $\Omega_K < 0$, the argument of the hyperbolic sine is imaginary, and we can use $\sinh$(ix) = i$\sin$(x). Using Eq. 4.7.31 in Eq. 4.7.13 gives the luminosity distance of a source observed with redshift z as follows.

$$\begin{aligned} d_L(z) &= a_0 r(z)(1+z) \qquad (4.7.32) \\ &= \frac{1+z}{H_0 {\Omega_K}^{\frac{1}{2}}} \times \sinh\left({\Omega_K}^{\frac{1}{2}}\int_{\frac{1}{1+z}}^{1} \frac{\mathrm{dx}}{x^2\sqrt{\Omega_\Lambda + \Omega_K x^{-2} + \Omega_M x^{-3} + \Omega_R x^{-4}}}\right) \end{aligned}$$

Chapter 5

Result and discussion

5.1 Observable cosmological distances

We have given formulas for distance measures and have described more detail in chapter three. Distance measures are used to tie some observable quantity to another quantity that is not directly observable. The distance measures we have discussed all are reduced to the euclidian distance at low redshift. At low redshift all are asymptotic to each other. In accord with our present understanding of cosmology, distance measures are calculated within the context of general relativity, where the freidmann-Lemaitre-Robertson-Walker solution is used to describe the universe. Distance measures we have described are which cosmologist more commonly used for measures of distances from observer to an object at redshift z. These are proper distance(d_p), luminosity distance(d_L), angular distance(d_A) and comoving distance(χ). Their equations are given as follows:

$$\chi = \int_0^r \frac{dr^2}{\sqrt{1-kr^2}} \tag{5.1.1}$$

$$d_L = {H_0}^{-1}\left[z + \frac{1}{2}\left(1-q_0\right)z^2 + \dots\right] \tag{5.1.2}$$

$$d_p = a(t)\int_0^r \frac{dr^2}{\sqrt{1-kr^2}} \tag{5.1.3}$$

$$d_A = (1+z)^{-2}{H_0}^{-1}\left[z + \frac{1}{2}\left(1-q_0\right)z^2 + \dots\right] \tag{5.1.4}$$

5.2 Lookback time and age of the universe

The lookback time t_L to an object is the difference between the age t_0 of the universe now (at observation) and the age t_e of the universe at the time the photons were emitted [39]. Quantitatively, we can calculate the age of the universe from Friedmann equation. To be completely explicit about the time-dependence of each term, we write

$$\begin{aligned}\rho_m(t) &= \left(\frac{a(t_0)}{a(t)}\right)^3 \rho_{m,0} \\ \rho_r(t) &= \left(\frac{a(t_0)}{a(t)}\right)^4 \rho_{r,0} \\ \rho_{vac}(t) &= \rho_{vac,0}\end{aligned} \tag{5.2.1}$$

Here we are using the convention that a subscript 0 denotes the present value of any quantity. Each of the above equations reflects the known dependence on a(t) for each contribution to the mass density, with the constant of proportionality written so that $\rho_X(t_0) = \rho_{X,0}$, for each type of matter X. Mass densities are usually tabulated as fractions of the critical density.

$$\rho_c = \frac{3H^2}{8\pi G} \tag{5.2.2}$$

using the convention that for each type of mass density X,

$$\Omega_X = \frac{\rho_X}{\rho_c} \tag{5.2.3}$$

So, we rewrite Eqs. 5.2.1 by replacing each $\rho_{X,0}$ by $\Omega_{X,0} * \rho_{c,0}$:

$$\begin{aligned}\rho_m(t) &= \frac{3H^2}{8\pi G}\left(\frac{a(t_0)}{a(t)}\right)^3 \Omega_{m,0} \\ \rho_r(t) &= \frac{3H^2}{8\pi G}\left(\frac{a(t_0)}{a(t)}\right)^4 \Omega_{r,0} \\ \rho_{vac}(t) &= \frac{3H^2}{8\pi G}\Omega_{vac,0}\end{aligned} \tag{5.2.4}$$

Defining

$$x = \frac{a(t)}{a(t_0)} \tag{5.2.5}$$

so that x varies from 0 to 1 as the universe evolves from the big bang to the present,Friedmann equation can be rewritten as

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2\left(\frac{\Omega_{m,0}}{x^3} + \frac{\Omega_{rad,0}}{x^4} + \Omega_{vac}\right) - \frac{kc^2}{a^2} \tag{5.2.6}$$

It is convenient to rewrite the curvature term in the same form as the other terms, by defining

$$\Omega_{k,0} = -\frac{kc^2}{H_0^2 a^2} \tag{5.2.7}$$

So

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{H_0^2}{x^4}\left(\Omega_{m,0}x + \Omega_{rad,0} + \Omega_{vac,0}x^4 + \Omega_{k,0}x^2\right) \tag{5.2.8}$$

The present age of the universe can then be found by taking the square root of Eq. 5.2.8.

$$x\frac{dx}{dt} = \frac{H_0}{x^2}\sqrt{\Omega_{m,0}x + \Omega_{rad,0} + \Omega_{vac,0}x^4 + \Omega_{k,0}x^2} \tag{5.2.9}$$

This equation can be rearranged as

$$dt = \frac{1}{H_0}\frac{xdx}{\sqrt{\Omega_{m,0}x + \Omega_{rad,0} + \Omega_{vac,0}x^4 + \Omega_{k,0}x^2}} \tag{5.2.10}$$

which can be integrated over the range of x from the big bang to the present to give

$$t_0 = \frac{1}{H_0}\int_0^1 \frac{xdx}{\sqrt{\Omega_{m,0}x + \Omega_{rad,0} + \Omega_{vac,0}x^4 + \Omega_{k,0}x^2}} \tag{5.2.11}$$

The above form is probably the easiest to integrate, but for some purposes it is useful to rewrite it by changing variables of integration to z, where

$$1 + z = \frac{a(t_0)}{a(t)} = \frac{1}{x} \tag{5.2.12}$$

The integral then becomes

$$t_0 = \frac{1}{H_0}\int_0^\infty \frac{dz}{(1+z)\sqrt{\Omega_{m,0}(1+z)^3 + \Omega_{rad,0}(1+z)^4 + \Omega_{vac,0} + \Omega_{k,0}(1+z)^2}} \tag{5.2.13}$$

In this form one could also find the "look-back time" to any particular redshift z by stopping the integration at that point.

$$t_L = \frac{1}{H_0}\int_0^z \frac{dz^{'}}{(1+z^{'})\sqrt{\Omega_{m,0}(1+z^{'})^3 + \Omega_{rad,0}(1+z^{'})^4 + \Omega_{vac,0} + \Omega_{k,0}(1+z^{'})^2}} \tag{5.2.14}$$

5.3 Age of the universe constraining Hubble and density parameters of the universe

According to the dust model cosmology, the universe is filled with matter whose equation of state is given as

$$\frac{\ddot{a}}{a} = \frac{-4\pi G}{3}\rho(1+3w) \tag{5.3.1}$$

Accelerating universe needs $\ddot{a} > 0$, so that $w < -1/3$ in the Friedmann equation. This implies that the universe has some sort of negative energy which is considerably different from matter filled universe with positive energy.

Another problem of the dust model is the age of the universe, where in the FRW universe its solution is given by

$$a = \left(\frac{t}{t_0}\right)^{\frac{2}{3}}. \tag{5.3.2}$$

By setting $a_0 = 1$ at present, and using the Hubble parameter equation

$$H = \frac{\dot{a}}{a} = \frac{2}{3t}, \tag{5.3.3}$$

the age of the universe is estimated to

$$t_0 = \frac{2}{3} H_0^{-1} = 6.51 h^{-1} \text{ billion years} \tag{5.3.4}$$

Observational data from WMAP7 gives h = 0.702 [9], thus the age of the universe is estimated to be 9.27 Gyr. Carretta et al. [40] estimated the age of globular cluster in the Milky Way galaxy to be 12.9 $\pm$ 2.9 Gyr, whereas Jimenez et al. [41] found the value 13.5 $\pm$ 2 Gyr. We see that the age of globular clusters are larger than 11 Gyr. Therefore the age of the universe estimated by Eq. 5.3.4 is inconsistent with the age of globular clusters mentioned above. As stated before,the age of the universe in lambda cold dark matter model is given by equation 5.2.13 and 5.2.14. The general case of the integrals in Eqs. 5.2.13 and 5.2.14 can be computed only by numerical integration, but various special cases can be carried out analytically. The case of a flat universe composed of non-relativistic matter and vacuum energy (i.e., $\Omega_{rad} = \Omega_k = 0, \Omega_m + \Omega_{vac} = 1$) can also be integrated analytically, yielding

$$t_L = \frac{1}{H_0} \int_0^z \frac{dz'}{(1+z')\sqrt{\Omega_{m,0}(1+z')^3 + \Omega_{vac,0}}} \tag{5.3.5}$$

Substituting $\Omega_{m,0} = 1 - \Omega_{\Lambda,0}$ and $x = (1+z')^{-\frac{3}{2}}$ reduces equation 5.3.5 to

$$t_L = \frac{2}{3\Omega_{vac,0}^{\frac{1}{2}} H_0} \int_{(1+z)^{-\frac{3}{2}}}^1 \frac{dx}{\sqrt{\frac{1-\Omega_{\Lambda,0}}{\Omega_{\Lambda,0}} + x^2}} \tag{5.3.6}$$

The indefinite integral

$$\int \frac{dx}{\sqrt{a^2 + x^2}} = \ln(x + \sqrt{a^2 + x^2}) + C \tag{5.3.7}$$

can be found in integral tables or evaluated via the trigonometric substitution $x = a\tan(u)$. Thus

$$t_L = \frac{2}{3\Omega_{vac,0}^{\frac{1}{2}} H_0} \ln\left[\frac{1+\Omega_{\Lambda,0}^{\frac{-1}{2}}}{(1+z)^{\frac{-3}{2}} + \sqrt{(1+z)^{-3} + \frac{1-\Omega_{\Lambda,0}}{\Omega_{\Lambda,0}}}}\right] \tag{5.3.8}$$

In the limit $z \to \infty$, t_L becomes the present age of the universe t_0:

$$t_0 = \frac{2}{3\Omega_{vac,0}^{\frac{1}{2}} H_0} \ln\left(\frac{1+\Omega_{\Lambda,0}^{\frac{1}{2}}}{(1-\Omega_{\Lambda,0})^{\frac{1}{2}}}\right) \tag{5.3.9}$$

The $\Omega_m - \Omega_\Lambda$ contour plots of eq. 5.3.9 for the scaled Hubble parameter: $h = 55,\ 60,\ 65,\ 70,\ 75$ are shown as in Fig. 5.1. In the plots, the right hand diagonal line represents the total density of

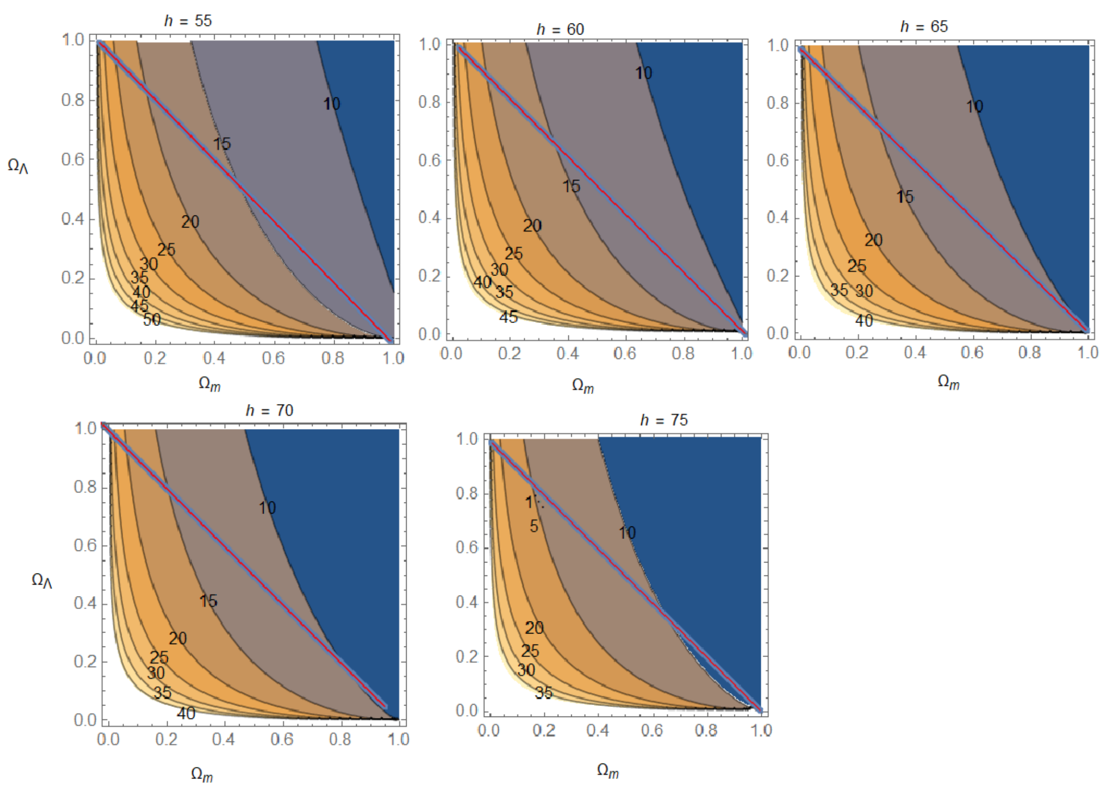

Figure 5.1: Age, densities and Hubble parameter constraints of the universe

the universe ($\Omega_{rad} = \Omega_k = 0, \Omega_m + \Omega_\Lambda = 1$). To fit this line, as we observe from the contours relatively low h prefers matter dominant universe while higher h prefers dark energy dominant universe. Comparing to the present observational constrain of cosmic age $13.7 \leq t_0 \leq 13.9$ Gyrs by WMAP 7-year data [9], the numerical result of our work fits to $0.265 \leq \Omega_{m0} \leq 0.281$. This means that dark energy contributed in our universe amounts to about 72% of the cosmic components of the universe.

5.4 Estimating mass of the universe in the flat ΛCDM model

Using the constant critical density ρ_c of the universe and the general relativistic volume element with spherical symmetry, we can estimate the mass of the universe given by:

$$m = \int \rho_c \sqrt{-g}\, r^2 \sin^2\theta dr d\theta d\phi, \tag{5.4.1}$$

where g is the determinant of the FLRW metric given as:

$$g = -a(t)^6 r^4 \sin^2\theta \tag{5.4.2}$$

Now integrating eq. 5.4.1, with appropriate dimensions restored from Hubble flow and the metric element equation, we obtain

$$m = \frac{4\pi\rho_c {H_0}^2 a^3}{15\, c^2} r^5 \tag{5.4.3}$$

Here r is the physical distance given by eq. 4.7.10. As $z \to \infty$, $r \to \frac{2ca}{H_0}$. Using this approximation, the critical density given by eq. 5.2.2 and the present value of $a = 1$, the mass of the universe is estimated to be given as:

$$m = \frac{32\, c^3}{5G\, H_0} \sim 10^{54}\text{kg};\ \text{h = 72} \tag{5.4.4}$$

Using the current observational data by WMAP where h = 72, the mass of the universe is

$$m \sim 10^{54}\text{kg}. \tag{5.4.5}$$

Our result well agrees with the one obtained from dimensional analysis [42] within a difference of the order of factor 10 pertaining to boundary conditions setup.

Chapter 6

Summary and Conclusion

In this thesis we were using the equation of the freidmann and observations from Wilkinson Microwave Anisotropy Probe(WMAP) to constraining the present density parameters of the universe. One of the key features of this work is that we used different value of H(z) with age of the globular clusters from Wilkinson Microwave Anisotropy Probe (WMAP)data to constraining age of the universe and density parameters of the universe. General theory of relativity and principle of cosmology are used to derive the appropriate Friedmann equations. The resulting Freidmann equations and boundary conditions are used to develop equations of cosmological distances and equation for calculate age of the universe. At larger values of time the matter density becomes less and the dark energy becomes dominant. The domination of dark energy causing an accelerating expansion. Because at the current time both the matter density and dark energy densities are playing the largest role in the evolution and acceleration of the universe.The more matter density of the universe is the younger than the more dark energy density of the universe.This implies that the present universe is dominated by dark energy density. The solution of friedmann also shows this result. The density of the universe and physical distance of the universe are used to estimate the mass of the universe to 10^{53}kg in general relativity.

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