NON STANDARD FINITE DIFFERENCE METHOD FOR SINGULARLY PERTURBED BOUNDARY VALUE PROBLEMS WITH NEGATIVE SHIFT PARAMETER

JIMMA UNIVERSITY COLLEGE OF NATURAL SCIENCES

DEPARTMENT OF MATHEMATICS

A Thesis Submitted to the Department of Mathematics Jimma University in Partial Fulfillment of the Requirements for the Degree of Master of Science in Mathematics.

(Numerical Analysis)

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February, 2022

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Declaration

I here by declare that the work which is being presented in this thesis entitled **"Non standard finite difference method for solving singularly perturbed boundary value problem with negative shift parameter"** in partial fulfillment of the requirement for the degree of Masters of Science in Mathematics, submitted to Jimma University, department of Mathematics is my original work and it has not been submitted for the award of any academic degree or the like in any other institution or university, and that all the sources I have used or quoted have been indicated and acknowledged as complete references.

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Acknowledgment

First of all, I would like to thank my omnipotent GOD for His good will and giving me life to do this thesis. Next to this, it is of great pleasure and proud privilege to express my deepestand sincere gratitude to my advisor Prof. Gemechis File. I am grateful for his indispensable encouragement, professionalism, valuable guidance, motivation, persistent help, and constructive suggestions throughout the period of my thesis work. Without his active guidance and good natured,it would have not been possible to complete this work. Then, my heartfelt gratitude goes to my co-advisor Habtamu Garoma (PhD), for his effective cooperation and great care. I also, express my sincere thank with gratitude to my wife Gadise Begna,My mother Tejitu Dugassa, and other my family members for their extended support.

Contents

Acronyms

* SPPs- Singularly Perturbed Problems.

* DDE – Delay differential equation

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* SPDDE - Singularly perturbed delay differential equation

* SPDCDE - Singularly perturbed delay convection diffusion equation

* BVPs- Boundary Value Problems.

* NSFDM- Non Standard Finite Difference Method.

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Abstract

In this thesis, we consider singularly perturbed differential equation containing negative shift parameter on the convection term. The considered problem exhibits boundary layer on the left or right side of the domain, depending on the sign of the coefficient of convective term. The terms with the negative shift treated using Taylor's approximation. The resulting singularly perturbed boundary value problem is solved using the technique of non-standard finite difference method. The formulated scheme converges uniformly with order of convergence $O(h)$. The theoretical finding is validated using numerical examples and observed to be more accurate than the results in the literature.

Chapter 1

INTRODUCTION

1.1 Background of the Study

Numerical analysis is a branch of mathematics concerned with theoretical foundations of numerical algorithms for the solution of problems arising in scientific applications, Wasow (1942).

The problems in which the highest order derivative is multiplied by a small positive parameter are known to be singularly perturbed problems and the parameter is known as the perturbation parameter. Depending on the solution behavior of the problem in the limiting case when perturbation parameter goes to zero, such type of problems are classified into two classes, namely; (i) regularly perturbed problems (ii) singularly perturbed problems. If the solution of the original problem tends to the solution of the reduced problem (i.e., the problem which is obtained by putting $\varepsilon = 0$ in the original problem) as the perturbation parameter tends to zero, the problem is known as regularly perturbed otherwise, it is known as singularly perturbed.

The classification of singularly perturbed higher order problems depends on how the order of the original equation is affected when small positive parameter ε is multiplying the highest derivative occurring in the differential equation. If the order is reduced by one, we say that the problem is of convection-diffusion type and reaction-diffusion type if the order is reduced by two. Singularly perturbed differential equations with negative shift are special cases of functional differential equations, where the evolution of a system at a certain time, depends on the present state of the system as well as the state of the system at an earlier time.

In general, a singularly perturbed differential equation with negative shift parameter is an ordinary differential equation in which the highest derivative is multiplied by a small parameter and involving at least one delay term. In recent years, many researchers have tried to develop different numerical methods for solving singularly perturbed delay differential equations. For examples, finite difference method of various orders and approaches (Phaneendra and Soujanya, 2014; Gemechis et al., 2017; Gashu et al., 2018), Galerkin method (Swamy et al., 2016), and Differential quadrature method are presented for solving singularly perturbed delay differential equations. However, the issue of accuracy and convergence of the scheme still needs attention and improvement. In this thesis, we present a stable and convergent method and more accurate than the stated methods for solving singularly perturbed delay convection-diffusion equations of the type under consideration.

1.2 Statement of the problem

Any differential equation in which the highest order derivative is multiplied by a small positive parameter and containing at least one negative/positive shift parameter is known as a singularly perturbed differential-difference equation. Such types of problem have a variety of applications in the mathematical modeling of various physical and biological phenomena. However, the computation of its solution has been a great challenge and has been of great importance due to the versatility of such equations in the mathematical modeling of processes in various application fields. The numerical treatment of such problems presents severe difficulties that have to be addressed to ensure accurate numerical solutions as (Roos et al.,1996) states that, the accuracy of the problem increased by increasing the resolution of the grid, which might be impractical in some cases like higher dimensions. A variety of different numerical approaches have been suggested in an attempt to obtain accurate and reliable schemes for the treatment of boundary value problems of singularly perturbed differential-difference equations with a small negative shift in the convection term (Gadisa and File, 2019). They also tried to discuss the effect of small shifts on the solution profile of the problem.

Recently, Duressa (2021) was presented the problem under consideration using exponential fitted operator method. But, still there is a room to increase the accuracy. Owing to this, the present study attempt to answer the following questions:
1. How does this study was describe the numerical method for singularly perturbed boundary value problem with negative shift parameter?
2. To what extent the proposed method converges?
3. To what extent the present method approximate the exact solution?

1.3 Objectives of the study

1.3.1 General Objective

The general objective of this study is to develop non- standard finite difference method for solving singularly perturbed boundary value problem with negative shift parameter

1.3.2 Specific Objectives

The specific objectives of the present study are:

- To formulate non-standard finite difference method for solving singularly perturbed boundary value problem with negative shift parameter.
- To establish the convergence of the scheme.
- To investigate the accuracy of the scheme

1.4 Significance of the study

The results obtained in this research may

- Serve as a reference material for scholars who works on this area.
- Give an idea about the application of numerical methods in different field of studies.
- Help the graduate students to acquire research skills and scientific procedure.
- Provide a numerical method for solving singularly perturbed convection diffusion equation with negative shift parameter.

1.5 Delimitation of the study

The singularly perturbed delay differential equations perhaps arise in variety of applied mathematics that contributes for the advancement of science and technology. Though, singularly perturbed delay differential equations are vast topics and have many applications in the real world, this study is delimited to singularly perturbed delay convection-diffusion equation of the form :

$$\varepsilon y^{''}(x) + a(x)y^{'}(x-\delta) + b(x)y(x) = f(x), x \in (0,1), \tag{1.5.1}$$

with the interval and boundary conditions

$$y(x) = \phi(x), -\delta \leq x \leq 0, y(1) = \gamma, \tag{1.5.2}$$

with small perturbation parameter, $0 < \varepsilon \ll 1$, $\delta < \varepsilon$ and δ is small negative shift parameter; $a(x)$, $b(x)$, $\phi(x)$ and $f(x)$ are bounded smooth functions in $(0, 1)$ and γ is given constant.

Further, the study is delimited to non standard finite difference method for solving singularly perturbed convection-diffusion equations with negative shift parameter, though there are varieties of methods for solving the problems under the study.

Chapter 2

RIVIEW OF RELATED LITERATURE

2.1 Singular perturbation Theory

The term "singular perturbations" was first used by (Friedrichs and Wasow, 1946) in a paper presented at a seminar on non-linear vibrations at New York University. Perturbation theory is a vast collection of mathematical methods used to obtain approximate solution to problems that have no closed form of analytical solution. Perturbation problems depend on a small positive parameter(s). These parameters affect the problem in such a way that the solution varies rapidly in some region of the problem domain and slowly in other parts. The study of many theoretical and applied problems in science and technology leads to boundary value problems for singularly perturbed differential equations that have a multi-scale character.perturbation theory is a subject, which studies the effect of small parameter in the mathematical model problems in ordinary differential equations. In mathematics, more precisely in perturbation theory, a singular perturbation problem is a problem containing a small parameter that cannot be approximated by setting the parameter value to zero. The boundary value problems for ordinary differential equations in which one or more small positive parameter(s) multiplying the second derivative(s) are known as singularly perturbed problems.

2.2 Singularly Perturbed Delay Differential Equation

Delay differential equations (DDEs) model problems where there is after effect affecting the variable of the problem as compared to differential equations which model the problem to current conditions. DDEs is said to be retarded type if the delay argument does not occur in the highest order derivative term, otherwise it is known as neutral DDEs. A singularly perturbed delay differential equations is differential equations in which its highest order derivative is multiplied by small perturbation parameter and having delay parameter(s) on the terms different from the highest order derivative, Gopalsamy(2013). Singularly perturbed DDEs arise in the mathematical modeling of various physical phenomena.

2.3 Boundary Value Problem

A boundary value problem is a problem, typically an ordinary or partial differential equation that has values assigned on physical boundary of the domain in which the problem is specified. A boundary value problem for a given differential equation consists of finding a solution of the given differential equation subject to a given set of boundary conditions, Kumar(2012). Finding the numerical solution of a boundary value problem is more difficult than that of corresponding initial value problem. There is a wide class of asymptotic expansion methods available for solving the two small parameters singularly perturbed boundary value problems.

But there can be difficulties in applying these asymptotic expansion methods, such as finding the appropriate asymptotic expansions in the inner and outer regions, which are not routine exercises but require skill, insight and experimentation.

2.4 Non standard Finite Difference Method

Non-standard finite difference schemes (NSFD) have emerged as an alternative method for solving a wide range of problems whose mathematical models involve algebraic, differential and biological models as well as chaotic systems, Mickens (2005). These techniques have many advantages over classical techniques and provide an efficient numerical solution. In fact, the non-standard finite difference method is an extension of the standard finite difference method. Non-standard schemes as introduced by Mickens, (1990) are used to resolve some of the issues related to numerical instabilities. Furthermore, Mickens, (2005,2000,1999) introduced certain rules for obtaining the best difference equations.

Mickens, (1994) mentions in details about these construction rules in his reference book Non standard Finite Difference Models of Differential Equations. Non-standard method is more stable than the standard finite methods and the domain of h for stability in the non-standard is larger than those of the standard method. If the denominator functions are chosen in appropriate form the non-standard methods produce better results,YOghoubi(2015)

Chapter 3

METHODOLOGY

3.1 Study Area and Period

The study was conducted in Jimma University, department of Mathematics from August 2020 to February 2022.

3.2 Study Design

The study was employed mixed-design (documentary review and experiment).

3.3 Source of Information

The relevant sources of information for this study are books, published articles on reputable journal and related studies from internet services.

3.4 Mathematical Procedure of the study

In order to achieve the above mentioned objectives, the study was follows the following steps:

1. Defining(or describing) the problems.

2.Discretizing the solution domain.

3.Formulating the numerical scheme for the governing problem under consideration.

4. Establishing the stability and convergence analysis of the scheme.

5. Write MATLAB code for the obtained numerical scheme.

6. Validating the scheme using numerical experment

7. Presenting the results in tables and graphs.

Chapter 4

DESCRIPTION OF THE METHODS, EXAMPLES AND RESULTS

4.1 Description of the method

In this section, the description of second order finite difference methods and their stability and convergence analysis is discussed. Consider singularly perturbed delay convection – diffusion equation of the standard form:

$$\varepsilon y^{''}(x) + a(x)y^{'}(x-\delta) + b(x)y(x) = f(x), x \in (0,1), \quad (4.1.1)$$

subject to the interval boundary conditions

$$y(x) = \phi(x), -\delta \leq x \leq 0, y(1) = \gamma, \quad (4.1.2)$$

with small perturbation parameter,$0 < \varepsilon \ll 1$ and δ is small delay parameter. The functions $a(x)$, $b(x)$ and $f(x)$ are assumed to be sufficiently smooth with $a(x) \geq$

$a > 0$ and $b(x) \geq b > 0$ for $x \in [0, 1]$. When the shift parameter δ is smaller than ε the use of Taylor's series expansion for the term containing shift argument is valid ,Tian (2002). In this work the case when $\delta < \varepsilon$ is considered. Thus , to approximate the term with delay parameter, Taylor's series expansion is applied as follows:

$$y'(x - \delta) = y'(x) - \delta y''(x) + O(\delta^2). \tag{4.1.3}$$

Now substituting Eq.(4.1.3) in to Eq.(4.1.1),

we get

$\varepsilon y''(x) + a(x)y'(x) - a(x)\delta y''(x) + b(x)y(x) = f(x), x \in (0, 1)$.

We obtain an asymptotically equivalent singularly perturbed two point boundary value problem of the form:

$$c_\varepsilon y^{''}(x) + a(x)y^{'}(x) + b(x)y(x) = f(x), \tag{4.1.4}$$

subject to the boundary conditions

$$y(0) = \phi(0), y(1) = \gamma, \tag{4.1.5}$$

where $c_\varepsilon = \varepsilon - \delta a(x)$,and assumed to be positive throughout the interval $[0, 1]$ since δ is smaller than ε, the effect of the value of the truncated term with $O(\delta^2)$ is negligible.

Hence, the solution of the asymptotically equivalent problem is equivalent to that of the original problem. Further, its error bound is also equivalent to that of the original problem .

4.2 Properties of continuous function

The following lemmas are necessary for the existence and uniqueness of the solution and for the problem to be well-posed .

Lemma 1 (Continuous minimum principle)

Assume that $v(x) \in C^2(\bar{\Omega})$ be any function satisfying $v(0) \geq 0$, $v(l) \geq 0$ and $Lv(x) \leq 0$, $\forall x \in \Omega = (0,l)$.then $v(x) > 0, \forall x \in \bar{\Omega} = [0,l]$.

Proof : Let x^* be such that $v(x^*) = \min_{x \in [0,l]} v(x)$ and assume that $v(x^*) < 0$. Clearly $x^* \notin \{0,l\}$, therefore $v'(x^*) = 0$ and $v''(x^*) \geq 0$. Moreover, $Lv(x^*) = \varepsilon v''(x^*) + a(x^*)v'(x^*) \geq 0$, which is a contradiction. It follows that $v(x^*) > 0$ and thus $v(x) \geq 0, \forall x \in [0,l]$.

The uniqueness of the solution is implied by this minimum principle. Its existence follows trivially (as for linear problems, the uniqueness of the solution implies its existence). This principle is now applied to prove that the solution of Eqs.$(4.1.4)-(4.1.5)$ is bounded.

The following lemma shows the bound for the derivatives of the solution.

Lemma 2 (Boundedness of the solution) Let $u(x)$ be the solution of the Eqs. $(4.1.1)-(4.1.2)$, then we obtain the bound

$|u(x)| \leq \frac{||f||}{b} + \{\max|\phi|, |\gamma|\}$, for $b(x) \geq b > 0$,

where b is lower bound of $b(x)$.

Proof : Defining barrier function

$\vartheta_{\pm}(x,t)$ as $u_{\pm}(x,t) = |u(x)| \leq \frac{||f||}{b} + \{\max|\phi|, |\gamma|\} \pm y(x)$ and applying the maximum principle , we obtain the required bound. At the boundary points.

$y_{\pm}(0) = \frac{||f||}{b} + \{\max|\phi|, |\gamma|\} \pm y(0) \geq 0,$

$y_{\pm}(1) = \frac{||f||}{b} + \{\max|\phi|, |\gamma|\} \pm y(1) \geq 0,$

on the differential operator

$$L\vartheta_{\pm}(x) = c_{\varepsilon}\vartheta_{\pm}(x) + a(x)\vartheta'_{\pm}(x) + b(x)\vartheta_{\pm}(x)$$
$$= c_{\varepsilon}(0 \pm u''(x) + a(x)(0 \pm u'(x) + b(x)(\frac{||Lu||}{b} + \{\max|\phi|, |\gamma|\} \pm y(x))$$

$=b(x)(\frac{||Lu||}{b} + \{\max|\phi|, |\gamma|\}) \pm f(x))$

$=0$, since $b(x) \geq b > 0$,

which implies

$$L_{\pm}\vartheta(x) \geq 0.$$

Hence, by maximum principle we obtain

$\vartheta(x) \geq 0, \forall x \in \overline{\Omega}$

. **Lemma 3**

Let y_{ε} be the solution of (P_{ε}). Then, for $k = 0, 1, 2, 3$,

$$| y_{\varepsilon}^{(k)}(x) | \leq C(1 + \varepsilon^{-k} \exp(\frac{-a}{\varepsilon}x)), \forall x \in [0, l].$$

Proof: For the proof refer (Woldaregay and Duressa, 2020).

4.3 Formulation of the Numerical Method

The theoretical basis of non-standard discrete numerical method is based on the development of exact finite difference method. Mickens(2005) presented techniques and rules for developing non-standard finite difference methods for different problem types. In Mickens's rules, to develop a discrete scheme, denominator function for the discrete derivatives must be expressed in terms of more complicated functions of step sizes than those used in the standard procedures. These complicated functions constitute a general property of the schemes, which is useful while designing reliable schemes for such problems.we consider separately for left and right boundary layer problems and develop individual schemes for each. First let us consider the right boundary layer problem.

Case (1): Right boundary layer problems

For the problem of the form in $(4.3.4) - (4.3.5)$, in order to construct exact finite difference scheme we follow the procedures of Bansal and Sharma, (2017).

Consider the constant coefficient sub equations from (4.3.4) – (4.4.5) as

$$c_\varepsilon y^{''}(x) + a(x)y^{'}(x) + by(x) = 0, \tag{4.3.6}$$

$$c_\varepsilon y^{''}(x) + a(x)y^{'}(x) = 0, \tag{4.3.7}$$

where $a(x) \geq a$ and $b(x) \geq b$

Thus, Eq. (4.3.6) has two independent solutions namely $\exp(\lambda_1 x)$ and $\exp(\lambda_2 x)$ with $\lambda_{1,2} = \frac{-a\pm\sqrt{(a)^2-4c_\varepsilon b}}{2c_\varepsilon}$.

We discretize the domain $(0,1)$, using uniform mesh length $x = h$ such that $\omega_1^N = \{x_i = x_o + ih, 1, 2, ..., N, x_o = 0, x_N = 1, h = \frac{1}{N}\}$,

where N is the number of mesh points. We denote Y_i as the approximate solution of $y(x)$ at mesh point x_i.

The target is to calculate a difference equation which has the same general solution as the differential equation in (4.3.7) has at the mesh pointx_i is given by $Y_i = A_1 \exp(\omega_1 x_i) + A_2 \exp(\omega_2 x_i)$.

Using the theory of difference equations for second order linear difference equations in (Bansal and Sharma, 2017), we obtain

$$\det \begin{pmatrix} Y_{i-1} & \exp(\lambda_1 x_{i-1}) & \exp(\lambda_2 x_{i-1}) \\ Y_i & \exp(\lambda_1 x_i) & \exp(\lambda_2 x_i) \\ Y_{i+1} & \exp(\lambda_1 x_{i+1}) & \exp(\lambda_2 x_{i+1}) \end{pmatrix} = 0, \tag{4.3.8}$$

Substituting the values of λ_1 andλ_2 gives

$$-\exp(\frac{-ah}{2c_\varepsilon})Y_{i-1} + 2cosh(h\frac{\sqrt{(a)^2 - 4c_\varepsilon b}}{2c_\varepsilon})Y_i - \exp(\frac{ah}{2c_\varepsilon})Y_{i+1}, \tag{4.3.9}$$

is an exact difference scheme for Eq. (4.3.7). For $c_\varepsilon \to 0$, we use the approximation $h\frac{\sqrt{(a)^2-4c_\varepsilon b}}{2c_\varepsilon}) \approx \frac{ah}{2c_\varepsilon}$ in (4.3.9).

Multiplying both sides by $\exp(\frac{ah}{2c_\varepsilon})$ and simplifying, we obtain

$$Y_{i-1} - 2Y_i + Y_{i+1} = (\exp(\frac{ah}{c_\varepsilon}) - 1)(Y_i - Y_{i-1}), \tag{4.3.10}$$

Rearranging Eq.(4.3.10) , we obtain

$$c_\varepsilon \frac{Y_{i-1} - 2Y_i + Y_{i+1}}{\frac{hc_\varepsilon}{a}(\exp(\frac{ah}{c_\varepsilon}) - 1}) + a\frac{Y_i - Y_{i-1}}{h} = 0. \tag{4.3.11}$$

The required denominator function for second derivative discretezation becomes

$$\omega^R = \frac{hc_\varepsilon}{a}(\exp(\frac{ah}{c_\varepsilon}) - 1). \tag{4.3.12}$$

Adopting ω^R for the variable coefficient problem we write as

$$\omega_i^R = \frac{hc_\varepsilon}{a(x_i)}(\exp(\frac{ah(x_i)}{c_\varepsilon}) - 1) \tag{4.3.13}$$

Using the denominator function ω_i^R in to the scheme (4.3.4), the difference scheme becomes

$$L_\omega^R Y_i \equiv c_\varepsilon \frac{[y_{i+1} - 2y_i + y_{i-1}]}{\omega_i^R} + a(x_i)\frac{[y_i - y_{i-1}]}{h} + b(x_i)y_i = f(x_i), i = 1, 2, 3, .., N-1. \tag{4.3.14}$$

This can be written as three term recurrence relation of the form:

$$E_i y_{i-1} + F_i y_i + G_i y_{i+i} = H_i, i = 1, 2, 3........, N-1, \tag{4.3.15}$$

where $E_i = \frac{c_\varepsilon}{\omega_i^R} - \frac{a_i}{h}$, $F_i = -\frac{2c_\varepsilon}{\omega_i^R} + \frac{a_i}{h} + b_i$, $G_i = \frac{c_\varepsilon}{\omega_i^R}$, and $H_i = f_i$.

Case (2): Left boundary layer problems

In this case $-a(x) \leq -a < 0$ in(4.3.4) – (4.3.5), we consider the constant coefficient sub- equations from (4.3.4) – (4.3.5) as

$$c_\varepsilon y^{''}(x) + a(x)y^{'}(x) + \beta y(x) = 0, \tag{4.3.16}$$

$$c_\varepsilon y^{''}(x) + a(x)y^{'}(x) = 0, \tag{4.3.17}$$

where $b(x) \geq b$,

Thus, Eq. (4.3.17) has two independent solutions namely $exp(\lambda_1 x)$ and $exp(\lambda_2 x)$ with $\lambda_{1,2} = \frac{a \mp \sqrt{(a)^2 - 4c_\varepsilon b}}{2c_\varepsilon}$

We discretize the domain (0, 1), using uniform mesh length $x = h$ such that $\omega_1^N = \{x_i = x_o + ih, 1, 2, ..., N, x_o = 0, x_N = 1, h = \frac{1}{N}\}$,

where N is the number of mesh intervals. We denote Y_i as the approximate solution of $y(x)$ at mesh point x_i.

The target is to calculate a difference equation which has the same general solution as the differential equation in (4.3.17) whose solution at the mesh point x_i is given by $Y_i = A_1 exp(\omega_1 x_i) + A_2 exp(\omega_2 x_i\ )$.

Using the theory of difference equations for second order linear difference equations in (Bansal and Sharma, 2017) , we obtain

$$\det \begin{pmatrix} Y_{i-1} & \exp(\lambda_1 x_{i-1}) & \exp(\lambda_2 x_{i-1}) \\ Y_i & \exp(\lambda_1 x_i) & \exp(\lambda_2 x_i) \\ Y_{i+1} & \exp(\lambda_1 x_{i+1}) & \exp(\lambda_2 x_{i+1}) \end{pmatrix} = 0, \tag{4.3.18}$$

Substituting the values of λ_1 and λ_2 and simplfying , we obtain

$$-\exp(\frac{ah}{2c_\varepsilon})Y_{i-1} + 2cosh(h\frac{\sqrt{(a)^2 - 4c_\varepsilon b}}{2c_\varepsilon})Y_i - \exp(\frac{-ah}{2c_\varepsilon})Y_{i+1} \tag{4.3.19}$$

is an exact difference scheme for Eq. (4.3.15). For $c_\varepsilon \to 0$, we use the approximation $h\frac{\sqrt{(a)^2-4c_\varepsilon b}}{2c_\varepsilon}) \approx \frac{ah}{2c_\varepsilon}$. After doing the arithmetic adjustment, we obtain

$$c_\varepsilon \frac{Y_{i-1} - 2Y_i + Y_{i+1}}{\frac{hc_\varepsilon}{a}(1 - \exp(\frac{ah}{c_\varepsilon})}) + a\frac{Y_i + 1 - Y_i}{h} = 0, \tag{4.3.20}$$

The denominator function becomes

$\omega^L = \frac{hc_\varepsilon}{a}(1 - \exp(\frac{ah}{c_\varepsilon}))$.

adopting it for the variable coefficient problem , we write as

$$\omega_i^L = \frac{hc_\varepsilon}{a(x_i)}(1 - \exp)(\frac{ah(x_i)}{c_\varepsilon}). \tag{4.3.21}$$

The required finite difference schemes becomes

$$L_\omega^L Y_i \equiv c_\varepsilon \frac{[y_{i+1} - 2y_i + y_{i-1}]}{\omega_i^L} + a(x_i)\frac{[y_{i+1} - y_i]}{h} + b(x_i)y_i = f(x_i), i = 1, 2, 3, .., N - 1, \tag{4.3.22}$$

This can be written as three term recurrencerelation as of the form:

$$E_i y_{i-1} + F_i y_i + G_i y_{i+i} = H_i, i = 1, 2, 3........, N - 1, \tag{4.3.23}$$

where $E_i = \frac{c_\varepsilon}{\omega_i^L}$, $F_i = -\frac{2c_\varepsilon}{\omega_i^L} - \frac{a_i}{h} + b_i$, $G_i = \frac{c_\varepsilon}{\omega_i^L} + \frac{a_i}{h}$ and $H_i = f_i$.

Therefore, on the whole domain, $[0, 1]$, the basic schemes to solve Eq.(4.1.4) and Eq. (4.1.5) are the scheme given in Eq. (4.3.15) and Eq. (4.3.23) using Thomas algorithm.

4.4 Uniform convergence analysis

In this section,we need to show the discrete scheme satisfying the discrete minimum principle and uniform convergence.

Lemma 4 : (Discrete Minimum Principle) Let v_i be any mush function that satisfies $v_0 \geq 0, v_N \geq 0$ and $L^h v_i \leq 0$,$i = 1, 2, ...N - 1$,then $v_i \geq 0$, $i = 0, 1, 2.....N$

proof : The proof is obtained by contradiction.Let f be such that $v_j = min v_i$ and suppose that $v_j < 0$.clearly, $j \notin \{0, N\}, v_{j+1} - v_j \geq 0$ and $v_j - v_{j-1} \leq 0$ Therefore

$L^h v_j = \frac{c_\varepsilon}{\omega_i^R}(v_{j+1} - 2v_j + v_{j-1}) + \frac{a_j}{h}(v_{j+1} - v_j) - b_j v_j$

$$=\frac{c_\varepsilon}{\omega_i^R}[(v_{j+1}-v_j)-(v_j-v_{j-1})+\frac{a_j}{h}(v_{j+1}-v_j)-b_jv_j\geq 0$$

where the strict inequality holds if $v_{j+1}-v_j>0$. This is a contradiction and therefore $v_j\geq 0$. since j is arbitrary,we have$v_i\geq 0$ $i=1,2,3...,N$

We proved above that the discrete operator L^h satisfy the minimum principle.

Next we analyze the uniform convergence analysis.Let us define the forward,backward and cenral finite difference operators as:

$D^+v_j=\frac{v_{j+1}-v_j}{h}$,$D^-v_j=\frac{v_j-v_{j-1}}{h}$,$\delta^2v_j=D^+D^-v_j=\frac{D^+v_j-D^-v_j}{h}$

Lemma 5 For a fixed mesh and for $c_\varepsilon\to 0$, it holds

$$\lim_{c_\varepsilon\to 0}\max_{1\leq i\leq N-1}\left(\frac{\exp(\frac{-ax_i}{c_\varepsilon})}{c_\varepsilon^m}\right)=0,\quad m=1,2,3,...$$

$$\lim_{c_\varepsilon\to 0}\max_{1\leq i\leq N-1}\left(\frac{\exp(\frac{-a(1-x_i)}{c_\varepsilon})}{c_\varepsilon^m}\right)=0,\quad m=1,2,3,...$$

where $x_i=ih,h=\frac{1}{N},i=1,2,...,N-1$.

proof : Consider the partition $[0,1]:=\{0=x_0<x_1<....<x_{N-1}<x_N=1\}$ for the interior grid points, we have

$$\max_{1\leq i\leq N-1}\frac{\exp\left(\frac{-ax_i}{c_\varepsilon}\right)}{c_\varepsilon^m}\leq\frac{\exp\left(\frac{-ax_1}{c_\varepsilon}\right)}{c_\varepsilon^m}=\frac{\exp\left(\frac{-ah}{c_\varepsilon}\right)}{c_\varepsilon^m},$$

$$\max_{1\leq i\leq N-1}\frac{\exp\left(\frac{-a(1-x_i)}{c_\varepsilon}\right)}{c_\varepsilon^m}\leq\frac{\exp\left(\frac{-a(1-x_{N-1})}{c_\varepsilon}\right)}{c_\varepsilon^m}=\frac{\exp\left(\frac{-ah}{c_\varepsilon}\right)}{c_\varepsilon^m},$$

$as\ \ x_1=1-x_{N-1}=\ h.$

Then, by the application of L'Hospital's rule m times gives

$$\lim_{c_\varepsilon\longrightarrow 0}\frac{\exp\left(\frac{-ah}{c_\varepsilon}\right)}{c_\varepsilon^m}=\lim_{r=\frac{1}{c_\varepsilon}\longrightarrow\infty}\frac{r^m}{\exp(ahr)}=\lim_{r=\frac{1}{c_\varepsilon}\longrightarrow\infty}\frac{m!}{(ah)^m\exp(ahr)}=0.$$

Hence, the proof is completed.

Theorem 1: Let the coefficients functions $a(x)$ and the source function $f(x)$ in Eqs.

(4.4.17) – (4.4.18) of the domain Ω be sufficiently smooth, so that $y(x) \in C^4[0,1]$. Then, the discrete solution Y_i satisfies

$$|L^N(y_i - Y_i)| \leq Ch\left(1 + \sup_{x\in(0,1)}\left(\frac{\exp(\frac{-ax_i}{c_\varepsilon})}{c_\varepsilon^3}\right)\right).$$

proof: We consider the truncation error discretization as

$$\begin{aligned}
|L^N(y_i - Y_i)| =& |L^N y_i - L^N Y_i|,\\
\leq& C|c_\varepsilon y_i'' + a_i y_i' - \{c_\varepsilon \frac{D^+D^-h^2}{\omega_i^R} y_i + a_i D^+ y_i\}|,\\
\leq& C|c_\varepsilon (y_i'' - \frac{D^+D^-h^2}{\omega_i^R} y_i) + a_i(y_i' - D^+ y_i)|,\\
\leq& Cc_\varepsilon |y_i'' - D^+D^- y_i| + Cc_\varepsilon |(\frac{h^2}{\omega_i^R} - 1)D^+D^- y_i| + Ch|y_i''|,\\
\leq& Cc_\varepsilon h^2 |y_i^{(4)}| + Ch|y_i''| + Ch|y_i''|,\\
\leq& Cc_\varepsilon h^2 |y_i^{(4)}| + Ch|y_i''|.
\end{aligned}$$

We used the estimate $c_\varepsilon|\frac{h^2}{\omega_i^R} - 1| \leq Ch$. Indeed, define $\rho = \dfrac{a_i h}{c_\varepsilon}, \rho \in (0,\infty)$ .Then,

$$c_\varepsilon|\frac{h^2}{\omega_i^R} - 1| = a_i h|\frac{1}{\exp(\rho) - 1} - \frac{1}{\rho}| =: a_i h Q(\rho).$$

By simplifying and writing explicitly, we obtain

$$Q(\rho) = \frac{\exp(\rho) - \rho - 1}{\rho(\exp(\rho) - 1)},$$

and we obtain the limit is bounded as

$$\lim_{\rho\longrightarrow 0} Q(\rho) = \frac{1}{2}, \quad \lim_{\rho\longrightarrow\infty} Q(\rho) = 0.$$

Hence, for all $\rho \in (0,\infty)$, we have $Q(\rho)C$. So, the error estimate in the discretization is bounded as

$$|L^N(y_i - Y_i)| C c_\varepsilon h^2 |y_i^{(4)}| + Ch|y_i''|. \tag{4.4.24}$$

From Eq. (4.4.24) and boundedness of derivatives of solution , we obtain

$$
\begin{aligned}
||L^N(y(x_i) - Y_i)|| \leq Cc_\varepsilon h^2 & \left| \left(1 + c_\varepsilon^{-4} \exp\left(\frac{-ax_i}{c_\varepsilon}\right)\right) \right|, \\
+ Ch & \left| \left(1 + c_\varepsilon^{-2} \exp\left(\frac{-ax_i}{c_\varepsilon}\right)\right) \right|, \\
\leq Ch^2 & \left| \left(c_\varepsilon + c_\varepsilon^{-3} \exp\left(\frac{-ax_i}{c_\varepsilon}\right)\right) \right|, \\
+ Ch & \left| \left(1 + c_\varepsilon^{-2} \exp\left(\frac{-ax_i}{c_\varepsilon}\right)\right) \right|, \\
\leq Ch & \left(1 + \sup_{x \in (0,1)} \left(\frac{\exp(\frac{-ax_i}{c_\varepsilon})}{c_\varepsilon^3}\right)\right),
\end{aligned}
$$

since $c_\varepsilon^{-3} > c_\varepsilon^{-2}$.

Theorem 2: Under the hypothesis of boundness of discrete solution (i.e., it satisfies the discrete minimum principle), Lemma 5 and Theorem 1, the discrete solution satisfies the following bound.

$$
\sup_{0 \leq \varepsilon \leq 1} \max_i |y_i - Y_i| \leq CN^{-1}. \tag{4.4.25}
$$

Proof: Results from boundness of solution, Lemma 5 and Theorem 1 gives the required estimates. Hence the proof.

4.5 Numerical Example and Results

To validate the established theoretical results, we perform numerical experiments using the model problems of the form in Eqs.(4.1.1) – (4.1.2).

Example 1:

$$
\begin{cases}
\varepsilon y''(x) - \exp(x) y'(x - \delta) - xy(x) = 0, \\
y(x) = 1, -\delta \leq x \leq 0, \quad y(1) = 1.
\end{cases}
$$

Example 2:

$$\begin{cases} \varepsilon y''(x) - (1+x)y'(x-\delta) - \exp(-x)y(x) = 1, \\ y(x) = 1, -\delta \le x \le 0, \quad y(1) = 1. \end{cases}$$

Example 3:

$$\begin{cases} \varepsilon y''(x) + y'(x-\delta) + y(x) = 0, \\ y(x) = 1, -\delta \le x \le 0, \quad y(1) = 1. \end{cases}$$

Having $y_j \equiv y_j^N$ (the approximated solution obtained via fitted operator finite difference method) for different values of h and ε, the maximum errors. Since the exact solution is not available, the maximum errors (denoted by E_ε^N) are evaluated using the double mesh principle for fitted operator finite difference methods using formula

$$E_\varepsilon^N := \max_{0 \le j \le n} |y_j^N - y_{2j}^{2N}|.$$

Further, we tabulate the ε- uniform error

$$E^N = \max_{0 < \varepsilon \le 1} E_\varepsilon^N.$$

The numerical rate of convergence are computed as

$$r_\varepsilon^N := \frac{\log(E_\varepsilon^N) - \log(E_\varepsilon^{2N})}{\log(2)}.$$

and the ε- uniform rate of convergence is computed using

$$R^N = \frac{\log(E^N) - \log(E^{2N})}{\log(2)}.$$

Table 4.1: Maximum absolute errors and rate of convergent for different values of ε, $\delta = 0.5\varepsilon$ and number of mesh size, N for Example 1.

ε	N=16	N=32	N= 64	N= 128	N= 256
10^{-2}	1.0815e-03	2.9684 e-04	7.6113e-05	1.9154e-05	4.7962e-06
10^{-4}	2.0031e-03	1.0180 e-03	5.1327e-04	2.5766e-04	2.2891e-04
10^{-6}	2.0031e-03	1.0180 e-03	5.1327e-04	2.5766e-04	2.2891e-04
10^{-8}	2.0031e-03	1.0180 e-03	5.1327e-04	2.5766e-04	2.2891e-04
10^{-10}	2.0031e-03	1.0180 e-03	5.1327e-04	2.5766e-04	2.2891e-04
E^N	2.0031e-03	1.0180 e-03	5.1327e-04	2.5766e-04	2.2891e-04
R^N	0.9765	0.9879	0.9942	0.9991	

Table 4.2: Comparison of maximum absolute errors and ε=0.1 for Example 1 at number of mesh points N.

δ	N=8	N=32	N=128	N=8	N=32	N=128
	Present M			Duressa(2021)		
0.03	5.1683e-04	3.2995e-05	2.0653e-06	1.8773e-03	1.247e-04	7.8243e-06
0.05	4.2381e-04	2.6999e-05	1.6891e-06	1.5524e-03	1.0187e-04	6.3843e-06
0.07	3.5805e-04	2.6663e-05	1.4172e-06	1.3187e-03	8.5539e-05	5.3586e-06
0.09	3.0779e-04	1.9404e-05	1.2137e-06	1.1409e-03	7.3473e-05	4.5998e-06

Table 4.3: Maximum absolute errors and rate of convergent for different values of ε, $\delta = 0.5\varepsilon$ and number of mesh size, N for Example 2.

ε	16	32	64	128	256
10^{-2}	4.7914e-03	1.3185 e-03	3.3984e-04	8.5532e-05	2.1428e-05
10^{-4}	8.6731e-03	4.4133 e-03	2.2256e-03	1.1175e-03	5.5925e-04
10^{-6}	8.6731e-03	4.4133 e-03	2.2256e-03	1.1175e-03	5.5925e-04
10^{-8}	8.6731e-03	4.4133 e-03	2.2256e-03	1.1175e-03	5.5925e-04
10^{-10}	8.6731e-03	4.4133 e-03	2.2256e-03	1.1175e-03	5.5925e-04
E^N	8.6731e-03	4.4133 e-03	2.2256e-03	1.1175e-03	5.5925e-04
R^N	0.9747	0.9877	0.9939	0.9987	

Table 4.4: Comparison of maximum absolute errors and ε=0.1 for Example 2 at number of mesh points N.

δ	N=8	N=32	N=128	N=8	N=32	N=128
	Present M			Duressa (2021)		
0.03	2.3518e-03	1.4946e-04	9.3492e-06	7.8120e-03	5.1772e-04	3.2466e-05
0.05	1.9088e-03	1.2077e-04	7.5557e-06	6.4652e-03	4.2158e-04	2.6415e-05
0.07	1.5765e-03	1.0057e-04	6.2894e-06	5.4621e-03	3.5314e-04	2.2121e-05
0.09	1.3529e-03	8.5462e-05	5.3440e-06	4.6929e-03	3.0211e-04	1.8924e-05

Table 4.5: Maximum absolute errors and rate of convergent for different values of ε, $\delta = 0.5\varepsilon$ and number of mesh size, N for Example 3.

ε	16	32	64	128	256
10^{-2}	1.7887e-02	6.6606 e-03	1.8827e-03	4.9114e-04	1.2389e-04
10^{-4}	1.9521e-02	1.0186 e-02	5.2007e-03	2.6274e-03	1.3205e-03
10^{-6}	1.9521e-02	1.0186 e-02	5.2007e-03	2.6274e-03	1.3205e-03
10^{-8}	1.9521e-02	1.0186 e-02	5.2007e-03	2.6274e-03	1.3205e-03
10^{-10}	1.9521e-02	1.0186 e-02	5.2007e-03	2.6274e-03	1.3205e-03
E^N	1.9521e-02	1.0186 e-02	5.2007e-03	2.6274e-03	1.3205e-03
R^N	1.0001	1.0000	1.0000	1.0001	

Table 4.6: Comparison of maximum absolute errors and ε=0.1 for Example 3 at number of mesh points N.

δ	N=8	N=32	N=128	N=8	N=32	N=128
	Present M			Duressa (2021)		
0.03	6.7217e-03	4.3369e-04	2.7141e-05	2.4155e-02	1.6227e-03	1.0172e-04
0.05	1.0288e-02	6.5201e-04	4.0814e-05	3.4122e-02	2.4758e-03	1.5568e-04
0.07	1.7230e-02	1.1693e-03	7.3343e-05	5.3192e-02	4.4352e-03	2.8433e-04
0.09	3.2744e-02	3.6524e-03	2.4050e-04	7.0034e-02	1.2727e-02	9.4159e-04

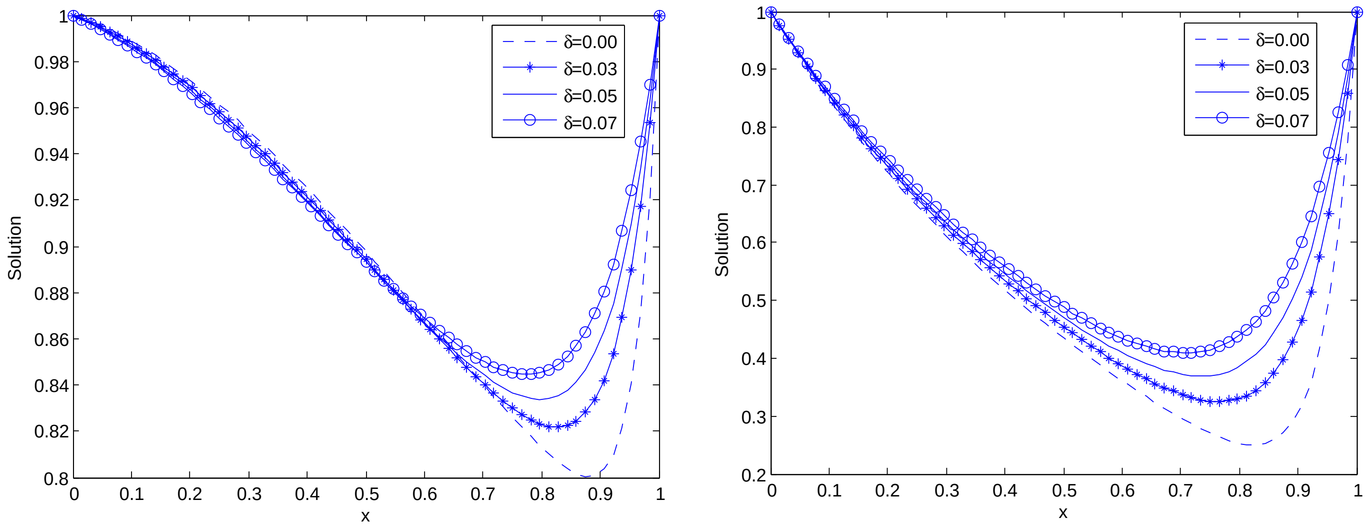

Figure 4.1: The behavior of Numerical Solution at $\varepsilon = 10^{-8}$ and different values of N for Example 1 and Example 2 respectively.

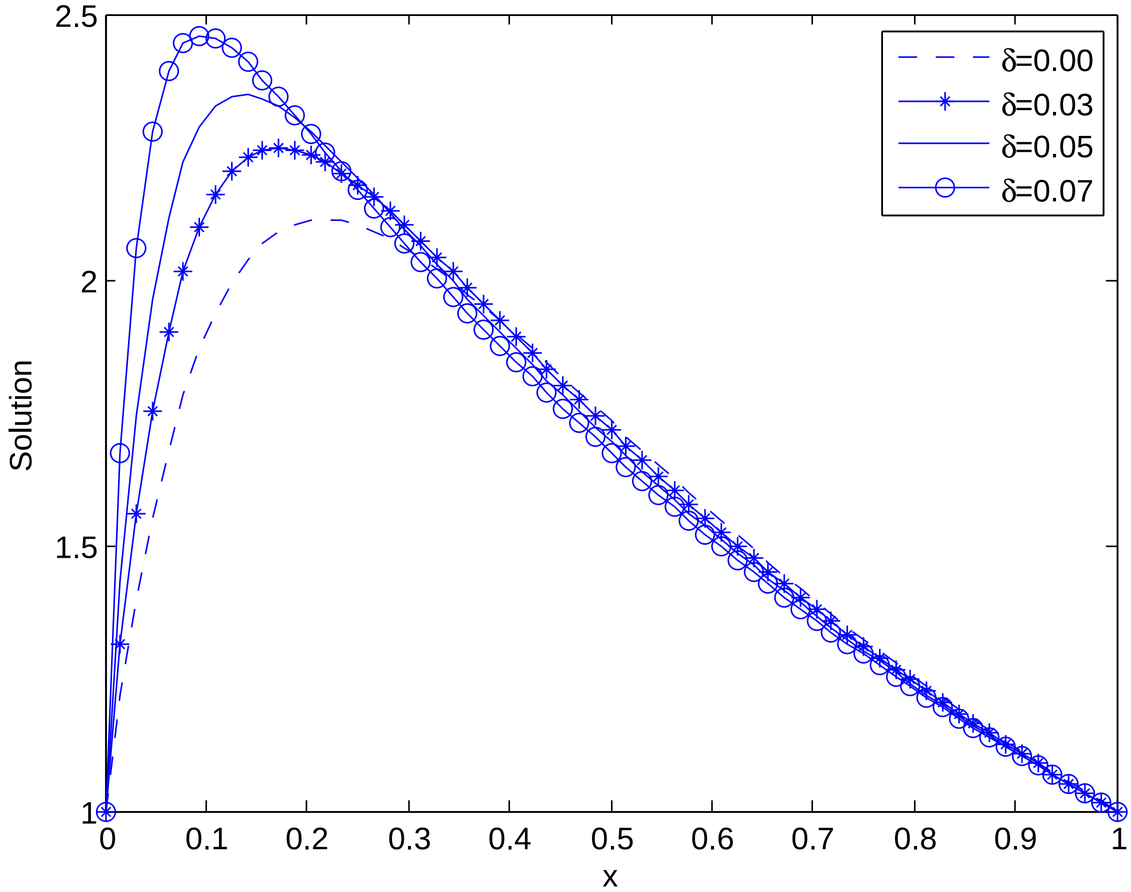

Figure 4.2: Point wise absolute error plot at $\varepsilon = 10^{-8}$ and different values of N for Example 3.

Chapter 5

Discussion,Conclusion and Scope for future Work

5.1 Discussion and Conclusion

In this thesis, we consider three examples exhibiting boundary layer. Example 3 exhibit left boundary layer and Examples 1 and 2 exhibit right boundary layer. In the computed solutions we used the perturbation parameter ε very small compared to the number of mesh points N. For each Examples, we computed the maximum absolute error, parameter uniform error and uniform rate of convergence. In Tables 4.1, 4.3 and 4.5 one can observe that the maximum absolute error is independent of the perturbation parameter as goes small. This means that, as the perturbation parameter goes small, the maximum absolute error of the scheme is bounded and it becomes uniformly convergent. On the last two rows of these tables the parameter uniform error and the parameter uniform rate of convergence are given. In Table 4.2,4.4 and 4.6 we give the comparison of the obtained result with the result given in Example 1,Example 2 and Example 3 respectively paper (Duressa,2021). As one can see, the obtained result is more accurate than the one in (Duressa,2021).

For left boundary layer problems, one can observe from Figure 4.2 for Example 3 as the values of the delay parameters increases the size of the boundary layer decreases. For the case of the right boundary layer problems as the values of the delay parameter increases the size of the boundary layer increases as it is seen on Figure 4.1 for Example 1 and 2.

5.2 Scope for future Work

In this thesis, non standard finite difference method for solving singularly perturbed boundary value problem with negative shift parameter is introduced.

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