ANALYTICAL SOLUTION OF TWO DIMENSIONAL FOURTH ORDER PARABOLIC PARTIAL DIFFERENTIAL EQUATION USING TRIPLE LAPLACE TRANSFORM COUPLED WITH ITERATIVE METHOD.

JIMMA UNIVERSITY COLLEGE OF NATURAL SCIENCES

DEPARTMENT OF MATHEMATICS

A thesis submitted to the department of mathematics, college of natural sciences in partial fulfillment for the requirements of the degree of masters of science in mathematics.

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Declaration

I, undersigned, declare that " Analytical solution of two dimensional fourth order parabolic equation using triple laplace transform coupled with iterative method " is original and it has not been submitted to any institution elsewhere for the award of any academic degree or like and that all the sources I have used or quoted have been indicated and acknowledged as complete references.

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Contents

Acronyms

PPDE- Parabolic Partial differential equation.

NPPDE-Non-linear Parabolic Partial differential equation.

LTM- Laplace transforms method.

TLTM-Triple Laplace Transform method.

PDE- Partial differential equation.

ODE -Ordinary differential equation.

IM -Iteration method

BC -Boundary condition

Abstract

This study presents triple Laplace transform coupled with iterative method to obtain the exact solution of two dimensional nonlinear fourth order parabolic partial differential equation subject to the appropriate initial and boundary conditions. The noise term in this equation is vanished by successive iterative method. The proposed technique has the advantage of producing exact solution and it is easily applied to the given problems analytically. Two test problems from mathematical physics are taken to show the accuracy, convergence and the efficiency of the proposed method. Furthermore, the results indicate that the introduced method is promising for solving other type nonlinear partial differential equations.

Chapter 1

Introduction

1.1 Background of the study

Partial differential equation has big importance in mathematics and other fields of sciences such as: economics, engineering, physics, dynamical models, and so on(Keskin and Oturanc, 2010). PDEs are categorized into different types, including elliptic, parabolic, and hyperbolic PDEs. One of the most important classes of partial differential equations occurring in applied mathematics is that associated with name of parabolic partial differential equation. The Laplace transform can be helpful in solving ordinary differential equation and partial differential equations because it can replace an ODE with an algebraic equation or replace a PDE with an ODE. Another reason that the Laplace transform is useful is that it can help deal with the boundary conditions of a PDE on an infinite domain. One of the most known methods to solve partial differential equation is the integral transform method (Atangana, 2014).

The Laplace Transform Method (LTM) is one of the integral transform methods that have been intensively used to solve linear and nonlinear equations (Hazewinkel, 2001) Performing this calculation in Laplace space turns the convolution into a multiplication; the latter is easier to solve because of its algebraic form. The Laplace transform can also be used to solve differential equations and is use extensively in electrical engineering . The Laplace transform reduces a linear differential equation to an algebraic equation, which can then be to solve by the formal rules of algebra. The original differential equation can then be to solve by applying the inverse Laplace transform (Atangana, 2013). The concept of single Laplace transform is extended to double Laplace transform to solve some kind of dierential equations and fractional dierential equations (Ranjit and Waghamare, 2016a).

To find the solution of nonlinear differential equation, an Iterative Method (IM) is a fascinating task in applied scientific branches. Over the last years, IMs have been to apply in many diverse elds like economics, engineering, physics, dynamical models, and so on. An IM is used to find a solution with degree of accuracy. The iterative procedure leads to a series, which can be to summed up to find an analytical formula, or it can form a suitable approximation (Bhalekar and DaftardarGejji, 2011).

As a result of this, Triple Laplace Transform (TLT) has been developed to solve two dimensional partial differential equations (Atangana, 2013). Besides, to obtain analytical solution of nonlinear differential equation (Dafrartar-GejjiansJafari, 2006) have introduced a new iterative method and used it to solve nonlinear differential equations (Bhalekar and Daftardar-Gejji, 2011).

The purpose of this work is to apply Triple Laplace transform coupled with iterative method developed in to find the analytic solution of two dimensional fourth order parabolic partial differential equations.

$$u_{tt}(x,y,t)+\mu(x,y)u\frac{\partial^4 u}{\partial x^4}+\lambda(x,y)u\frac{\partial^4 u}{\partial y^4}=f(x,y,t),\quad a<x,y<b,t>0, \tag{1.1}$$

with initial conditions:

$$u(x,y,0)=f_0(x,y),u_t(x,y,0)=f_1(x,y),u_{tt}(x,y,0)=f_2(x,y). \tag{1.2}$$

and boundary conditions:

$$
\begin{aligned}
&(a,y,t)=g_0(y,t), && u(b,y,t)=g_1(y,t),\\
&u(x,a,t)=g_2(x,t), && u(x,b,t)=g_3(x,t)\\
&u_x(a,y,t)h_0(y,t), && u_x(b,y,t)=h_1(y,t),\\
&u_y(x,a,t)=h_2(x,t), && u_y(x,b,t)=h_3(x,t)\\
&\frac{\partial^2 u}{\partial x^2}(a,y,t)=k_0(y,t), && \frac{\partial^2 u}{\partial x^2}(b,y,t)=k_1(y,t)\\
&\frac{\partial^2 u}{\partial y^2}(x,a,t)=k_2(y,t), && \frac{\partial^2 u}{\partial y^2}(x,b,t)=k_3(y,t),
\end{aligned}
\tag{1.3}
$$

where $\mu(x,y)$ and $\lambda(x,y)$ are positive, u is a function of x,y,t and $f(x,y,t)$ is a known analytic function and the functions $fi, gi, ki, hi, i=0,1,2,3$ are continuous. In this work, the fourth-order parabolic partial differential equation (1.1) was solved analytically to obtain the exact solution by using triple Laplace transform coupled with iterative method.

1.2 Statement of the problem

Triple Laplace Transform (TLT) has been developed to solve two dimensional partial differential equations (Atangana,2013). Besides, to obtain analytical solution of nonlinear differential equation (Dafrartar-Gejji ans Jafari,2006) have introduced a new iterative method and used it to solve nonlinear differential equations (Bhalekar andDaftardar-Gejji, 2011) As a result, this study mainly focuses on the following ideas related to two dimensional fourth order parabolic partial differential equation using triple Laplace transform coupled with iterative method given by (1.1).

- Applying Triple Laplace transform method coupled with iterative method to obtain the analytical solutions of the two dimensional fourth order parabolic differential equations.
- Verify the applicability of the method by using specific examples.
- Compare the result obtained in this method with some other results in the literature.

1.3 OBJECTIVE OF THE STUDY.

1.3.1 General objective

The general objective of this study is to investigate the analytical solution of fourth order parabolic partial differential equation in two dimensions by using Triple Laplace transform coupled with iterative method.

1.3.2 Specific objectives

The specific objects of this study are:

- To determine the solution of two dimensional fourth order parabolic partial differential equation by using Triple Laplace transform coupled with iterative method(IM).

- To demonstrate the applicability of triple Laplace transform coupled with iterative method(IM) by using some specific examples.

1.4 Significance of the study

The outcome of the study is believed to have the following significances.

- It provides technique to find the solution of two - dimensional fourth order parabolic partial differential equations of Triple Laplace transform method coupled with IM.

- It may familiarize the researcher with the scientific communication in mathematics and develops the skills of mathematical research.

- It can be used as reference material for other researches in the same area.

1.5 Delimitation of the study

The study is delimited to fourth order nonlinear parabolic partial differential equation in two dimensions and is focusing on finding analytical solution by using Triple Laplace transform coupled with iterative method.

Chapter 2

REVIEW OF RELATED LITERATURE

Many of the phenomena that arise in mathematical physics and engineering fields can be described by partial differential equations (PDEs). In physics for example, the heat flow and the wave propagation phenomena are well described by partial differential equations. Moreover, most physical phenomena of fluid dynamics, quantum mechanics, electricity, plasma physics, propagation of shallow water waves, and many other models are formulated by partial differential equations (Majeed.A.W,2018). PDE s have numerous essential applications in various fields of science and engineering such as fluid thermodynamics, heat transfer, physics (Keskin and Oturanc, 2010).In Mathematics, a PDE is a differential equation that contains unknown multi-variable functions and their partial derivatives. PDEs are used to formulate problems involving functions of several variables, and are either solved by hand, or used to create a relevant computer model. A special case is ordinary differential equations (ODEs), which deal with functions of a single variable and their derivatives. PDEs can be used to describe a wide variety of phenomena such as sound, heat, electrostatics, electrodynamics, fluid dynamics, elasticity, or quantum mechanics. These seemingly distinct physical phenomena can be formalized similarly in terms of PDEs (Zauderer, 1989). Many real life problems are represented by differential equation some of those are transport equation ($u_x + u_y = 0$), Laplace equation ($u_{xx} + u_{yy} = 0$), Heat equation ($u_t - ku_{xx} = 0$, Wave equation($u_{tt} - c^2 u_{xx} = 0$), parabolic equation in one dimension and two dimension. In addition to this Differential equations are the mathematical expressions of some real life problems arising out of the real world around us such as physical, biological, engineering, financial or sociological fields.

Differential equation can be catagorized as partial Differential equation (PDE) and Ordinary differential equation (ODE) ,Where PDE is a differential equation that contains unknown multi-variable functions and their partial derivatives and ODE represents a function of a single variable and their derivatives.There are many problems arising in science and engineering are modeled using linear or nonlinear partial differential equations (PDEs). Boundary and initial value problems in PDEs occur in fluid mechanics, mathematical physics, astrophysics, biology, material science, electromagnetism, image processing, computer graphics, etc. PDEs are categorized into different types, including elliptic, parabolic, and hyperbolic PDEs. In this study, the triple laplace transform method will be applied to solve non-linear two dimensional parabolic partial differential equation of the form eq (1.1) ,since Triple Laplace transform method (TLTM) coupled with iterative method is one of the modern method used to solve such differential equation.

The Laplace Transform Method (LTM) is one of the integral transform methods that have been intensively used to solve linear and nonlinear equations. The Laplace transform method is used frequently in engineering and physics; the output of a linear time invariant system can be calculated by convolving its unit impulse response with the input signal. Performing this calculation in Laplace space turns the convolution into a multiplication; the latter is easier to solve because of its algebraic form.The Laplace transform can also be used to solve differential equations and is used extensively in electrical engineering .The Laplace transform reduces a linear differential equation to an algebraic equation, which can then be solved by the formal rules of algebra. The original differential equation can then be solved by applying the inverse Laplace transform (Deresse, A. T., Mussa, Y. O., and Gizaw, A. K). (2021). Recently, the concept of single Laplace transform is extended to double Laplace transform to solve some kind of differential equations and fractional differential equations such as linear/nonlinear space-time fractional telegraph equations, functional, integral and partial differential equations. Dhunde and Waghmare applied the Double Laplace Transform method for solving a one dimensional Boundary Value Problems. Through this method the boundary value problem is solved without converting it into Ordinary Differential equation, therefore no need to find complete solution of Ordinary Differential equation. This is the biggest advantage of the proposed method.

Furthermore, different scholars were extended the double Laplace transform method to Triple Laplace Transform (TLT) to solve two dimensional nonlinear partial differential equations arising in various natural phenomena. Mathematical approaches to PDEs are divided into two methods called analytical method which strives to find exact formulae for the dependent variable as a function of independent variables numerical method which result in approximate values of dependent variables (Saravanan and Magesh, 2013).

Chapter 3

METHODOLOGY

3.1 Study area and period

This study was conducted to find the analytic solution of two dimensional fourth order parabolic partial differential equations by using triple Laplace transform method coupled with iterative method under Department of Mathematics, college of natural science ,Jimma university from August 2019 up to February, 2022.

3.2 Study Design

This study was designed to be done analytically.

3.3 Source of information

The source of information for this study are collected from related documentary materials such as reference books, internets, published and unpublished articles.

3.4 Mathematical Procedures

To attain the objective of the study, the following steps are carried out.

1. Applying triple Laplace transform on both sides of equation (1.1).

2. Employing double Laplace transform to equations (1.2) and (1.3).

3. Implementing the triple inverse Laplace transformation to the results obtained in steps 1 and 2.

4. Assume that $u(x, y, t) = \sum_{i=1}^{\infty} u_i(x, y, t)$ is the solution of equation (1.1).

5. Implementing the new iterative method to the nonlinear term of equation (1.1).

6. Using Triple Laplace transform coupled with new iterative method the recursive relations are developed.

7. Finally, the solution of equations (1.1) - (1.3) will be constructed to have the following series form: $u(x, y, t) = u_0(x, y, t) + u_1(x, y, t) + u_2(x, y, t) + ...u_n(x, y, t) + ...$

Chapter 4

RESULT AND DISCUSSION

4.1 Premilinaries

In this section, we give some essential definitions, properties and theorems of Triple Laplace transform of partial differential equation, which should be used in the present study.

Definition 1: Let $f(x)$ be a piecewise continuously differentiable function in every finite interval and is absolutely integrable function on the whole real line (Debtnath, 2007).

The Fourier integral formula of $f(x)$ is defined by:

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} e^{-ikt} f(t) dt \right] e^{-ikt} dk, \tag{4.1}$$

where k is a complex number and is taken as the variable of the transform.

Definition 2: Euler gamma function $\Gamma(s)$ is defined by the uniformly convergent integral and is given by:

$$\Gamma(s) = \int_{0}^{\infty} x^{(s-1)} e^{-x} dx \qquad s > 0. \tag{4.2}$$

Note that $\Gamma(s) = (s-1)!$ for a positive integer s (Debtnath, 2015).

4.1.1 Triple Laplace Transform

Definition 3: Let $f(x,y,t)$ be a function of three variables x,y and t defined in the positive xyt in space. The triple Laplace transform of the function $f(x,y,t)$ is defined by

$$L_{xyt}[f(x,y,t)] = F(k,p,s) = \int_0^\infty e^{-kx}\int_0^\infty e^{-py}\int_0^\infty e^{-st} f(x,y,t)dxdydt, \tag{4.3}$$

whenever the integral exist (Thakuri and Shrikume,2015).

Here $L_{xyt}[f(x,y,t)]$ denote $L_xL_yL_t[f(x,y,t)]$,and k,p,s are complex numbers.

From this definition, we deduce that:

$$L_{xyt}\{f(x)g(y)h(t)\} = F(k)G(p)H(s) = L_x\{f(x)\}L_y\{g(y)\}L_t\{h(t)\}. \tag{4.4}$$

Definition 4: The inverse triple Laplace transform of$F(k,p,s)$, $L^{-1}_{xyt}[F(k,p,s)] = f(x,y,t)$ is given by the complex triple integral formula.

$$L^{-1}_{xyt}[F(k,p,s)] = f(x,y,t) = \frac{1}{2\pi i}\int_{q-i\infty}^{q+i\infty} e^{-kx}\left(\frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} e^{-px}\Big[\frac{1}{2\pi i}\int_{z-i\infty}^{z+i\infty} e^{-sx}ds\Big]dp\right)dk, \tag{4.5}$$

where $L^{-1}_{xyt}[F(k,p,s)]$ denote $L^{-1}_xL^{-1}_yL^{-1}_t[F(k,p,s)]$,and $F(k,p,s)$ must be an analytic function for all k,p and s in the region defined by the inequality $Re(k) > q, Re(p) > r$and $Re(s) > z$ where q,r and z are real constant to be chosen suitably.

Property 1: The triple Laplace transform of second order partial derivatives are given by:

$$L_{xyt}[\frac{\partial^2}{\partial x^2}f(x,y,t)] = k^2F(k,p,s) - kF(0,p,s) - F_x(0,p,s),$$

$\mathrm{L}_{xyt}[\frac{\partial^2}{\partial y^2}f(x,y,t)] = p^2F(k,p,s) - pF(k,0,s) - F_y(0,p,s),$

$\mathrm{L}_{xyt}[\frac{\partial^2}{\partial t^2}f(x,y,t)] = s^2F(k,p,s) - sF(k,p,0) - F_t(k,p,0),$

$\mathrm{L}_{xyt}[\frac{\partial^2}{\partial x\partial y}f(x,y,t)] = kpF(k,p,s) - F(0,p,s) - F(k,0,s).$(4.6)

Furthermore, the triple Laplace transform of first order partial derivatives are given by:

$$Ł_{xyt}[\frac{\partial}{\partial x} f(x,y,t)] = kF(k,p,s) - F(0,p,s),$$

$Ł_{xyt}[\frac{\partial}{\partial y} f(x,y,t)] = pF(k,p,s) - F(k,0,s),$

$Ł_{xyt}\frac{\partial}{\partial t} f(x,y,t) = sF(k,p,s) - F(k,p,0).$ (4.7)

4.1.2 Existence and uniqueness of the Triple Laplace Transform

. **Theorem 1 (Existence)**: Let $f(x,y,t)$ be a continuous function on the interval $[0,\infty)$ wich is of exponential order that is for some a ,b,c $\in R$.

$$Consider \sup_{x,y,t>0} |\frac{f(x,y,t)}{exp[ax+by+ct]}| < 0. \tag{4.8}$$

Under this condition, the triple transform,

$F(k,p,s) = \int_0^\infty \int_0^\infty \int_0^\infty e^{-kx} e^{-py} e^{-st} f(x,y,t) dxdydt$

Exists for all $k > a, p > b$ and $s > c$ and is in actuality infinitely differentiable with respect to $k > a, p > b$ and $s > c$.

Proof: we have

$$\begin{aligned} |F(k,p,s)| =& |\int_0^\infty \int_0^\infty \int_0^\infty e^{-kx-py-st}| \\ & \leq k \int_0^\infty e^{-(k-a)x} dx \int_0^\infty e^{-(p-b)y} dy \int_0^\infty e^{-(s-c)t} dt, \\ =& \frac{k}{(k-a)(p-b)(s-c)}. \end{aligned} \tag{4.9}$$

Then it follows that , $\lim_{k,p,s\to\infty} |F(k,p,s)| = 0.$

and hence $\lim_{k,p,s\to\infty} F(k,p,s) = 0.$

This proves the existence of triple Laplace transform.

Theorem 2 (Uniqueness): Let $f(x,y,t)$ and $g(x,y,t)$ be continuous functions defined for $x,y,t \geq 0$ and having Laplace transforms, $F(k,p,s)$ and $G(k,p,s)$, respectively.

If $F(k,p,s) = G(k,p,s)$, then $f(x,y,t) = g(x,y,t)$.

Proof: By definition 3 the inverse triple Laplace transform of$f(x,y,t)$ is given by the integral expression $L_{xyt}^{-1}[F(k,p,s)] = f(x,y,t) = \frac{1}{2\pi i}\int_{q-i\infty}^{q+i\infty} e^{-kx}\left(\frac{1}{2\pi i}\int_{-i\infty}^{r+i\infty} e^{-px}\left[\frac{1}{2\pi i}\int_{z-i\infty}^{z+i\infty} e^{-sx}ds\right]dp\right)dk$

$f(x,y,t) = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} e^{kx}\left(\frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} e^{px}\left[\frac{1}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty} e^{sx}ds\right]dp\right)dk = g(x,y,t),$

where α, β,and μ, are sufficiently large. By hypothesis, F(k,p,s) = G(k,p,s), then replacing this in the previous expression, we get

$f(x,y,t) = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} e^{kx}\left(\frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} e^{px}\left[\frac{1}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty} e^{sx}ds\right]dp\right)dk = g(x,y,t)$

and this proves the uniqueness of the triple Laplace transform.

4.1.3 Some properties of triple Laplace transform

Property 2 (Linearity of Triple Laplace Transform(TLT)):

If $f(x,y,t)$ and $g(x,y,t)$ are two functions of x, y and t such that

$L_{xyt}f(x,y,t)$=$F(k,p,s)$and$L_{xyt}g(x,y,t) = G(k,p,s)$then,

$L_{xyt}\alpha f(x,y,t) + \beta g(x,y,t) = \alpha L_{xyt}f(x,y,t) +_{xyt} g(x,y,t) = \alpha F(k,p,s) + \beta G(k,p,s),$

where α and β are constants.

Property 3(Change of Scale Property):

$Let L_{xyt}f(x,y,t) = F(k,p,s)$, then $L_{xyt}f(ax,by,ct) = \frac{1}{abc}F(\frac{k}{a},\frac{p}{b},\frac{s}{c})$,

where a,b and c are non-zero constants .

Proof: we have

$L_xL_yL_t[f(ax,by,ct)] = \int_0^\infty e^{-kx}\int_0^\infty e^{-py}\int_0^\infty e^{-st}f(x,y,t)dxdydt$

Applying change of variables $u = ax, v = by, w = ct$, we get

$$\begin{aligned} L_xL_yL_t[f(ax,by,ct)] &= \int_0^\infty e^{-k(\frac{u}{a})}\int_0^\infty e^{-p(\frac{v}{b})}\int_0^\infty e^{-s(\frac{w}{c})}f(u,v,w)\frac{du}{a}\frac{dv}{b}\frac{dw}{c} \\ &= \frac{1}{abc}\int_0^\infty e^{-k(\frac{u}{a})}\int_0^\infty e^{-p(\frac{v}{b})}\int_0^\infty e^{-s(\frac{w}{c})}f(u,v,w)dudvdw \qquad (4.10) \\ &= \frac{1}{abc}F(\frac{k}{a},\frac{p}{b},\frac{s}{c}), \end{aligned}$$

where a, b, and c are non-zero constants and hence property 3 is proved.

Property 4(First Shifting Property):

If $L_{xyt}f(x,y,t) = F(k,p,s)$,then $L_{xyt}[e^{ax+by+ct}f(x,y,t)] = F(k-a,p-b,s-c)$.

Proof: By definition,

$L_xL_yL_t[e^{ax+by+ct}f(x,y,t)] = \int_0^\infty e^{-kx}\int_0^\infty e^{-py}\int_0^\infty e^{ax+by+ct}f(x,y,t)dxdydt$

$= \int_0^\infty e^{-(k-a)x}\int_0^\infty e^{-(p-b)y}\int_0^\infty e^{-(s-c)t}f(x,y,t)dxdydt = F(k-a,p-b,s-c)$

Hence property 4 is proved

Property 5(Second Shifting Property):

If $L_{xyt}f(x,y,t) = F(k,p,s)$, then $L_{xyt}f(x-a,y-b,t-c)H(x-a,y-b,t-c)$

$= e^{-ak-bp-cs}F(k,p,s)$,

where H(x,y,t) is the Heaviside unit step function defined by

$$H(x-a,y-b,t-c) = \begin{cases} 1: & x>a, y>b, t>c \\ 0: & x<a, y<b, t<c. \end{cases} \tag{4.11}$$

Proof: by definition of triple Laplace transform,

$\mathrm{L}_xL_yL_t[f(x-a,y-b,t-c)H(x-a,y-b,t-c)]$

$= \int_0^\infty\int_0^\infty\int_0^\infty e^{-kx-py-st}f(x-a,y-b,t-c)H(x-a,y-b,t-c)dxdydt.$

$=\int_0^\infty\int_0^\infty\int_0^\infty e^{-kx-py-st}f(x-a,y-b,t-c)dxdydt$

Taking $u = x-a$, $v = y-b$ and $w = t-c$ we have,

$\int_0^\infty\int_0^\infty\int_0^\infty e^{-kx-py-st}f(x-a,y-b,t-c)$ dxdydt

$= e^{-ka-pb-sc}\int_0^\infty\int_0^\infty\int_0^\infty e^{-ku-pv-sw}f(u,v,w)dudvdw = e^{-ka-pb-sc}F(k,p,s)$.

Hence property 5 is proved.

Property 6: If$L_{xyt}f(x,y,t) = F(k,p,s)$, then

$L_xL_yL_t(x^my^nt^r) = \frac{m!n!r!}{k^{m+1}p^{n+1}s^{r+1}}$,$(m>-1)$, $(n>-1)$ and $(r>-1)$ are real numbers.

Proof: the proof follows by extending the concept double Laplace transform to triple Laplace transform (Debtnat,2015). By Definition 2,

$L_xL_yL_t(x^my^nt^r) = \int_0^\infty\int_0^\infty\int_0^\infty e^{-kx-py-st}x^my^nt^r dxdydt = \int_0^\infty e^{-kx}x^m dx \int_0^\infty e^{-py}y^n dy \int_0^\infty e^{-st}t^r dt$

put $kx = u$ $py = v$ and $st = w$ we obtain,

$L_xL_yL_t(x^my^nt^r) = \frac{1}{k^{m+1}}\int_0^\infty u^u e^{-u}du \frac{1}{p^{n+1}}\int_0^\infty v^n e^{-v}dv \frac{1}{s^{r+1}}\int_0^\infty w^r e^{-w}dw$

$= \frac{\Gamma(m+1)\Gamma(n+1)\Gamma(r+1)}{k^{m+1}p^{n+1}s^{r+1}}$

$= \frac{m!n!r!}{k^{m+1}p^{n+1}s^{r+1}}.$

is proves property 6

4.1.4 Triple Laplace transforms of some functions of three variables

Based on the definition and properties of triple Laplace transform table 4.1 display triple laplace transform of some special functions.

Table 4.1 Triple Laplace transform for some function of three variables (Atangana, 2013)

Functions of f(x,y,t)	Triple Laplace transform $F(k, ps)$
abc	$\frac{abc}{kps}$
xyt	$\frac{1}{k^2p^2s^2}$
$x^my^nt^r$	$\frac{m!n!r!}{k^{m+1}p^{n+1}s^{r+1}}$
$e^{-ax-by-ct}$	$\frac{1}{(k+a)(p+b)(s+c)}$
$cos(x)cos(y)cos(t)$	$\frac{kps}{(k^2+1)(p^2+1)(s^2+1)}$
$sin(x)sin(y)sin(t)$	$\frac{1}{(k^2+1)(p^2+1)(s^2+1)}$
$cos(x+y+t)$	$\frac{k+p+s-kps}{(k^2+1)(p^2+1)(s^2+1)}$
$sin(x+y+t)$	$\frac{-1+ps+k(p+s)}{(k^2+1)(p^2+1)(s^2+1)}$

4.1.5 The new iterative method

Consider the following general functional equation.

$$u = N(u) + f \tag{4.12}$$

where N is a nonlinear operator from a Banach space $f : B \to B$ and f is a known function. We are looking for a solution u of equation (4.12) having the series from:

$$u = \sum_{i=0}^{\infty} u_i \tag{4.13}$$

The nonlinear operator N can be decomposed as :

$$N(\sum_{i=0}^{\infty} u_i) = N(u_0) + \sum_{i=1}^{\infty}[N(\sum_{j=0}^{i} u_j) - N(\sum_{j=0}^{i-1} u_j)] \tag{4.14}$$

From equation (4.13)and (4.14),equation (4.12) is equivalent to

$$\sum_{i=0}^{\infty} u_i = f + N(u_0) + \sum_{i=1}^{\infty}[N(\sum_{j=0}^{i} u_j) - N(\sum_{j=0}^{i-1} u_j)], \tag{4.15}$$

where we have used equations (4.13) and (4.14) to get (4.15). Consider now the sequences given by:

$$G_n = N(\sum_{i=0}^{n} u_i) - N(\sum_{i=0}^{n-1} u_i), n = 1, 2, 3... \tag{4.16}$$

We define the recurrence relation: $G_0 = u_0 = f$

$$G_1 = u_1 = N(u_0) \tag{4.17}$$

$$G_n = u_{n+1} = N(u_0 + ...u_n) - N(u_0 + ... + u_{n-1}), n \geq 1 \tag{4.18}$$

Then,

$$(u_1 + ... + u_{n+1}) = N(u_1 + ... + u_n), n \geq 1 \tag{4.19}$$

and

$$u(x) = f + \sum_{i=0}^{\infty} u_i \tag{4.20}$$

The n-term approximate solution of equation (4.19) is given by:

$$u = u_0 + u_1 + ... + u_{n-1}. \tag{4.21}$$

4.1.6 Convergence of the new iterative method

Theorem 3: If N is a continuously differentiable functional in a neighborhood of u_0 and $\|N^{(n)}(u_0)\| \leq M \leq e^{-1}$ for all n, then the series $\sum_{i=0}^{\infty} G_n$ is absolutely convergent (Bhalekar and Daftarder, 2011), and where G_n is as indicated in equation (4.16).

Proof: Consider the recurrence relation.

$$\xi_n = \xi_0 exp(\xi_n - 1, n = 1, 2, 3...where \xi_0 = M \tag{4.22}$$

Define,

$$\beta_n = \xi_n - \xi_{n-1} \tag{4.23}$$

using equation (4.16) - (4.23), and the hypothesis of Theorem (3), we observe that

$$G_n \leq \beta_n, for n = 1, 2, 3, ... \tag{4.24}$$

Let

$$\alpha_n = \sum_{i=1}^{n} \beta_i \leq \xi_n - \xi_0 \tag{4.25}$$

Note that ,$\xi_o = e^{-1} > 0, \xi_1 = \xi_0 exp(\xi_0) > \xi_0, \xi_2 = \xi_0 exp(\xi_1) > \xi_0 exp(\xi_0) = \xi_1$

In general ,$\xi_n > \xi_{n-1} > 0$ and hence $\sum_{n=1}^{\infty} \beta_n$ is a serie of positive real numbers.

$$0 \leq \xi_0 = M = e^{-1} < 1.0 < \xi_1 = \xi_0 exp(\xi_0) < \xi_0 e^1 = e^{-1} e^1 = 1.0 < \xi_2 = \xi_0 exp(\xi_1) < \xi_0 e^1 = 1 \tag{4.26}$$

It follows that,$0 \leq \xi_n < 1$ hence,$a_n = \xi_n - \xi_0 < 1$.

This implies that the sequence is bounded, and hence convergent.

Therefore,$\sum_{n=0}^{\infty} G_n$ is absolutely convergent by comparison test.

On the other hand this proves the existence of analytical solution by the help of the new iterative method.

4.2 Main result

The aim of this study is to obtain analytical solution of two dimensional fourth order parabolic partial differential equation using Triple Laplace transfom coupled with Iterative Method. This is done by extending the work of (Ranjit and Waghamar, 2016) that was used to solve two dimensional NLPDE by using double Laplace transform and iterative method. So, the definitions, theorems and some derivations related to triple Laplace transform mentioned in the preceding section be applied here. Consider the two-dimensional nonlinear fourth order parabolic equation

$u_{tt}(x,y,t) + \mu(x,y)u\frac{\partial^4 u}{\partial x^4} + \lambda(x,y)u\frac{\partial^4 u}{\partial y^4} = f(x,y,t)$, $a < x, y < b, t > 0$.

with initial conditions:

$u(x,y,0) = f_0(x,y), u_t(x,y,0) = f_1(x,y), u_{tt}(x,y,0) = f_2(x,y)$.

and boundary conditions:

$(a,y,t) = g_0(y,t),$ $\quad u(b,y,t) = g_1(y,t),$

$u(x,a,t) = g_2(x,t),$ $\quad u(x,b,t) = g_3(x,t)$

$u_x(a,y,t)h_0(y,t),$ $\quad u_x(b,y,t) = h_1(y,t),$

$u_y(x,a,t) = h_2(x,t),$ $\quad u_y(x,b,t) = h_3(x,t)$

$\frac{\partial^2 u}{\partial x^2}(a,y,t) = k_0(y,t),$ $\quad \frac{\partial^2 u}{\partial x^2}(b,y,t) = k_1(y,t)$

$\frac{\partial^2 u}{\partial y^2}(x,a,t) = k_2(y,t),$ $\quad \frac{\partial^2 u}{\partial y^2}(x,b,t) = k_3(y,t),$

where $\mu(x,y)$ and$\lambda(x,y)$ are positive and $f(x,y,t)$ is a source function.

In order to solve such problem we first decompose the source function $f(x,y,t)$ into $f_1(x,y,t)$and

$f_2(x,y,t)$. The part $f_1(x,y,t)$ with the terms in equation (1.1) is always leads to the simple algebraic expression while applying the inverse triple Laplace transform. The portion $h_2(x,y,t)$ is combined with the nonlinear term of equation (1.1) to avoid complicated terms in the iteration process.

Step 1: Applying triple Laplace transform on both sides of equation (1.1), we get

$$s^2\bar{u}(k,p,s)-s\bar{u}(k,p,0)-\bar{u}_t(k,p,0) = L_{xyt}(f(x,y,t))-L_{xyt}\left(\mu(x,y)u\frac{\partial^4 u}{\partial x^4}(x,y,t)+\lambda(x,y)u\frac{\partial^4 u}{\partial y^4}(x,y,t)\right), \tag{4.27}$$

where $f(x,y,t)$ is the source of functions of equation (1.1) .

Note that the existence of Laplace transform of functions of three variables .

Step 2. Now employing double Laplace transform to equations (1.2) and (1.3), we have

$u(x,y,0) = f_0(x,y), u_t(x,y,0) = f_1(x,y), u_{tt}(x,y,0) = f_2(x,y)$

$u(a,y,t) = g_0(y,t), u(b,y,t) = g_1(y,t), u(x,a,t) = g_2(x,t), u(x,b,t) = g_3(x,t)$

$u(x,y,0) = f_0(x,y), u_t(x,y,0) = f_1(x,y), u_{tt}(x,y,0) = f_2(x,y)$

$u_x(a,y,t) = h_0(y,t), u_x(b,y,t) = h_1(y,t), u_y(x,a,t) = h_2(x,t), u_y(x,b,t) = h_3(x,t)$

$u_{xx}(a,y,t) = k_0(y,t), u_{xx}(b,y,t) = k_1(y,t), u_{yy}(x,a,t) = k_2(x,t), u_{yy}(x,b,t) = k_3(x,t).$

By substituting equations (1.2) and (1.3) into equation (4.27) and simplifying, we obtain

$$\bar{u}(k,p,s) = \frac{1}{s^2}[s\bar{u}(k,p,0)+\bar{u}_t(k,p,0)+L_{xyt}(f(x,y,t))-L_{xyt}(\mu(x,y)u\frac{\partial^4 u}{\partial x^4}(x,y,t)+\lambda(x,y)u\frac{\partial^4 u}{\partial y^4}(x,y,t)]. \tag{4.28}$$

Step 3. By implementing the triple inverse Laplace transformation of equation (4.28), we obtain:

$$\bar{u}(k,p,s) = L^{-1}_{xyt}\frac{1}{s^2}\left\{\begin{array}{r} s\bar{u}(k,p,0)+\bar{u}_t(k,p,0) \\ -L_{xyt}(\mu(x,y)u\frac{\partial^4 u}{\partial x^4}(x,y,t)+\lambda(x,y)u\frac{\partial^4 u}{\partial y^4}(x,y,t) \\ +L_{xyt}(f(x,y,t)) \end{array}\right\}. \tag{4.29}$$

Step 4. Assume that u(x,y,t)=$\sum_{i=0}^{\infty} u_i(x,y,t)$ is the solution of equation (1.1). Then substituting equation (4.29) into equation (4.28), we obtain:

$$\sum_{i=0}^{\infty} u_i(x,y,t) = L_{xyt}^{-1}\left\{ \frac{1}{s^2}\left[\begin{array}{r} s\bar{u}(k,p,0)+\bar{u}_t(k,p,0) \\ -L_{xyt}\left(\mu(x,y)u\frac{\partial^4 u}{\partial x^4}(x,y,t)+\lambda(x,y)u\frac{\partial^4 u}{\partial y^4}(x,y,t)\right) \\ +L_{xyt}(f(x,y,t)) \end{array} \right]\right\}. \quad (4.30)$$

Step5. By implementing the new iterative method in equation (4.30) is decomposed as the nonlinear term $\mu(x,y)u\frac{\partial^4 u}{\partial x^4}(x,y,t)+\lambda(x,y)u\frac{\partial^4 u}{\partial y^4}(x,y,t)$ in equation (4.30) is decomposed as:

$$\sum_{i=0}^{\infty} u_i(x,y,t) = L_{xyt}^{-1}\left\{ \frac{1}{s^2}\left[\begin{array}{r} s\bar{u}(k,p,0)+\bar{u}_t(k,p,0) \\ +L_{xyt}\left(f(x,y,t)-\mu(x,y)u\frac{\partial^4 u}{\partial x^4}(x,y,t)+\lambda(x,y)u\frac{\partial^4 u}{\partial y^4}(x,y,t)\right) \end{array} \right]\right\}. \quad (4.31)$$

Step6. Using Triple Laplace Transform coupled with new iterative method, we introduce the recursive relations and get:

$$u_0(x,y,t) = L_{xyt}^{-1}\left\{ \frac{1}{s^2}\left[s\bar{u}(k,p,0)+\bar{u}_t(k,p,0) \right]\right\}. \quad (4.32)$$

$$u_1(x,y,t) = L_{xyt}^{-1}\left\{ \frac{1}{s^2}L_{xyt}\left[f(x,y,t)-\mu(x,y)u\frac{\partial^4 u}{\partial x^4}(x,y,t)+\lambda(x,y)u\frac{\partial^4 u}{\partial y^4}(x,y,t \right]\right\}. \quad (4.33)$$

$$u_{n+1}(x,y,t) = -L_{xyt}^{-1}\left\{ \frac{1}{s^2}L_{xyt}\left[\begin{array}{r} \sum_{r=0}^{n}\left(\mu(x,y)u_r\frac{\partial^4 u_r}{\partial x^4}(x,y,t)+\lambda(x,y)u_r\frac{\partial^4 u_r}{\partial y^4}(x,y,t)\right)- \\ \sum_{r=0}^{n-1}\left(\mu(x,y)u_r\frac{\partial^4 u_r}{\partial x^4}(x,y,t)+\lambda(x,y)u_r\frac{\partial^4 u_r}{\partial y^4}(x,y,t)\right) \end{array} \right]\right\}, n \geq 1. \quad (4.34)$$

Step7. The solution of equations (1.1) - (1.3) in series form is given by:

$$u(x,y,t) = u_0(x,y,t)+u_1(x,y,t)+u_2(x,y,t)+...+u_n(x,y,t)+... \quad (4.35)$$

4.2.1 Illustrative examples

In order to show the validity and effectiveness of the method under consideration some examples are presented here.

Example 1: Consider the following two dimensional nonlinear fourth-order parabolic partial differential equation :

$$\frac{\partial^2 u}{\partial t^2} + xu\frac{\partial^4 u}{\partial x^4} + yu\frac{\partial^4 u}{\partial y^4} = 2x^4y^4 + 24t^4x^8y^5 + 24t^4x^5y^8, \frac{1}{2} < x, y < 1, t > 0 \tag{4.36}$$

with initial conditions

$$u(x, y, 0) = 0, u_t(x, y, 0) = 0. \tag{4.37}$$

and boundary conditions:

$$\begin{aligned} u(\frac{1}{2}, y, t) = (\frac{1}{2})^4y^4t^2, u(1, y, t) = y^4t^2. \\ \frac{\partial^2 u}{\partial x^2}(\frac{1}{2}, y, t) = 12(\frac{1}{2})^2y^4t^2, \frac{\partial^2 u}{\partial x^2}(1, y, t) = 12y^4t^2. \\ \frac{\partial^2 u}{\partial y^2}(x, \frac{1}{2}, t) = 12(\frac{1}{2})^2x^4t^2, \frac{\partial^2 u}{\partial y^2}(1, y, t) = 12x^4t^2. \end{aligned} \tag{4.38}$$

solution:Rewrite the equation(4.36)

$\frac{\partial^2 u}{\partial t^2} - 2x^4y^4 = 24t^4x^8y^5 + 24t^4x^5y^8 - (xu\frac{\partial^4 u}{\partial x^4} + yu\frac{\partial^4 u}{\partial y^4})$.

Applying properties of triple Laplace transform on both sides of equation (4.36) , we get,

$$s^2\bar{u}(k, p, s) - s\bar{u}(k, p, 0) - \bar{u}_t(k, p, 0) - \frac{2(4!)(4!)}{sp^5k^5} = L_{xyt}\left[24t^4x^8y^5 + 24t^4x^5y^8 - (xu\frac{\partial^4 u}{\partial x^4} + yu\frac{\partial^4 u}{\partial y^4})\right]. \tag{4.39}$$

Applying double Laplace transform to equations (4.37) and (4.38), we obtain.

$$\begin{aligned}\bar{u}(k,p,0)=0 \; and \; \bar{u}_t(k,p,0)=0 and \bar{u}(\frac{1}{2},p,s)=(\frac{1}{2})^4\frac{(3)!(1)!}{y^4t^2}\\ \bar{u}(1,p,s)=\frac{(3)!(1)!}{y^4t^2},\bar{u}_{xx}(\frac{1}{2},p,s)=12(\frac{1}{2})^2\frac{(3)!(1)!}{y^4t^2},\bar{u}_{xx}(1,p,s)=12\frac{(3)!(1)!}{y^4t^2},\\ \bar{u}_{yy}(\frac{1}{2},p,s)=12(\frac{1}{2})^2\frac{(3)!(1)!}{x^4t^2}\bar{u}_{yy}(k,1,s)=12\frac{(3)!(1)!}{x^4t^2}.\end{aligned} \tag{4.40}$$

By substituting equations (4.40) into equation (4.39), we get

$\bar{u}(k,p,s)=\frac{1}{s^2}\left(\frac{2(4!)(4!)}{sp^5k^5}+L_{xyt}[24t^4x^8y^5+24t^4x^5y^8-(xu\frac{\partial^4u}{\partial x^4}+yu\frac{\partial^4u}{\partial y^4})]\right)$

$$\bar{u}(k,p,s)=\frac{2(4!)(4!)}{s^3p^5k^5}+\frac{1}{s^2}\left\{L_{xyt}[24t^4x^8y^5+24t^4x^5y^8-(xu\frac{\partial^4u}{\partial x^4}+yu\frac{\partial^4u}{\partial y^4})]\right\}. \tag{4.41}$$

Applying inverse triple Laplace transform to equation (4.41), we get

$u(x,y,t)=x^4y^4t^2+L^{-1}_{xyt}\left\{\frac{1}{s^2}L_{xyt}[24t^4x^8y^5+24t^4x^5y^8-(xu\frac{\partial^4u}{\partial x^4}+yu\frac{\partial^4u}{\partial y^4})]\right\}.$

Now ,applying the new iterative method,

$$u_0(x,y,t)=x^4y^4t^2. \tag{4.42}$$

$\mathrm{u}_1(x,y,t)=L^{-1}_{xyt}\left\{\frac{1}{s^2}L_{xyt}[24t^4x^8y^5+24t^4x^5y^8-(xu_0\frac{\partial^4u_0}{\partial x^4}+yu_0\frac{\partial^4u_0}{\partial y^4})]\right\}.$

$$u_{n+1}(x,y,t)=-L^{-1}_{xyt}\left\{\frac{1}{s^2}L_{xyt}\left[\sum_{r=0}^{n}\left(xu_r\frac{\partial^4u_r}{\partial x^4}+yu_r\frac{\partial^4u_r}{\partial y^4}\right)-\sum_{r=0}^{n-1}(xu_r\frac{\partial^4u_r}{\partial x^4}+yu_r\frac{\partial^4u_r}{\partial y^4})\right]\right\},n\geq 1 \tag{4.43}$$

Similarly, we obtain $u_2(x,y,t)=u_3(x,y,t)=0$ and so on,

Therefore, the solution of Example 1 by using equation (4.35) is

$u(x,y,t)=x^4y^4t^2$.

Let us now test the convergence of the obtained series solution. From equation (4.42), we have,

$u_0(x,y,t) = x^4y^4t^2$ and

N(u(x,y,t))=$\mathrm{L}_{xyt}^{-1}\left\{\frac{1}{s^2}L_{xyt}[24t^4x^8y^5 + 24t^4x^5y^8 - (xu\frac{\partial^4 u}{\partial x^4} + yu\frac{\partial^4 u}{\partial y^4})]\right\}$.

Thus for $x, y, t \geq 0$, we have

$$\mathrm{N}(\mathrm{u}_0(x,y,t)) = L_{xyt}^{-1}\left\{\frac{1}{s^2}L_{xyt}[24t^4x^8y^5 + 24t^4x^5y^8 - (xu_0\frac{\partial^4 u_0}{\partial x^4} + yu_0\frac{\partial^4 u_0}{\partial y^4})]\right\}$$

$$= L_{xyt}^{-1}\left\{\frac{1}{s^2}L_{xyt}[24t^4x^8y^5 + 24t^4x^5y^8 - (xu_0\frac{\partial^4 u_0}{\partial x^4} + yu_0\frac{\partial^4 u_0}{\partial y^4})]\right\} = 0.$$

Therefore,$\|N(u_0(x,y,t))\| = \|0\| = 0 < \frac{1}{e}$.

$N'(u(x,y,t))$=$L_{xyt}^{-1}\left\{\frac{1}{s^2}L_{xyt}[(24t^4x^8y^5 + 24t^4x^5y^8)' - (xu\frac{\partial^4 u}{\partial x^4} + yu\frac{\partial^4 u}{\partial y^4})']\right\}$.

where $N'(u(x,y,t))$ represents the partial derivatives $\frac{\partial}{\partial x}u(x,y,t)$ $\frac{\partial}{\partial y}u(x,y,t)$ or $\frac{\partial}{\partial t}u(x,y,t)$.

$N(\frac{\partial}{\partial t}u(x,y,t))$= $L_{xyt}^{-1}\left\{\frac{1}{s^2}L_{xyt}[96t^3x^8y^5 + 96t^3x^5y^8 - (xu\frac{\partial^4 u}{\partial x^4} + yu\frac{\partial^4 u}{\partial y^4})']\right\}$.

Then,

$N(\frac{\partial}{\partial t}u_0(x,y,t))$= $L_{xyt}^{-1}\left\{\frac{1}{s^2}L_{xyt}[96t^3x^8y^5 + 96t^3x^5y^8 - (96t^3x^5y^8 + 96t^3x^8y^5)]\right\}$=0.

$N(\frac{\partial}{\partial x}u_0(x,y,t))$= $L_{xyt}^{-1}\left\{\frac{1}{s^2}L_{xyt}[192t^4x^7y^5 + 120t^4x^4y^8 - (120t^4x^4y^8 + 192t^4x^7y^5)]\right\}$=0.

$N(\frac{\partial}{\partial y}u_0(x,y,t))$= $L_{xyt}^{-1}\left\{\frac{1}{s^2}L_{xyt}[120t^4x^8y^4 + 192t^4x^5y^7 - (192t^4x^5y^7 + 120t^4x^8y^4)]\right\}$=0.

Therefore,$\|N(u_0'(x,y,t))\| = \|0\| = 0 < \frac{1}{e}$.

$N(\frac{\partial^2}{\partial t^2}u_0(x,y,t))$= $L_{xyt}^{-1}\left\{\frac{1}{s^2}L_{xyt}[288t^2x^8y^5 + 288t^2x^5y^8 - (288t^2x^5y^8 + 288t^2x^8y^5)]\right\}$=0.

$N(\frac{\partial^2}{\partial x^2}u_0(x,y,t))$= $N(\frac{\partial^2}{\partial y^2}u_0(x,y,t))$=$N(\frac{\partial^2}{\partial x\partial y}u_0(x,y,t))$=0.

$N(\frac{\partial^2}{\partial x\partial t}u_0(x,y,t))$= $N(\frac{\partial^2}{\partial y\partial t}u_0(x,y,t))$=0.

Therefore,$\|N(u_0'{}'(x,y,t))\| = \|0\| = 0 < \frac{1}{e}$.

where $N(u_0'{}'(x,y,t))$ represents all the second order partial derivatives of $u_0(x,y,t)$.

Similarly by principle of Mathematical induction we have ,

$\|N(u_0^{(3)}(x,y,t))\|$=$\|N(u_0^{(4)}(x,y,t))\| = ... =$$\|N(u_0^{(n)}(x,y,t))\| = \|0\| = 0 < \frac{1}{e}$,for all $n \geq 3$.

As the condition ,If N is a continuously differentiable functional in a neighborhood of u_0 and$\|N(u_0^{(n)}(x,y,t))\| \leq M \leq e^{-1}$ for all n, then the series solution, $u(x,y,t) = \sum_{i=0}^{\infty} u_i(x,y,t)$is absolutely convergent and hence the solution obtained by the new iterative method is convergent on the domain of interest.

The numerical solution representing the solution behavior of Example 1 is depicted in Figure 1.

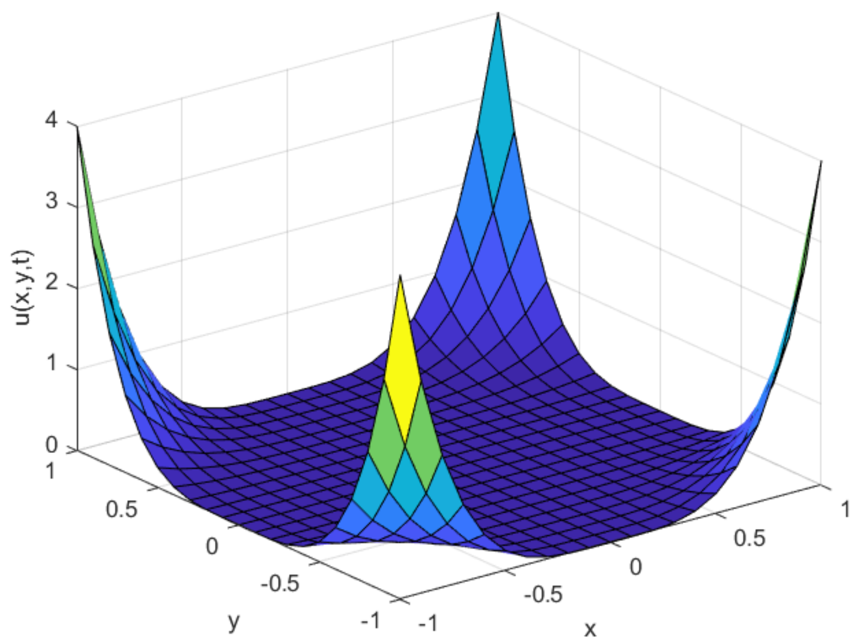

Fig. 1 Solution behavior of Example 1 for $t = 2$

Example 2: Consider the following three dimensional non-homogeneous nonlinear fourth-order parabolic partial differential equation with variable coefficients :

$$\frac{\partial^2 u}{\partial t^2} + xu\frac{\partial^4 u}{\partial x^4} + yu\frac{\partial^4 u}{\partial y^4} = 6tx^4y^4 + 24t^6x^8y^5 + 24t^6x^5y^8, \frac{1}{2} < x, y < 1, t > 0. \tag{4.44}$$

with initial conditions:

$$u(x, y, 0) = 0, u_t(x, y, 0) = 0. \tag{4.45}$$

and boundary conditions:

$$\begin{gathered} u(\frac{1}{2}, y, t) = (\frac{1}{2})^4 y^4 t^3, u(1, y, t) = y^4 t^3. \\ \frac{\partial u}{\partial x}(\frac{1}{2}, y, t) = 4(\frac{1}{2})^3 y^4 t^3, \frac{\partial u}{\partial x}(1, y, t) = 4y^4 t^3. \\ \frac{\partial u}{\partial y}(x, \frac{1}{2}, t) = 4(\frac{1}{2})^3 x^4 t^3, \frac{\partial u}{\partial y}(1, y, t) = 4x^4 t^3. \end{gathered} \tag{4.46}$$

Solution:Rewrite the equation(4.44)

$\frac{\partial^2 u}{\partial t^2} - 6tx^4y^4 = 24t^6x^8y^5 + 24t^6x^5y^8 - (xu\frac{\partial^4 u}{\partial x^4} + yu\frac{\partial^4 u}{\partial y^4}),$

Applying properties of triple Laplace transform on both sides of equation (4.44), we get,

$$s^2\bar{u}(k, p, s) - s\bar{u}(k, p, 0) - \bar{u}_t(k, p, 0) - \frac{6(4!)(4!)}{s^2p^5k^5} = L_{xyt}\Big[24t^6x^8y^5 + 24t^6x^5y^8 - (xu\frac{\partial^4 u}{\partial x^4} + yu\frac{\partial^4 u}{\partial y^4})\Big]. \tag{4.47}$$

Applying double Laplace transform to equations (4.45) and (4.46), we obtain

$$\begin{aligned}\bar{u}(k,p,0) &= 0 \ and \ \bar{u}_t(k,p,0) = 0and\\ \bar{u}(\frac{1}{2},p,s) = (\frac{1}{2})^4\frac{(3)!(2)!}{y^4t^3}, \bar{u}(1,p,s) &= \frac{(3)!(2)!}{y^4t^3}\\ \bar{u}_x(\frac{1}{2},p,s) = 4(\frac{1}{2})^3\frac{(3)!(2)!}{y^4t^3}\bar{u}_x(1,p,s) &= 4\frac{(3)!(2)!}{y^4t^3}\\ \bar{u}_y(\frac{1}{2},p,s) = 4(\frac{1}{2})^3\frac{(3)!(2)!}{x^4t^3}, \bar{u}_y(1,p,s) &= 4\frac{(3)!(2)!}{x^4t^3}\end{aligned} \tag{4.48}$$

By substituting equations (4.48) into equation (4.47)), we get

$\bar{u}(k,p,s)= \frac{1}{s^2}\left\{\frac{6(4!)(4!)}{s^2p^5k^5} + L_{xyt}\left[24t^6x^8y^5 + 24t^6x^5y^8 - (xu\frac{\partial^4 u}{\partial x^4} + yu\frac{\partial^4 u}{\partial y^4})\right]\right\} \bar{u}(k,p,s) = \frac{(3!)(4!)(4!)}{s^4p^5k^5} + \frac{1}{s^2}\left\{L_{xyt}\left[24t^6x^8y^5 + 24t^6x^5y^8 - (xu\frac{\partial^4 u}{\partial x^4} + yu\frac{\partial^4 u}{\partial y^4})\right]\right\}.(4.49)$

Applying inverse triple Laplace transform to equation (4.49), we get:

$$u(x,y,t) = x^4y^4t^3 + L^{-1}_{xyt}\left\{\frac{1}{s^2}\left\{L_{xyt}\left[24t^6x^8y^5 + 24t^6x^5y^8 - (xu\frac{\partial^4 u}{\partial x^4} + yu\frac{\partial^4 u}{\partial y^4})\right]\right\}\right\}. \tag{4.50}$$

Now, applying the new iterative method,

$$u_0(x,y,t) = x^4y^4t^3 \tag{4.51}$$

$$\mathrm{u}_1(x,y,t) = L^{-1}_{xyt}\left\{\frac{1}{s^2}\left\{L_{xyt}\left[24t^6x^8y^5 + 24t^6x^5y^8 - (xu_0\frac{\partial^4 u_0}{\partial x^4} + yu_0\frac{\partial^4 u_0}{\partial y^4})\right]\right\}\right\}.$$

$$u_{n+1}(x,y,t) = -L^{-1}_{xyt}\left\{\frac{1}{s^2}\left\{L_{xyt}\left[\sum_{r=0}^{n}(xu_r\frac{\partial^4 u_r}{\partial x^4} + yu_r\frac{\partial^4 u_r}{\partial y^4}) - \sum_{r=0}^{n-1}(xu_r\frac{\partial^4 u_r}{\partial x^4} + yu_r\frac{\partial^4 u_r}{\partial y^4})\right]\right\}\right\}, n \geq 1.$$

Similarly, we obtain $u_2(x,y,t) = u_3(x,y,t) = 0$ and so on.

Therefore, the solution of Example 2 by using equation (4.35) is

$$u(x,y,t) = x^4y^4t^3.$$

Let us now test the convergence of the obtained series solution.

From equation (4.51), we have,

$$u_0(x,y,t) = x^4y^4t^3 and$$

$$N(u(x,y,t)) = L^{-1}_{xyt}\left\{\frac{1}{s^2}\left\{L_{xyt}\left[24t^6x^8y^5 + 24t^6x^5y^8 - (xu\frac{\partial^4 u}{\partial x^4} + yu\frac{\partial^4 u}{\partial y^4})\right]\right\}\right\}.$$

Thus for all $x, y, t \geq 0$, we have,

$$N(u_0(x,y,t)) = L^{-1}_{xyt}\left\{\frac{1}{s^2}\left\{L_{xyt}\left[24t^6x^8y^5 + 24t^6x^5y^8 - (xu_0\frac{\partial^4 u_0}{\partial x^4} + yu_0\frac{\partial^4 u_0}{\partial y^4})\right]\right\}\right\}$$

$$=L^{-1}_{xyt}\left\{\frac{1}{s^2}\left\{L_{xyt}\left[24t^6x^8y^5 + 24t^6x^5y^8 - (xu_0\frac{\partial^4 u_0}{\partial x^4} + yu_0\frac{\partial^4 u_0}{\partial y^4})\right]\right\}\right\} = 0.$$

Therefore,$\|N(u_0(x,y,t))\| = \|0\| = 0 < \frac{1}{e}$.

$$N(u'(x,y,t)) = L^{-1}_{xyt}\left\{\frac{1}{s^2}\left\{L_{xyt}\left[(24t^6x^8y^5 + 24t^6x^5y^8)' - (xu\frac{\partial^4 u}{\partial x^4} + yu\frac{\partial^4 u}{\partial y^4})'\right]\right\}\right\}.$$

where $N'(u(x,y,t))$ represents the partial derivatives $\frac{\partial}{\partial x}u(x,y,t)$ $\frac{\partial}{\partial y}u(x,y,t)$ or $\frac{\partial}{\partial t}u(x,y,t)$.

$$N(\frac{\partial}{\partial t}u(x,y,t))= L^{-1}_{xyt}\left\{\frac{1}{s^2}L_{xyt}[144t^5x^8y^5 + 144t^5x^5y^8 - (xu\frac{\partial^4 u}{\partial x^4} + yu\frac{\partial^4 u}{\partial y^4})']\right\}.$$

Then,

$$N(\frac{\partial}{\partial t}u_0(x,y,t))= L^{-1}_{xyt}\left\{\frac{1}{s^2}L_{xyt}[144t^5x^8y^5 + 144t^5x^5y^8 - (144t^5x^5y^8 + 144t^5x^8y^5)]\right\}=0.$$

$$N(\frac{\partial}{\partial x}u_0(x,y,t))= L^{-1}_{xyt}\left\{\frac{1}{s^2}L_{xyt}[192t^6x^7y^5 + 120t^6x^4y^8 - (120t^6x^4y^8 + 192t^6x^7y^5)]\right\}=0.$$

$$N(\frac{\partial}{\partial y}u_0(x,y,t))= L^{-1}_{xyt}\left\{\frac{1}{s^2}L_{xyt}[120t^6x^8y^4 + 192t^6x^5y^7 - (192t^6x^5y^7 + 120t^6x^8y^4)]\right\}=0.$$

Therefore,$\|N(u_0'(x,y,t))\| = \|0\| = 0 < \frac{1}{e}$.

$$N(\frac{\partial^2}{\partial t^2}u_0(x,y,t))= L^{-1}_{xyt}\left\{\frac{1}{s^2}L_{xyt}[720t^4x^8y^5 + 720t^4x^5y^8 - (720t^4x^5y^8 + 720t^4x^8y^5)]\right\}=0.$$

$$N(\frac{\partial^2}{\partial x^2}u_0(x,y,t))= N(\frac{\partial^2}{\partial y^2}u_0(x,y,t))=N(\frac{\partial^2}{\partial x\partial y}u_0(x,y,t))=0.$$

$$N(\frac{\partial^2}{\partial x\partial t}u_0(x,y,t))= N(\frac{\partial^2}{\partial y\partial t}u_0(x,y,t))=0.$$

Therefore,$\|N(u_0''(x,y,t))\|=\|0\| = 0 < \frac{1}{e}$.

where $N(u_0''(x,y,t))$ represents all the second order partial derivatives of$u_0(x,y,t)$.

Similarly by principle of Mathematical induction we have ,

$\|N(u_0^{(3)}(x,y,t))\|=\|N(u_0^{(4)}(x,y,t))\| = ... = \|N(u_0^{(n)}(x,y,t))\| = \|0\| = 0 < \frac{1}{e}$,for all $n \geq 3, 4, 5, ...$

As the condition ,If N is a continuously differentiable functional in a neighborhood of u_0 and $\|N(u_0^{(n)}(x,y,t))\| \leq M \leq e^{-1}$ for all n, then the series solution, $u(x,y,t) = \sum_{i=0}^{\infty} u_i(x,y,t)$ is absolutely convergent and hence the solution obtained by the new iterative method is convergent on the domain of interest.

The numerical solution representing the solution behavior of Example 2 is depicted in Figure 2.

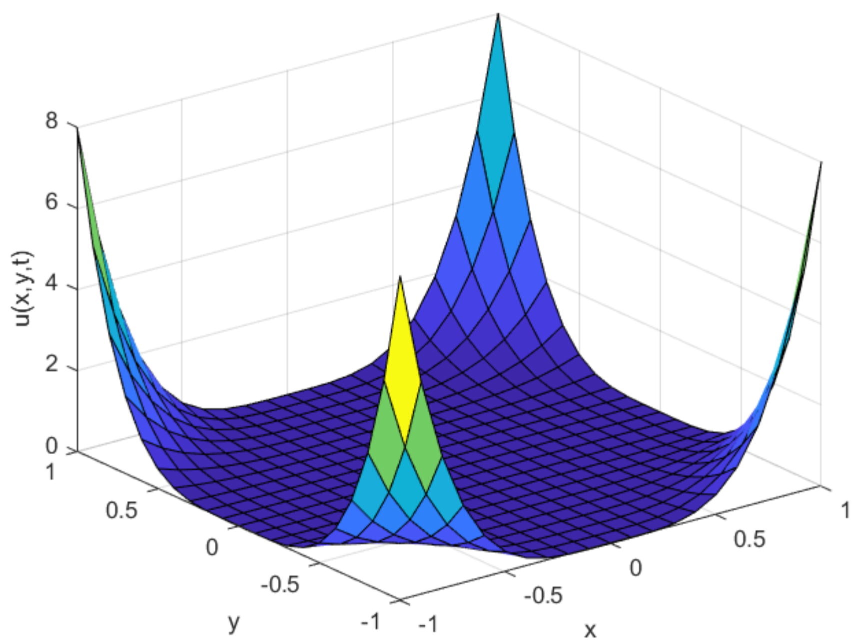

Fig. 2 Solution behavior of Example 2 for $t = 2$

Chapter 5

CONCLUSION AND FUTURE SCOPE

5.1 CONCLUSION

In this study, triple Laplace transform coupled with iterative method is applied to obtain exact solution of two dimensional nonlinear fourth order parabolic equation subject to initial and boundary conditions.Illustrative Examples are presented to shows the validity of the method under consideration.The solutions of Examples 1 - 2 obtained by the proposed method is solution of the form $u(x,y,t) = \sum_{n=0}^{\infty} u_n(x,y,t) = u_0(x,y,t) + u_1(x,y,t) + u_2(x,y,t) + .. + u_n(x,y,t) + ...$ and consists of only one term i.e., $u(x,y,t) = u_0(x,y,t)$.

From this study we concluded that triple Laplace transform coupled with iterative method finds quite practical analytical results with less computational work.

5.2 FUTURE SCOPE

Triple Laplace transform coupled with iterative method can be successfully applied to solve nonlinear higher dimensional parabolic partial differential equation.

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