A Fitted Mesh Numerical Method for Solving Singularly Perturbed Burger-Fisher Equation

COLLEGE OF NATURAL SCIENCES
DEPARTMENT OF MATHEMATICS

By

Nayilot Ejara

Under the supervision of

Prof. Gemechis File Duressa

Fasika Wondimu Gelu (MSc)

A Thesis Submitted to the Department of Mathematics, Jimma University in Partial Fulfillment for the Requirements of the Degree of Master of Science in Mathematics

(Numerical Analysis)

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Declaration

I undersigned declare that this thesis entitled ”A fitted mesh numerical method for solving singularly perturbed Burger-Fisher equation ” is my own original work and it has not been submitted for the award of any academic degree or the like in any other institution or university and that all the sources I have used or quoted have been indicated and acknowledged as complete references.

Researcher

Name: ____________________ Sign: ____________________ Date: ____________________

The work has been done under the supervision and approval of the advisor

Supervisor

Name: ____________________ Sign: ____________________ Date: ____________________

Co-supervisor

Name: ____________________ Sign: ____________________ Date: ____________________

Acknowledgment

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Abstract

In this thesis, we presented a fitted mesh numerical method for solving singularly perturbed Burger-Fisher equation. Since the problem is nonlinear, we apply quasi-linearizion technique on the non-linear part of the equation. Then, the resulting linearized problem is discretized using an implicit second-order finite difference approximation in the time direction on uniform mesh.The numerical scheme formulated using both forward and backward finite difference methods are applied in the space direction on a piecewise uniform Shishkin mesh.The error analysis has been established for the method. As a perturbation parameter goes to small values, a boundary layer is produced in the neighborhood of left lateral surface. Applicability of the proposed method is demonstrated by numerical experiments.

Chapter 1

Introduction

1.1 Background of the Study

Numerical analysis is a branch of mathematics concerned with theoretical foundations of numerical algorithms for the solution of problems arising in science and engineering. In real life, we often encounter many problems which are described by parameter dependent differential equations. The behavior of the solutions of these types of differential equations depends on the magnitude of the parameters. Nonlinear partial differential equations are important in most fields of science and engineering. One of the nonlinear partial differential equations is Burger-Fisher equation. Burgers-Fisher equation is a very important in fluid dynamic model and the study of this model has been considered by many authors both for conceptual understanding of physical flows and testing various numerical methods. Burgers- Fisher equation is a highly nonlinear equation because it is a combination of reaction, convection and diffusion mechanisms, this equation is called Burgers-Fisher because it has the properties of convective phenomenon from Burgers equation and having diffusion transport as well as reactions kind of characteristics from Fisher equation.

Nonlinear partial differential equations are important in most fields of science and engineering. One of the nonlinear partial differential equations is Burger-Fisher equation. To solve the Burger-Fisher equation various classical numerical techniques were applied by different researchers, for example, spectral collocation method by Javidi (2006), a compact finite difference method by Sari (2010), collocation of cubic B-splines by Mittal (2015), direct discontinuous Galerkin method by

Wei (2012), cubic B-spline quasi-interpolation by Zhu (2010), etc.

There exists a class of nonlinear partial differential equations in which the coefficient of the highest order derivative term is very small. A member of this class is termed as singularly perturbed nonlinear partial differential equation and the small coefficient specifying the problem is identified as singular perturbation parameter $\varepsilon(0 < \varepsilon << 1)$.

Burger-Fisher equation is singularly perturbation problem by multiplying the perturbation parameter $\varepsilon(0 < \varepsilon << 1)$ in its highest spatial derivative. Due to the presence of boundary layer(s), the above presented methods are in question and known to be inadequate to approximate the exact solution. Therefore, the main purpose of this study is to formulate fitted mesh numerical method for singularly perturbed Burger-Fisher equation.

1.2 Statement of the Problem

Kaushik (2007) and Kaushik (2008) presented the initial boundary value problem for the numerical solution of singularly perturbed Burger-Huxley equation. Hybrid finite difference methods for solving singularly perturbed modified Burgers and Burgers-Huxley equations was designed (Kadalbajoo, 2010). Numerical study for the singularly perturbed generalized Burgers-Huxley equation using a three-step Taylor-Galerkin finite element method was developed (Kumar, 2011). Higher order numerical scheme for singularly perturbed Burger-Huxley equation was developed (Jiwari, 2011). A singular perturbation approach to solve Burgers-Huxley equation via monotone finite difference scheme on layer-adaptive mesh was developed (Gupta, 2011). Kamboj (2013) and Liu (2020) presented the numerical solution for singularly perturbed Burger-Huxley equation. All the aforementioned scholars proposed different numerical methods for singularly perturbed Burgers-Huxley equation. The main purpose of this study is to develop fitted mesh numerical method for solving singularly perturbed Burger-Fisher equation.

As a result, this study attempted to answer the following questions:

- How does the fitted mesh numerical method be described for solving singularly perturbed Burger-Fisher equation?

- To establish error analysis of the present schemes.

- How to check the accuracy of the proposed method using numerical example?

1.3 Objectives of the Study

1.3.1 General Objective

The general objective of this study is to formulate Fitted Mesh Numerical Method for Solving Singularly Perturbed Burger-Fisher Equation.

1.3.2 Specific Objectives

The specific objectives of the present study are:

- To describe forward and backward finite difference methods for solving singularly perturbed Burger-Fisher equation.

- To establish error analysis of the present schemes.

- To check the computational accuracy of the proposed methods using numerical experiments.

1.4 Significance of the Study

The outcomes of this study may have the following importance:

- Provide some background information for other researchers who work on this area.

- Help the graduate students to acquire research skills and scientific procedures.

- To provide the numerical methods in solving singularly perturbed Burger-Fisher equation.

1.5 Delimitation of the Study

This study is delimited to fitted mesh numerical method for solving singularly perturbed Burger-Fisher equation (using forward and backward finite difference methods) of the form:

$$L_\varepsilon u = u_t + \alpha u u_x - \varepsilon u_{xx} - \beta(1-u)u = 0, \quad (x,t) \in D = \Omega_x \times \Omega_t = (0,1) \times (0,T], \tag{1.1}$$

subject to the initial-boundary conditions

$$
\begin{aligned}
&u(x,0) = \phi(x), \;\; x \in \Omega \equiv (0,1), \\
&\begin{cases} u(0,t) = f(t), \quad t \in (0,T], \\ u(1,t) = g(t), \quad t \in (0,T], \end{cases}
\end{aligned} \tag{1.2}
$$

where $\alpha,\ \beta \geq 0$ are small parameters and $0 < \varepsilon \ll 1$ is the diffusion coefficient.

Chapter 2

Review of Related Literatures

2.1 Singular Perturbation Theory

Lundwing Prandtl was the first introduce the concept layer in 1904 at the Third International Congruence of Mathematics in Heidelberg Germany. His hypothesis was in the setting of fluid dynamics, fluid adjacent to the boundary sticks to the edge in a thin boundary layer due friction but this friction has no effect to the flow on the interior. The term singular perturbation appears to have been first coined (Friedrichs and Wasow, 1946). Wasow continued to the contribute to the area of asymptotic methods over many years and his book " Asymptotic expansion for ordinary differential equation " Vasil'eva (1981), attracted much interest in the area of singular perturbed boundary value problems. In Russia, mainly at Moscow State University, research activity on singular perturbations for ordinary differential equations, originated and developed by Tikhonov (1952) and his students, especially Vasil'eva (1981) continues to be vigorously pursued even today. A brief survey for the historical development of singular perturbation problems is covered in the recent book by (O'Malley, 1991) and (Roos et al., 2008). More precisely, a perturbation problem is problem that contains a small parameter ε, called perturbation parameter. If the solution of the problem can be approximated by setting the value of the perturbation parameter equal to zero, then the problem is called regular perturbation problem, otherwise it is called singular perturbation problem. That is, if it is impossible to approximate the solution by asymptotic expansion as the perturbation parameter tends to zero, then the problem is called singular. Some numerical methods for solving singularly perturbed problems have been studied extensively in the literature. Singularly perturbed differential equations are characterized by the presence of a small parameter multiplying the highest-order derivatives. Such problems arise in many areas of applied mathematics and engineering. Among these are the Navier-Stokes equations of fluid flow at high Reynolds number, mathematical models of liquid crystal materials and chemical reactions, control theory, reaction-diffusion processes, quantum mechanics, and electrical networks. An overview of some existence and uniqueness results and applications of singularly perturbed problems can be found (Roos et al., 2008).

Very few researchers have proposed some numerical schemes to solve generalized Burger-Fisher equation for $\varepsilon = 1$. Javidi (2011) gave numerical solution of generalized Burger-Fisher equation for the case $\varepsilon = 1$ using spectral collocation method. Sari et al. (2010) proposed a compact finite difference scheme for solving generalized Burger-Fisher equation. Zhu and Kang (2010) used cubic B-splines quasi-interpolation to solve Burger-Fisher equation. Mittal et al. (2015) proposed a numerical scheme based on cubic splines for solving generalized Burger-Fisher and Burger-Huxley problems. Wei at al. (2012) proposed direct discontinuous Galerkin method to solve singularly perturbed generalized Burger-Fisher equation for $\varepsilon << 1$. Since the problem under consideration is singularly perturbed in nature, traditional finite element methods cannot be relied to capture sharp boundary layers. Therefore, we need some special treatment to capture these sharp boundary layers as $\varepsilon \to 0$. Motivation of the present work is to propose an accurate numerical method for capturing the boundary layers occurring in solution of singularly perturbed Burger-Fisher equation.

2.2 Quasilinearzation Technique

The nonlinear partial differential equation is linearized around a nominal solution of the nonlinear partial differential equation which satisfies the boundary conditions. Suppose $u^{(k)}(x)$ is the nominal solution of the nonlinear partial differential equation. The quasilinearization process yields a sequence $< u^{(k)} >$ of linear equations. Bellman and Kalaba (1965) developed the quasilinearzation technique which is used to reduce the given nonlinear boundary value problem into the corresponding sequence of linear boundary value problem. The quasilinearzation technique of reducing nonlinear boundary value problem into a sequence of linear boundary value problem involves some steps. First, we linearize the semi-linear ordinary differential equation around a nominal solution, which satisfies the specified boundary conditions. Second, we solve a sequence of boundary value problems in which the solution of the linear boundary value problem satisfies the specified boundary conditions and is taken as the nominal profile for the linear boundary value problem. Quasilinearzation technique is used to linearize the original semi-linear singular perturbation problem into a sequence of linear singular perturbation problems.

2.3 Numerical versus Analytical Methods

Suppose we have a differential equation and we want to find a solution of the differential equation. The best is when we can find out the exact solution using calculus, trigonometry and other techniques. The techniques used for calculating the exact solution are known as analytic methods because we used the analysis to figure it out. Analytical solution is continuous. The exact solution is also referred to as a closed form solution or analytical solution. But this tends to work only for simple differential equations with simple coefficients, but for higher order or non-linear differential equations with complex coefficient, it becomes very difficult to find exact solution. Therefore, we need numerical methods for solving the equations. Numerical methods are commonly used for solving mathematical problems that are formulated in science and engineering where it is difficult or even impossible to obtain exact solutions. Numerical solution is discrete. Numerical methods, on the other hand, can give an approximate solution to any equation.

2.4 Finite Difference Methods

Most problems cannot be solved analytically, henceforth finding good approximation solutions using numerical methods will be very useful. From different classification of numerical methods such as finite difference method, spectral method, finite element method, finite volume method, spline method, finite difference method seems to be the simplest approach for the numerical solution of boundary value problems (Roos et al., 2008). Finite difference methods are widely used by the scientific community and it is always a convenient choice for solving boundary value problems because of their simplicity. In finite difference methods, derivatives appearing in the differential equations are replaced by finite difference approximations obtained by Taylor series expansions at the grid points. This gives a large algebraic system of equations to be solved by Thomas Algorithm in place of the differential equation to give the solution value at the grid points and hence the solution is obtained at grid points. Some of the finite difference methods include forward approximation, backward approximation, central difference approximation.

Chapter 3

Research Methodology

3.1 Study Area and Period

This study is conducted at Jimma University department of Mathematics from September 2020 to August 2021.

3.2 Study Design

This study employed both documentary review design and experimental design.

3.3 Source of Informations

The relevant source of information for this study are books and published articles.

3.4 Mathematical Procedures

This study was conducted based on the following mathematical procedures

1. Defining the problem.
2. Applying the quasilinearization technique to linearize the equation.
3. Discretizing the problem with implicit second-order in time & fitted mesh methods in space.
4. Establishing error analysis for the developed schemes.
5. Writing MATLAB code for the developed schemes.
6. Validation of the schemes using numerical computations.

Chapter 4

Formulation of the Method, Error Analysis and Numerical Computations

4.1 Formulation of the Method

In this chapter, we deal first with the quasilinearization technique and then time semi-discretization was made and finally we derive the numerical scheme. In this study, we consider the following singularly perturbed nonlinear Burger-Fisher equation with the initial-boundary conditions

$$L_\varepsilon u(x,t) \equiv u_t - \varepsilon u_{xx} + \alpha u u_x - \beta(1-u)u = 0, \quad (x,t) \in D = \Omega_x \times \Omega_t = (0,1) \times (0,T], \tag{4.1}$$

$$\begin{cases} u(x,0) = \varphi(x), & 0 \le x \le 1, \\ u(0,t) = f(t), \quad u(1,t) = g(t), & 0 \le t \le T, \end{cases} \tag{4.2}$$

where ε is the perturbation parameter and the initial-boundary functions $\varphi(x)$, $f(t)$, $g(t)$ are smooth and bounded.

Re-writing Eq. (4.1) we obtain:

$$L_\varepsilon u(x,t) \equiv u_t - \varepsilon u_{xx} = F(x,t,u,u_x), \quad (x,t) \in D = (0,1) \times (0,T], \tag{4.3}$$

where $F(x,t,u,u_x) = -\alpha u u_x + \beta(1-u)u$ is the nonlinear function of x, t, u and u_x.

4.1.1 Quasilinearization Technique

The quasilinearization technique is the generalized Newton-Raphson-Kantorovich technique for the nonlinear differential equation (Bellman and Kalaba, 1965). On expanding the nonlinear term $F(x,t,u,u_x)$ using Taylor series up to first order, we have

$$\begin{aligned} F\left(x,t,u^{(i+1)},u_x^{(i+1)}\right) \cong & F\left(x,t,u^{(i)},u_x^{(i)}\right) + \left(u^{(i+1)} - u^{(i)}\right) F_u \mid_{(x,t,u^{(i)},u_x^{(i)})} \\ & + \left(u_x^{(i+1)} - u_x^{(i)}\right) F_{u_x} \mid_{(x,t,u^{(i)},u_x^{(i)})} + \cdots \end{aligned} \tag{4.4}$$

Substituting Eq. (4.4) into (4.3), we have

$$\begin{aligned} L_\varepsilon u^{(i+1)} \equiv & u_t^{(i+1)} - \varepsilon u_{xx}^{(i+1)} = F\left(x,t,u^{(i)},u_x^{(i)}\right) + \left(u^{(i+1)} - u^{(i)}\right) F_u \mid_{(x,t,u^{(i)},u_x^{(i)})} \\ & + \left(u_x^{(i+1)} - u_x^{(i)}\right) F_{u_x} \mid_{(x,t,u^{(i)},u_x^{(i)})} + \cdots . \end{aligned} \tag{4.5}$$

Simplifying Eq. (4.5), we have the following linearized differential equation of the form

$$L_\varepsilon u^{(i+1)} \equiv \left(u_t^{(i+1)} - \varepsilon u_{xx}^{(i+1)} + a^{(i)} u_x^{(i+1)} + b^{(i)} u^{(i+1)}\right)(x,t) = f^{(i)}(x,t), \tag{4.6}$$

with the following initial-boundary conditions, respectively

$$\begin{cases} u^{(i+1)}(x,0) = \varphi(x), & 0 \le x \le 1, \\ u^{(i+1)}(0,t) = f(t), \quad u^{(i+1)}(1,t) = g(t), & 0 \le t \le T, \end{cases} \tag{4.7}$$

where $f^{(i)}(x,t) = F\left(x,t,u^{(i)},u_x^{(i)}\right) - u^{(i)} F_u \mid_{(x,t,u^{(i)},u_x^{(i)})} - u_x^{(i)} F_{u_x} \mid_{(x,t,u^{(i)},u_x^{(i)})}$, $a^{(i)}(x,t) = -F_{u_x} \mid_{(x,t,u^{(i)},u_x^{(i)})}$, $b^{(i)}(x,t) = -F_u|_{(x,t,u^{(i)},u_x^{(i)})}$. Hence Eq. (4.6)-(4.7) is linear for each $u^{(i+1)}(x,t)$. Starting with initial approximation $u^{(0)}(x,t) = \varphi(x)\exp(-ct)$ which satisfies the given initial condition and the constant c is the combinations of the coefficients of boundary conditions, we solve problem (4.6)-(4.7) for $i = 0$, that is, taking the first iteration. Therefore, Eq. (4.6)-(4.7) becomes

$$L_\varepsilon u^{(1)} \equiv \left(u_t^{(1)} - \varepsilon u_{xx}^{(1)} + a^{(0)} u_x^{(1)} + b^{(0)} u^{(1)}\right)(x,t) = f^{(0)}(x,t), \tag{4.8}$$

with the following the initial-boundary conditions

$$\begin{cases} u^{(1)}(x,0) = \varphi(x), & 0 \le x \le 1, \\ u^{(1)}(0,t) = f(t), \quad u^{(1)}(1,t) = g(t), & 0 \le t \le T, \end{cases} \tag{4.9}$$

where the coefficients and source functions are give as follows

$$\begin{aligned} &a^{(0)}(x,t) = -F_{u_x}\big|_{(x,t,u^{(0)},u_x^{(0)})}, \quad b^{(0)}(x,t) = -F_u\big|_{(x,t,u^{(0)},u_x^{(0)})}, \\ &f^{(0)}(x,t) = F\left(x,t,u^{(0)},u_x^{(0)}\right) - u^{(0)}F_u\big|_{(x,t,u^{(0)},u_x^{(0)})} - u_x^{(0)}F_{u_x}\big|_{(x,t,u^{(0)},u_x^{(0)})}. \end{aligned}$$

4.1.2 Time Semi-discretization

We divide the time interval $[0,T]$ with uniform step length Δt. Hence, the interval $[0,T]$ is partitioned into N equal sub-intervals with each nodal points satisfying $0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T$. The time nodal points are generated by $t_n = n\Delta t$, $\Delta t = \frac{T}{N}$, $n = 0, \cdots, N$, where N denotes the number of mesh intervals. Re-writing Eq. (4.8) at the nodal point $(x, t_{n+\frac{1}{2}})$, we have

$$L_\varepsilon^N u^{n+\frac{1}{2}}(x) \equiv u_t^{n+\frac{1}{2}}(x) - \varepsilon u_{xx}^{n+\frac{1}{2}}(x) + p^{n+\frac{1}{2}}(x) u_x^{n+\frac{1}{2}}(x) + q^{n+\frac{1}{2}}(x) u^{n+\frac{1}{2}}(x) = r^{n+\frac{1}{2}}(x), \tag{4.10}$$

where $p^{n+\frac{1}{2}}(x) = a^{(0)}(x, t_{n+\frac{1}{2}})$, $q^{n+\frac{1}{2}}(x) = b^{(0)}(x, t_{n+\frac{1}{2}})$, $r^{n+\frac{1}{2}}(x) = f^{(0)}(x, t_{n+\frac{1}{2}})$ and $u = u^{(1)}$. From Taylor series expansion, we have

$$u^{n+1}(x) = u^{n+\frac{1}{2}}(x) + \frac{\Delta t}{2} u_t^{n+\frac{1}{2}} + \frac{\Delta t^2}{8} u_{tt}^{n+\frac{1}{2}} + O(\Delta t^3) \tag{4.11}$$

$$u^{n}(x) = u^{n+\frac{1}{2}}(x) - \frac{\Delta t}{2} u_t^{n+\frac{1}{2}} + \frac{\Delta t^2}{8} u_{tt}^{n+\frac{1}{2}} + O(\Delta t^3) \tag{4.12}$$

From Eqs. (4.11) and (4.12), we obtain

$$u_t^{n+\frac{1}{2}}(x) = \frac{u^{n+1}(x) - u^n(x)}{\Delta t} + TE, \tag{4.13}$$

where $TE = -\frac{\Delta t^2}{24} u_{ttt}^{n+\frac{1}{2}}$. This truncation error shows that the time semi-discretization is bounded and the global error estimate is given by

$$\|E\|_\infty \leq C\Delta t^2, \tag{4.14}$$

for the arbitrary constant $C = \frac{1}{24}|u_{ttt}^{n+\frac{1}{2}}|$. We conclude that time semi-discretization is second-order uniformly convergent. Taking the averages of all the terms in Eq. (4.10) except the time derivative term, we have

$$-\varepsilon u_{xx}^{n+\frac{1}{2}}(x) + p^{n+\frac{1}{2}}(x) u_x^{n+\frac{1}{2}}(x) + q^{n+\frac{1}{2}}(x) u^{n+\frac{1}{2}}(x) - r^{n+\frac{1}{2}}(x) = \frac{1}{2}\left[L_\varepsilon^{n+1} + L_\varepsilon^n - (r^{n+1}(x) + r^n(x))\right] \tag{4.15}$$

where $L_\varepsilon^{n+1} = -\varepsilon u_{xx}^{n+1}(x) + p^{n+1}(x) u_x^{n+1}(x) + q^{n+1}(x) u^{n+1}(x)$ and $L_\varepsilon^n = -\varepsilon u_{xx}^n(x) + p^n(x) u_x^n(x) + q^n(x) u^n(x)$. Putting Eqs. (4.13) and (4.15) into Eq. (4.10) and rearranging gives

$$L_\varepsilon^{\Delta t} u^{n+1}(x) \equiv -\varepsilon u_{xx}^{n+1}(x) + p^{n+1}(x) u_x^{n+1}(x) + c^{n+1}(x) u^{n+1}(x) = z^{n+1}(x), \tag{4.16}$$

subject to the initial-boundary conditions, respectively

$$\begin{cases} u(x,0) = \varphi(x), & 0 \leq x \leq 1, \\ u(0,t_{n+1}) = f(t), \quad u(1,t_{n+1}) = g(t), & 0 \leq t \leq T, \end{cases} \tag{4.17}$$

where $c^{n+1}(x) = q^{n+1}(x) + \frac{2}{\Delta t}$, $z^{n+1}(x) = \varepsilon u_{xx}^n(x) - p^n(x) u_x^n(x) - (q^n(x) - \frac{2}{\Delta t}) u^n(x) + r^{n+1}(x) + r^n(x)$. We assume that the functions $p^{n+1}(x)$ and $c^{n+1}(x)$ are sufficiently smooth functions satisfying the conditions $p^{n+1}(x) \leq \alpha < 0$, $c^{n+1}(x) \geq c_0 > 0$. The differential operator in Eq. (4.16) satisfies the following continuous maximum principle.

Lemma 4.1 *Assume $\Phi^{n+1}(x) \in C^2(\bar{D})$ be a smooth function such that $\Phi^{n+1}(0) \geq 0$, $\Phi^{n+1}(1) \geq 0$. Then, $L_\varepsilon^{\Delta t} \Phi^{n+1}(x) \leq 0$, $\forall x$ implies that $\Phi^{n+1}(x) \geq 0$, $\forall x$.*

Proof: Let x^* be such that $\Phi^{n+1}(x^*) = \min_{x\in\bar{\Omega}} \Phi^{n+1}(x)$ and assume that $\Phi^{n+1}(x^*) < 0$. It is clear that $x^* \notin \{0,1\}$. Therefore, we have $(\Phi^{n+1})_x = 0$ and $(\Phi^{n+1})_{xx} \geq 0$. Then,

$$L_\varepsilon^{\Delta t}\Phi^{n+1}(x^*) = -\varepsilon\Phi_{xx}^{n+1}(x^*) + p^{n+1}(x^*)\Phi_x^{n+1}(x^*) + c^{n+1}(x^*)\Phi^{n+1}(x^*) > 0,$$

which contradicts. It follows that $\Phi^{n+1}(x^*) \geq 0$ and thus $\Phi^{n+1}(x) \geq 0, \ \forall x \in \bar{D}$. □

Bounds for the solution of the semi-discretized problem in Eq. (4.7) and its derivatives are established in the following theorem.

Theorem 4.2 *The bounds of the solution $u^{n+1}(x)$ and its derivatives satisfy*

$$\left\|\frac{\partial^j u^{n+1}(x)}{\partial x^j}\right\|_{\bar{D}} \leq C\left(1+\varepsilon^{-j}\exp(\frac{-\alpha x}{\varepsilon})\right), \quad j = 1,2,3,$$

where the constant C is independent of ε.

Proof: For the proof see (Stynes, 1988). □

We decompose the solution as the sum $u^{n+1}(x) = v^{n+1}(x) + w^{n+1}(x)$, where $v^{n+1}(x)$ is the solution of the regular component and $w^{n+1}(x)$ is the singular component solution.

Theorem 4.3 *The bound of the regular component $v^{n+1}(x)$ and its derivatives satisfies*

$$|v_{(j)}^{n+1}(x)|_{\bar{D}} \leq C\big(1+\varepsilon^{2-j}\big), \quad j = 0,1,2,3,$$

and the bound of the singular component satisfies and its derivatives are given, respectively

$$|w^{n+1}(x)|_{\bar{D}} \leq Ce^{-\alpha x/\varepsilon}, \quad \forall x \in \bar{D},$$
$$|w_{(j)}^{n+1}(x)|_{\bar{D}} \leq C\varepsilon^{-j}e^{-\alpha x/\varepsilon}, \quad j = 1,2,3,$$

where C is a constant independent of ε and mesh points.

Proof: The detailed and descriptive proof of this theorem is established (Cai, 2007). □

4.1.3 Spatial Discretization

To discretize in the spatial direction, we use Shishkin mesh which will condense large number of mesh points in the boundary layer region as $\varepsilon \to 0$. Then, our main target is to make fine mesh in the layer region and coarse everywhere else. Let M be a positive integer such that $M = 2^r$ with $r \geq 3$. On Shishkin mesh, the spatial interval $x \in [0,1]$ is partitioned into two sub-intervals $[0,\tau]$ and $[\tau,1]$ with uniform mesh elements of $\frac{M}{2}$. The transition parameter τ is defined as $\tau = \min\left\{\frac{1}{2}, 2\varepsilon \ln M\right\}$. The spatial mesh points is given by

$$x_m = \begin{cases} mh, & \text{if } m = 0,\cdots,M/2, \\ \tau + (m - \frac{M}{2})H, & \text{if } m = M/2+1,\cdots,M, \end{cases}$$

where the spatial mesh widths for $m = 0,1,2,\cdots,M$ is given by $h_m = x_{m+1} - x_m$. Here $h = \frac{2\tau}{M}$, for $m = 1,\cdots,M/2$ and $H = 2(1-\tau)/M$, for $m = M/2+1,\cdots,M$ are the spatial step size in $[0,\tau]$ and $[\tau,1]$, respectively. We use $\hat{h}_m = h_m + h_{m+1}$ for $m = 1,2,\cdots,M-1$. When $\tau = \frac{1}{2}$, the mesh is uniform and the analysis proceed in the classical way. Throughout the error analysis, we assume $\tau = 2\varepsilon \ln M$, in which case the mesh is non-uniform.

4.1.4 Scheme I

To develop the upwind finite difference schemes, we use the following finite difference operators

$$D^+ u_m^{n+1} = \frac{u_{m+1}^{n+1} - u_m^{n+1}}{h_{m+1}}, \quad D^- u_m^{n+1} = \frac{u_m^{n+1} - u_{m-1}^{n+1}}{h_m}$$

We fully discretize Eqs. (4.16)-(4.17) using the forward finite difference operator for first derivative for $m = 1,2,\cdots,M-1$ and $n = 0,1,\cdots,N$ as follows

$$\begin{aligned} L_\varepsilon^{M,\Delta t} u_m^{n+1} &\equiv \frac{-2\varepsilon}{\hat{h}_m}(D_x^+ u_m^{n+1} - D_x^- u_m^{n+1}) + p_m^{n+1} D_x^+ u_m^{n+1} + c_m^{n+1} u_m^{n+1} \\ &= \frac{2\varepsilon}{\hat{h}_m}(D_x^+ u_m^n - D_x^- u_m^n) - p_m^n D_x^+ u_m^n - (q_m^n - \frac{2}{\Delta t})u_m^n + r_m^{n+1} + r_m^n, \end{aligned} \tag{4.18}$$

with the following fully discrete intitial-boundary conditions

$$\begin{cases} u_m^0 = \varphi(x_m), & x_m \in [0,1], \\ u_0^{n+1} = f(t_{n+1}), \qquad u_M^{n+1} = g(t_{n+1}), & t_{n+1} \in [0,T], \end{cases} \tag{4.19}$$

where $p^{n+1}(x_m) = p_m^{n+1}$, $q^{n+1}(x_m) = q_m^{n+1}$, $r_m^{n+1} = r^{n+1}(x_m)$.
Now, using $D^+u_m^{n+1} = \frac{u_{m+1}^{n+1} - u_m^{n+1}}{h_{m+1}}$ for the first derivative term, we have

$$\begin{aligned} &\frac{-2\varepsilon}{\hat{h}_m}\left(\frac{u_{m+1}^{n+1} - u_m^{n+1}}{h_{m+1}} - \frac{u_m^{n+1} - u_{m-1}^{n+1}}{h_m}\right) + p_m^{n+1}\frac{u_{m+1}^{n+1} - u_m^{n+1}}{h_{m+1}} + c_m^{n+1}u_m^{n+1} \\ &= \frac{2\varepsilon}{\hat{h}_m}\left(\frac{u_{m+1}^{n} - u_m^{n}}{h_{m+1}} - \frac{u_m^{n} - u_{m-1}^{n}}{h_m}\right) - p_m^{n}\frac{u_{m+1}^{n} - u_m^{n}}{h_{m+1}} - (q_m^n - \frac{2}{\Delta t})u_m^n + r_m^{n+1} + r_m^n, \end{aligned} \tag{4.20}$$

Equivalently, Eq. (4.20) can be re-written as the three term recurrence relation of the form

$$\begin{aligned} &u_m^0 = \varphi(x_m), \quad m = 0,1,\cdots,M, \\ &E_m u_{m-1}^{n+1} + F_m u_m^{n+1} + G_m u_{m+1}^{n+1} = H_m^n, \quad m = 1,2,\cdots,M-1, \;\; n = 1,\cdots,N-1, \\ &u_0^{n+1} = f(t_{n+1}), \qquad u_M^{n+1} = g(t_{n+1}), \;\; t_{n+1} \in [0,T], \end{aligned} \tag{4.21}$$

where the coefficients are given by

$$\begin{aligned} E_m &= \frac{-2\varepsilon}{h_m\hat{h}_m}, \qquad F_m = \frac{2\varepsilon}{h_m h_{m+1}} - \frac{p_m^{n+1}}{h_{m+1}} + q_m^{n+1} + \frac{2}{\Delta t}, \qquad G_m = \frac{-2\varepsilon}{h_{m+1}\hat{h}_m} + \frac{p_m^{n+1}}{h_{m+1}}, \\ H_m^n &= r_m^{n+1} + r_m^n + \frac{2\varepsilon}{\hat{h}_m}\left(\frac{u_{m+1}^{n} - u_m^{n}}{h_{m+1}} - \frac{u_m^{n} - u_{m-1}^{n}}{h_m}\right) - p_m^{n}\frac{u_{m+1}^{n} - u_m^{n}}{h_{m+1}} - (q_m^n - \frac{2}{\Delta t})u_m^n. \end{aligned}$$

4.1.5 Scheme II

We fully discretize Eqs. (4.16)-(4.17) using the backward finite difference operator for first derivative for $m = 1, 2, \cdots, M-1$ and $n = 0, 1, \cdots, N$ as follows

$$\begin{aligned} L_{\varepsilon}^{M,\Delta t} u_m^{n+1} &\equiv \frac{-2\varepsilon}{\hat{h}_m}(D_x^+ u_m^{n+1} - D_x^- u_m^{n+1}) + p_m^{n+1} D_x^- u_m^{n+1} + c_m^{n+1} u_m^{n+1} \\ &= \frac{2\varepsilon}{\hat{h}_m}(D_x^+ u_m^n - D_x^- u_m^n) - p_m^n D_x^- u_m^n - (q_m^n - \frac{2}{\Delta t}) u_m^n + r_m^{n+1} + r_m^n, \end{aligned} \tag{4.22}$$

where $p^{n+1}(x_m) = p_m^{n+1},\ q^{n+1}(x_m) = q_m^{n+1},\ r_m^{n+1} = r^{n+1}(x_m)$. Now, using $D^- u_m^{n+1} = \frac{u_m^{n+1} - u_{m-1}^{n+1}}{h_m}$ for the first derivative term, we have

$$\begin{aligned} &\frac{-2\varepsilon}{\hat{h}_m}\left(\frac{u_{m+1}^{n+1} - u_m^{n+1}}{h_{m+1}} - \frac{u_m^{n+1} - u_{m-1}^{n+1}}{h_m}\right) + p_m^{n+1}\frac{u_m^{n+1} - u_{m-1}^{n+1}}{h_m} + c_m^{n+1} u_m^{n+1} \\ &= \frac{2\varepsilon}{\hat{h}_m}\left(\frac{u_{m+1}^n - u_m^n}{h_{m+1}} - \frac{u_m^n - u_{m-1}^n}{h_m}\right) - p_m^n \frac{u_m^n - u_{m-1}^n}{h_m} - (q_m^n - \frac{2}{\Delta t}) u_m^n + r_m^{n+1} + r_m^n, \end{aligned} \tag{4.23}$$

Similarly, Eq. (4.23) can be re-written as the three term recurrence relation of the form

$$\begin{aligned} &u_m^0 = \varphi(x_m), \quad m = 0, 1, \cdots, M, \\ &E_m u_{m-1}^{n+1} + F_m u_m^{n+1} + G_m u_{m+1}^{n+1} = H_m^n, \quad m = 1, 2, \cdots, M-1,\ \ n = 1, \cdots, N-1, \\ &u_0^{n+1} = f(t_{n+1}), \quad u_M^{n+1} = g(t_{n+1}), \quad t_{n+1} \in [0, T], \end{aligned} \tag{4.24}$$

where the coefficients are given by

$$\begin{aligned} E_m =& \frac{-2\varepsilon}{h_m \hat{h}_m} - \frac{p_m^{n+1}}{h_m}, \quad F_m = \frac{2\varepsilon}{h_m h_{m+1}} + \frac{p_m^{n+1}}{h_m} + q_m^{n+1} + \frac{2}{\Delta t}, \quad G_m = \frac{-2\varepsilon}{h_{m+1}\hat{h}_m}, \\ H_m^n =& r_m^{n+1} + r_m^n + \frac{2\varepsilon}{\hat{h}_m}\left(\frac{u_{m+1}^n - u_m^n}{h_{m+1}} - \frac{u_m^n - u_{m-1}^n}{h_m}\right) - p_m^n \frac{u_m^n - u_{m-1}^n}{h_m} - (q_m^n - \frac{2}{\Delta t}) u_m^n. \end{aligned}$$

4.2 Error Analysis

In this section, we establish error estimate for the discrete forward scheme by decomposing the numerical solution u_m^n. We have the following discrete maximum principle.

Lemma 4.4 *Assume that for any mesh function $Z(x_m,t_n)$ defined on $\bar{D}^{M,\Delta t}$ such that if $Z(x_0,t_n) \geq 0$, $Z(x_M,t_n) \geq 0$ and $L_\varepsilon^{M,\Delta t} Z(x_m,t_n) \geq 0,\ \forall (x_m,t_n) \in D$, then $Z(x_m,t_n) \geq 0$, $(x_m,t_n) \in \bar{D}$.*

The numerical solution has the following decomposition

$$u^{M,\Delta t}(x_m,t_n) = v^{M,\Delta t}(x_m,t_n) + w^{M,\Delta t}(x_m,t_n), \quad \forall (x_m,t_n) \in \bar{D},$$

where the regular component $v^{M,\Delta t}(x_m,t_n)$ and singular part $w^{M,\Delta t}(x_m,t_n)$ satisfies

$$\begin{aligned}
&L_\varepsilon^{M,\Delta t} v^{M,\Delta t}(x_m,t_n) = H(x_m,t_n), \quad \forall (x_m,t_n) \in D, \\
&v^{M,\Delta t}(x_m,t_n) = v(x_m,t_n), \\
&L_\varepsilon^{M,\Delta t} w^{M,\Delta t}(x_m,t_n) = 0, \quad \forall (x_m,t_n) \in D, \\
&w^{M,\Delta t}(x_m,t_n) = w(x_m,t_n).
\end{aligned}$$

Therefore, we have

$$(u^{M,\Delta t} - u)(x_m,t_n) = (v^{M,\Delta t} - v)(x_m,t_n) + (w^{M,\Delta t} - w)(x_m,t_n), \ \forall (x_m,t_n) \in \bar{D}.$$

Now, we estimate the error bound in the regular and singular components separately.

Theorem 4.5 *The error in the regular component $v^{M,\Delta t}(x_m,t_n)$ satisfies*

$$|(v^{M,\Delta t} - v)(x_m,t_n)| \leq C(M^{-1} + \Delta t^2), \quad m = 0,1,\cdots,M, \quad n\Delta t \leq T.$$

Proof: The estimate of local truncation error is obtained from the differential and difference equations as follows

$$\begin{aligned}
L_\varepsilon^{M,\Delta t}(v^{M,\Delta t} - v)(x_m,t_n) &= f - L_\varepsilon^{M,\Delta t} \\
&= (L_\varepsilon - L_\varepsilon^{M,\Delta t}) v(x_m,t_n) \\
&= \left[-\varepsilon\left(\frac{\partial^2}{\partial x^2} - \delta_x^2\right) + p_m^{n+1}\left(\frac{\partial}{\partial x} - D_x^+\right) - \left(\frac{\partial}{\partial t} - D_t^-\right)\right] v(x_m,t_n).
\end{aligned}$$

It follows that the truncation error associated with the regular component v of the solution $\bar{u}$ satisfies the following estimate

$$|L_\varepsilon^{M,\Delta t}(v^{M,\Delta t}-v)(x_m,t_n)| \leq \frac{\varepsilon}{3}(x_{m+1}-x_{m-1})\left\|\frac{\partial^3 v}{\partial x^3}\right\|_{\bar{D}} + \frac{p_m^{n+1}}{2}(x_{m+1}-x_m)\left\|\frac{\partial^2 v}{\partial x^2}\right\|_{\bar{D}} + \frac{\Delta t^2}{24}\left\|\frac{\partial^3 v}{\partial t^3}\right\|_{\bar{D}}.$$

Since $x_{m+1}-x_{m-1} \leq 2M^{-1}$, $x_{m+1}-x_m \leq M^{-1}$, $\Delta t = \frac{T}{N}$ and using the bounds on the derivatives of v given in Theorem 4.3, we get

$$\begin{aligned}
|L_\varepsilon^{M,\Delta t}(v^{M,\Delta t}-v)(x_m,t_n)| &\leq C(M^{-1}+\Delta t^2) \leq C(\varepsilon M^{-1}(1+\varepsilon^{-1})+M^{-1}+\Delta t^2),\\
&\leq C(M^{-1}(\varepsilon+1)+M^{-1}+\Delta t^2),\\
&\leq C(M^{-1}+\Delta t^2), \quad \text{since } \varepsilon << 1,
\end{aligned}$$

Application of the discrete maximum principle to the mesh function $(v^{M,\Delta t}-v)(x_m,t_n)$ yields to the estimate

$$|(v^{M,\Delta t}-v)(x_m,t_n)| \leq C(M^{-1}+\Delta t^2), \quad (x_m,t_n) \in D,$$

□

The following technical results will be used to prove ε-uniform convergence for singular component.

Lemma 4.6 *Let ψ be a smooth function defined on $[0,1]$. Then the following estimates for the truncation error hold true*

$$|L_\varepsilon^{M,N}\psi - L_\varepsilon\psi| \leq C\left[\varepsilon\int_{x_{m-1}}^{x_{m+1}}|\psi'''(x)|dx + \int_{x_{m-1}}^{x_m}|\psi''(x)|dx\right],$$

for $0<m<M$.

Lemma 4.7 *For $m=0,1,\cdots,M$, define the mesh function*

$$S_m = \prod_{j=1}^{m}\left(1+\frac{\alpha h_j}{\varepsilon}\right)^{-1}, \quad m=1,2,\cdots,M,$$

with the usual convection that $S_0 = 1$ for $m = 0$. Then, the following estimate hold

$$L_\varepsilon^{M,\Delta t} S_m \geq \frac{CS_m}{\max\{\varepsilon, h_{m+1}\}}, \quad \text{for } 1 \leq m \leq M-1.$$

Lemma 4.8 *For each m and $0 < \alpha < \frac{m}{2}$, the mesh function $S_m^n(\alpha)$ satisfies the following inequality*

$$\exp\left(\frac{-\alpha x_m}{\varepsilon}\right) \leq S_m^n(\alpha), \quad \text{for all } 0 \leq m \leq M,$$

and on Shishkin mesh, mesh function $S_m^n(\alpha)$ also satisfies the following inequality

$$S_m^n(\alpha) \leq CM^{-\alpha} \ln^2 M, \quad 1 \leq i \leq M-1.$$

Theorem 4.9 *Let $w^{M,\Delta t}$ be the numerical solution of the homogeneous problem and w be the bound in Theorem (4.3). The error estimate in the singular component $w^{M,\Delta t}(x_m,t_n)$ satisfies*

$$|(w^{M,\Delta t} - w)(x_m,t_n)| \leq C(M^{-1} \ln^2 M + \Delta t^2), \ m = 1, \cdots, M, \quad n\Delta t \leq T.$$

Proof: To prove this theorem, consider the error in outer region $[\sigma,1] \times (0,T]$. In this case, we consider the following mesh functions

$$\Psi^{\pm}(x_m,t_n) = CS_m^n(\alpha) \pm w^{M,\Delta t}(x_m,t_n), \ \forall (x_m,t_n) \in \Omega^{M,\Delta t},$$

where $C = |w(x_0,t_n)|$. We have $\Psi^{\pm}(x_0,t_n) = CS_0^n(\alpha) \pm w(x_0,t_n) = C \pm w(x_0,t_n) \geq 0$, $\Psi^{\pm}(x_m,t_0) = CS_m^0(\alpha) \geq 0$, $\Psi^{\pm}(x_M,t_n) = CS_M^n(\alpha) \geq 0$,
$L_\varepsilon^{M,\Delta t}\Psi^{\pm}(x_m,t_n) = CL_\varepsilon^{M,\Delta t} S_m^n(\alpha) \pm L_\varepsilon^{M,\Delta t} w(x_m,t_n) = CL_\varepsilon^{M,\Delta t} S_m^n(\alpha) \geq 0$. Now, using the discrete minimum principle, we have

$$|w^{M,\Delta t}(x_m,t_n)| \leq CS_m^n(\alpha) = C \prod_{j=m+1}^{M} \left(1 + \frac{\alpha h_j}{\varepsilon}\right)^{-1}, \ m = 0, \cdots, M, n\Delta t \leq T,$$

Using triangle inequality and the bound of singular component, we have

$$\begin{aligned}
|(w^{M,\Delta t}-w)(x_m,t_n)| &\leq |w^{M,\Delta t}(x_m,t_n)|+|w(x_m,t_n)|,\\
&\leq C\prod_{j=m+1}^{M}\left(1+\frac{\alpha h_j}{\varepsilon}\right)^{-1}+Ce^{-\frac{\alpha x_m}{\varepsilon}},\\
&\leq C\prod_{j=m+1}^{M}\left(1+\frac{\alpha h_j}{\varepsilon}\right)^{-1}=CS_m^n(\alpha).
\end{aligned}$$

With the choice of $\rho=1$ and using Eq. (4.12), we get the following error bound in the outer layer region:

$$|(w^{M,\Delta t}-w)(x_m,t_n)| \leq C(M^{-1}\ln^2 M+\Delta t^2),\;\; 1\leq m\leq M-1, n\Delta t\leq T. \tag{4.25}$$

Considering the inner region $(0,\sigma]\times(0,T]$, the truncation error becomes

$$\begin{aligned}
|L_\varepsilon^{M,\Delta t}(w^{M,\Delta t}-w)(x_m,t_n)| &\leq C\left[\varepsilon\int_{x_{m-1}}^{x_{m+1}}|w'''(x)|dx+\int_{x_{m-1}}^{x_m}|w''(x)|dx+\Delta t^2\right],\\
&\leq C\big[\frac{h_m}{\varepsilon^3}\int_{x_{m-1}}^{x_{m+1}}\exp\big(-\frac{\alpha x}{\varepsilon}\big)dx+\Delta t^2\big],\\
&\leq C\big[\frac{h}{m\varepsilon}\exp\big(\frac{-\alpha x_m}{\varepsilon}\big)\big\{\exp\big(\frac{\alpha h}{\varepsilon}\big)-\exp\big(\frac{-\alpha}{\varepsilon}\big)\big\}+\Delta t^2\big],\\
&\leq C\big[\frac{h}{\alpha\varepsilon}\exp\big(\frac{-\alpha x_m}{\varepsilon}\big)\sinh\big(\frac{\alpha h}{\varepsilon}\big)+\Delta t^2\big],
\end{aligned}$$

We assume $\sinh(\xi)\leq C\xi$, for $0\leq\xi\leq 1$. So, $\sinh\big(\frac{\alpha h}{\varepsilon}\big)\leq\frac{C\alpha h}{\varepsilon}$. Thus, error estimate reduces to

$$\begin{aligned}
|L_\varepsilon^{M,\Delta t}(w^{M,\Delta t}-w)(x_m,t_n) &\leq C\big[\frac{h^2}{\varepsilon^2}\exp\big(\frac{-\alpha x_m}{\varepsilon}\big)\big)+\Delta t^2\big],\\
&\leq C\big((M^{-2}\ln^2 M)S_m^n(\alpha)+\Delta t^2\big)\\
&\leq C\big(M^{-1}\ln^2 M+\Delta t^2\big).
\end{aligned} \tag{4.26}$$

From equation 4.26 and the discrete maximum principle, we obtain the error estimate at the singular component

$|(w^{M,\Delta t}-w)(x_m,t_n)|\leq C(M^{-1}\ln^2 M+\Delta t^2)$. □

Theorem 4.10 *Let $u(x,t)$ be the continuous solution of problem in Eq.* (4.1) *and $u^{M,\Delta t}(x_m,t_n)$ be* *the solution of the totally discrete problem in Eq.* (4.21)*. Then, we have the following error bound*

$$\|(u^{M,\Delta t} - u)(x_m,t_n)\|_{\bar{D}^{M,\Delta t}} \leq C\big(M^{-1}\ln^2 M + \Delta t^2\big), \quad m = 1,\cdots,M, \quad n\Delta t \leq T,$$

where C is a constant independent of ε and the mesh parameters.

Proof: The proof follows from Theorem (4.5) and Theorem (4.9). □

A very similar techniques establish the error estimate for the discrete backward scheme.

4.3 Numerical Computations

To check the applicability of the proposed method, numerical experiments were made for our problem. Since the test example has the exact solution, absolute errors and rate of convergences are computed at the point (x_m,t_n) for different parameter values α and β by

$$e_\varepsilon^{M,\Delta t} = \max_{0\leq m\leq M;\ t\in[0,T]} \big|u(x,t) - u(x_m,t_n)\big|, \qquad r_\varepsilon^{M,\Delta t} = \log_2\big(\tfrac{e_\varepsilon^{M,\Delta t}}{e_\varepsilon^{2M,\Delta t/2}}\big).$$

where $u(x,t)$ is the exact solution and $u(x_m,t_n)$ is the numerical solution.

Example 4.1 *Consider the following singularly perturbed Burger-Fisher equation*

$$\begin{cases} u_t - \varepsilon u_{xx} + \alpha u u_x - \beta(1-u)u = 0, & (x,t) \in (0,1)\times(0,1], \\ u(x,0) = \frac{1}{2} + \frac{1}{2}\tanh(\theta_1 x), & x \in [0,1], \\ u(0,t) = \frac{1}{2} + \frac{1}{2}\tanh(-\theta_1\theta_2 t), & t \in [0,1], \\ u(1,t) = \frac{1}{2} + \frac{1}{2}\tanh(\theta_1 - \theta_1\theta_2 t), & t \in [0,1]. \end{cases}$$

where $\theta_1 = -\frac{\alpha}{4\varepsilon}$ and $\theta_2 = \frac{\alpha}{2} + \frac{2\varepsilon\beta}{\alpha}$, the exact solution is given by

$$u(x,t) = \frac{1}{2} + \frac{1}{2}\tanh(\theta_1 x - \theta_1\theta_2 t).$$

Table 4.1: Maximum absolute errors for equal parameters $\alpha = \beta = 0.01$.

$\varepsilon \downarrow$	$M = N = 16$	32	64	128	256
Forward					
2^{-4}	4.4040e-09	2.1449e-09	1.0512e-09	5.0731e-10	2.3788e-10
2^{-6}	7.1593e-07	3.4246e-07	1.6507e-07	7.9576e-08	3.8294e-08
2^{-8}	2.7655e-05	1.3121e-05	6.3625e-06	3.1224e-06	1.5403e-06
2^{-10}	5.4252e-04	2.3595e-04	1.1028e-04	5.2468e-05	2.5522e-05
2^{-12}	8.7337e-04	3.7502e-03	1.9697e-03	8.9009e-04	4.0935e-04
Backward					
2^{-4}	9.7359e-09	2.6592e-09	1.0850e-09	5.6110e-10	2.9664e-10
2^{-6}	6.2790e-07	3.2227e-07	1.6115e-07	7.9636e-08	3.9295e-08
2^{-8}	2.2361e-05	1.1843e-05	6.0578e-06	3.0537e-06	1.5305e-06
2^{-10}	2.3502e-04	1.5836e-04	8.8981e-05	4.7287e-05	2.4305e-05
2^{-12}	1.2545e-03	9.0584e-04	8.9662e-04	5.8946e-04	2.4322e-04

Table 4.2: Maximum absolute errors for equal parameters $\alpha = \beta = 0.001$.

$\varepsilon \downarrow$	$M = N = 16$	32	64	128	256
Forward					
2^{-4}	6.6357e-12	5.8453e-13	1.1213e-13	5.5678e-14	2.8422e-14
2^{-6}	4.7729e-10	1.2204e-10	2.9069e-11	9.1598e-12	4.4463e-12
2^{-8}	1.1379e-08	3.2471e-09	1.5432e-09	7.6424e-10	3.7898e-10
2^{-10}	3.4355e-07	1.7010e-07	8.4676e-08	4.2158e-08	2.0994e-08
2^{-12}	7.6718e-06	3.5673e-06	1.7148e-06	8.4045e-07	4.1589e-07
Backward					
2^{-4}	7.2666e-12	9.4219e-13	1.0719e-13	5.4567e-14	2.6756e-14
2^{-6}	4.9269e-10	1.3152e-10	3.4603e-11	9.3154e-12	4.4351e-12
2^{-8}	1.1661e-08	3.4261e-09	1.5026e-09	7.5439e-10	3.7676e-10
2^{-10}	3.0012e-07	1.5911e-07	8.1954e-08	4.1480e-08	2.0828e-08
2^{-12}	5.4210e-06	3.0166e-06	1.5783e-06	8.0629e-07	3.9612e-07

Table 4.3: Maximum absolute errors for different parameters $\alpha = 0.001$ and $\beta = 0.01$.

$\varepsilon \downarrow$	$M = N = 16$	32	64	128	256
Forward					
2^{-4}	3.2477e-11	8.3912e-12	2.2081e-12	5.3324e-13	1.1824e-13
2^{-6}	4.6181e-10	1.1956e-10	2.9387e-11	6.8362e-12	2.0253e-12
2^{-8}	1.1500e-08	3.3037e-09	1.4273e-09	6.7683e-10	3.0627e-10
2^{-10}	3.4113e-07	1.6851e-07	8.3527e-08	4.1220e-08	2.0163e-08
2^{-12}	7.6776e-06	3.5680e-06	1.7152e-06	8.3970e-07	4.1469e-07
Backward					
2^{-4}	3.2687e-11	8.5130e-12	2.2827e-12	5.7176e-13	1.3989e-13
2^{-6}	4.6406e-10	1.2166e-10	3.0926e-11	9.3012e-12	6.0622e-12
2^{-8}	1.1619e-08	3.3961e-09	1.4971e-09	7.8225e-10	4.2054e-10
2^{-10}	2.9889e-07	1.5886e-07	8.2236e-08	4.1984e-08	2.1442e-08
2^{-12}	5.4328e-06	3.0217e-06	1.5810e-06	8.0851e-07	4.0923e-07

Table 4.4: Maximum absolute errors for equal parameters $\alpha = 0.01$ and $\beta = 0.001$.

$\varepsilon \downarrow$	$M = N = 16$	32	64	128	256
Forward					
2^{-4}	4.9495e-09	2.4182e-09	1.1985e-09	5.9630e-10	2.9727e-10
2^{-6}	7.2752e-07	3.4847e-07	1.6840e-07	8.1515e-08	3.9561e-08
2^{-8}	2.7630e-05	1.3114e-05	6.3620e-06	3.1238e-06	1.5433e-06
2^{-10}	5.4241e-04	2.3609e-04	1.1019e-04	5.2449e-05	2.5529e-05
2^{-12}	8.5624e-04	3.7280e-03	1.9704e-03	8.8832e-04	4.0883e-04
Backward					
2^{-4}	1.0220e-08	2.9091e-09	1.1768e-09	5.9145e-10	2.9662e-10
2^{-6}	6.3836e-07	3.2704e-07	1.6323e-07	8.0321e-08	3.9307e-08
2^{-8}	2.2357e-05	1.1825e-05	6.0478e-06	3.0469e-06	1.5248e-06
2^{-10}	2.3455e-04	1.5820e-04	8.8814e-05	4.7205e-05	2.4255e-05
2^{-12}	1.2558e-03	9.1016e-04	8.9533e-04	5.8916e-04	3.3412e-04

4.3.1 Discussions

Computational results in Tables (4.1) and Table (4.2) confirms that the present method gives more accurate results for the Example (4.1). From these table of values, we observe that when the value of parameters is small, we obtain small errors as demonstrated in Table (4.2) with the comparison in Table (4.1). As demonstrated in Tables (4.3) and (4.4), when $\alpha < \beta$ and $\alpha > \beta$, we compared the maximum absolute errors using the proposed schemes. Table (4.5) shows the rate of convergences for different parameter values using Table values (4.1) and (4.2). Numerical solution using surface

Table 4.5: Rate of convergences for equal parameters using Tables (4.1) and (4.2).

	$\alpha=\beta=0.01$				$\alpha=\beta=0.001$			
$\varepsilon \downarrow$	16	32	64	128	16	32	64	128
Forward								
2^{-4}	1.0379	1.0289	1.0511	1.0926	3.5049	2.3821	1.0391	0.9701
2^{-6}	1.0639	1.0529	1.0527	1.0552	1.9675	2.0698	1.6661	1.0427
2^{-8}	1.0757	1.0442	1.0269	1.0194	1.7671	1.1891	0.9941	1.0017
2^{-10}	1.2012	1.0973	1.0717	1.0397	1.0141	1.0064	1.0061	1.0058
2^{-12}	2.1023	0.9290	1.1460	1.1206	1.1047	1.0568	1.0288	1.0150
Backward								
2^{-4}	1.8723	1.2933	0.9514	0.9195	2.9472	3.1358	0.9741	1.0282
2^{-6}	0.9623	0.9999	1.0169	1.0191	1.9054	1.9263	1.8932	1.0707
2^{-8}	0.9170	0.9672	0.9882	0.9966	1.7671	1.1891	0.9941	1.0017
2^{-10}	0.5696	0.8316	0.9121	0.9602	0.9155	0.9571	0.9824	0.9939
2^{-12}	0.4698	0.0148	0.6051	1.2771	0.8456	0.9346	0.9690	1.0254

plot for Example (4.1) is plotted in Figure (4.1)-Figure (4.3). From the graphs plotted in Figures (4.1)-(4.5) for Example (4.1), we conclude that the problem (4.1) has boundary layer near $x=0$. In Figure (4.4), the behavior of singular perturbation parameter ε has been depicted as $\varepsilon \to 0$. The solution plots have been drawn for various values of ε. In this plot, it is very clear that as singular perturbation parameter gets smaller and smaller, sharper boundary layers appear in the solution and the proposed numerical scheme is efficient enough to capture the boundary layer near the boundary $x=0$. Figure (4.5) depicts the comparison of exact and numerical solution profile for reasonable values of parameters. Figure (4.6) shows that the stated problem has the left boundary layer so that maximum absolute errors exists in the layer region. From all the Figures, we can easily see that the problem has left boundary layer which confirms the theoretical findings.

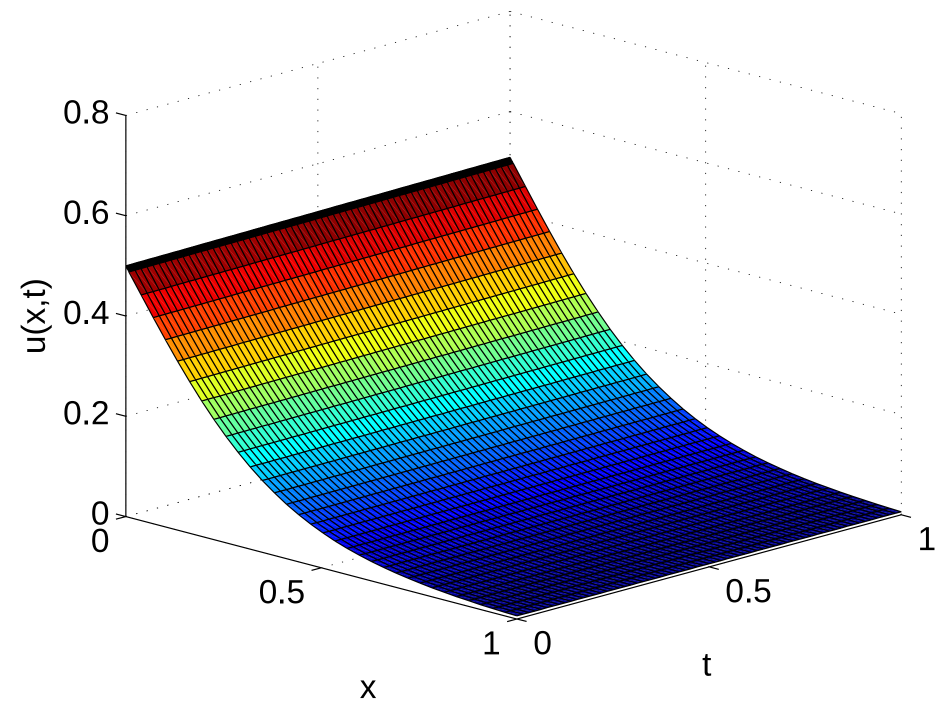


Figure 4.1: Surface plot at $M = N = 64,\ \varepsilon = 2^{-10},\ \alpha = \beta = 0.01$.

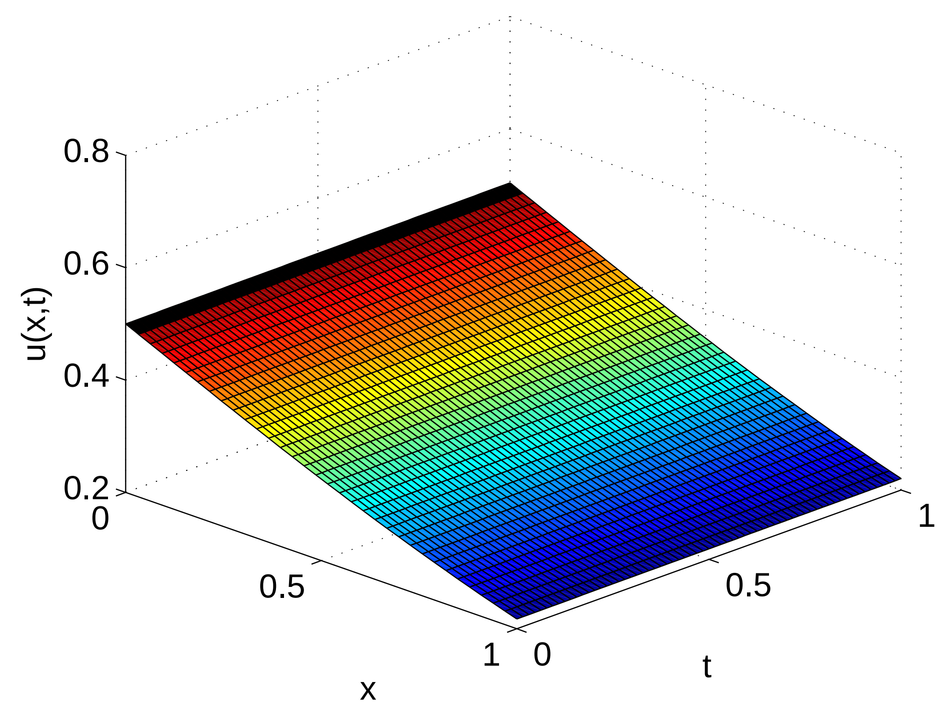


Figure 4.2: Surface plot at $M = N = 64,\ \varepsilon = 2^{-8},\ \alpha = \beta = 0.001$.

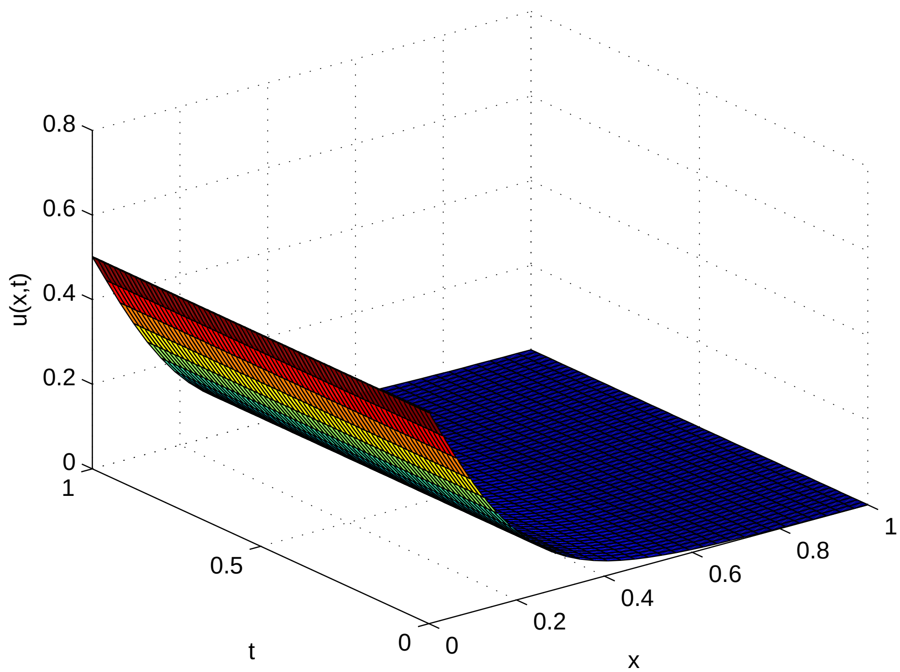
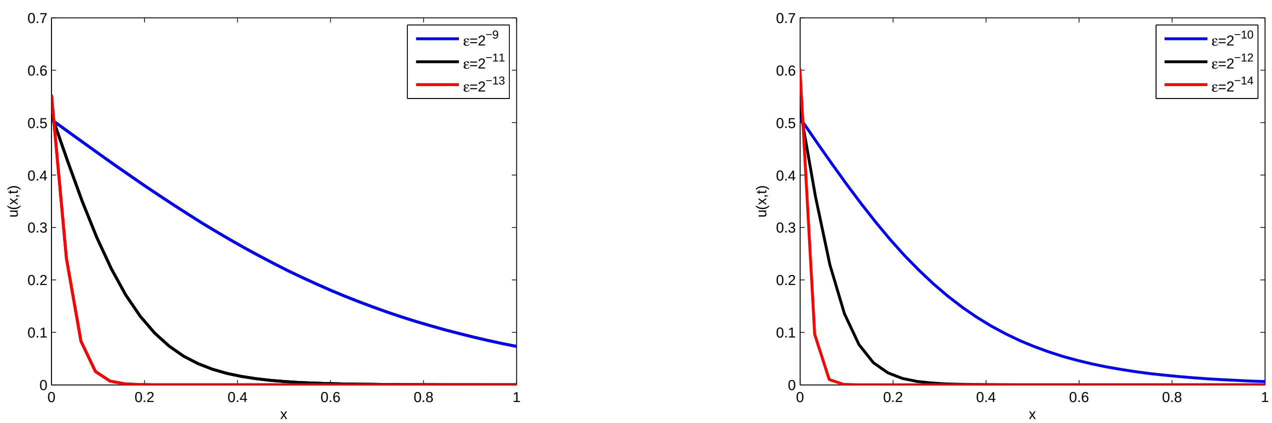


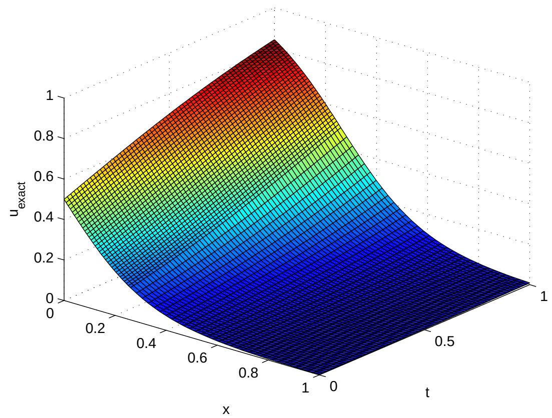
Figure 4.3: Surface plot at $M = N = 64,\ \varepsilon = 2^{-12},\ \alpha = \beta = 0.01.$



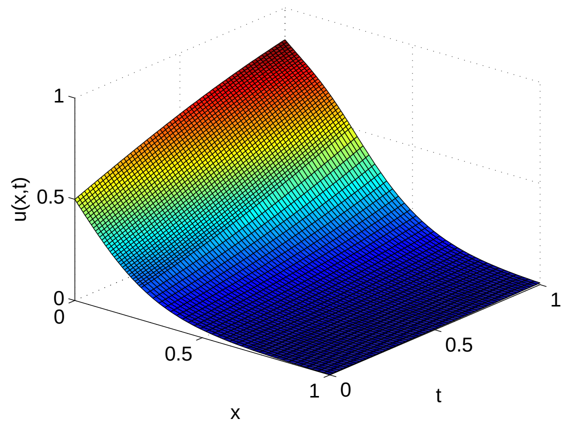
(a) $\varepsilon = 2^{-9},\ \varepsilon = 2^{-11}$ and $\varepsilon = 2^{-13}$.

(b) $\varepsilon = 2^{-10},\ \varepsilon = 2^{-12}$ and $\varepsilon = 2^{-14}$.

Figure 4.4: Effect of the perturbation parameter ε at $M = N = 64,\ \alpha = 0.01 = \beta$.



(a) Exact Solution



(b) Numerical Solution

Figure 4.5: Surface plot at $M = N = 64,\ \varepsilon = 2^{-5},\ \alpha = 0.4 = \beta$.

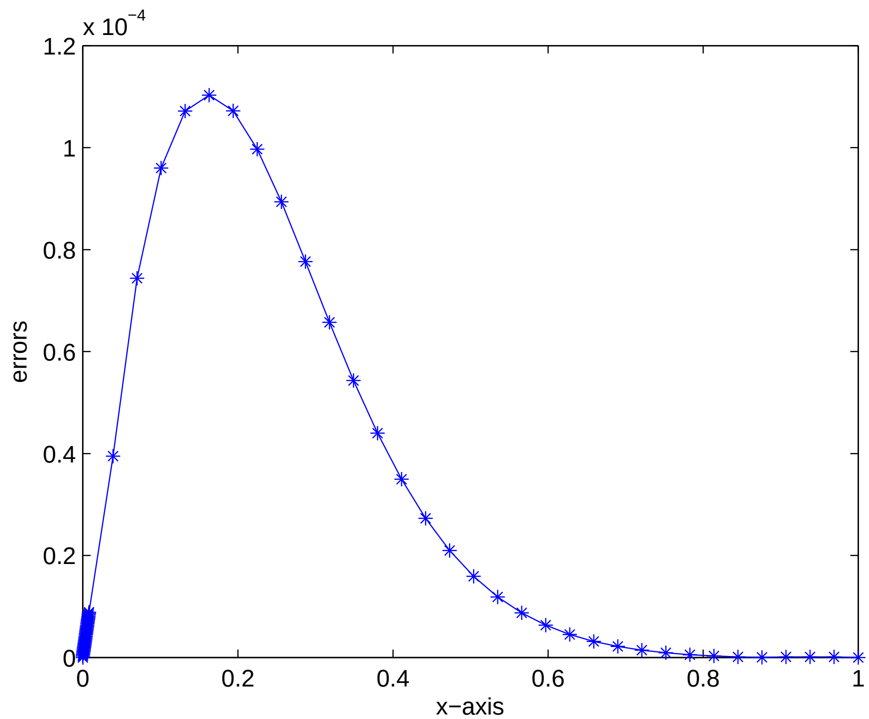


Figure 4.6: Pointwise absolute errors at $M = N = 64,\ \alpha = 0.01 = \beta,\ \varepsilon = 2^{-10}$.

Chapter 5

Conclusion and Future Scope

5.1 Conclusion

In this thesis, we proposed more accurate numerical scheme for solving singularly perturbed Burger-Fisher equation based on Shishkin mesh in the finite difference framework. Since Burger-Fisher equation is a non-linear problem, quasilinearization process has been used to tackle the nonlinearity occurring in the problem. Time discretization has been performed using implicit second-order finite difference approximation. Then spatial discretization has been carried out using forward and backward finite difference methods based on a piecewise uniform Shishkin mesh. The convergence analysis of the proposed numerical scheme has been discussed. At the end, it has been shown numerically that the proposed method is effective for capturing sharp boundary layers arising in the solution as singular perturbation parameter $\varepsilon \to 0$.

5.2 Scope for Future Work

In the present thesis, the forward and backward finite difference methods are developed for solving singularly perturbed nonlinear Burger-Fisher equation. With the help of the techniques presented in this thesis, we may construct other more accurate numerical methods for Burger-Fisher equation and other non-linear problems.

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