Stability and Hopf Bifurcation Analysis of Prey-Predator Mathematical Model with Delay

A Thesis Submitted to the Department of Mathematics, Jimma University in Partial Fulfillment for the Requirements of the Degree of Masters of Science in Mathematics (Differential Equation)

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Declaration

Here, I submit the thesis entitled "**Stability and Hopf Bifurcation Analysis of Prey-Predator Mathematical Model with Delay"** for the award of degree of Master of Science in Mathematics. I, the undersigned declare that, this study is original and it has not been submitted to any institution elsewhere for the award of any academic degree or the like, where other sources of information have been used, they have been acknowledged.

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Abstract

The dynamic relationship between predators and their prey has long been and will continue to be one of the dominant themes of research in applied mathematics and ecology. Stability and Hopf Bifurcation Analysis of Prey-Predator Mathematical Model with Delay. In this thesis, mathematical model of prey predator with delay was studied. To show Positivity of the solution for the model given. The equilibrium points for the system were calculated. The model under consideration was nonlinear so that it was linearized by Jacobian matrix at the positive equilibrium point. The local stability conditions were proved by using Routh Huwertiz stability criteria and local stability of the model in the absence and presence of delay was studied at the positive equilibrium point by linearizing the model. Finally, Hopf bifurcation condition was well spelled out. Generally the end result is stability by positive equilibrium points.

Keywords: Jacobian Matrix, Routh Huwertiz Stability Criteria, Hopf Bifurcation.

List of Variables and Parameters

δ_1 - Migration rate coefficient of the prey species from the unreserved to reserved zone,

δ_2 - Migration rate coefficient of the prey species from the reserved to unreserved zone;

$\mathbb{k}$ -Carrying capacities of the prey in the unreserved zone,

l - Carrying capacities of the prey in the reserved zone.

r -Intrinsic growth rates of the prey in the unreserved zone.

s -Intrinsic growth rates of the prey in the reserved zone.

β_1 - Attack rate of the predator on the prey in the unreserved region area.

β_2 - Conversion rate of the prey in the unreserved zone to a predator.

γ_1 - Attack rate of the top predator on the predator;

γ_2 - Conversion rate of the predator to the top predator.

β_0 - Predator natural death rate.

α - Top predator's natural death rate.

τ - Conversion rate of the prey in the unreserved zone to a predator.

$x(t)$ - Denotes the biomass density of prey in unreserved zone

$y(t)$ - Denotes the biomass density of prey in reserved zone

$z(t)$ - Denotes the biomass density of the predator

$w(t)$ - Denotes the biomass density of to the top predator

Contents

CHAPTER ONE

INTRODUCTION

1.1 Background of the Study

Predator-prey model is the first model to illustrate the interaction between predators and prey. It's a topic of great interest for many ecologists and mathematicians. This model assumes that the predator populations have negative effects on the prey populations. The dynamic relationship between predators and their prey has long been and will continue to be one of the dominant themes in both ecology and mathematical ecology due to its universal existence and importance (Berryman, 1992). Population dynamics deals with the time-dependent behavior of modeled ecological systems. These models provide significant insights into the behavior of nature. The mathematical equations can govern the time evolution of interacting species. The evolution and growth of the species depend on many factors, such as overcrowding, age structure, past population size, sources of food supply, interactions with other species, topographical, ecological and environmental conditions in the habitat, including seasonal and climatic variations. In recent years, predator-prey models are arguably the most fundamental building blocks of any biological and ecosystems as all biomasses are grown out of their resource masses. Species compete, evolve and disperse often simply for the purpose of seeking resources to sustain their struggle for their very existence. Their extinctions are often the results of their failure in obtaining the minimum level of resources needed for their subsistence. Mathematical models inter of ordinary differential equation (ODE) have been widely used to model physical phenomena, engineering systems, and economic behavior, biological and biomedical processes. In particular, ODE models have recently played a prominent role in describing the dynamic behavior of predator-prey systems. The study of population phenomena or growth phenomena or competition between two species is really dominated problem in the biological system. Prey-predator interactions abound in the biological world, and are one of the most important topics in theoretical ecology (Sinha 2018).

The study of predation has long history, beginning with the work of Lotka and Volterra and continuing to be of interest today. In most ecological models the growth rate of species does not only depend on the instantaneous population size but also on the past history of the population.

For example, in the prey-predator model the loss of prey by predator will affect the growth rate of predators at the future time (Kolmanovskii and Myshkis, 1999). In 2018, Shireen and Matthias proposed prey-predator mathematical modelas follows:

$$\begin{aligned}
\frac{dx}{dt} &= rx(1-\frac{x}{k}) - \delta_1 x + \delta_2 y - \beta_1 xz \\
\frac{dy}{dt} &= sy(1-\frac{y}{l}) + \delta_1 x - \delta_2 y \\
\frac{dz}{dt} &= \beta_2 xz - \beta_0 z - \gamma_1 zw \\
\frac{dw}{dt} &= \gamma_2 zw - \alpha w
\end{aligned} \tag{1.1}$$

Here, the model (1.1) has been analyzed with the initial conditions. All parameters of the model (1.1) are assumed to be positive and described as follows $\mathbb{k}$ and l are the carrying capacities of the prey in the unreserved and reserved zone, respectively, with intrinsic growth rates r and s ; δ_1 is the migration rate coefficient of the prey species from the unreserved to reserved area and δ_2 the migration rate coefficient of the prey species from the reserved to unreserved zone; β_1 is the attack rate of the predator on the prey in the unreserved region; β_2 is the conversion rate of the prey in the unreserved zone to a predator; γ_1 is the attack rate of the top predator on the predator; γ_2 is the conversion rate of the predator to the top predator; and finally, β_0 and α represent the predator and top predator's natural death rate, respectively .

In the real world, there is sometimes a need to control population at a reasonable level because otherwise this population may cause increase or even extinction of other populations.

Bearing this in mind, if an average of time delays, taking into account some purposeful action of various factors on the system acts only on predators, the model (1.1) modified to the following model.

$$
\begin{aligned}
\frac{dx}{dt} &= rx(1-\frac{x}{k})-\delta_1 x+\delta_2 y-\beta_1 xz \\
\frac{dy}{dt} &= sy(1-\frac{y}{l})+\delta_1 x-\delta_2 y \\
\frac{dz}{dt} &= \beta_2 x(t-\tau)z-\beta_0 z-\gamma_1 zw \\
\frac{dw}{dt} &= \gamma_2 zw-\alpha w,
\end{aligned}
\tag{1.2}
$$

τ - is conversion rate of the prey in the unreserved zone to a predator.

Stability of dynamical systems plays a very important role in control system analysis and design. Unlike the case of linear systems, proving stability of equilibrium points of nonlinear systems is more complicated. For example Chernet Tuge and Mitiku Daba (2017) investigated the stability analysis of delayed nonlinear cournot model in the sense of Lyapunov. One of the finding of this investigation indicates that the presence of equal information time delay in the given model causes oscillation process in the system and doesn't affect the qualitative behavior of the solution (no effect on the stability of the equilibrium point), but only changes the transition process. In other words, it delays stability as delay parameter increases. On the other hand, when one of the firms has implementation delay and the rival player makes decision without delay, it leads to instability of the dynamic system at least locally.

Bifurcation theory is the mathematical study of changes in the qualitative or topological structure of a given dynamical systems. Local bifurcation occurs when a parameter change causes the stability of equilibrium to change. To date, many authors have studied the dynamics of predator-prey models with time delay and obtained complex dynamic behavior, such as stability of equilibrium, Hopf bifurcation. For example, Song and Wei (2005) investigated further the dynamics of the system prey-predator model by considering the time delay as the bifurcation parameter and they obtained that, under certain conditions, the unique positive equilibrium of the model is absolute stable while it is conditionally stable and there exist switches from stability to instability under other conditions.

In 2011, Xu*et al.* studied stability and Hopf bifurcation analysis for a Lotka-Volterra predator–prey model with two-times delays. In 2015, Soliman and Jarallah, studied asymptotic stability of solutions of Lotka-Volterra predator-prey model for four species. In 2014, Yue and Qingling studied stability and bifurcation analysis of a singular delayed predator-prey bio-economic model with stochastic fluctuations. In 2012, Mukherjee studied the bifurcation and stability analysis of prey-predator model with a reserved area. In 2013, Liu *et al*., studied global stability analysis and optimal control of a harvested eco-epidemiological prey predator model with vaccination and taxation. In 2016, Jana, *et a* studied on the stability and Hopf bifurcation of a prey-generalist predator system with independent age-selective harvesting.

In 2016, Shiva Reddy studied Dynamics in harvested prey–predator mathematical model with noise and diffusion. In2016,Naji and Jawad studied the dynamics of prey-predator model with are served zone. In 2017, Ahmed Buseri studied global asymptotic stability analysis of predator-prey model. Also, in 2017, Peng*et al*. studied hybrid control of Hopf bifurcation in a Lotka-Volterra predator-prey model with two-time delays. In 2018, Shireen and watthias studied Modeling, Dynamics and Analysis of Multi-Species Systems with Prey Refuge. The researcher proposed the model as well as conducted necessary analysis. In 2018, Dawit Getachew studied Stability Analysis of Prey-Predator Mathematical Model with Delay and Control of the Prey. In 2020, Li and Zhao studied Periodic Solution of a Neutral Delay Leslie Predator-Prey Model and the Effect of Random Perturbation on the Smith Growth Model.In2017, Ali, *et al.* studied Dynamics of a three species ratio-dependent food chain model with intra-specific competition within the top predator.

However, to the best knowledge of the author, the stability and Hopf bifurcation analysis of the mathematical model of prey-predator with delays represented by equation (1.2) is not yet investigated. Therefore, the central goal of this study is to investigate the dynamic behavior such as positivity, local stability Analysis and Hopf bifurcation of prey-predator model with delay represented by equation (1.2).

1.2 Statement of the Problem

A slight change in the environment could have a profound influence on all living species. In particular, a food chain might lose one of their components. These problems will create an imbalance in the ecosystem. For species protection, some strategies and appropriate measures that will diminish interaction by species including the creation of reserved zones, restricting harvesting, etc. need to be deployed (Mukherjee, 2012). Several scholars conducted research on prey predator mathematical model as pointed out in introduction part. For example, Shireen and Matthias proposed and analyzed mathematical model of prey predator represented by Eq. (1.1). However, those scholars didn't take time delay into consideration in the mathematical model they developed. As a result, it sounds to incorporate time delay into the mathematical model to get more realistic information.

Therefore, this research mainly focuses on the following problems related to prey predator mathematical model with delay given by Eq. (1.2).

- Positivity of the solution of the model given by (1.2),
- Local stability analysis of the model given by Eq.(1.2),
- Hopf bifurcation condition of model given by Eq.(1.2).

1.3 Objective of the Study

1.3.1 General Objective

The general objective of this study is to investigating stability and Hopf bifurcation analysis of prey-predator mathematical model with delay represented by Eq. (1.2).

1.3.2 Specific Objectives

The specific objectives of the present study are:

- To show positivity solution of model given by Eq. (1.2),
- To determine local stability analysis of model given by Eq.(1.2),,
- To establish Hop bifurcation of model given by Eq. (1.2).

1.4Significance of the Study

Many species become extinct and many others are at the verge of extinction due to several reasons like, over exploitation, over predation, environmental pollution, and mismanagement of natural resources etc. As a result, this study enables policy makers of ecosystem for co-existence different species by providing theme necessary conditions which guarantee for co-existence.

1.5 Delimitation of the Study

This study is delimited to stability and Hopf bifurcation analysis of the prey-predator mathematical model with delay given by Eq. (1.2

CHAPTER TWO

LITERATURE REVIEW

2.1 Historical background

Mathematical models in terms of ordinary differential equations (ODE's) have been widely used to model physical phenomena, engineering systems, economic behavior, biological and biomedical processes. In particular, ODE models have recently played a prominent role in describing the dynamic behavior of predator-prey systems. To study the dynamic behavior of model, mathematical modeling is used as an effective tool to describe and analyze the model. Mathematical population models have been used to study the dynamics of prey predator systems since Lotka and Volterra proposed the simple model of prey-predator interactions now called the Lotka-Volterra model. Since then, many mathematical models, some reviewed in this study, have been constructed based on more realistic explicit and implicit biological assumptions. Modeling is a frequently evolving process, to gain a deep understanding of the mathematical aspects of the problem and to yield non trivial biological insights; one must carefully construct biologically meaningful and mathematically tractable population models (Kuang, 2002).

Inter species or Intra species competition models have been the subjects of central discussions in ecological and biological systems. Among the competition models, Lotka-Volterra inter-specific competition model occupies the top role to discuss the competitive behavior of the biological species which determines the present state in terms of past state and changes with the period of time. The competition models are used in forecasting of species growth rate, maximum and minimum consumption of resource, food pre- serving, environment capacities and many others applications. The study of population phenomena or growth phenomena or competition between two species is really dominated problem in the biological system. Volterra (1926) first developed competition model between a predator and a prey (Brauer and Soudack, 1979). In the ecological system, the predator-prey model is among the oldest studies and also the first model to illustrate the interaction between predators and prey.

This model assume that the predator populations have negative effects on the prey population sand this system was formulated by VitoVolterra who is an Italian mathematician and Alfred Lotka who is an American mathematical biologist in 1925 (Boyce, 2010).

2.2 Models of Prey Predator with time delay

In nature, populations do not reproduce instantaneously; rather it is mediated by certain time delay required for gestation, maturation time, capturing time, or other reasons. Thus, time delays of one type or another have been incorporated into mathematical models of population dynamics. Delay is a general concept that can represent different phenomena such as the time it takes for the progenitor to reach maturity or the finite gestation period of a species. Mathematical delays are input in model to correct the classical logistic model, which assumes that the growth rate of a population at time is determined by the number of individuals at that time. Of course, biological delays are complex and the mathematical representation is often a simplification of reality. The time delay is considered into the population dynamics when the rate of change of the population is not only a function of the present population but also depends on the past population. This relationship/interaction between two or more species has been essential in theoretical ecology since the famous Lotka–Volterra equations (Volterra, 1926), which are a system of first which is a system of first order, nonlinear differential equations that describe the dynamics and interactions between two or more species of biological systems. Of course, the qualitative properties of a prey-predator system such as stability of the steady states, bifurcation analysis, and oscillation of the solutions usually depend on the system parameters (Kaung, 1993).

2.3 Recent studies

In the context of predator-prey interaction, some studies that treat population can be extended by Martin and Ruan have analyzed generalized cause predator prey models where the prey is harvested with constant rate while Kar considered the predator-prey model with the predator harvested and suggested that it is ideal to study the combined harvesting of predator and prey population models (Kar, 2003).

In 2018, Teshale Fikre studies Stability and Bifurcation Analysis of Prey-Predator Mathematical Model with Delay and Control of the Predator. They showed how to classify the possibilities and determine the region of stability. They found that if the equilibrium point is asymptotically stable, which is determined by a local linearization, then every solution whose initial value is in some neighborhood of the stable equilibrium point tends to it as the time approaches infinity. There exists an asymptotically stable limit cycle when the constant rate is small and the equilibrium point is unstable. In 2019, Srinivasarao studied Prey-Predator Model for Awash National Park, Oromia, Ethiopia and its Stability Analysis with Simulations. The study is based on formulation of a mathematical model to study the dynamics of the population densities and analyzing the stability of equilibrium points of the prey-predator model.

In 2020,Wang and Zou, studied On a Predator–Prey System with Digestion Delay and Anti-predation Strategy. In 2020, Rihan, *et al*, studied Stability and Hopf Bifurcation of Three-Species Prey-Predator System with Time Delays and Allee Effect. In 2021, San-Xing and Xin-You studied Dynamics of a delayed predator-prey system with fear effect, herd behavior and disease in the susceptible prey. Then taking time delay as the bifurcation parameters, the existence of Hopf bifurcation of the system at the positive equilibrium is given. Thirdly, the global asymptotic stability of the equilibrium is discussed by constructing a suitable Lyapunov function. Next, the direction of Hopf bifurcation and the stability of the periodic solution are analyzed based on the center manifold theorem and normal form theory.

Although the above studies were conducted by different researchers, still there is a room for further study. Consequently, this study is going to contribute on mathematical rigorous analysis of mathematical model represented by Eq. (1.2).

CHAPTER THREE

METHODOLOGY

3.1. Study Area and Period

The study was conducted at Jimma University under the department of Mathematics from December, 2020 to February, 2022 G.C.

3.2 Study Design

This study will employ mixed-design (documentary review design and experimental design) on prey-predator model given by equation (1.2).

3.3. Source of Information

The relevant sources of information for this study are books, published articles & related studies from internet.

3.4. Mathematical Procedures

In order to achieve the stated objectives, the study will follow the following mathematical procedures:

1. Showing positivity of the solution of the model,
2. Determining the steady state points of the model,
3. Linearizing the mathematical model of prey-predator under consideration,
4. Determining the local stability analysis condition of the model,
5. Establishing Hopf-bifurcation conditions.

CHAPTER FOUR

RESULT AND DISCUSSION

4.1 Preliminaries

Definition 4.1: Consider non-linear system $\frac{dx}{dt} = f(x)$, where $f : R^n \to R^n$. A point $x^* \in R^n$ is an equilibrium point if $\frac{dx}{dt}(x^*) = f(x^*) = 0$

Definition 4.2: For a linear system $\frac{dx}{dt} = AX$ the stability of equilibrium point can be completely determined by location of Eigen values of A. This is expressed as follows;

I. If the all Eigen-values of the Jacobean matrix have real parts less than zero, then the linear system is locally asymptotically stable and
II. If at least one of the Eigenvalues of Jacobean matrix has real part greater than zero, then the system is unstable (Khalil, 2002).

Definition 4.3: Routh-Hurwitz Stability Criterion (Katsuhiko, 1970)

Given characteristic polynomial of the form $a_0 m^n + a_1 m^{n-1} + ... + a_n = 0$

Where $a_0 \neq 0$ and $a_n > 0$, then the Routh-Hurwitz array or table is given as follows.

$$
\begin{array}{r|lllll}
m^n & a_0 & a_2 & a_4 & a_6 & \cdots \\
m^{n-1} & a_1 & a_3 & a_5 & a_7 & \cdots \\
m^{n-2} & b_1 & b_2 & b_3 & b_4 & \cdots \\
m^{n-3} & c_1 & c_2 & c_3 & c_4 & \cdots \\
m^{n-4} & d_1 & d_2 & d_3 & d_4 & \cdots \\
\vdots & \vdots & \vdots & \vdots & & \\
m^2 & e_1 & e_2 & & & \\
m^1 & f_1 & & & & \\
m^0 & g_0 & & & &
\end{array}
$$

Where

$$b_1 = \frac{a_1a_2 - a_0a_3}{a_1} \quad c_1 = \frac{b_1a_3 - a_1b_2}{b_1}$$
$$b_2 = \frac{a_1a_4 - a_0a_5}{a_1}, \quad c_2 = \frac{b_1a_5 - a_1b_3}{b_1}, \quad d_1 = \frac{c_1b_2 - b_1c_2}{c_1}$$
$$b_3 = \frac{a_1a_6 - a_0a_7}{a_1} \quad c_3 = \frac{b_1a_7 - a_1b_4}{b_1} \quad d_2 = \frac{c_1b_3 - b_1c_3}{c_1}$$

4.1.2 Descartes rule of signs

Let p(x) defines a characteristic polynomial with real coefficients and a non- zero constant term, with the terms being in descending powers of x. Hence, the number of positive real roots of p(x) = 0 either equals the number of variances in sign occurring in the coefficients of p(x), or less than the number of variations by a positive even number.

4.2 Positive of the Solution of the Model

Let, $X = (x, y, z, w) \in \mathbb{R}_+{}^4$ and $f(X) = [f_1(X), f_2(X), f_3(X), f_4(X)]$

Where, $f(X): \mathbb{R}_+^4 \to \mathbb{R}_+^4$ and $f \in \mathbb{R}_+^4$ Then system (1.2) becomes:

$$\begin{aligned}
\frac{dx}{dt} &= rx(1 - \frac{x}{k}) - \delta_1 x + \delta_2 y - \beta_1 xz = f(x, y, z, w) \\
\frac{dy}{dt} &= sy(1 - \frac{y}{l}) + \delta_1 x - \delta_2 y = f(x, y, z, w) \\
\frac{dz}{dt} &= \beta_2 x(t - \tau)z - \beta_0 z - \gamma_1 zw = f(x, y, z, w) \\
\frac{dw}{dt} &= \gamma_2 zw - \alpha w = f(x, y, z, w)
\end{aligned} \tag{4.1}$$

From the first equation of Eq. (4.1)

$$\frac{dx}{dt} = rx(-\frac{rx}{k}) - \delta_1 x + \delta_2 y - \beta_1 zx$$
$$\frac{dx}{dt} = \left(r - \frac{rx}{k} - \delta_1) + \frac{\delta_2 y}{x} - \beta_1 z \right) x$$
$$\frac{dx}{x} = \left(r - \frac{rx}{k} - \delta_1) + \frac{\delta_2 y}{x} - \beta_1 z \right) dt$$

Where

$$\frac{dx}{x} = p(x, y, z, w)dt \tag{4.2}$$

'Where $p(x, y, z.w,) = r - \frac{rx}{k} - \delta_1 + {}^{\delta_2 y}\!/_{x} - \beta_1 z$

Integrating Eq. (4.2) from $[0, t]$

$$\int \frac{dx}{x} = \int_0^t p(x(s), y(s), z(s)w(s))ds$$

$$\ln x = \int_0^t p(x(s), y(s), z(s)w(s))ds$$

$$e^{\ln x} = e^{\int_0^t p(x(s), y(s), z(s)w(s))ds}$$

$$x = c_1 e^{\int_0^t p(x(s), y(s), z(s)w(s))ds} \tag{4.3}$$

Where c_1 is an integrating constant? Apply initial condition at $t = 0$

$$x(0) \geq 0$$
$$x(0) = c_1$$

Putting the value of c_1 in to Eq. (4.3)

$$x(t) = x(0)e^{\int_0^t p(x(s), y(s), z(s)w(s))ds} \geq 0, \qquad \forall t \geq 0$$

Therefore, $x_1(t)$ is positive $\forall t \geq 0$.

From the second equation of Eq. (4.1)

$$\frac{dy}{dt} = sy(1 - \frac{y}{l}) + \delta_1 x - \delta_2 y$$
$$\frac{dy}{dt} = y\left(s - \frac{sy}{l} + {}^{\delta_1 x}\!/_{y} - \delta_2\right)$$
$$\frac{dy}{y} = \left(s - \frac{sy}{l} + {}^{\delta_1 x}\!/_{y} - \delta_2\right)dt$$

Where

$$\frac{dy}{y} = k(x, y, z, w)dt \tag{4.4}$$

Where $k(x, y, z.w,) = s - \frac{sy}{l} + {}^{\delta_1 x}\!/_{y} - \delta_2$

Then integrating Eq. (4.4) from $[0, t]$

$$\int \frac{dy}{y} = \int_0^t k(x(s), y(s), z(s)w(s))ds$$

$$\ln y = \int_0^t k(x(s), y(s), z(s)w(s))ds$$

$$e^{\ln y} = e^{\int_0^t k(x(s),y(s),z(s)w(s))ds}$$

$$y = ce^{\int_0^t k(x(s),y(s),z(s)w(s))ds} \qquad (4.5)$$

Where c is an integrating constant. Apply initial condition at $t = 0$

$$y(0) \geq 0$$
$$y(0) = c$$

Putting the value of c in to Eq. (4.5)

$$y(t) = y(0)e^{\int_0^t k(x(s),y(s),z(s)w(s)ds} \geq 0, \qquad \forall t \geq 0$$

Therefore, $y(t)$ is positive $\forall t \geq 0$

From the third equation of Eq. (4.1)

$$\frac{dz}{dt} = \beta_2 x(t-\tau)z - \beta_0 z - \gamma_1 zw$$
$$\frac{dx}{dt} = \left(\beta_2 x(t-\tau) - \beta_0 - \gamma_1 w\right)z$$
$$\frac{dx}{z} = \left(\beta_2 x(t-\tau) - \beta_0 - \gamma_1 w\right)dt$$

$$\frac{dz}{z} = m(x, y, z, w)dt \qquad (4.6)$$

Where $m(x, y, z, w) = \beta_2 x(t-\tau) - \beta_0 - \gamma_1 w$

Then integrating Eq. (4.6) from $[0, t]$

$$\int \frac{dz}{z} = \int_0^t m(x(s-\tau_1), y(s), z(s)w(s))ds$$

$$\ln z = \int_0^t m(x(s-\tau_1), y(s), z(s)w(s))ds$$

$$e^{\ln z} = e^{\int_0^t m(x(s-\tau_1),y(s),z(s)w(s))ds}$$

$$z = ce^{\int_0^t m(x(s-\tau_1),y(s),z(s)w(s))ds} \qquad (4.7)$$

Where c is an integrating constant. Apply initial condition at $t = 0$

$$z(0) \geq 0$$
$$z(0) = c$$

Putting the value of c in to Eq. (4.7)

$$z(t) = z(0)e^{\int_0^t m(x(s-\tau_1),y(s),z(s)w(s))ds} \geq 0, \qquad \forall t \geq 0$$

Therefore, $z(t)$ is positive $\forall t \geq 0$

From the fourth equation of Eq. (4.1)

$$\frac{dz}{dt} = \gamma_2 zw - \alpha w$$
$$\frac{dx}{dt} = \left(\gamma_2 z - \alpha\right) w$$
$$\frac{dx}{w} = \left(\gamma_2 z - \alpha\right) dt$$
$$\frac{dw}{w} = n(x, y, z, w)dt \qquad (4.8)$$

Where $N(x, y, z, w) = \gamma_2 z - \alpha$

Then integrating Eq. (4.8) from $[0, t]$

$$\int \frac{dw}{w} = \int_0^t N(x(s), y(s), z(s)w(s))ds$$

$$\ln w = \int_0^t N(x(s), y(s), z(s)w(s))ds$$

$$e^{\ln w} = e^{\int_0^t N(x(s), y(s), z(s)w(s))ds}$$

$$w = ce^{\int_0^t N(x(s), y(s), z(s)w(s))ds} \tag{4.9}$$

Where c is an integrating constant. Apply initial condition at $t = 0$

$$w(0) \geq 0$$
$$w(0) = c$$

Putting the value of c in to Eq. (4.9)

$$w(t) = w(0)e^{\int_0^t N(x(s), y(s), z(s)w(s))ds} \geq 0, \qquad \forall t \geq 0$$

Therefore, $w(t)$ is positive $\forall t \geq 0$

Therefore, all solutions of system (1.2) are non- negative.

4.3 Equilibrium point

In dynamical system theory, equilibrium solutions are solutions which do not change with time (Meiss, 2007). Studying equilibrium solutions is important in mathematical biology because it predicts long-term behaviors of a system. In the following, the existence of the equilibrium points of the system (1.2) will be elucidated. To find equilibrium point, equate the right hand side of equation (1.2) with zero

$$\begin{cases} rx(1-\frac{x}{k})-\delta_1 x+\delta_2 y-\beta_1 xz=0 \\ sy(1-\frac{y}{l})+\delta_1 x-\delta_2 y=0 \\ \beta_2 x(t-\tau)z-\beta_0 z-\gamma_1 zw=0 \\ \gamma_1 zw-\alpha w=0 \end{cases} \tag{4.10}$$

Since the time delay has no effect on the equilibrium point, firstly

$$rx(1-\frac{x}{k})-\delta_1 x+\delta_2 y-\beta_1 xz=0 \tag{4.11}$$

$$sy(1-\frac{y}{l})+\delta_1 x-\delta_2 y=0 \tag{4.12}$$

$$(\beta_2 x-\beta_0-\gamma_1 w)z=0 \tag{4.13}$$

$$(\gamma_2 z-\alpha)w=0 \tag{4.14}$$

From Eq. (4.14)

$$(\gamma_2 z-\alpha)w=0 \;\; ,w=0,\;\; \gamma_2 z-\alpha=0,\;\; z=\frac{\alpha}{\gamma_2}$$

For $w=0$ Eq. (4.14) gives:-

$$(\beta_2 x-\beta_0-\gamma_1 w)z=0,\;\;\; z=0,\; \beta_2 x-\beta_0=0,\;\; x=\frac{\beta_0}{\beta_2}$$

For $w=0$ and $z=0$

$$r(1-\frac{x}{k})-\delta_1+\frac{\delta_2 y}{x}=0 \tag{4.15}$$

$$s(1-\frac{y}{l})-\delta_2+\frac{\delta_1 x}{y}=0 \tag{4.16}$$

form Eq.(4.15)

$$y = \frac{1}{\delta_2}[\frac{rx^2}{k} - (r-\delta_1)x] \qquad (4.17)$$

$$s\left(\frac{1}{\delta_2}[\frac{rx^2}{k} - (r-\delta_1)x] \right) - s\left(\frac{\frac{1}{\delta_2}[\frac{rx^2}{k} - (r-\delta_1)x]}{l} \right)^2 - [\frac{rx^2}{k} - (r-\delta_1)x] + \delta_1 x = 0$$

$$\frac{srx^2}{\delta_2 k} - \frac{s}{\delta_2}(r-\delta_1)x - s\left(\frac{rx^2}{l\delta_2 k} - \frac{1}{l\delta_2}(r-\delta_1)x \right)^2 - \frac{rx^2}{k} + (r-\delta_1)x + \delta_1 x = 0$$

$$\frac{srx^2}{\delta_2 k} - \frac{s}{\delta_2}(r-\delta_1)x - s\left(\frac{rx^2}{l\delta_2 k} - \frac{1}{l\delta_2}(r-\delta_1)x \right)^2 - \frac{rx^2}{k} + (r-\delta_1)x + \delta_1 x = 0$$

$$\frac{srx^2}{\delta_2 k} - \frac{s}{\delta_2}(r-\delta_1)x - \frac{sr^2x^4}{l\delta_2{}^2k^2} + \frac{2sr(r-\delta_1)x^3}{l\delta_2 k} - \frac{(r-\delta_1)^2x^2}{l\delta_2} - \frac{rx^2}{k} + (r-\delta_1)x + \delta_1 x = 0$$

$$\left[\frac{srx}{\delta_2 k} - \frac{s}{\delta_2}(r-\delta_1) - \frac{sr^2x^3}{l\delta_2{}^2k^2} + \frac{2sr(r-\delta_1)x^2}{l\delta_2{}^2k} - \frac{(r-\delta_1)^2x}{l\delta_2} - \frac{rx}{k} + (r-\delta_1) + \delta_1 \right] x = 0$$

$$x = 0 \text{ or } \begin{cases} \frac{srx}{\delta_2 k} - \frac{s}{\delta_2}(r-\delta_1) - \frac{sr^2x^3}{l\delta_2{}^2k^2} + \frac{2sr(r-\delta_1)x^2}{l\delta_2{}^2k} \\ + \frac{(r-\delta_1)^2x}{l\delta_2} - \frac{rx}{k} + (r-\delta_1) + \delta_1 = 0 \end{cases}$$

$$-\frac{srx}{\delta_2 k} + \frac{s}{\delta_2}(r-\delta_1) + \frac{sr^2x^3}{l\delta_2{}^2k^2} - \frac{2sr(r-\delta_1)x^2}{l\delta_2{}^2k} + \frac{(r-\delta_1)^2x}{l\delta_2} + \frac{rx}{k} - (r-\delta_1) - \delta_1 = 0$$

$$\frac{sr^2}{l\delta_2^2k^2}x^3 - \frac{2rs(r-\delta_1)}{l\delta_2^2k}x^2 + \left(\frac{s(r-\delta_1)^2}{l\delta_2^2} - \frac{r(r-\delta_2)}{\delta_2 k} \right)x + \frac{(r-\delta_1)(s-\delta_2)}{\delta_2} - \delta_1 = 0$$

For $x = 0$ Eq. (4.17) givens $y = 0$

Therefore $E_1 = (0,0,0,0)$

Now, substituting the value of y into Eq. (4.16), and after a little algebraic manipulation yields:

$$ax^3 + bx^2 + cx + d = 0 \qquad (4.18)$$

Where,

$$a = \frac{sr^2}{l\delta_2^2 k^2} > 0 \tag{4.19}$$

$$b = \frac{-2rs(r-\delta_1)}{l\delta_2^2 k} < 0 \tag{4.20}$$

$$c = \frac{s(r-\delta_1)^2}{l\delta_2^2} - \frac{r(r-\delta_2)}{\delta_2 k}, \tag{4.21}$$

$$d = \frac{(r-\delta_1)(s-\delta_2)}{\delta_2} - \delta_1. \tag{4.22}$$

Hence, by using Descartes rule of signs, Eq (4.18) has positive solution, if the following inequalities hold:

$$\frac{s(r-\delta_1)^2}{l\delta_2^2} < \frac{r(r-\delta_2)}{\delta_2 k}$$

$$(r-\delta_1)(s-\delta_2) < \delta_2\delta_1$$

Knowing the value of x^* , the value of y^* is computed from Eq. (4.17). It should also be noted that for y^* to be positive, the following must be the case

$$y^* = \frac{1}{\delta_2}[\frac{rx^{*2}}{k} - (r-\delta_1)x^*] > 0, y^* = 0$$

$$\frac{1}{\delta_2}[\frac{rx^*}{k} - (r-\delta_1)]x^* > 0, x^* = 0, \text{or } \frac{rx^*}{k} - (r-\delta_1) > 0$$

$$\frac{rx^*}{k} > (r-\delta_1)$$

$$x^* > \frac{k}{r}(r-\delta_1) \tag{4.23}$$

Similarly, the value of x^* can be determined from Eq. (4.16) as:

$$x^* = \frac{1}{\delta_1}[\frac{sy^*}{k} - (s-\delta_2)y^*] \tag{4.24}$$

While, y^* is a positive root that can be determined from Eq. (4.24), so that:

$$x^* = \frac{1}{\delta_1}[\frac{sy^{*2}}{k} - (s-\delta_2)y^*]$$

$$x^* = \frac{1}{\delta_1}[\frac{sy^{*2}}{k} - (s-\delta_2)y^*] > 0, \ \ x^* = 0, \text{ or } \ \frac{1}{\delta_1}[\frac{sy^*}{k} - (s-\delta_2)]y^* > 0$$

$$y^* > 0, \frac{sy^*}{k} - (s-\delta_2) > 0,$$

$$y^* > \frac{l}{s}(s-\delta_2) \tag{4.25}$$

$$E_2 = (x^*, y^*, 0, 0)$$

Where x^* and y^* positive. Conditions (4.24) and (4.25) represent the necessary conditions for the existence of the planar equilibrium point in the interior of $\mathbb{R}_+^2$ of the $xy-$plane.

The equilibrium point $\left(x^*, y^*, z^* 0,\right)$ exists in the interior of $\mathbb{R}_+^3$ of the $xyz-$plane, if and only if, x^* , y^* and z^* are the positive roots of the following set of algebraic equations:

$$r(1-\frac{x}{k}) - \delta_1 + \frac{\delta_2 y}{x} - \beta_1 xz = 0 \tag{4.26}$$

$$s(1-\frac{y}{l}) - \delta_2 + \frac{\delta_1 x}{y} = 0 \tag{4.27}$$

$$\beta_2 x - \beta_0 = 0 \tag{4.28}$$

Solving the above equations, gives that:

$$x^* = \frac{\beta_0}{\beta_2}$$

$$y^* = \frac{l}{2s\beta_2}[l\beta_2(s-\delta_2) + \sqrt{l^2\beta_2^{\,2}(s-\delta_2)^2 + \_\, 4sl\beta_0\beta_1\delta_1}\,]$$

$$z^* = \frac{\beta_2}{\beta_0\beta_1}[\frac{\beta_0(r-\delta_1)}{\beta_2} - \frac{r\beta_0^{\,2}}{k\beta_2^{\,2}} + \delta_2 y^*$$

For, $\bar{z}$ to be positive, the following condition must holds:

$$\frac{\beta_2}{\beta_0\beta_1}\left(\frac{\beta_0(r-\delta_1)}{\beta_2}\right) + \delta_2 y^* \geq \frac{r\beta_0^{\,2}}{k\beta_2^{\,2}} \tag{4.29}$$

The equilibrium point $\left(x^*, y^*, z^* 0,\right)$ exists in the interior of R_+^3 of the $xyz-$plane, if and only if, x^* , y^* and z^* are the positive roots of the as followed equilibrium point:

.

$$E_3 = \left(\frac{\beta_0}{\beta_2}, \frac{l}{2s\beta_2}[l\beta_2(s-\delta_2) + \sqrt{l^2\beta_2{}^2(s-\delta_2)^2 + \_\ 4sl\beta_0\beta_1\delta_1}], \frac{\beta_2}{\beta_0\beta_1}\left[\frac{\beta_0(r-\delta_1)}{\beta_2} - \frac{r\beta_0{}^2}{k\beta_2{}^2}\right] + \delta_2 y^*, 0 \right)$$

Then to find all positive value

$$r(1-\frac{x}{k}) - \delta_1 + \frac{\delta_2 y}{x} - \beta_1 z = 0 \qquad (4.30)$$

$$s(1-\frac{y}{l}) - \delta_2 + \frac{\delta_1 x}{y} = 0 \qquad (4.31)$$

$$\beta_2 x - \beta_0 - \gamma_1 w = 0 \qquad (4.32)$$

$$\gamma_2 z - \alpha = 0 \qquad (4.33)$$

From the above list of equations, the following is obtained:

$$y^* = \frac{l}{2s}[(s-\delta_2) + \sqrt{(s-\delta_2)^2 \frac{4s\delta_1 x^*}{l}}] \qquad (4.34)$$

$$z^* = \frac{\alpha}{\gamma_2} \qquad (4.35)$$

$$w^* = \frac{\beta_2 x^* - \beta_0}{\gamma_1} \qquad (4.36)$$

By substituting the values of y^* and z^* in Eq. (4.30), a little algebraic manipulation yields:

$$ax^3 + bx^2 + cx + d = 0 \qquad (4.37)$$

Where,

$$a = \left(\frac{-r}{k}\right)^2 > 0$$

$$b = \frac{2r}{k}\left((r-\delta_1) - \frac{\beta_1\alpha}{\gamma_2}\right)$$

$$c = \frac{-2r}{k}\left(\frac{\delta_1 l}{2s}(s-\delta_2)\right) + \left((r-\delta_1) - \frac{\beta_1\alpha}{\gamma_2}\right)^2$$

$$d = -\left[\frac{\delta_2 l}{s}(s-\delta_2)((r-\delta_1) - \frac{\beta_1\alpha}{\gamma_2}) + \frac{\delta_1\delta_2{}^2 l}{s}\right]$$

By using Descartes rule of signs, Eq (4.37) has positive solution, if the following inequalities hold:

$$(r-\delta_1) > \frac{\beta_1\alpha}{\gamma_2} \qquad (4.38)$$

Knowing the value of x^*, the values of y^* and w^* can be computed from Eq. (4.37). It should also be noted that for w^* to be positive, the following must be true.. $\beta_2 x^* > \beta_0$

$$E_4 = \left(x^*, \frac{l}{2s}[(s-\delta_2) + \sqrt{(s-\delta_2)^2 + \frac{4s\delta_1 x^*}{l}}], \frac{\alpha}{\gamma_2}, \frac{\beta_2 x^* - \beta_0}{\gamma_1} \right)$$

4.4 Linearization

In mathematics, linearization is finding the linear approximation to a function at a given point. In the study of dynamical systems, linearization is a method for assessing the local stability of an equilibrium point of a system of nonlinear differential equations or discrete dynamical systems. Linearization can be used to give important information about how the system behaves in the neighborhood of equilibrium points. Linearization makes it possible to use tools for studying linear systems to analyze the behavior of a nonlinear function near a given point. The small perturbation of the homogeneous equilibrium point develops in the large time limit

Let

$$\begin{aligned}
&x_1(t)=x(t)-x^* \quad, x(t)=x_1(t)+x^* \quad \Rightarrow x'(t)=x_1'(t)\\
&y_1(t)=y(t)-y^* \quad, y(t)=y_1(t)+y^* \quad \Rightarrow y'(t)=y_1'(t)\\
&z_1(t)=z(t)-z^* \quad, z(t)=z_1(t)+z^* \quad \Rightarrow z'(t)=z_1'(t)\\
&w_1(t)=w(t)-w^* \quad w(t)=w_1(t)+w^* \quad \Rightarrow w'(t)=w_1'(t)
\end{aligned} \tag{4.39}$$

Plugging Eq. (4.39) into Eq. (1.2). Lead to

$$\begin{cases}
x_1'(t)=(x_1(t)+x^*)r(1-\dfrac{(x_1(t)+x^*)}{k})-\delta_1(x_1(t)+x^*)+\delta_2(y_1(t)+y^*)-\beta_1(x_1(t)+x^*)(z_1(t)+z^*)\\
y_1'(t)=(y_1(t)+y^*)s(1-\dfrac{(y_1(t)+y^*)}{l})-\delta_2(y_1(t)+y^*)+\delta_1(x_1(t)+x^*)\\
z_1'(t)=\beta_2(x_1(t-\tau)+x^*)(z_1(t)+z^*)-\beta_0(z_1(t)+z^*)-\gamma_1(z_1(t)+z^*)(w_1(t)+w^*)\\
w_1'(t)=\gamma_1(z_1(t)+z^*)(w_1(t)+w^*)-\alpha(w_1(t)+w^*)
\end{cases}$$

$$\begin{cases}
x_1'(t)=\dfrac{(x_1(t)rk-x_1^2(t)r-x_1(t)rx^*+rkx^*-x_1(t)rx^*-r(x^*)^2}{k}-\delta_1x_1(t)-\delta_1x^*\\
\qquad +\delta_2y_1(t)+\delta_2y^*-\beta_1x_1(t)z_1(t)-\beta_1z_1(t)x^*-\beta_1x_1(t)-\beta_1z^*x^*\\
y_1'(t)=\dfrac{(y_1(t)sl-y_1^2(t)s-y_1(t)sy^*+sly^*-y_1(t)sy^*-s(y^*)^2}{l}-\delta_2y_1(t)-\delta_2y^*+\delta_1x_1(t)+\delta_1x^*\\
z_1'(t)=\beta_2(x_1(t-\tau)z_1(t)+z_1(t)x^*+x_1(t-\tau)z^*+x^*z^*)-\beta_0z_1(t)-\beta_0z^*\\
\qquad -\gamma_1z_1(t)w_1(t)-\gamma_1w_1(t)z^*-\gamma_1z_1(t)w^*-\gamma_1z^*w^*\\
w_1'(t)=\gamma_2z_1(t)w_1(t)+\gamma_2w_1(t)z^*+\gamma_2z_1(t)w^*+\gamma_2z^*w^*)-\alpha w_1(t)-\alpha w^*
\end{cases}$$

$$\begin{cases} x_1'(t) = \dfrac{(x_1(t)rk - x_1^2(t)r - x_1(t)rx^* + rkx^* - x_1(t)rx^* - r(x^*)^2}{k} - \delta_1 x_1(t) - \delta_1 x^* \\ \qquad + \delta_2 y_1(t) + \delta_2 y^* - \beta_1 x_1(t) z_1(t) - \beta_1 z_1(t) x^* - \beta_1 x_1(t) - \beta_1 z^* x^* \\ y_1'(t) = \dfrac{(y_1(t)sl - y_1^2(t)s - y_1(t)sy^* + sly^* - y_1(t)sy^* - s(y^*)^2}{l} - \delta_2 y_1(t) - \delta_2 y^* + \delta_1 x_1(t) + \delta_1 x^* \\ z_1'(t) = \beta_2 (x_1(t-\tau) z_1(t) + z_1(t) x^* + x_1(t-\tau) z^* + x^* z^*) - \beta_0 z_1(t) - \beta_0 z^* \\ \qquad - \gamma_1 z_1(t) w_1(t) - \gamma_1 w_1(t) z^* - \gamma_1 z_1(t) w^* - \gamma_1 z^* w^* \\ w_1'(t) = \gamma_2 z_1(t) w_1(t) + \gamma_2 w_1(t) z^* + \gamma_2 z_1(t) w^* + \gamma_2 z^* w^*) - \alpha w_1(t) - \alpha w^* \end{cases}$$

$$\begin{cases} x_1'(t) = rx^* - \dfrac{r(x^*)^2}{k} - \delta_1 x^* + \delta_2 y^* - \beta_1 z^* x^* + \dfrac{(x_1(t)rk - x_1^2(t)r - 2x_1(t)rx^*}{k} \\ \qquad - \delta_1 x_1(t) + \delta_2 y_1(t) - \beta_1 x_1(t) z_1(t) - \beta_1 z_1(t) x^* - \beta_1 x_1(t) z^* \\ y_1'(t) = sy^* - \dfrac{s(y^*)^2}{l} - \delta_2 y^* + \delta_1 x^* + y_1(t)sl - \dfrac{y_1^2(t)s - 2y_1(t)sy^*}{l} - l\delta_2 y_1(t) + l\delta_1 x_1(t) \\ z_1'(t) = \beta_2 x^* z^* - \beta_0 z^* - \gamma_1 z^* w^* + \beta_2 x_1(t-\tau) z_1(t) + \beta_2 z_1(t) x^* + \beta_2 x_1(t-\tau) z^* - \beta_0 z_1(t) \\ \qquad - \gamma_1 z_1(t) w_1(t) - \gamma_1 w_1(t) z^* - \gamma_1 z_1(t) w^* \\ w_1'(t) = \gamma_2 z^* w^* - \alpha w^* + \gamma_2 z_1(t) w_1(t) + \gamma_2 w_1(t) z^* + \gamma_2 z_1(t) w^* - \alpha w_1(t) \end{cases}$$

Since (x^*, y^*, z^*, w^*) is an equilibrium point

$$\frac{rkx^* - r(x^*)^2 - k\delta_1 x^* + k\delta_2 y^* - k\beta_1 z^* x^*}{k} = 0$$

$$\frac{sly^* - s(y^*)^2 - l\delta_2 y^* + l\delta_1 x^*}{l} = 0$$

$$\beta_2 x^* z^* - \beta_0 z^* - \gamma_1 z^* w^* = 0$$

$$\gamma_2 z^* w^* - \alpha w^* = 0$$

Hence, the following equation.

$$\begin{cases} x_1'(t) = \dfrac{(x_1(t)rk - x_1^2(t)r - 2x_1(t)rx^*}{k} \dfrac{-k\delta_1 x_1(t) + k\delta_2 y_1(t) - k\beta_1 x_1(t)z_1(t) - k\beta_1 z_1(t)x^* - k\beta_1 x_1(t)z^*}{k} \\ y_1'(t) = \dfrac{(y_1(t)sl - y_1^2(t)s - y_1(t)sy^* - y_1(t)sy^* - l\delta_2 y_1(t) + l\delta_1 x_1(t)}{l} \\ z_1'(t) = +\beta_2 x_1(t-\tau_1)z_1(t) + \beta_2 z_1(t)x^* + \beta_2 x_1(t-\tau_1)z^* - \beta_0 z_1(t) \\ \qquad -\gamma_1 z_1(t)w_1(t) - \gamma_1 w_1(t)z^* - \gamma_1 z_1(t)w^* \\ w_1'(t) = \gamma_2 z_1(t)w_1(t) + \gamma_2 w_1(t)z^* + \gamma_2 z_1(t)w^* - \alpha w_1(t) \end{cases}$$

$$\begin{cases} x_1'(t) = \dfrac{\left(rk - 2rx^* - k\beta_1 z^* - k\delta_1\right)}{k} x_1(t) - \dfrac{r}{k} x_1^2(t) + \delta_2 y_1(t) - \beta_1 x_1(t)z_1(t) - \beta_1 z_1(t)x^* \\ y_1'(t) = \dfrac{\left(sl - 2sy^* - l\delta_2\right)}{l} y_1(t - \dfrac{s}{l} y_1^2(t) + \delta_1 x_1(t) \\ z_1'(t) = \left(\beta_2 x^* - \beta_0 - \gamma_1 w^*\right) z_1(t) + \beta_2 x_1(t-\tau_1)z_1(t) + \beta_2 x_1(t-\tau_1)z^* - \gamma_1 z_1(t)w_1(t) - \gamma_1 w_1(t)z^* \\ w_1'(t) = \left(\gamma_2 z^* - \alpha\right) w_1(t) + \gamma_2 z_1(t)w_1(t) + \gamma_2 z_1(t)w^* \end{cases}$$

$$\begin{aligned} x'(t) &= a_1 x_1(t) - a_2 x_1^2(t) + a_3 y_1(t) - a_4 z_1(t)x_1(t) - a_5 z_1(t) \\ y'(t) &= b_1 y_1(t) - b_2 y_1^2(t) + b_3 x_1(t) \\ z'(t) &= cz_1(t) - c_2 x_1(t-\tau)z_1(t) + c_3 x_1(t-\tau) - c_4 z_1(t)w_1(t) \\ w'(t) &= d_1 w_1(t) + d_2 z_1(t)w_1(t) + d_3 z_1(t) \end{aligned} \tag{4.40}$$

Where

$$a_1 = r - \frac{2x^* r}{k} - \delta_1 - \beta_1 z^*,\ a_2 = -\frac{r}{k},\ a_3 = \delta_2,\quad a_4 = -\beta_1 z^*,\ a_5 = -\beta_1 x^*$$

$$b_1 = s - \frac{2y^* s}{l} - \delta_2,\ b_2 = s,\qquad b_3 = \delta_1$$

$$c_1 = \beta_2 x^* - \beta_0,\quad c_2 = \beta_2,\quad c_3 = -\beta_2 x^*,\quad c_4 = \gamma_1$$

$$d_1 = \gamma_2 z^* - \alpha,\quad d_2 = \gamma_2,\quad d_3 = -\gamma_2 w^*$$

However $x_1(t), y_1(t), z_1(t)$ and $w_1(t)$ are small perturbations hence its products as well as any higher order greater or equal to two goes to zero.

$$x_1^2(t) \to 0,\quad z_1(t)x_1(t) \to 0,\quad y_1^2(t) \to 0,\ x_1(t-\tau_1)z_1(t) \to 0$$

$$z_1(t)w_1(t) \to 0,\qquad z_1(t-\tau_2)w_1(t) \to 0$$

Therefore equation (4.40) reduce to

$$
\begin{aligned}
x'(t) &= a_1x_1 + a_3y_1 - a_5z_1 \\
y'(t) &= b_1y_1 + b_3x_1 \\
z'(t) &= c_1z_1 + c_3x_1(t-\tau) \\
w'(t) &= d_1w_1 + d_3z_1(t)
\end{aligned}
\tag{4.41}
$$

This is the linearized form.

4.5 local stability analysis

Local stability of the model is predicated from the linearized part

$$
\begin{aligned}
x'(t) &= a_1x_1 + a_3y_1 - a_5z_1 \\
y'(t) &= b_1y_1 + b_3x_1 \\
z'(t) &= c_1z_1 + c_3x_1(t-\tau) \\
w'(t) &= d_1w_1 + d_3z_1(t)
\end{aligned}
\tag{4.42}
$$

The characteristic equation.

$$
\begin{cases}
x_1(t) = ne^{\lambda t} \quad then \;\; x_1'(t) = n\lambda e^{\lambda t} \\
y_1(t) = me^{\lambda t} then \;\; y_1'(t) = m\lambda e^{\lambda t} \\
z_1(t) = fe^{\lambda t} then \;\; z_1'(t) = f\lambda e^{\lambda t} \\
w_1(t) = ge^{\lambda t} then \;\; w_1'(t) = g\lambda e^{\lambda t}
\end{cases}
\tag{4.43}
$$

Plugging equation (4.43) into equation (4.42)

$$
\begin{cases}
n\lambda e^{\lambda t} = a_1ne^{\lambda t} + a_3me^{\lambda t} - a_5fe^{\lambda t} \\
m\lambda e^{\lambda t} = b_1me^{\lambda t} + b_3ne^{\lambda t} \\
f\lambda e^{\lambda t} = c_1fe^{\lambda t} + c_3ne^{\lambda(t-\tau)} \\
g\lambda e^{\lambda t} = d_1ge^{\lambda t} + d_3fe^{\lambda(t)}
\end{cases}
$$

$$
\text{Since } e^{\lambda t} \neq 0 \quad
\begin{aligned}
n\lambda &= a_1n + a_3m - a_5f \\
m\lambda &= b_1m + b_3n \\
f\lambda &= c_1f + c_3ne^{-\lambda\tau} \\
g\lambda &= d_1g + d_3f
\end{aligned}
$$

$$\begin{cases} a_1n+a_3m-a_5f-n\lambda=0, \\ b_1m+b_3n-m\lambda=0, \\ c_1f+c_3ne^{-\lambda\tau}-f\lambda=0, \\ d_1g+d_3f-g\lambda=0, \end{cases}$$

$$\begin{cases} (a_1-\lambda)n+a_3m-a_5f=0, \\ (b_1-\lambda)m+b_3n=0, \\ (c_1-\lambda)f+c_3ne^{-\lambda\tau}=0, \\ (d_1-\lambda)g+d_3f=0. \end{cases} \tag{4.44}$$

For Eq. (4.44) to have non-trivial solution the determinant of coefficients matrix must be zero

$$\begin{vmatrix} a_1-\lambda & a_3 & -a_5 & 0 \\ b_3 & b_1-\lambda & 0 & 0 \\ c_3e^{-\lambda\tau} & 0 & c_1-\lambda & 0 \\ 0 & 0 & d_3 & d_1-\lambda \end{vmatrix}=0$$

$$(a_1-\lambda)\begin{vmatrix} b_1-\lambda & 0 & 0 \\ 0 & c_1-\lambda & 0 \\ 0 & d_3 & d_1-\lambda \end{vmatrix}-a_3\begin{vmatrix} b_3 & 0 & 0 \\ c_3e^{-\lambda\tau} & c_1-\lambda & 0 \\ 0 & d_3 & d_1-\lambda \end{vmatrix}+a_5\begin{vmatrix} b_3 & b_1-\lambda & 0 \\ c_3e^{-\lambda\tau} & 0 & 0 \\ 0 & 0 & d_1-\lambda \end{vmatrix}=0$$

$$(a_1-\lambda)(b_1-\lambda)\big((c_1-\lambda)(d_1-\lambda)-(d_3)(0)\big)-a_3(b_3)\big((c_1-\lambda)(d_1-\lambda)-(d_3)(0)\big)$$
$$+a_5\Big[b_3(0)-(b_1-\lambda)\big((c_3e^{-\lambda\tau})(d_1-\lambda)-0\big)\Big]=0$$
$$(a_1-\lambda)(b_1-\lambda)\big((c_1-\lambda)(d_1-\lambda)\big)-a_3(b_3)\big((c_1-\lambda)(d_1-\lambda)\big)$$
$$+a_5\Big(-(b_1-\lambda)\big((c_3e^{-\lambda\tau})(d_1-\lambda)\big)\Big)=0$$

$$(a_1-\lambda)(b_1-\lambda)\big((c_1-\lambda)(d_1-\lambda)\big)-a_3(b_3)\big((c_1-\lambda)(d_1-\lambda)\big)$$
$$-a_5\left(b_1d_1c_3e^{-\lambda\tau}-b_1\lambda c_3e^{-\lambda\tau}-d_1\lambda c_3e^{-\lambda\tau}+\lambda^2c_3e^{-\lambda\tau}\right)=0$$
$$\lambda^4-a_1\lambda^3-b_1\lambda^3-c_1\lambda^3-d_1\lambda^3+a_1b_1\lambda^2+a_1c_1\lambda^2+a_1d_1\lambda^2+c_1d_1\lambda^2+c_1b_1\lambda^2+d_1b_1\lambda^2$$
$$-a_1b_1c_1\lambda-a_1b_1d_1\lambda-b_1d_1c_1\lambda-a_1d_1c_1\lambda+a_1b_1c_1d_1-a_3b_3c_1d_1+a_3b_3c_1\lambda+a_3b_3c_1d_1\lambda-a_3b_3\lambda^2$$
$$-a_5b_1d_1c_3e^{-\lambda\tau}+a_5b_1\lambda c_3e^{-\lambda\tau}+a_5d_1\lambda c_3e^{-\lambda\tau}-a_5\lambda^2c_3e^{-\lambda\tau}=0$$

$$\begin{cases} \lambda^4 - \left(a_1 + b_1 + c_1 + d_1\right)\lambda^3 + \left(a_1b_1 + a_1c_1 + a_1d_1 + c_1d_1 + c_1b_1 + d_1b_1 - a_3b_3 - a_5c_3e^{-\lambda\tau}\right)\lambda^2 \\ -\left(a_1b_1c_1 + a_1b_1d_1 + b_1d_1c_1 + a_1d_1c_1 - a_3b_3c_1 - a_3b_3d_1 - a_5b_1c_3e^{-\lambda\tau} - a_5d_1c_3e^{-\lambda\tau}\right)\lambda \\ +a_1b_1c_1d_1 - a_3b_3c_1d_1 - a_5b_1d_1c_3e^{-\lambda\tau} = 0 \end{cases} \quad (4.45)$$

Which is the characteristic equation of Eq. (4.42)

Case 1 if $\tau = 0$ then the characteristic equation become

$$\begin{cases} \lambda^4 - \left(a_1 + b_1 + c_1 + d_1\right)\lambda^3 + \begin{pmatrix} a_1b_1 + a_1c_1 + a_1d_1 + c_1d_1 + c_1b_1 \\ +d_1b_1 - a_3b_3 - a_5c_3 \end{pmatrix}\lambda^2 \\ -\begin{pmatrix} a_1b_1c_1 + a_1b_1d_1 + b_1d_1c_1 + a_1d_1c_1 - a_3b_3c_1 \\ -a_3b_3d_1 - a_5b_1c_3 - a_5d_1c_3 \end{pmatrix}\lambda \\ +a_1b_1c_1d_1 - a_3b_3c_1d_1 - a_5b_1d_1c_3 = 0 \end{cases} \quad (4.46)$$

$$\lambda^4 + A_1\lambda^3 + A_2\lambda^2 + A_3\lambda + A_4 = 0 \quad (4.47)$$

Where

$$A_1 = -\left(a_1 + b_1 + c_1 + d_1\right)$$
$$A_2 = a_1b_1 + a_1c_1 + a_1d_1 + c_1d_1 + c_1b_1 + d_1b_1 - a_3b_3 - a_5c_3$$
$$A_3 = -\left(a_1b_1c_1 + a_1b_1d_1 + b_1d_1c_1 + a_1d_1c_1 - a_3b_3c_1 - a_3b_3d_1 - a_5b_1c_3 - a_5d_1c_3\right)$$
$$A_4 = a_1b_1c_1d_1 - a_3b_3c_1d_1 - a_5b_1d_1c_3$$

Theorem: The positive equilibrium point of the system given by Eq. (4.38) is locally stable in the absence of time delay. If the following condition.

Proof:

$$\begin{array}{c|ccc} & 1 & A_2 & A_4 \\ \lambda^4 & A_1 & A_3 & 0 \\ \lambda^3 & \dfrac{A_1A_2 - A_3}{A_1} & A_4 & 0 \\ \lambda^2 & & & \\ \lambda^1 & \dfrac{A_1A_2A_3 - A_3^2 - A_1^2A_4}{A_1A_2 - A_3} & 0 & 0 \\ \lambda^0 & & & \\ & A_4 & 0 & 0 \end{array}$$

Where,

$$A_1 > 0 \quad (4.48)$$
$$A_4 > 0 \quad (4.49)$$
$$A_1A_2A_3 > A_4A_1^2 + A_3^2 \quad (4.50)$$

$$A_1 = \left(a_1 + b_1 + c_1 + d_1\right) > 0$$
$$A_1 = \left(r - \frac{2x^* r}{k} - \delta_1 - \beta_1 z^* + s - \frac{2y^* s}{l} - \delta_2 + \beta_2 x^* - \beta_0 + \gamma_2 z^* - \alpha \right) > 0$$

For condition (4.23) and (4.25) equilibrium positive

Therefore $A_1 > 0$

For Eq. (4.49)

$$A_4 > 0$$
$$A_4 = \left(a_1 b_1 c_1 d_1 - a_3 b_3 c_1 d_1 - a_5 b_1 d_1 c_3\right) > 0$$
$$A_4 = \begin{pmatrix} (r - \frac{2x^* r}{k} - \delta_1 - \beta_1 z^*)(s - \frac{2y^* s}{l} - \delta_2)(\beta_2 x^* - \beta_0)(\gamma_2 z^* - \alpha) \\ -(\beta_2 x^* - \beta_0)(\gamma_2 z^* - \alpha)\delta_2 \delta_1 + \beta_1 z^* (s - \frac{2y^* s}{l} - \delta_2)(\gamma_2 z^* - \alpha)\beta_2 x^* \end{pmatrix} > 0$$

The positive equilibrium points (x^*, y^*, z^*, w^*) is locally asymptotically stable when condition (4.48) and (4.49) is satisfied

$$\Delta = A_1 A_2 A_3 - A_3^2 - A_1^2 A_4$$

Now, according to the Routh-Hurwitz criteria, all the eigenvalues of positive equilibrium points have roots with negative real parts,

Case 2: If $\tau \neq 0$ then the characteristic equation becomes

$$\lambda^4 - \left(a_1 + b_1 + c_1 + d_1\right)\lambda^3 + \left(a_1 b_1 + a_1 c_1 + a_1 d_1 + c_1 d_1 + c_1 b_1 + d_1 b_1 - a_3 b_3\right)\lambda^2$$
$$-\left(a_1 b_1 c_1 + a_1 b_1 d_1 + b_1 d_1 c_1 + a_1 d_1 c_1 - a_3 b_3 c_1 - a_3 b_3 d_1\right)\lambda$$
$$+a_1 b_1 c_1 d_1 - a_3 b_3 c_1 d_1 + \left(-a_5 c_3 \lambda^2 - (a_5 b_1 c_3 + a_5 d_1 c_3)\lambda - a_5 b_1 d_1 c_3\right)e^{-\lambda\tau} = 0$$
$$\lambda^4 - \left(a_1 + b_1 + c_1 + d_1\right)\lambda^3 + \left(a_1 b_1 + a_1 c_1 + a_1 d_1 + c_1 d_1 + c_1 b_1 + d_1 b_1 - a_3 b_3 - a_5 c_3 e^{-\lambda\tau}\right)\lambda^2$$
$$-\left(a_1 b_1 c_1 + a_1 b_1 d_1 + b_1 d_1 c_1 + a_1 d_1 c_1 - a_3 b_3 c_1 - a_3 b_3 d_1 - a_5 b_1 c_3 e^{-\lambda\tau} - a_5 d_1 c_3 e^{-\lambda\tau}\right)\lambda$$
$$+a_1 b_1 c_1 d_1 - a_3 b_3 c_1 d_1 - a_5 b_1 d_1 c_3 e^{-\lambda\tau} = 0$$

$$\lambda^4 + M_1 \lambda^3 + M_2 \lambda^2 + M_3 \lambda + M_4 + \left(N_1 \lambda^2 + N_2 \lambda + N_3\right)e^{-\lambda\tau} = 0 \quad (4.51)$$

Where,

$$M_1 = -\left(a_1 + b_1 + c_1 + d_1\right)$$
$$M_2 = \left(a_1b_1 + a_1c_1 + a_1d_1 + c_1d_1 + c_1b_1 + d_1b_1 - a_3b_3\right)$$
$$M_3 = -\left(a_1b_1c_1 + a_1b_1d_1 + b_1d_1c_1 + a_1d_1c_1 - a_3b_3c_1 - a_3b_3d_1\right)$$
$$M_4 = a_1b_1c_1d_1 - a_3b_3c_1d_1$$
$$N_1 = -a_5c_3$$
$$N_2 = -\left(a_5b_1c_3 - a_5d_1c_3\right)$$
$$N_3 = -a_5b_1d_1c_3$$

For $\omega > 0$ suppose $\lambda = \omega i$ is a root of Eq. (4.51) it follows

$$(\omega i)^4 + M_1(\omega i)^3 + M_2(\omega i)^2 + M_3(\omega i) + M_4 + \left(N_1(\omega i)^2 + N_2\omega i + N_3\right)e^{-\omega i\tau} = 0$$
$$\omega^4 - M_1\omega^3 i - M_2\omega^2 + M_3\omega i + M_4 + \left(-N_1\omega^2 + N_2\omega i + N_3\right)(\cos\omega\tau - i\sin\omega\tau) = 0$$
$$\omega^4 - M_1\omega^3 i - M_2\omega^2 + M_3\omega i + M_4 + \begin{pmatrix} -N_1\omega^2\cos\omega\tau + N_1\omega^2 i\sin\omega\tau + N_2\omega i\cos\omega\tau \\ +N_2\omega\sin\omega\tau + N_3\cos\omega\tau - N_3 i\sin\omega\tau \end{pmatrix} = 0$$

Equating real and imaginary parts.

$$\begin{cases} \omega^4 - M_2\omega^2 + M_4 = N_1\omega^2\cos\omega\tau - N_2\omega\sin\omega\tau - N_3\cos\omega\tau \\ -M_1\omega^3 + M_3\omega = -N_1\omega^2\sin\omega\tau - N_2\omega\cos\omega\tau + N_3\sin\omega\tau \end{cases} \tag{4.52}$$

Squaring both side of Eq. (4.49)

$$\left(\omega^4 - M_2\omega^2 + M_4\right)^2 = \left(N_1\omega^2\cos\omega\tau - N_2\omega\sin\omega\tau - N_3\cos\omega\tau\right)^2 \tag{4.53}$$
$$\left(-M_1\omega^3 + M_3\omega\right)^2 = \left(-N_1\omega^2\sin\omega\tau - N_2\omega\cos\omega\tau + N_3\sin\omega\tau\right)^2 \tag{4.54}$$

$$\begin{cases} \omega^8 - 2M_2\omega^6 \\ +2M_4\omega^4 - 2M_4M_2\omega^2 + M_2{}^2\omega^4 + M_4{}^2 = \begin{pmatrix} N_1{}^2\omega^4\cos^2(\omega\tau) + N_2{}^2\omega^2\sin^2(\omega\tau) \\ +2N_2N_1\omega^3\cos(\omega\tau)\sin(\omega\tau) \\ -2N_2N_3\sin(\omega\tau)\cos(\omega\tau) \\ -2N_1{}^2N_3\omega^2\cos^2(\omega\tau) + N_3{}^2\cos^2(\omega\tau) \end{pmatrix} \end{cases} \tag{4.55}$$

$$\begin{cases} M_1\omega^6 - 2M_1M_3\omega^4 + M_3{}^2\omega^2 = \begin{pmatrix} N_1{}^2\omega^4\sin^2(\omega\tau) + N_2{}^2\omega^2\cos^2(\omega\tau) \\ -2N_2N_1\omega^3\cos(\omega\tau)\sin(\omega\tau) \\ +2N_2N_3\sin(\omega\tau)\cos(\omega\tau) \\ -2N_1{}^2N_3\omega^2\sin^2(\omega\tau) + N_3{}^2\sin^2(\omega\tau) \end{pmatrix} \end{cases} \tag{4.56}$$

Adding Eq. (4.55) and Eq. (4.56)

$$\begin{cases}\omega^8 - 2M_2\omega^6 + 2M_4\omega^4 - 2M_4M_2\omega^2 + \\ M_2{}^2\omega^4 + M_4{}^2 + M_1{}^2\omega^6 - 2M_1M_3\omega^4 + M_3{}^2\omega^2\end{cases} = N_1{}^2\omega^4 + N_2\omega^2 - 2N_1N_3\omega^2 + N_3{}^2$$

$$\begin{cases}\omega^8 - 2M_2\omega^6 + 2M_4\omega^4 - 2M_4M_2\omega^2 \\ +M_2{}^2\omega^4 + M_4{}^2 + M_1{}^2\omega^6 - 2M_1M_3\omega^4 \\ +M_3{}^2\omega^2 - N_1{}^2\omega^4 - N_2\omega^2 + 2N_1N_3\omega^2 - N_3{}^2\end{cases} = 0$$

$$\begin{cases}\omega^8 + (M_1{}^2 - 2M_2)\omega^6 + (M_2{}^2 - 2M_1M_3 + 2M_4 - N_1{}^2)\omega^4 \\ +(-2M_4M_2 + M_3{}^2 - N_2 + 2N_1N_3)\omega^2 + M_4{}^2 - N_3{}^2\end{cases} = 0$$

$$\omega^8 + P\omega^6 + Q\omega^4 + R\omega^2 + T = 0 \quad , \tag{4.57}$$

Where,

$$P = M_1^2 - 2M_2$$
$$Q = M_2{}^2 - 2M_1M_3 + 2M_4 - N_1^2$$
$$R = -2M_4M_2 + M_3{}^2 - N_2 + 2N_1N_3$$
$$T = M_4{}^2 - N_3{}^2$$

Remark: $P > 0$

Proof.

$$P = M_1^2 - 2M_2 \tag{4.58}$$
$$M_1 = -\left(a_1 + b_1 + c_1 + d_1\right) \tag{4.59}$$
$$M_2 = \left(a_1b_1 + a_1c_1 + a_1d_1 + c_1d_1 + c_1b_1 + d_1b_1 - a_3b_3\right) \tag{4.60}$$

Then for this $M_1^2 - 2M_2 > 0$

$$\left(-\left(a_1 + b_1 + c_1 + d_1\right)\right)^2 - 2\left(a_1b_1 + a_1c_1 + a_1d_1 + c_1d_1 + c_1b_1 + d_1b_1 - a_3b_3\right) > 0$$

$$\begin{cases}a_1{}^2 + b_1{}^2 + c_1{}^2 + d_1{}^2 + 2a_1b_1 + 2a_1c_1 + 2a_1d_1 + 2c_1d_1 + 2c_1b_1 + 2d_1b_1 \\ -2a_1b_1 - 2a_1c_1 - 2a_1d_1 - 2c_1d_1 - 2c_1b_1 - 2d_1b_1 + 2a_3b_3\end{cases} > 0$$

$$a_1{}^2 + b_1{}^2 + c_1{}^2 + d_1{}^2 + 2a_3b_3 > 0$$

$$\left(r - \frac{2x^*r}{k} - \delta_1 - \beta_1 z^*\right)^2 + \left(s - \frac{2y^*s}{l} - \delta_2\right)^2 + \left(\beta_2 x^* - \beta_0\right)^2 + \left(\gamma_2 z^* - \alpha\right)^2 + 2\delta_1\ \delta_2 > 0$$

Therefore $P>0$ for this parameter is positives

To find the minimum values of τ for which the stability of the system lost substitute ω_0 in Eq. (4.51) we obtain.

$$\begin{cases} \omega_0^{\,4} - M_2\omega_0^{\,2} + M_4 = N_1\omega_0^{\,2}\cos(\omega_0\tau) - N_2\omega_0\sin(\omega_0\tau) - N_3\cos(\omega_0\tau) & (4.61) \\ -M_1\omega_0^{\,3} + M_3\omega_0 = -N_1\omega_0^{\,2}\sin(\omega_0\tau) - N_2\omega_0\cos(\omega_0\tau) + N_3\sin(\omega_0\tau) & (4.62) \end{cases}$$

Hence, Eq. (4.61) and Eq. (4.62) both side multiple by $N_2\omega_0$ and $(N_1\omega_0^{\,2} + N_3)$ respectively.

$$N_2\omega_0(\omega_0^{\,4} - M_2\omega_0^{\,2} + M_4) = N_2\omega_0(N_1\omega_0^{\,2} - N_3)\cos(\omega_0\tau) - N_2^{\,2}\omega_0^{\,2}\sin(\omega_0\tau)$$
$$(N_1\omega_0^{\,2} - N_3)(-M_1\omega_0^{\,3} + M_3\omega_0) = (N_1\omega_0^{\,2} - N_3)(-N_1\omega_0^{\,2} + N_3)\sin(\omega_0\tau) - (N_1\omega_0^{\,2} - N_3)N_2\omega_0\cos(\omega_0\tau)$$

Then the add two equation.

$$N_2\omega_0(\omega_0^{\,4} - M_2\omega_0^{\,2} + M_4) + (N_1\omega_0^{\,2} - N_3)(-M_1\omega_0^{\,3} + M_3\omega_0) = \left(N_1^{\,2}\omega_0^{\,4} - N_3^{\,2} - N_2^{\,2}\omega_0^{\,2}\right)\sin(\omega_0\tau)$$
$$\left(N_1^{\,2}\omega_0^{\,4} - N_3^{\,2} - N_2^{\,2}\omega_0^{\,2}\right)\sin(\omega_0\tau) = \begin{cases} N_2\omega_0(\omega_0^{\,4} - M_2\omega_0^{\,2} + M_4) + \\ (N_1\omega_0^{\,2} - N_3)(-M_1\omega_0^{\,3} + M_3\omega_0) \end{cases}$$
$$\sin(\omega_0\tau) = \frac{N_2\omega_0(\omega_0^{\,4} - M_2\omega_0^{\,2} + M_4) + (N_1\omega_0^{\,2} - N_3)(-M_1\omega_0^{\,3} + M_3\omega_0)}{\left(N_1^{\,2}\omega_0^{\,4} - N_3^{\,2} - N_2^{\,2}\omega_0^{\,2}\right)}$$

$$\sin(\omega_0\tau - 2\pi k) = \frac{N_2\omega_0(\omega_0^{\,4} - M_2\omega_0^{\,2} + M_4) + (N_1\omega_0^{\,2} - N_3)(-M_1\omega_0^{\,3} + M_3\omega_0)}{\left(N_1^{\,2}\omega_0^{\,4} - N_3^{\,2} - N_2^{\,2}\omega_0^{\,2}\right)} \qquad (4.63)$$

For Eq. (4.61) and Eq. (4.62) both side multiple by $(N_1\omega_0^{\,2} - N_3)$ and $N_2\omega_0$ respectively

$$(-N_1\omega_0^{\,2} + N_3)(\omega_0^{\,4} - M_2\omega_0^{\,2} + M_4) = (-N_1^{\,2}\omega_0^{\,4} - N_3^{\,2})\cos(\omega_0\tau) - (-N_1\omega_0^{\,2} + N_3)N_2\omega_0\sin(\omega_0$$
$$N_2\omega_0(-M_1\omega_0^{\,3} + M_3\omega_0) = N_2\omega_0(-N_1\omega_0^{\,2} + N_3)\sin(\omega_0\tau) - N_2^{\,2}\omega_0^{\,2}\cos(\omega_0\tau)$$

Then add two equation

$$\begin{pmatrix}(-N_1\omega_0^{\,2}+N_3)(\omega_0^{\,4}-M_2\omega_0^{\,2}+M_4)\\ +N_2\omega_0(-M_1\omega_0^{\,3}+M_3\omega_0)\end{pmatrix}=\left(-N_1^{\,2}\omega_0^{\,4}-N_3^{\,2}-N_2^{\,2}\omega_0^{\,2}\right)\cos(\omega_0\tau)$$

$$\left(-N_1^{\,2}\omega_0^{\,4}-N_3^{\,2}-N_2^{\,2}\omega_0^{\,2}\right)\cos(\omega_0\tau)=\begin{cases}(N_1\omega_0^{\,2}-N_3)(\omega_0^{\,4}-M_2\omega_0^{\,2}+M_4)+\\ N_2\omega_0(-M_1\omega_0^{\,3}+M_3\omega_0)\end{cases}$$

$$\cos(\omega_0\tau)=\frac{(N_1\omega_0^{\,2}-N_3)(\omega_0^{\,4}-M_2\omega_0^{\,2}+M_4)+N_2\omega_0(-M_1\omega_0^{\,3}+M_3\omega_0)}{\left(-N_1^{\,2}\omega_0^{\,4}-N_3^{\,2}-N_2^{\,2}\omega_0^{\,2}\right)}$$

$$\cos(\omega_0\tau-2\pi k)=\frac{N_2\omega_0(\omega_0^{\,4}-M_2\omega_0^{\,2}+M_4)+(N_1\omega_0^{\,2}-N_3)(-M_1\omega_0^{\,3}+M_3\omega_0)}{\left(-N_1^{\,2}\omega_0^{\,4}-N_3^{\,2}-N_2^{\,2}\omega_0^{\,2}\right)} \qquad (4.64)$$

Dividing Eq. (4.63) by Eq. (4.64).

$$\frac{\sin(\omega_0\tau-2\pi k)}{\cos(\omega_0\tau-2\pi k)}=\frac{\dfrac{N_2\omega_0(\omega_0^{\,4}-M_2\omega_0^{\,2}+M_4)+(N_1\omega_0^{\,2}-N_3)(-M_1\omega_0^{\,3}+M_3\omega_0)}{\left(N_1^{\,2}\omega_0^{\,4}-N_3^{\,2}-N_2^{\,2}\omega_0^{\,2}\right)}}{\dfrac{N_2\omega_0(\omega_0^{\,4}-M_2\omega_0^{\,2}+M_4)+(N_1\omega_0^{\,2}-N_3)(-M_1\omega_0^{\,3}+M_3\omega_0)}{\left(-N_1^{\,2}\omega_0^{\,4}-N_3^{\,2}-N_2^{\,2}\omega_0^{\,2}\right)}}$$

$$\tan(\omega_0\tau-2\pi k)=\frac{\begin{pmatrix}N_2\omega_0(\omega_0^{\,4}-M_2\omega_0^{\,2}+M_4)+\\(N_1\omega_0^{\,2}-N_3)(-M_1\omega_0^{\,3}+M_3\omega_0)\end{pmatrix}\left(-N_1^{\,2}\omega_0^{\,4}-N_3^{\,2}-N_2^{\,2}\omega_0^{\,2}\right)}{\left(N_1^{\,2}\omega_0^{\,4}-N_3^{\,2}-N_2^{\,2}\omega_0^{\,2}\right)\begin{pmatrix}N_2\omega_0(\omega_0^{\,4}-M_2\omega_0^{\,2}+M_4)+\\(N_1\omega_0^{\,2}-N_3)(-M_1\omega_0^{\,3}+M_3\omega_0)\end{pmatrix}}$$

$$\omega_0\tau-2\pi k=\arctan\left\{\frac{\begin{pmatrix}N_2\omega_0(\omega_0^{\,4}-M_2\omega_0^{\,2}+M_4)+\\(N_1\omega_0^{\,2}-N_3)(-M_1\omega_0^{\,3}+M_3\omega_0)\end{pmatrix}\left(-N_1^{\,2}\omega_0^{\,4}-N_3^{\,2}-N_2^{\,2}\omega_0^{\,2}\right)}{\left(N_1^{\,2}\omega_0^{\,4}-N_3^{\,2}-N_2^{\,2}\omega_0^{\,2}\right)\begin{pmatrix}N_2\omega_0(\omega_0^{\,4}-M_2\omega_0^{\,2}+M_4)+\\(N_1\omega_0^{\,2}-N_3)(-M_1\omega_0^{\,3}+M_3\omega_0)\end{pmatrix}}\right\}$$

$$\omega_0\tau=\arctan\left\{\frac{\begin{pmatrix}N_2\omega_0(\omega_0^{\,4}-M_2\omega_0^{\,2}+M_4)+\\(N_1\omega_0^{\,2}-N_3)(-M_1\omega_0^{\,3}+M_3\omega_0)\end{pmatrix}\left(-N_1^{\,2}\omega_0^{\,4}-N_3^{\,2}-N_2^{\,2}\omega_0^{\,2}\right)}{\left(N_1^{\,2}\omega_0^{\,4}-N_3^{\,2}-N_2^{\,2}\omega_0^{\,2}\right)\begin{pmatrix}N_2\omega_0(\omega_0^{\,4}-M_2\omega_0^{\,2}+M_4)+\\(N_1\omega_0^{\,2}-N_3)(-M_1\omega_0^{\,3}+M_3\omega_0)\end{pmatrix}}+2\pi k\right\}$$

$$\tau=\frac{1}{\omega_0}\arctan\left\{\frac{\begin{pmatrix}N_2\omega_0(\omega_0^{\,4}-M_2\omega_0^{\,2}+M_4)+\\(N_1\omega_0^{\,2}-N_3)(-M_1\omega_0^{\,3}+M_3\omega_0)\end{pmatrix}\left(-N_1^{\,2}\omega_0^{\,4}-N_3^{\,2}-N_2^{\,2}\omega_0^{\,2}\right)}{\left(N_1^{\,2}\omega_0^{\,4}-N_3^{\,2}-N_2^{\,2}\omega_0^{\,2}\right)\begin{pmatrix}N_2\omega_0(\omega_0^{\,4}-M_2\omega_0^{\,2}+M_4)+\\(N_1\omega_0^{\,2}-N_3)(-M_1\omega_0^{\,3}+M_3\omega_0)\end{pmatrix}}+\frac{1}{\omega_0}2\pi k\right\}$$

If $k=0, \tau_0 = \frac{1}{\omega_0} \arctan \left\{ \frac{\begin{pmatrix} N_2\omega_0(\omega_0{}^4 - M_2\omega_0{}^2 + M_4) + \\ (N_1\omega_0{}^2 - N_3)(-M_1\omega_0{}^3 + M_3\omega_0) \end{pmatrix}\left(-N_1{}^2\omega_0{}^4 - N_3{}^2 - N_2{}^2\omega_0{}^2\right)}{\left(N_1{}^2\omega_0{}^4 - N_3{}^2 - N_2{}^2\omega_0{}^2\right)\begin{pmatrix} N_2\omega_0(\omega_0{}^4 - M_2\omega_0{}^2 + M_4) + \\ (N_1\omega_0{}^2 - N_3)(-M_1\omega_0{}^3 + M_3\omega_0) \end{pmatrix}} \right\}$

The smallest out off value at which stability of the equilibrium points is lost and never be regained in the future time.

4.6 Hopf Bifurcation

Bifurcation study is a powerful tool in understanding an ecological community because bifurcation implies an abrupt change from one state to the other. For predator-prey systems, the population of prey and predators may stay at a steady state or oscillate periodically. The bifurcation parameter considered in the model is time delay.

1. From the characteristic equation (4.51), suppose it has a simple pair of pure imaginary Eigenvalues $\lambda = i\omega, \omega > 0$. By the same analysis made for local stability with delay, there exist when $\omega > 0$ condition (4.58) is satisfied.

2. Transeversality condition

$$\lambda^4 + M_1\lambda^3 + M_2\lambda^2 + M_3\lambda + M_4 + \left(N_1\lambda^2 + N_2\lambda + N_3\right)e^{-\lambda\tau} = 0 \qquad (4.65)$$

Differentiate both sides of (4.51) with respect to τ where λ is a function of τ

$$4\lambda^3 \frac{d\lambda}{d\tau} + 3M_1\lambda^2 \frac{d\lambda}{d\tau} + 2M_2\lambda \frac{d\lambda}{d\tau} + M_3 \frac{d\lambda}{d\tau} + \frac{d\lambda}{d\tau}\left(N_1\lambda^2 + N_2\lambda + N_3\right)e^{-\lambda\tau} = 0$$

$$\begin{aligned} &4\lambda^3 \frac{d\lambda}{d\tau} + 3M_1\lambda^2 \frac{d\lambda}{d\tau} + 2M_2\lambda \frac{d\lambda}{d\tau} + M_3 \frac{d\lambda}{d\tau} + 2N_1\lambda e^{-\lambda\tau} \frac{d\lambda}{d\tau} \\ &\quad - N_1\lambda^2\tau e^{-\lambda\tau} \frac{d\lambda}{d\tau} - N_1\lambda^3 e^{-\lambda\tau} + N_2 e^{-\lambda\tau} \frac{d\lambda}{d\tau} - N_2\tau\lambda e^{-\lambda\tau_1} \frac{d\lambda}{d\tau} \\ &\quad - N_2\lambda^2 e^{-\lambda\tau} - N_3\tau e^{-\lambda\tau} \frac{d\lambda}{d\tau} - N_3\lambda e^{-\lambda\tau} = 0 \end{aligned}$$

$$\begin{aligned} &\begin{pmatrix} 4\lambda^3 + 3M_1\lambda^2 + 2M_2\lambda + M_3 + 2N_1\lambda e^{-\lambda\tau} \\ -N_1\lambda^2\tau e^{-\lambda\tau} + N_2 e^{-\lambda\tau} - N_2\tau\lambda e^{-\lambda\tau} - N_3\tau_1 e^{-\lambda\tau} \end{pmatrix}\frac{d\lambda}{d\tau} \\ &\qquad - N_1\lambda^3 e^{-\lambda\tau} - N_2\lambda^2 e^{-\lambda\tau} - N_3\lambda e^{-\lambda\tau} = 0 \end{aligned}$$

$$\begin{aligned} &\left(4\lambda^3 + 3M_1\lambda^2 + 2M_2\lambda + M_3 + \left(2\lambda N_1 - \lambda^2 N_1\tau + N_2 - \tau\lambda N_2 - N_3\tau\right)e^{-\lambda\tau}\right)\frac{d\lambda}{d\tau} \\ &\qquad = \lambda^2 N_2 e^{-\lambda\tau} + \lambda^3 N_1 e^{-\lambda\tau} + \lambda N_3 e^{-\lambda\tau} \end{aligned}$$

$$\frac{d\lambda}{d\tau}=\frac{N_1\lambda^3e^{-\lambda\tau}+N_2\lambda^2e^{-\lambda\tau}+N_3\lambda e^{-\lambda\tau}}{\left(4\lambda^3+3M_1\lambda^2+2M_2\lambda+M_3+(2N_1\lambda-N_1\lambda^2\tau+N_2-N_2\tau\lambda-N_3\tau_1)e^{-\lambda\tau}\right)}$$

$$\left(\frac{d\lambda}{d\tau}\right)^{-1}=\frac{4\lambda^3+3M_1\lambda^2+2M_2\lambda+M_3+(2N_1\lambda-N_1\lambda^2\tau+N_2-N_2\tau\lambda-N_3\tau)e^{-\lambda\tau}}{N_1\lambda^3e^{-\lambda\tau}+N_2\lambda^2e^{-\lambda\tau}+N_3\lambda e^{-\lambda\tau}}$$

$$\left(\frac{d\lambda}{d\tau}\right)^{-1}=\frac{4\lambda^3+3M_1\lambda^2+2M_2\lambda+M_3+(2N_1\lambda-N_1\lambda^2\tau+N_2-N_2\tau\lambda-N_3\tau)e^{-\lambda\tau}}{\left(N_2\lambda^2+N_1\lambda^3+N_3\lambda\right)e^{-\lambda\tau}}$$

$$\left(\frac{d\lambda}{d\tau}\right)^{-1}=\frac{4\lambda^3+3M_1\lambda^2+2M_2\lambda+M_3}{\left(N_2\lambda^2+N_1\lambda^3+N_3\lambda\right)e^{-\lambda\tau}}+\frac{(2N_1\lambda-N_1\lambda^2\tau+N_2-N_2\tau\lambda-N_3\tau)e^{-\lambda\tau}}{\left(N_2\lambda^2+N_1\lambda^3+N_3\lambda\right)e^{-\lambda\tau}}$$

$$\left(\frac{d\lambda}{d\tau}\right)^{-1}=\frac{4\lambda^3+3M_1\lambda^2+2M_2\lambda+M_3}{\left(N_2\lambda^2+N_1\lambda^3+N_3\lambda\right)e^{-\lambda\tau}}+\frac{\left(2N_1\lambda+N_2\right)e^{-\lambda\tau}}{\left(N_2\lambda^2+N_1\lambda^3+N_3\lambda\right)e^{-\lambda\tau}}-\frac{\tau(N_1\lambda^2+N_2\lambda+N_3)e^{-\lambda\tau}}{\lambda\left(N_2\lambda+N_1\lambda^2+N_3\right)e^{-\lambda\tau}}$$

$$\left(\frac{d\lambda}{d\tau}\right)^{-1}=\frac{4\lambda^3+3M_1\lambda^2+2M_2\lambda+M_3}{\left(N_2\lambda^2+N_1\lambda^3+N_3\lambda\right)e^{-\lambda\tau}}+\frac{2N_1\lambda+N_2}{\left(N_2\lambda^2+N_1\lambda^3+N_3\lambda\right)}-\frac{\tau}{\lambda}$$

$$\left(\frac{d\lambda}{d\tau}\right)^{-1}=\frac{4\lambda^3+3M_1\lambda^2+2(M_2+N_1)\lambda+N_2+M_3}{\left(N_2\lambda^2+N_1\lambda^3+N_3\lambda\right)e^{-\lambda\tau}}-\frac{\tau}{\lambda}$$

$$\left(\frac{d\lambda}{d\tau}\right)^{-1}=\frac{4\lambda^3}{\lambda\left(N_2\lambda+N_1\lambda^2+N_3\right)e^{-\lambda\tau}}+\frac{3M_1\lambda^2}{\lambda\left(N_2\lambda+N_1\lambda^2+N_3\right)e^{-\lambda\tau}}+\frac{2(M_2+N_1)\lambda}{\lambda\left(N_2\lambda+N_1\lambda^2+N_3\right)e^{-\lambda\tau}}+\frac{N_2+M_3}{\lambda\left(N_2\lambda+N_1\lambda^2+N_3\right)e^{-\lambda\tau}}-\frac{\tau}{\lambda}$$

$$\left(\frac{d\lambda}{d\tau}\right)^{-1}=\frac{4\lambda^2}{\left(N_1\lambda^2+N_2\lambda+N_3\right)e^{-\lambda\tau}}+\frac{3M_1\lambda}{\left(N_1\lambda^2+N_2\lambda+N_3\right)e^{-\lambda\tau}}+\frac{2(M_2+N_1)}{\left(N_1\lambda^2+N_2\lambda+N_3\right)e^{-\lambda\tau}}+\frac{N_2+M_3}{\lambda\left(N_1\lambda^2+N_2\lambda+N_3\right)e^{-\lambda\tau}}-\frac{\tau}{\lambda}$$

$\lambda = \omega i (\omega > 0)$

$$\left(\frac{d\lambda}{d\tau}\right)^{-1}_{\lambda=\omega i} = \frac{4(\omega i)^2}{\left(N_1(\omega i)^2 + N_2\omega i + N_3\right)e^{-\omega i\tau}} + \frac{3M_1\omega i}{\left(N_1(\omega i)^2 + N_2\omega i + N_3\right)e^{-\omega i\tau}}$$
$$+ \frac{2(M_2 + N_1)}{\left(N_1(\omega i)^2 + N_2\omega i + N_3\right)e^{-\omega i\tau}} + \frac{N_2 + M_3}{\omega i\left(N_1(\omega i)^2 + N_2\omega i + N_3\right)e^{-\omega i\tau}} - \frac{\tau}{\omega i}$$

$$\left(\frac{d\lambda}{d\tau}\right)^{-1}_{\lambda=\omega i} = \frac{-4\omega^2}{\left(-N_1\omega^2 + N_2\omega i + N_3\right)e^{-\omega i\tau}} + \frac{3M_1\omega i}{\left(N_1(\omega i)^2 + N_2\omega i + N_3\right)e^{-\omega i\tau}}$$
$$+ \frac{2(M_2 + N_1)}{\left(-N_1\omega^2 + N_2\omega i + N_3\right)e^{-\omega i\tau}} + \frac{N_2 + M_3}{\omega i\left(-N_1\omega^2 + N_2\omega i + N_3\right)e^{-\omega i\tau}} - \frac{\tau}{\omega i}$$

$$\left(\frac{d\lambda}{d\tau}\right)^{-1}_{\lambda=\omega i} = \frac{-4\omega^2 e^{\omega i\tau}}{\left(-N_1\omega^2 + N_2\omega i + N_3\right)} + \frac{3(M_1\omega i)e^{\omega i\tau}}{\left(-N_1\omega^2 + N_2\omega i + N_3\right)}$$
$$+ \frac{2(M_2 + N_1)e^{\omega i\tau}}{\left(-N_1\omega^2 + N_2\omega i + N_3\right)} + \frac{(N_2 + M_3)e^{\omega i\tau}}{\left(-iN_1\omega^3 - N_2\omega^2 + \omega i N_3\right)} - \frac{\tau}{\omega i}$$

$$\left(\frac{d\lambda}{d\tau}\right)^{-1}_{\lambda=\omega i} = \frac{-4\omega^2(\cos(\omega\tau) + i\sin(\omega\tau))}{\left(-N_1\omega^2 + N_2\omega i + N_3\right)} + \frac{3(M_1\omega i)(\cos(\omega\tau) + i\sin(\omega\tau))}{\left(-N_1\omega^2 + N_2\omega i + N_3\right)}$$
$$+ \frac{2(M_2 + N_1)(\cos(\omega\tau) + i\sin(\omega\tau))}{\left(-N_1\omega^2 + N_2\omega i + N_3\right)} + \frac{(N_2 + M_3)(\cos(\omega\tau) + i\sin(\omega\tau))}{\left(-iN_1\omega^3 - N_2\omega^2 + \omega i N_3\right)} - \frac{\tau}{\omega i}$$

$$\left(\frac{d\lambda}{d\tau}\right)^{-1}_{\lambda=\omega i} = \frac{-4\omega^2\cos(\omega\tau) - 4\omega^2 i\sin(\omega\tau)}{\left(-N_1\omega^2 + N_2\omega i + N_3\right)} + \frac{3M_1\omega i\cos(\omega\tau) - 3M_1\omega\sin(\omega\tau)}{\left(-N_1\omega^2 + N_2\omega i + N_3\right)}$$
$$+ \frac{2(M_2 + N_1)(\cos(\omega\tau) + 2(M_2 + N_1)i\sin(\omega\tau)}{\left(-N_1\omega^2 + N_2\omega i + N_3\right)}$$
$$+ \frac{(N_2 + M_3)\cos(\omega\tau) + (N_2 + M_3)i\sin(\omega\tau)}{\left(-iN_1\omega^3 - N_2\omega^2 + \omega i N_3\right)} - \frac{\tau}{\omega i}$$

$$\left(\frac{d\lambda}{d\tau}\right)^{-1}_{\lambda=\omega i} = \frac{-4\omega^2\cos(\omega\tau) - 4\omega^2 i\sin(\omega\tau)\left(-N_1\omega^2 + N_2\omega i + N_3\right)}{\left(-N_1\omega^2 + N_2\omega i + N_3\right)\left(-N_1\omega^2 + N_2\omega i + N_3\right)}$$
$$+ \frac{(3M_1\omega i\cos(\omega\tau) - 3M_1\omega\sin(\omega\tau))\left(-N_1\omega^2 + N_2\omega i + N_3\right)}{\left(-N_1\omega^2 + N_2\omega i + N_3\right)\left(-N_1\omega^2 + N_2\omega i + N_3\right)}$$
$$+ \frac{2(M_2 + N_1)(\cos(\omega\tau) + 2(M_2 + N_1)i\sin(\omega\tau)\left(-N_1\omega^2 + N_2\omega i + N_3\right)}{\left(-N_1\omega^2 + N_2\omega i + N_3\right)\left(-N_1\omega^2 + N_2\omega i + N_3\right)}$$
$$+ \frac{(N_2 + M_3)\cos(\omega\tau) + (N_2 + M_3)i\sin(\omega\tau)\left(-iN_1\omega^3 - N_2\omega^2 + N_3\omega i\right)}{\left(-iN_1\omega^3 - N_2\omega^2 + N_3\omega i\right)\left(-iN_1\omega^3 - N_2\omega^2 + N_3\omega i\right)} + \frac{\tau\omega i}{\omega^2}$$

$$\left(\frac{d\lambda}{d\tau}\right)^{-1}_{\lambda=\omega i} = \frac{-4\omega^2\cos(\omega\tau) - 4\omega^2 i\sin(\omega\tau)\left(-N_1\omega^2 + N_2\omega i + N_3\right)}{\left(N_1^{\,2}\omega^4 + N_2^{\,2}\omega - 2N_1N_2\omega^2 + N_3^{\,2}\right)}$$

$$+\frac{(3M_1\omega i\cos(\omega\tau) - 3M_1\omega\sin(\omega\tau))\left(-N_1\omega^2 + N_2\omega i + N_3\right)}{\left(N_1^{\,2}\omega^4 + N_2^{\,2}\omega - 2N_1N_2\omega^2 + N_3^{\,2}\right)}$$

$$+\frac{2(M_2 + N_1)((\cos(\omega\tau) + i\sin(\omega\tau))\left(-N_1\omega^2 + N_2\omega i + N_3\right)}{\left(N_1^{\,2}\omega^4 + N_2^{\,2}\omega - 2N_1N_2\omega^2 + N_3^{\,2}\right)}$$

$$+\frac{(N_2 + M_3)\cos(\omega\tau) + (N_2 + M_3)i\sin(\omega\tau)\left(iN_1\omega^3 - N_2\omega^2 + N_3\omega i\right)}{\left(-N_1^{\,2}\omega^6 - 2N_1N_3\omega^4 + N_2^{\,2}\omega^4 + N_3^{\,2}\omega^2\right)} + \frac{\tau\omega i}{\omega^2}$$

$$\left(\frac{d\lambda}{d\tau}\right)^{-1}_{\lambda=\omega i} = \frac{\begin{array}{l}-4\omega^2\cos(\omega\tau)(-N_1\omega^2 + N_2\omega i + N_3)\\ -4\omega^2 i\sin(\omega\tau)\left(-N_1\omega^2 + N_2\omega i + N_3\right)\end{array}}{\left(N_1^{\,2}\omega^4 + N_2^{\,2}\omega - 2N_1N_2\omega^2 + N_3^{\,2}\right)}$$

$$+\frac{\begin{array}{l}(-3N_1M_1\omega^2 - 3M_1N_2\omega^2 + 3N_3M_1\omega i)\cos(\omega\tau)\\ -3M_1\omega\sin(\omega\tau))\left(-3N_1\omega^2 - 3M_1N_2\omega^2 + 3N_3M_1\omega i\right)\end{array}}{\left(N_1^{\,2}\omega^4 + N_2^{\,2}\omega - 2N_1N_2\omega^2 + N_3^{\,2}\right)}$$

$$+\frac{\begin{array}{l}2(M_2 + N_1)(\cos(\omega\tau)\left(-N_1\omega^2 + N_2\omega i + N_3\right)\\ +2(M_2 + N_1)i\sin(\omega\tau)\left(-N_1\omega^2 + N_2\omega i + N_3\right)\end{array}}{\left(N_1^{\,2}\omega^4 + N_2^{\,2}\omega - 2N_1N_2\omega^2 + N_3^{\,2}\right)}$$

$$+\frac{\begin{array}{l}(N_2 + M_3)\cos(\omega\tau)\left(iN_1\omega^3 - N_2\omega^2 + N_3\omega i\right)\\ +(N_2 + M_3)i\sin(\omega\tau)\left(iN_1\omega^3 - N_2\omega^2 + N_3\omega i\right)\end{array}}{\left(-N_1^{\,2}\omega^6 - 2N_1N_3\omega^4 + N_2^{\,2}\omega^4 + N_3^{\,2}\omega^2\right)} + \frac{\tau\omega i}{\omega^2}$$

Identify the real part of $\left(\frac{d\lambda}{d\tau}\right)^{-1}_{\lambda=\omega i}$ and after some simplification we get

$$\operatorname{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}_{\lambda=\omega i} = \frac{-4\omega^2\cos(\omega\tau)(-N_1\omega^2 + N_3) - 4N_2\omega^3\sin(\omega\tau)}{\left(N_1{}^2\omega^4 + N_2{}^2\omega - 2N_1N_2\omega^2 + N_3{}^2\right)}$$

$$+\frac{(-3N_1M_1\omega^2 - 3M_1N_2\omega^2)\cos(\omega\tau)3N_3M_1\omega^2\sin(\omega\tau))}{\left(N_1{}^2\omega^4 + N_2{}^2\omega - 2N_1N_2\omega^2 + N_3{}^2\right)}$$

$$+\frac{2(M_2+N_1)(\cos(\omega\tau)\left(-N_1\omega^2 + N_3\right) + 2(M_2+N_1)N_2\omega\sin(\omega\tau)}{\left(N_1{}^2\omega^4 + N_2{}^2\omega - 2N_1N_2\omega^2 + N_3{}^2\right)}$$

$$+\frac{-(N_2+M_3)N_2\omega^2\cos(\omega\tau) + (N_2+M_3)\sin(\omega\tau)\left(-N_1\omega^3 - N_3\omega\right)}{\left(-N_1{}^2\omega^6 - 2N_1N_3\omega^4 + N_2{}^2\omega^4 + N_3{}^2\omega^2\right)}$$

$\operatorname{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1} \neq 0$, provided that

$$\frac{-4\omega^2\cos(\omega\tau)(-N_1\omega^2 + N_3) - 4N_2\omega^3\sin(\omega\tau)}{\left(N_1{}^2\omega^4 + N_2{}^2\omega - 2N_1N_2\omega^2 + N_3{}^2\right)}$$

$$+\frac{(-3N_1M_1\omega^2 - 3M_1N_2\omega^2)\cos(\omega\tau)3N_3M_1\omega^2\sin(\omega\tau))}{\left(N_1{}^2\omega^4 + N_2{}^2\omega - 2N_1N_2\omega^2 + N_3{}^2\right)}$$

$$+\frac{2(M_2+N_1)(\cos(\omega\tau)\left(-N_1\omega^2 + N_3\right) + 2(M_2+N_1)N_2\omega\sin(\omega\tau)}{\left(N_1{}^2\omega^4 + N_2{}^2\omega - 2N_1N_2\omega^2 + N_3{}^2\right)}$$

$$+\frac{-(N_2+M_3)N_2\omega^2\cos(\omega\tau) + (N_2+M_3)\sin(\omega\tau)\left(-N_1\omega^3 - N_3\omega\right)}{\left(-N_1{}^2\omega^6 - 2N_1N_3\omega^4 + N_2{}^2\omega^4 + N_3{}^2\omega^2\right)} \neq 0 \qquad (4.66)$$

Transversality condition hold ,that is the eigenvalue cross the imaginary axis with non-zero speed, As a result, the systems undergo Hopf bifurcation at when $\tau = \tau_0$ condition (4.51) and (4.66) .

CHAPTER FIVE

CONCLUSION AND FUTURE WORK

5.1. Conclusions

In this thesis, mathematical model of prey predator with delay was studied. The result of the study indicates that the positive equilibrium point in the absence of delay is stable with certain conditions. In the presence of delay the system becomes stable with certain conditions and loses its stability at cutoff value. Finally, by considering the bifurcation parameter as a time delay the system undergo Hopf bifurcation at cutoff value with certain condition stated by equation (4.51).

5.2 Future Work

One can investigate the Hopf bifurcation of the system by considering other parameters involved in the model different from time delay. Global stability with delay, direction of stability and Hopf bifurcation, Persistence of the prey predator, global existence of periodic solution of the model are also further investigation. Furthermore, it is possible to consider control that adds with time parameter and different time delay on the four equations.

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