EXISTENCE AND UNIQUENESS OF A POSITIVE SOLUTION FOR FOUR-POINT BOUNDARY VALUE PROBLEMS IN A CONE.

A Thesis submitted to the Department of Mathematics in Partial Fulfillment for the Requirement of the Degree of Master of Science in Mathematics

By: GutaDemessa

Under the supervision of:

Advisor: WesenLegesse (Ph.D.)

Co-Advisor:AbebechYifru(M.Sc.)

February, 2022

Jimma, Ethiopia

Declaration

I, the undersigned declare that, this research paper entitled ”Existence and Uniqueness of a Positive Solution for Four-Point Boundary Value Problems in a Cone.” is my own original work and it has not been submitted for the award of any academic degree or the like in any other institution or university, and that all the sources I have used or quoted have been indicated and acknowledged.

Name:GutaDemessa

Signature: ....

Date:...

The work has been done under the supervision of:

Name: WesenLegese (Ph.D.)

Signature:..................................

Date:...

Name: AbebechYifru (M.Sc.)

Signature: ....

Date: ....

Acknowledgement

First of all, I am indebted to my almighty God who gave me long life, strength and helped me to reach this time. Next, I would like to express my deepest gratitude to my advisor Dr.WesenLegesse and co-advisor Ms.AbebechYifru for their unreserved support, unlimited advice, constructive comments and immediate responses that helped me in developing this thesis. Lastly, I would like to thank my family for their strong stand in giving me financial and moral support in preparing this research.

Abstract

In this thesis, we constructed the Green's functionfor corresponding homogeneous equation by using its properties. Under the suitable conditions, we established the existence and uniqueness of positive solution for four-point boundary value problems by applying Krasnoselskii's fixed point theorem and Banach contraction principle respectively. This study was mostly dependent on secondary source of data such as journals, books which related to our study area and internet.

Acronym

Throughout this research, we denote the following notation.

R is the set of real numbers.

$\partial\Omega$ is boundary of omega.

BVPs is boundary value problems.

Table of Content

Contents **Page**

CHAPTER ONE
INTRODUCTION

1.1. Background of the study

Differential equations with deviated arguments are found to be important mathematical tools for the better understanding of several real-world problems in physics, mechanics, engineering, economics, etc. In fact, the theory of integer order differential equations with deviated arguments has found its extensive applications in realistic mathematical modeling of a wide variety of practical situations and has emerged as an important area of investigation.

Boundary value problems associated with linear as well as non-linear ordinary differential equations or finite difference equations have created a great deal of interest and play an important role in many fields of applied mathematics such as engineering design and manufacturing. Major industries like automobile, aerospace, pharmaceutical, petroleum, electronics and communications as well as emerging technologies like biotechnology and nanotechnology rely on the boundary value problems to simulate complex phenomena at different scales for designing and manufacturing of high-technological products. In these applied setting, positive solutions are meaningful.

Boundary conditions mean a condition that is required to be satisfied at all or part of the boundary of a region in which a set of differential conditions has to be solved. In the field of differential equations, a boundary value problem is a differential equation together with a set of additional constraints, called the boundary conditions. A solution to a boundary value problem is a solution to the differential equation which also satisfies the boundary conditions.

In analyzing nonlinear phenomena many mathematical models give rise to problems for which only positive solutions make sense.Since the publication of the monograph positive solutions of operator equations in the year 1964 by academician, M.A. Krasnoselskii, hundreds of research articles on the theory of positive solutions of nonlinear problems have appeared. The existence of positive solutions of boundary value problems was studied by many researchers. We list down few of them which are related to our particular problem.

Erbe(Erbe, L. H. and Wang, H., 1994), studied the existence of positive solutions of ordinary differential equations by using fixed point theorem in cone.

$$u''(t) + a(t)f\big(u(t)\big) = 0, \quad 0 < t < 1$$
$$\alpha u(0) - \beta u'(0) = 0$$
$$\gamma u(1) + \delta u'(1) = 0$$

where $f \in C([0,\infty),[0,\infty))$, $a \in C([0,1],[0,\infty))$ and $a(t) \neq 0$ on any interval of $[0,1]$, $\alpha,\beta,\gamma,\delta \geq 0$ and $\rho = \gamma\beta + \alpha\gamma + \alpha\delta > 0.$

Lian(Lian, Wong and Yeh, 1996), studied the existence of at least one positive solution and multiple positive solutions for the two-point boundary value problems.

$$u''(t) + f\big(t,u(t)\big) = 0, \quad 0 < t < 1$$
$$\alpha u(0) - \beta u'(0) = 0$$
$$\gamma u(1) + \delta u'(1) = 0.$$

For a parameter k$\neq$ *0*most of the authors focused on the existence of positive solutions for the second-order ordinary differential equations satisfying the Neumann boundary conditions.

Ruyum(Ruyun Ma, 1998), studied positive solutions of nonlinear three-point boundary value problem by using fixed point theorem in cone.

$$u'' + a(t)f(u) = 0.\ t \in (0,1),$$
$$u(0) = 0,\ \alpha u(\eta) = u(1),$$

where $0 < \eta < 1$, $0 < \alpha < \dfrac{1}{\eta}$ and $f, a \in C([0,\infty),[0,\infty))$.

Wang(Ma, R. and Wang, H., 2003), studied positive solutions of nonlinear three-point boundary value problems by applying fixed point theorem in cones.

$$u(0) = 0,\ \alpha u(\eta) = u(1),$$

where $0 < \eta < 1$ and $0 < \alpha\phi_1(\eta) < 1$ are given $h \in C([0,1],[0,\infty))$ and $f \in C([0,\infty),[0,\infty))$.

Liu (Liu, B., 2004), studied positive solutions of a nonlinear four-point boundary value problems by Krasnoselskii's theorems in a cone.

$$y''(t) + a(t)f(y(t)) = 0,\ 0 < t < 1,$$

$$y(0) = \alpha y(\xi),\ y(1) = \beta y(\eta),$$

where $0 < \xi \leq \eta < 1,\ 0 < \alpha < 1,$ and $\alpha\xi(1-\beta) + (1-a)(1-\beta\eta) > 0$ also $a, y \in C([0,\infty),[0,\infty))$.

Zhang (Zhang, X. and Liu, L., 2007), studied positive solutions of fourth-order four-point boundary value problems with p-Laplacian operator by using the upper and lower solution method and fixed-point theorems.

$[\varphi_p(u''(t))]'' = f(t,u(t)),\ 0<t<1,$ with the four-point boundary conditions

$u(0)=0,\ u(1)=au(\xi),\ u''(o)=0,\ u''(1)=bu''(\eta),$

where $\varphi_p(t)=|t|^{p-2}t,\ p>1, 0<\xi,\eta<1,\ f\in C((0,1)\times(0,+\infty),[0,+\infty)).$

Dong and Bai (Dong, X. and Bai, Z., 2008), studied the existence of one or two positive solutions for the fourth-order boundary value problem with variable parameters.

$$u^{(4)}(t)+B(t)u''(t)-A(t)u(t)=f\big(t,u(t),u''(t)\big),\qquad 0<t<1$$
$$u(0)=u(1)=u''(0)=u''(1)=0.$$

where $A(t),\ B(t)\in C[0,1]$ and $f(t,u,v)\colon [0,1]\times[0,\infty)\times\mathbb{R}\to[0,\infty)$ is continuous.

Benaicha and Haddouchi(Benaicha, S. and Haddouchi, F., 2016), studied positive solutions of a nonlinear fourth-order integral boundary value problem.

$$u''''(t)+f(u(t))=0,\ t\in(0,1),$$
$$u'(0)=u'(1)=u''(0)=0,\ u(0)=\int_0^1 a(s)u(s)ds$$

where $f\in C([0,\infty),[0,\infty));a\in C([0,1],[0,\infty))$ and $0<\int_0^1 a(s)ds<1.$

Xu and Wang (Xu, W. and Wang, H., 2017), studied the Positive Solution of a Nonlinear Four-Point Boundary-Value Problem by usingfixed point theorem in cone.

$$u''(t)+q(t)f(t)=0,$$
$$u'(0)=0,u(1)=a_1u(\zeta)+a_2u(\eta),$$

where $0<\zeta,\eta<1,\ \ f,g\in C([0,\infty),[0,\infty))$ and $a_1+a_2\neq 1$.

Motivated by the above mentioned results, in this paper, we investigated the existence and uniqueness of positive solutions for four-point boundary value problems in cone.

$$-u''(t)+k^2u(t)=f\big((t,u(t)\big)\ \ 0\ \le t\le 1 \qquad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1.1)$$

$$u(0)-\alpha u(\zeta)=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.2)$$

$$u(1)-\beta u(\eta)=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.3),$$

where $0 < \zeta < \eta < 1,\ \alpha, \beta > 0\ \ k \geq 0$ is a constant and f is continuous function by applying Karsnosel'skiis fixed point theorem and Banach contraction theorem.

1.2 .Statement of the problem

Xu and Wang (Xu, W. and Wang, H., 2017), studied the Positive Solution of a Nonlinear Four-Point Boundary-Value Problem by usingfixed point theorem in cone.

$$u''(t) + q(t) f(t) = 0$$

$$u'(0) = 0, u(1) = a_1 u(\zeta) + a_2 u(\eta)$$

where $0 < \zeta, \eta < 1$, $f, g \in C([0,\infty),[0,\infty))$ and $a_1 + a_2 \neq 1$.

Liu (Liu, B., 2004), studied positive solutions of a nonlinear four-point boundary value Problems by Krasnoselskii's theorem in a cone.

$$y''(t) + a(t) f(y(t)) = 0,\ 0 < t < 1,$$

$$y(0) = \alpha y(\xi),\ y(1) = \beta y(\eta),$$

where $0 < \xi \leq \eta < 1,\ 0 < \alpha < 1,$ and $\alpha\xi(1-\beta) + (1-a)(1-\beta\eta) > 0$ also $a, y \in C([0,\infty),[0,\infty))$.

In this Research work, we concentrated in establishing the existence and uniqueness of positive solutions for four-point boundary value problems for (1.1) -(1.3) by using Kransnoselskii's fixed point theorem and Banach Contraction theorem in cone respectively.

1.3.1 General objective:

The general objective of this thesis is to study the existence and uniquenessof a positive solution for four-point boundary value problems in a cone(1.1) -(1.3).

1.3.2. Specific Objectives:

The specific objective of this study was:

- To construct Green's function for the corresponding homogeneous equation.
- To formulate operator equation for the given boundary value problem.
- Toproveexistence and uniqueness of a positive solution by using Krasnoselskii's fixed point theorem.

1.4. Significance of the study

The result of this thesis may have the following importance.

- The outcome of this thesis may give a better understanding about research for the researcher.
- It may contribute to research activities in the study area.
- It may provide some background information for other researchers who want toconduct a research on related topics.

- Furthermore, this thesis was useful for graduate program of the department of mathematics.

1.5 Delimitation of the study

This study was delimited to finding the existence and uniqueness of a positive solutionfor-four point boundary value problems in a cone from (1.1) - (1.3) by using Krasnoselskii's fixed point theorem.

CHAPTER TWO
REVIEW OF RELATED LITERATURE

2.1 Over View of Positive Solutions

Positive solution is very important in diverse disciplines of mathematics since it can be applied for solving various problems and it is one of the most dynamic research subjects in nonlinear analysis. The existence of positive solutions for boundary value problems has been studied by many researchers such as:

Yang (Yang, B. 2005), studied positive solutions for a fourth order boundary value problem

$$u''''(t) = g(t) f(u(t)),\ \ 0 \le t \le 1,$$

$$u''''(t) = g(t) f(u(t)),\ \ 0 \le t \le 1,$$

where $f:[0,\infty) \to [0,\infty)$ is continuous and $g:[0,1] \to [0,\infty)$ is a continuous function such that $\int_0^1 g(t)dt > 0.$

Bai and Gu (Z.Bai and Z.Gu, 2007), studied positive solutions for some second-order four-point boundary value problems.

$x''(t) + \lambda h(t) f(t, x(t)) = 0$, $0 < t < 1$,

$x(0) = \alpha x(\xi)$, $x(1) = \beta x(\eta)$.

Nieto (Nieto, J. J., 2013), studied existence of a solution for a three-point second order boundary value problem by using fixed point theorem.

$$-u''(t) = f(t, u(t)),\ 0 \le t \le T,$$

$$u(0) = \alpha u(\mu) = u(T),$$

where $T > 0,\ f:[0,T]\times R \to R$ is continuous function $\alpha \in R$ and $\eta \in (0,T)$.

Sveikete (Sveikate, N., 2016), studied the existence of solutions on three-point boundary value problem by using quasi linearization approach.

$$x'' + k^2 x = f(t, x, x'),$$

$$x(0) = 0, x(1) = \alpha x(\mu),$$

where $0 < \eta < 1,\ \alpha > 0$ and f may be unbounded.

2.2 Preliminaries

In this section, we provide some definitions, basic concepts on Green's function, definition of existence of positive solutions and statements of few standard fixed point theorems, which are frequently used in thesis.

Definition.2.1 Let X be a non-empty set. A map $T: X \to X$ is said to be a self-map withdomainof $T = D(T) = X$ and range of$T = R(T) = T(X) \subseteq X$.

Definition2.2 Let $T: X \to X$ be self-map. A point x in X is called fixed point of T if $Tx = x$.

Definition 2.3.Consider the second-order linear differential equation,

$$p_0(x)y'' + p_1(x)y' + p_2(x)y = r(x), \qquad x \in J = [\alpha, \beta]. \tag{2.1}$$

Where the functions $p_0(x), p_1(x), p_2(x)$ and $r(x)$ are continuous in J and boundary conditions of the form

$$\begin{aligned} l_1[y] &= a_0 y(\alpha) + a_1 y'(\alpha) + b_0 y(\beta) + b_1 y'(\beta) = A \\ l_2[y] &= c_0 y(\alpha) + c_1 y'(\alpha) + d_0 y(\beta) + d_1 y'(\beta) = B\,. \end{aligned} \tag{2.2}$$

Wherea_i, b_i, c_i, d_i, ($i = 0, 1$) and A, B are given constants.

The boundary value problem (2.1)-(2.2) are calleda non-homogeneous two-point linear boundary value problem, where asthe homogeneous differential equation

$$p_0(x)y'' + p_1(x)y' + p_2(x)y = 0, \qquad x \in J = [\alpha, \beta\ ] \tag{2.3}$$

Together with the homogeneous boundary conditions

$$l_1[y] = 0, \qquad l_2[y] = 0 \tag{2.4}$$

be called a homogeneous two-point linear boundary value problem. The function called a Green's function $G(x,t)$ for the homogeneous boundary value problems (2.3)-(2.4) and the solution of the nonhomogeneous boundary value problem (2.1)-(2.2) can be explicitly expressed in terms of $G(x,t)$.

Obviously, for the homogeneous problem (2.3)-(2.4) the trivial solution always exists. Green's function $G(x,t)$ for the boundary value problem(2.3)-(2.4) is defined in the square $[\alpha,\beta] \times [\alpha,\beta]$and possesses the following fundamental properties:

(i) $G(x,t)$ is continuous in $[\alpha,\beta] \times [\alpha,\beta]$,

(ii) $\partial G(x,t)/\partial x$ is continuous in each of the triangles$\alpha \le x \le t \le \beta$ and $\alpha \le t \le x \le \beta$; moreover,

$$\frac{\partial G(t^+,\ t)}{\partial x} - \frac{\partial G(t^-,\ t)}{\partial x} = -\frac{1}{p_0(t)}.$$

Where$\frac{\partial G(t^+,t)}{\partial x} = \lim_{\substack{x\to t \\ x>t}} \frac{\partial G(x,t)}{\partial x}$, $\quad \frac{\partial G(t^-,t)}{\partial x} = \lim_{\substack{x\to t \\ x<t}} \frac{\partial G(x,t)}{\partial x}$,

(iii) for every $t \in [\alpha, \beta]$, $z(x) = G(x, t)$is a solution of the differential equation(2.3) in each of the intervals $[\alpha, t)$ and $(t, \beta]$,

(iv) for every $t \in [\alpha, \beta]$, $z(x) = G(x, t)$satisfies the boundary conditions (2.4).

These properties completely characterize Green's function $G(x, t)$.

Definition 2.4 A normed linear space is a linear space X in which for each vector x there corresponds a real number, denoted by $\| x \|$called the norm of x and has the following properties:

(i) $\| x \| \geq 0$, for all $x \in X$ and $\| x \| = 0$ if and only if $x = 0$,

(ii) $\| x + y \| \leq \| x \| + \| y \|$, for all $x, y \in X$,

(iii) $\| \alpha x \| = |\alpha| \| x \|$, for all $x \in X and \alpha being a scalar.$

Definition 2.5 Let Xbe a normed linear space with norm denoted by $\|.\|$. A sequence of elements $\{x_n\}$ of X is a Cauchy sequence, if for every $\epsilon > 0$ there exists an integer N such that $\| x_n - x_m \| < \epsilon$, for all $m, n \geq N$.

Definition 2.6 Anormed linear space X is said to be complete, if every Cauchy sequence in X converges to a point in$X$.

Definition 2.7 ABanach space is a complete normed linear space.

Definition 2.8 Let Ebe a Banach space over R. A non-empty, convex, closed set $P \subset E$ is said to be a cone provided that

$(a) \alpha u + \beta v \in$ P for all $u, v \in$ P and all $\alpha, \beta \geq 0$ and

$(b) u, -u \in$ P implies $u = 0$

Theorem (Contraction Mapping Theorem). If T is a contraction mapping on a Banach space X with contraction constant α, with $0 \leq \alpha < 1$, then T has a unique fixed point $x_0 \in X$.

Definition 2.9 Let $X$and Y be Banach spaces and $T : X \to Y$, $T$is said to be completely continuous, if T is continuous and for each bounded sequence $\{x_n\} \subset X$, $\{Tx_n\}$ has a convergent subsequence.

Definition 2.10 A function $f(t, y)$ satisfies a Lipschitz condition in the variable y on a set $D \subset R^2$ if a constant L>0 exists with the absolute value of $f(t, y_1) - f(t, y_2) \leq L|y_1 - y_2|$, whenever, (t, y_1), (y, y_2) are in D and L is Lipchitz constant.

Theorem (Krasnoselskii, M.A, 1964), Let E be a Banach space, and let $P \subset E$ be a cone in E.AssumeΩ_1, Ω_2 are open subsets of E $with$ $0 \in \Omega_1$, $\overline{\Omega_1} \subset \Omega_2$, and let $T: P \cap \left(\overline{\Omega_2} \setminus \Omega_1\right) \to P$ is a completely continuous operator. Such that either

(i) $\| Tu \| \leq \| u \|$, u $\in P \cap \partial\Omega_1$, and $\| Tu \| \geq \| u \|$, u $\in P \cap \partial\Omega_2$, or

(ii) $\| Tu \| \geq \| u \|$, u $\in P \cap \partial\Omega_1$, and $\| Tu \| \leq \| u \|$, u $\in P \cap \partial\Omega_2$,

Then T has a fixed point in $P \cap (\overline{\Omega_2} \setminus \Omega_1)$.

CHAPTER THREE
METHODOLOGY

This chapter contains study period and site, study design, source of information and mathematical procedures.

3.1 Study period and site

The study was conducted from December 2021 to February 2022 in Jimma University under the department of mathematics.

3.2 Study design

In order to achieve the objective of the study we employed analytical method of design.

3.3 Source of Information

The relevant sources of information for this study were different mathematics books, published articles, journals and related studies from internet.

3.4 .Mathematical Procedure of the Study

The study follows the following steps:

- Existing the definition of second-order four-point boundary value problem.
- Constructing the Green's function for the corresponding homogeneous equation.
- Formulating the equivalent operator equation for the boundary value problem (1.1)- (1.3).
- Prove existence and uniqueness of positive solution for the given operator equation.

CHAPTER FOUR
MAIN RESULT AND DISCUSSION

4.1 Construction of Green's Function

In this section, we construct Green's function for the associated homogeneous boundary value problem corresponding to (1.1) - (1.3).

Let us consider the boundary value problem

$$-u''(t) + k^2u(t) = f(t, u(t))), \qquad 0 \le t \le 1,$$
$$u(0) - \alpha u(\xi) = 0,$$
$$u(1) - \beta u(\eta) = 0,$$

where $k > 0$ is a constant,$\alpha \ge 0, \beta \ge 0$ and $0 < \xi < \eta < 1$.

Lemma4.1 Let $y(t) \in C([0,1])$ and $y(t) \ge 0$. Then the boundary value problem

$$-u''(t) + k^2u(t) = y(t), \qquad 0 \le t \le 1, \tag{4.1}$$
$$u(0) = 0, u(1) = 0, \tag{4.2}$$

has a unique solution,

$$u(t) = \int_0^1 H(t,s)y(s)\, ds.$$

Where $H(t,s)$ is Green's function for the homogeneous problem

$$-u''(t) + k^2u(t) = 0, 0 \le t \le 1, \tag{4.3}$$

satisfying the boundary condition (4.2)and given by

$$H(t,s) = \begin{cases} \frac{\sinh kt \ \sinh k(1-s)}{k \sinh k}, & 0 \le t \le s \le 1 \\ \frac{\sinh ks \ \sinh k(1-t)}{k \sinh k}, & 0 \le s \le t \le 1 \end{cases}. \tag{4.4}$$

Proof: We prove by the properties of Green's function. Obviously thedifferential equation (4.3) with the boundary condition (4.2) has only trivial solution. Green's function $H(t,s)$ defined in the square $[0,1] \times [0,1]$. Let $u_1(t) = \cosh kt$ and $u_2(t) = \sinh kt$ are the two linearly independent solutions of (4.3).

Green function for second- order two point boundary value problem can be defined in the form

$$H(t,s) = \begin{cases} u_1(t)\lambda_1(s) + u_2(t)\lambda_2(s) & if\ 0 \le t \le s \le 1 \\ u_1(t)\mu_1(s) + u_2(t)\mu_2(s) & if\ 0 \le s \le t \le 1 \end{cases}, \tag{4.5}$$

where $\lambda_1(s), \lambda_2(s), \mu_1(s) and\ \mu_2(s)$ are functions to be determined. By applying properties of Green function (i) and (ii) we obtain

$$\begin{cases} u_1(t)\big(\mu_1(s) - \lambda_1(s)\big) + u_2(t)\big(\mu_2(s) - \lambda_2(s)\big) = 0 \\ u'_1(t)\big(\mu_1(s) - \lambda_1(s)\big) + u'_2(t)\big(\mu_2(s) - \lambda_2(s)\big) = -1 \end{cases}$$

Let $v_1(s) = \mu_1(s) - \lambda_1(s)$ and $v_2(s) = \mu_2(s) - \lambda_2(s)$ (*)

Then$\begin{cases} coshkt\, v_1(s) + sinhkt\, v_2(s) = 0 \\ ksinhkt\, v_1(s) + \, kcoshkt\, v_2(s) = -1 \end{cases}$

From this we get

$v_1(s) = \frac{sinhks}{k}andv_2(s) = \frac{-coshks}{k}$

From (*), we have $\mu_1(s) = \lambda_1(s) + \frac{sinhks}{k}$and $\mu_2(s) = \lambda_2(s) - \frac{coshks}{k}$.

Substituting the value of μ_1 and μ_2 in equation (4.5) we have

$$H(t,s) = \begin{cases} \cosh k(t)\lambda_1(s) + \sinh k(t)\lambda_2(s) \quad if\ 0 \le t \le s \le 1 \\ \cosh k(t)\lambda_1(s) + \cosh kt\frac{\sinh ks}{k} + \sinh k(t)\lambda_2(s) - \sinh kt\frac{\cosh ks}{k}\ if\ 0 \le s \le t \le 1. \end{cases}$$

$H(t,s)$Satisfies the boundary conditions with respect to t. Hence, $\lambda_1(s)$and$\lambda_2(s)$ uniquely determined as

$\lambda_1(s) = 0$ and $\lambda_2(s) = \frac{sinhk\, coshks - coshk\, sinhks}{ksinh\, k} = \frac{sinhk(1-s)}{ksinhk}$. Finally, substituting the value of $\mu_1, \mu_2. \lambda_1$, and λ_2 in equation (4.5) we arrive on the required Green's function

$$H(t,s) = \begin{cases} \frac{\sinh kt\ \sinh k(1-s)}{k \sinh k}, & 0 \le t \le s \le 1 \\ \frac{\sinh ks\ \sinh k(1-t)}{k \sinh k}, & 0 \le s \le t \le 1 \end{cases},$$

and the solution of the BVP

$$-u'' + k^2 u = f\big(t, u(s)\big),$$
$$u(0) = 0, u(1) = 0,$$

is given by

$$\omega(t) = \int_0^1 H(t,s) f(s, u(s)) ds, \tag{4.6}$$

and$\omega(0) = 0, \omega(1) = 0, \omega(\eta) = \int_0^1 H(\eta, s) f(s, u(s))) ds$.

Lemma 4.2 $H(t,s)$ has the following properties

i, $H(t,s) \le H(s,s)$,for all $t, s\ \in [0,1]$.

ii, $H(t,s) \ge NH(s,s)$, for all $t \in [\delta, 1-\delta], s \in [0,1], N = \frac{\sinh k\delta}{\sinh k}$.

Proof: we prove this lemma by splitting in to different cases.

i. $H(t,s)$is positive for all $t, s\ \in [0,1]$

for $0 \le s \le t \le 1$, we have

$$\frac{H(t,s)}{H(s,s)} = \frac{\sinh ks \sinh k(1-t)}{\sinh ks \sinh k(1-s)} = \frac{\mathrm{sinhk}(1-\mathrm{t})}{\mathrm{sinhk}(1-\mathrm{s})} \le 1$$
$$\Longrightarrow H(t,s) \le H(s,s), \qquad t, s\ \in [0,1]$$

For $0 \le t \le s \le 1$ we have

$$\frac{H(t,s)}{H(s,s)} = \frac{\sinh kt \sinh k(1-s)}{\sinh ks \sinh k(1-s)} = \frac{\text{sinhkt}}{\text{sinhks}} \le 1.$$

$\Longrightarrow\ H(t,s) \le H(s,s), \qquad t,s \in [0,1].$

ii. If $s \le t$ for $t \in [\delta, 1-\delta], s \in [0,1]$, we have

$$\frac{H(t,s)}{H(s,s)} = \frac{\sinh ks \sinh k(1-t)}{\sinh ks \sinh k(1-s)} \ge \frac{\sinh k\,\delta}{\sinh k}.$$

$\Longrightarrow\ H(t,s) \ge NH(s,s).$

If $t \le s$ for $t \in [\delta, 1-\delta], s \in [0,1]$, we have

$$\frac{H(t,s)}{H(s,s)} = \frac{\sinh kt \sinh k(1-s)}{\sinh ks \sinh k(1-s)} \ge \frac{\sinh k\,\delta}{\sinh k}.$$

$\Longrightarrow\ H(t,s) \ge NH(s,s).$

Lemma 4.3 Let$\alpha \ge 0, \beta \ge 0,\ 0 < \xi < \eta < 1$ and $f\big(t,u(t)\big) \in C\big([0,1] \times [0,\infty), [0,\infty)\big)$.

Then the boundary value problem (1.1)-(1.3) has a unique solution

$$u(t) = \int_0^1 G(t,s) f\big(s,u(s)\big) ds, \tag{4.7}$$

where

$$G(t,s) = H(t,s) + \frac{\{[\beta \sinh k - \alpha\beta \sinh k(1-\xi)] \sinh k\xi \sinh k(1-t)\}}{\Delta} H(\eta,s) + \frac{\{\alpha\beta \sinh k(1-\eta) \sinh kt + (\sinh k - \alpha\beta \sinh k\eta) \sinh k(1-t)\}}{\Delta} H(\xi,s). \tag{4.7a}$$

And $\Delta = \alpha\beta sinhk\ sinhk(\eta-\xi) + sinh^2 k - \alpha sinhk(1-\xi) - \beta sinhk\ sinhk\xi > 0.$

Proof: The four point boundary value problem (1.1)-(1.3) can be obtained by replacing $u(0) = 0$ by $u(0) = \alpha u(\xi)$ and $u(1) = 0$ by $u(1) = \beta u(\eta)$ in (4.2).Thus we suppose the solution of the four-point boundary value problem (1.1)-(1.3) can be expressed by$u(t) = \omega(t) + B_1 sinhkt + B_2 sinhk(1-t),$ (4.8)

whereB_1 and B_2 will be determined.

$$\begin{cases} u(0) = B_2 sinhk = \alpha u(\xi) \\ u(1) = B_1 sinhk = \beta u(\eta) \\ u(\xi) = \int_0^1 H(\xi,s) f(t,u(s)) ds + B_1 sinhk\xi + B_2 sinhk(1-\xi) \\ u(\eta) = \int_0^1 H(\eta,s) f(t,u(s)) ds + B_1 sinhk\eta + B_2 sinhk(1-\eta) \end{cases} \tag{4.9}$$

$$B_2 sinhk = \alpha \int_0^1 H(\xi,s) f(s,u(s))ds + \alpha B_1 sinhk\xi + \alpha B_2 sinhk(1-\xi)$$

$$-\alpha B_1 sinhk\xi + B_2\big(sinhk - \alpha sinhk(1-\xi)\big) = \alpha \int_0^1 Hf\big(s,y(s)\big)ds. \qquad (4.10)$$

$$B_1 sinhk = \beta \int_0^1 H(\eta,s) f(s,y(s))ds + \beta B_1 sinhk\eta + \beta B_2 sinhk(1-\eta)$$

$$-B_1(sinhk - \beta sinhk\eta) - \beta B_2 sinhk(1-\eta) = \beta \int_0^1 H(\eta,s) f\big(s,u(s)\big)ds. \qquad (4.11)$$

Solving equation (4.10) and (4.11) simultaneously we have

$$B_1 = \frac{\begin{vmatrix} \int_0^1 \beta H(\eta,s) f(s,u(s))ds & -\beta sinhk(1-\eta) \\ \int_0^1 \alpha H(\xi,s) f(s,u(s))ds & sinhk - \alpha sinhk(1-\xi) \end{vmatrix}}{\begin{vmatrix} sinhk - \beta sinhk\eta & -\beta sinhk(1-\eta) \\ -\alpha sinhk\xi & sinhk - \alpha sinhk(1-\xi) \end{vmatrix}};$$

and

$$B_2 = \frac{\begin{vmatrix} sinhk - \beta sinhk\eta & \int_0^1 \beta H(\eta,s) f(s,u(s))ds \\ -\alpha sinhk\xi & \int_0^1 \alpha H(\xi,s) f(s,u(s)ds \end{vmatrix}}{\begin{vmatrix} sinhk - \beta sinhk\eta & -\beta sinhk(1-\eta) \\ -\alpha sinhk\xi & sinhk - \alpha sinhk(1-\xi) \end{vmatrix}};$$

Computing for B_1 and B_2 we get

$$B_1 = \frac{[\beta sinhk - \alpha\beta sinhk(1-\xi)] \int_0^1 H(\eta,s) f\big(s,u(s)\big)ds + \alpha\beta sinhk(1-\eta) \int_0^1 H(\xi,s) f\big(s,u(s)\big)ds}{\Delta} \text{and}$$

$$B_2 = \frac{[\sinh k - \alpha\beta \sinh k\eta] \int_0^1 H(\xi,s) f(s,u(s))ds + \alpha\beta \sinh k\xi \int_0^1 H(\eta,s) f(s,u(s))ds}{\Delta}.$$

Substituting B_1 and B_2 in equation (4.8)

$$u(t) = \omega(t) + \frac{[\beta sinhk - \alpha\beta sinhk(1-\xi)] \int_0^1 H(\eta,s) f\big(s,u(s)\big)ds + \alpha\beta sinhk(1-\eta) \int_0^1 H(\xi,s) f\big(s,u(s)\big)ds}{\Delta} sinhkt + \frac{[sinhk - \beta\alpha sinhk\eta] \int_0^1 H(\xi,s) f\big(s,u(s)\big)ds + \alpha\beta sinhk\xi \int_0^1 H(\eta,s) f\big(s,u(s)\big)ds}{\Delta} sinhk(1-t).$$

Rearranging

$$\int_0^1 H(t,s)f(s,u(s))ds + \frac{\{[\beta\sinh k - \alpha\beta\sinh k(1-\xi)]\sinh kt + \alpha\beta\sinh k\xi\sinh k(1-t)\}\int_0^1 H(\eta,s)f(s,u(s))ds}{\triangle}$$
$$+\frac{\{\alpha\beta\sinh k(1-\eta)\sinh kt + (\sinh k - \alpha\beta\sinh k\eta)\sinh k(1-t)\}}{\triangle}\int_0^1 H(\xi,s)f(s,u(s))ds.$$

In closed form the above equation re-written as

$$u(t) = \int_0^1 [H(t,s) + \frac{\{[\beta sinhk - \alpha\beta sinhk(1-\xi)]sinhkt + \alpha\beta sinhk\xi sinhk(1-t)\}}{\Delta}H(\eta,s)$$
$$+\frac{\{\alpha\beta sinhk(1-\eta)sinhkt + (sinhk - \alpha\beta sinhk\eta)sinhk(1-t)\}}{\Delta}H(\xi,s)]f\big(s,u(s)\big)ds.$$

Before stating the results we make the following assumptions throughout the thesis.

A1. $f\big(t,u(t)\big) : [0,1]\times[0,\infty)\to[0,\infty)$, is continuous.

A2. $0<\xi\le\eta<1, sinhk > \alpha sinhk(1-\xi)andsinhk > \alpha\beta sinhk\eta$.

Lemma 4.4 The function $G(t,s)$ possesses the following properties.

$$G(t,s)\ge 0 \; for\ all\ t,s\in[0,1].$$

$$G(t,s)\le MH(s,s),\;\; M = \max\{M_1,M_2,M_3,M_4,M_5,M_6\}.$$

$$G(t,s)\ge \Pi H(s,s).$$

Proof;Consider$G(t,s)$ given by (4.7a).

From assumption A2 and the positivity of $H(t,s)$ for all $t,s\in[0,1]$ we assure that $G(t,s)$ is nonnegative.

We prove the second part of the Lemma by considering different cases

$$G(t,s) = H(t,s) + \frac{\{[\beta sinhk - \alpha\beta sinhk(1-\xi)]sinhkt + \alpha\beta sinhk\xi sinhk(1-t)\}}{\Delta}H(\eta,s)$$
$$+\frac{\{\alpha\beta sinhk(1-\eta)sinhkt + (sinhk - \alpha\beta sinhk\eta)sinhk(1-t)\}}{\Delta}H(\xi,s).$$

Case I: Let $0\le s\le \min\{t.\zeta\}\le 1$. We have the following.

$$G(t,s) \leq H(s,s) + \frac{\{[\beta sinhk - \alpha\beta sinhk(1-\xi)]sinhkt + \alpha\beta sinhk\xi sinhk(1-t)\}}{\Delta}H(s,s)$$
$$+ \frac{\{\alpha\beta sinhk(1-\eta)sinhkt + (sinhk - \alpha\beta sinhk\eta)sinhk(1-t)\}}{\Delta}H(s,s);$$

$$\leq\ H(s,s) + \frac{\{[\beta sinhk - \alpha\beta sinhk(1-\xi)]sinhk + \alpha\beta sinhk\xi sinhk\}}{\Delta}H(s,s)$$
$$+ \frac{\{\alpha\beta sinhk(1-\eta)sinhk + (sinhk - \alpha\beta sinhk\eta)sinhk\}}{\Delta}H(s,s);$$

$$G(t,s) \leq \left[1 + \frac{\{[\beta sinhk - \alpha\beta sinhk(1-\xi)]sinhk + \alpha\beta sinhk\xi sinhk\}}{\Delta}\right.$$
$$\left. + \frac{\{\alpha\beta sinhk(1-\eta)sinhk + (sinhk - \alpha\beta sinhk\eta)sinhk\}}{\Delta}\right]H(s,s).$$

Thus $G(t,s) \leq M_1 H(s,s)$. (4.12)

Case II: Let $0 \leq t \leq s \leq \xi \leq \eta$.

$$G(t,s) \leq H(s,s) + \frac{\{[\beta sinhk - \alpha\beta sinhk(1-\xi)]sinhkt + \alpha\beta sinhk\xi sinhk(1-t)\}}{\Delta}H(s,s)$$
$$+ \frac{\{\alpha\beta sinhk(1-\eta)sinhkt + (sinhk - \alpha\beta sinhk\eta)sinhk(1-t)\}}{\Delta}H(s,s);$$

$$\leq\ H(s,s) + \frac{\{[\beta sinhk - \alpha\beta sinhk(1-\xi)]sinhk\xi\ + \alpha\beta sinhk\xi sinhk\}}{\Delta}H(s,s)$$
$$+ \frac{\{\alpha\beta sinhk(1-\eta)sinhk\xi + (sinhk - \alpha\beta sinhk\eta)sinhk\}}{\Delta}H(s,s);$$

$$G(t,s) \leq \left[1 + \frac{\{[\beta sinhk - \alpha\beta sinhk(1-\xi)]sinhk\xi + \alpha\beta sinhk\boldsymbol{\xi} sinhk\}}{\Delta}\right.$$
$$\left. + \frac{\{\alpha\beta sinhk(1-\eta)sinhk\xi + (sinhk - \alpha\beta sinhk\eta)sinhk\}}{\Delta}\right]H(s,s).$$

Hence, $G(t,S) \leq M_2 H(s,s)$. (4.13)

Case III: Let $0 \leq \xi \leq s \leq \min\{\eta, t\}$.

$$G(t,s) \leq H(s,s) + \frac{\{[\beta sinhk - \alpha\beta sinhk(1-\xi)]sinhkt + \alpha\beta sinhk\xi sinhk(1-t)\}}{\Delta} H(s,s)$$
$$+ \frac{\{\alpha\beta sinhk(1-\eta)sinhkt + (sinhk - \alpha\beta sinhk\eta)sinhk(1-t)\}}{\Delta} H(s,s);$$

$$\leq\ H(s,s) + \frac{\{[\beta sinhk - \alpha\beta sinhk(1-\xi)]sinhk + \alpha\beta sinhk\xi sinhk(1-\xi)\}}{\Delta} H(s,s)$$
$$+ \frac{\{\alpha\beta sinhk(1-\eta)sinhk + (sinhk - \alpha\beta sinhk\eta)sinhk(1-\xi)\}}{\Delta} H(s,s);$$

$$G(t,s) \leq \left[1 + \frac{\{[\beta sinhk - \alpha\beta sinhk(1-\xi)]sinhk + \alpha\beta sinhk\xi sinhk(1-\xi)\}}{\Delta}\right.$$
$$\left. + \frac{\{\alpha\beta sinhk(1-\eta)sinhk + (sinhk - \alpha\beta sinhk\eta)sinhk(1-\xi)\}}{\Delta}\right] H(s,s).$$

Hence, $G(t,S) \leq M_3 H(s,s).$ (4.14)

Case IV: Letmax$\{t,\xi\} \leq s \leq \eta < 1.$

$$G(t,s) \leq H(s,s) + \frac{\{[\beta sinhk - \alpha\beta sinhk(1-\xi)]sinhkt + \alpha\beta sinhk\xi sinhk(1-t)\}}{\Delta} H(s,s)$$
$$+ \frac{\{\alpha\beta sinhk(1-\eta)sinhkt + (sinhk - \alpha\beta sinhk\eta)sinhk(1-t)\}}{\Delta} H(s,s);$$

$$\leq\ H(s,s) + \frac{\{[\beta sinhk - \alpha\beta sinhk(1-\xi)]sinhk\eta + \alpha\beta sinhk\xi sinhk\}}{\Delta} H(s,s)$$
$$+ \frac{\{\alpha\beta sinhk(1-\eta)sinhk\eta + (sinhk - \alpha\beta sinhk\eta)sinhk\}}{\Delta} H(s,s);$$

$$G(t,s) \leq \left[1 + \frac{\{[\beta sinhk - \alpha\beta sinhk(1-\xi)]sinhk\eta + \alpha\beta sinhk\xi sinhk\}}{\Delta}\right.$$
$$\left. + \frac{\{\alpha\beta sinhk(1-\eta)sinhk\eta + (sinhk - \alpha\beta sinhk\eta)sinhk\}}{\Delta}\right] H(s,s).$$

Hence, $G(t,S) \leq M_4 H(s,s).$ (4.15)

Case V:Let $0 \leq \xi \leq \eta \leq s \leq\ t.$

$$G(t,s) \leq H(s,s) + \frac{\{[\beta sinhk - \alpha\beta sinhk(1-\xi)]sinhkt + \alpha\beta sinhk\xi sinhk(1-t)\}}{\Delta} H(s,s)$$
$$+ \frac{\{\alpha\beta sinhk(1-\eta)sinhkt + (sinhk - \alpha\beta sinhk\eta)sinhk(1-t)\}}{\Delta} H(s,s);$$

$$\leq \ H(s,s) + \frac{\{[\beta sinhk - \alpha\beta sinhk(1-\xi)]sinhk + \alpha\beta sinhk\xi sinhk(1-\eta)\}}{\Delta} H(s,s)$$
$$+ \frac{\{\alpha\beta sinhk(1-\eta)sinhk + (sinhk - \alpha\beta sinhk\eta)sinhk(1-\eta)\}}{\Delta} H(s,s);$$

$$G(t,s) \leq \left[1 + \frac{\{[\boldsymbol{\beta sinhk - \alpha\beta sinhk(1-\xi)}]sinhk + \boldsymbol{\alpha\beta sinhk\xi} sinhk(1-\eta)\}}{\Delta} \right.$$
$$\left. + \frac{\{\alpha\beta sinhk(1-\eta)sinhk + (sinhk - \alpha\beta sinhk\eta)sinhk(1-\eta)\}}{\Delta} \right] H(s,s).$$

Thus, $G(t,s) \leq M_5 H(s,s)$. (4.16)

Case VI: Let$\max\{\eta,t\} \leq s \leq 1$.

$$G(t,s) \leq H(s,s) + \frac{\{[\beta sinhk - \alpha\beta sinhk(1-\xi)]sinhkt + \alpha\beta sinhk\xi sinhk(1-t)\}}{\Delta} H(s,s)$$
$$+ \frac{\{\alpha\beta sinhk(1-\eta)sinhkt + (sinhk - \alpha\beta sinhk\eta)sinhk(1-t)\}}{\Delta} H(s,s);$$

$$\leq \ H(s,s) + \frac{\{[\beta sinhk - \alpha\beta sinhk(1-\xi)]sinhk + \alpha\beta sinhk\xi sinhk\}}{\Delta} H(s,s)$$
$$+ \frac{\{\alpha\beta sinhk(1-\eta)sinhk + (sinhk - \alpha\beta sinhk\eta)sinhk\}}{\Delta} H(s,s);$$

$$G(t,s) \leq \left[1 + \frac{\{[\beta sinhk - \alpha\beta sinhk(1-\xi)]sinhk + \alpha\beta sinhk\xi sinhk\}}{\Delta} \right.$$
$$\left. + \frac{\{\alpha\beta sinhk(1-\eta)sinhk + (sinhk - \alpha\beta sinhk\eta)sinhk\}}{\Delta} \right] H(s,s).$$

Thus, $G(t,s) \leq M_6 H(s,s)$.(4.17)

Setting, $M = \max\{M_1, M_2, M_3, M_4, M_5, M_6\}$.

For all$t, s, \in [0,1]$, $G(t,s) \leq MH(s,s)$.

iii) We prove the third part of the Lemma by considering different cases as we proved (ii)

$$G(t,s) = H(t,s) + \frac{\{[\beta sinhk - \alpha\beta sinhk(1-\xi)]sinhkt + \alpha\beta sinhk\xi sinhk(1-t)\}}{\Delta} H(\eta,s)$$
$$+ \frac{\{\alpha\beta sinhk(1-\eta)sinhkt + (sinhk - \alpha\beta sinhk\eta)sinhk(1-t)\}}{\Delta} H(\xi,s).$$

Case I: Let$0 \leq s \leq \min\{t.\zeta\} \leq 1$. We have the following.

$$G(t,s) \geq N[H(s,s)$$
$$+ \frac{\{[\beta sinhk - \alpha\beta sinhk(1-\xi)]sinhkt + \alpha\beta sinhk\xi sinhk(1-t)\}}{\Delta} H(s,s)$$
$$+ \frac{\{\alpha\beta sinhk(1-\eta)sinhkt + (sinhk - \alpha\beta sinhk\eta)sinhk(1-t)\}}{\Delta} H(s,s)];$$

$$\geq N\, H(s,s).$$

Therefore, $G(t,s) \geq NH(s,s)$.

Case II: Let $0 \leq t \leq s \leq \xi \leq \eta$.

$$G(t,s) \geq N[H(s,s)$$
$$+ \frac{\{[\beta sinhk - \alpha\beta sinhk(1-\xi)]sinhkt + \alpha\beta sinhk\xi sinhk(1-t)\}}{\Delta} H(s,s)$$
$$+ \frac{\{\alpha\beta sinhk(1-\eta)sinhkt + (sinhk - \alpha\beta sinhk\eta)sinhk(1-t)\}}{\Delta} H(s,s)];$$

$$\geq N\, H(s,s) + \frac{\{\alpha\beta sinhk\xi sinhk(1-\xi)\}}{\Delta} NH(s,s)$$
$$+ \frac{\{(sinhk - \alpha\beta sinhk\eta)sinhk(1-\xi)\}}{\Delta} NH(s,s);$$

$$\geq \left[1 + \frac{\{\alpha\beta sinhk\xi sinhk(1-\xi)\}}{\Delta} + \frac{\{(sinhk - \alpha\beta sinhk\eta)sinhk(1-\xi)\}}{\Delta}\right] NH(s,s).$$

Hence, $G(t,S) \geq N_2 H(s,s)$.

Case III: Let $0 \leq \xi \leq s \leq \min\{\eta, t\}$.

$$G(t,s) \geq N[H(s,s) + \frac{\{[\beta sinhk - \alpha\beta sinhk(1-\xi)]sinhkt + \alpha\beta sinhk\xi sinhk(1-t)\}}{\Delta}H(s,s) + \frac{\{\alpha\beta sinhk(1-\eta)sinhkt + (sinhk - \alpha\beta sinhk\eta)sinhk(1-t)\}}{\Delta}H(s,s)];$$

$$\geq NH(s,s) + \frac{\{[\beta sinhk - \alpha\beta sinhk(1-\xi)]sinhk\xi\}}{\Delta}NH(s,s) + \frac{\{\alpha\beta sinhk(1-\eta)sinhk\xi\}}{\Delta}NH(s,s);$$

$$G(t,s) \geq \left[1 + \frac{\{[\beta sinhk - \alpha\beta sinhk(1-\xi)]sinhk\xi\}}{\Delta} + \frac{\{\alpha\beta sinhk(1-\eta)sinhk\xi\}}{\Delta}\right]NH(s,s).$$

Hence, $G(t,S) \geq N_3H(s,s)$.

Case IV: Let $\max\{t,\xi\} \leq s \leq \eta < 1$.

$$G(t,s) \geq N[H(s,s) + \frac{\{[\beta sinhk - \alpha\beta sinhk(1-\xi)]sinhkt + \alpha\beta sinhk\xi sinhk(1-t)\}}{\Delta}H(s,s) + \frac{\{\alpha\beta sinhk(1-\eta)sinhkt + (sinhk - \alpha\beta sinhk\eta)sinhk(1-t)\}}{\Delta}H(s,s)];$$

$$\geq NH(s,s) + \frac{\{\alpha\beta sinhk\xi sinhk(1-\eta)\}}{\Delta}NH(s,s) + \frac{\{(sinhk - \alpha\beta sinhk\eta)sinhk(1-\eta)\}}{\Delta}NH(s,s);$$

$$G(t,s) \geq \left[1 + \frac{\{\alpha\beta sinhk\xi sinhk(1-\eta)\}}{\Delta} + \frac{\{(sinhk - \alpha\beta sinhk\eta)sinhk(1-\eta)\}}{\Delta}\right]NH(s,s).$$

Hence, $G(t,S) \geq N_4H(s,s)$.

Case V: Let $0 \leq \xi \leq \eta \leq s \leq\ t$.

$$G(t,s) \geq NH(s,s) + \frac{\{[\beta sinhk - \alpha\beta sinhk(1-\xi)]sinhkt + \alpha\beta sinhk\xi sinhk(1-t)\}}{\Delta} NH(s,s) + \frac{\{\alpha\beta sinhk(1-\eta)sinhkt + (sinhk - \alpha\beta sinhk\eta)sinhk(1-t)\}}{\Delta} NH(s,s);$$

$$\geq NH(s,s) + \frac{\{[\beta sinhk - \alpha\beta sinhk(1-\xi)]sinhk\eta\}}{\Delta} NH(s,s) + \frac{\{\alpha\beta sinhk(1-\eta)sinhk\eta\}}{\Delta} NH(s,s);$$

$$G(t,s) \geq \left[1 + \frac{\{[\beta sinhk - \alpha\beta sinhk(1-\xi)]sinhk\eta\}}{\Delta} + \frac{\{\alpha\beta sinhk(1-\eta)sinhk\eta\}}{\Delta}\right] NH(s,s).$$

Thus, $G(t,s) \geq N_5 H(s,s)$.

Case VI:Letmax$\{\eta, t\} \leq s \leq 1$.

$$G(t,s) \geq NH(s,s) + \frac{\{[\beta sinhk - \alpha\beta sinhk(1-\xi)]sinhkt + \alpha\beta sinhk\xi sinhk(1-t)\}}{\Delta} NH(s,s) + \frac{\{\alpha\beta sinhk(1-\eta)sinhkt + (sinhk - \alpha\beta sinhk\eta)sinhk(1-t)\}}{\Delta} NH(s,s);$$

$$\geq NH(s,s) + \frac{\{[\beta sinhk - \alpha\beta sinhk(1-\xi)]sinhk\eta\}}{\Delta} NH(s,s) + \frac{\{\alpha\beta sinhk(1-\eta)sinhk\eta\}}{\Delta} NH(s,s);$$

$$G(t,s) \geq \left[1 + \frac{\{[\beta sinhk - \alpha\beta sinhk(1-\xi)]sinhk\eta\}}{\Delta} + \frac{\{\alpha\beta sinhk(1-\eta)sinhk\eta\}}{\Delta}\right] NH(s,s).$$

Thus, $G(t,s) \geq N_6 H(s,s)$.

Setting, $\Pi = \min\{N,\ N_2, N_3, N_4, N_5, N_6\}$. (4.18)

From the above six cases we conclude that for all $t, s, \in [0,1]$,

$G(t,s) \geq \Pi H(s,s)$.

4.2 Existence of One Positive Solution

In this section, we prove the existence of at least one positive solutions for second order four point boundary value problem (1.1)- (1.3) by applying Krasnoselskii'sFixed point theorem, To establish existence of at least one positive solutions for Four point BVP (1.1)-(1.3) we use the following Krasnoselskii'sfixed point theorem

Consider the Banach space $E = \{u : u \in C[0, 1]\}$ equipped with the norm

$$\|u\| = \max_{t\in[0,1]} |u(t)|.$$

Define a cone $P \subset E$ by

$$P = \left\{u \in E: u(t) \geq 0, t \in [0,1]\ and\ \min_{t\in(0,1]} u(t) \geq \Pi\|u\|\right\}. \quad (4.19)$$

whereΠ is given in Equation (4.18).

Let $T: P \longrightarrow P$ be the operator defined by

$$Tu(t) = \int_0^1 G(t,s)f\big(s,u(s)\big)ds, \quad t \in [0,1]. \quad (4.20)$$

Lemma 4.5: The operator Tdefined in (4.20) is a self-map on P and completely continuous.

Proof:$Tu(t) = \int_0^1 [G(t,s)f(s,u(s))]\,ds \leq \int_0^1 \big[G(s,s)f\big(s,u(s)\big)\big]\,ds$.

Note that, by the non-negativity of *G* and *f*, one has $\|Tu\| \leq \int_0^1 [G(s,s)f(s,u(s))]\,ds$ from which we have,

$$\min Tu(t) \geq \min \int_0^1 \big[G(s,s)f\big(s,u(s)\big)\big]\,ds = \pi \int_0^1 [G(s,s)f(s,u(s))]\,ds \quad t \in (0,1],$$

whereπ is given in eq.(4.18)

$\Longrightarrow \|Tu\| \geq \pi\|Tu\|$, where$\|u\| \in P$.

Therefore, $T: P \to P$ is a self-map.Since, $G(t,s) and\ f(t,u(s))$are continuous the map T is completely continuous.

In this thesis we consider the second order BVPs

$-u'' + k^2u(t) = f\big(t,u(t)\big), t \in [0,1]$, with boundary conditions:

$$u(0) - \alpha u(\zeta) = 0,$$

$$u(1) + \beta u(\eta) = 0,$$

where $0 < \zeta < \eta < 1,\ \alpha, \beta > 0\ and\ \ k \geq 0$ is a constant.

The following conditions will be assumed throughout.

$$C_1: 0 \leq \int_0^1 \left[G(t,s)f\big(s,u(s)\big)\right] ds \leq \infty.$$

$\boldsymbol{C_2}$: $f[0,1] \times [0,\infty) \to [0,\infty)$is a continuous function.

$\boldsymbol{C_3}$: $P = \gamma\beta + \alpha\gamma + \alpha\delta > 0, \alpha, \beta, \gamma, \delta \geq 0.$ By using Karsnosel'skiis fixed point theorem the existence of positive solution of (1.1)-(1.3) is obtained in the case when, f is either super linear or sublinear. To be precise, we define the nonnegative extended real number f_0, f^0, f_∞ *and* f^∞ as follow.

$f^0 = \lim_{u \to 0^+} \sup_{t\in[0,1]} \frac{f(t,u(t))}{u(t)} = 0, f_\infty = \lim_{u\to\infty} \min_{t\in[0,1]} \frac{f(t,u(t))}{u(t)} = \infty$ in super linear case.

$f_0 = \lim_{u \to 0^+} \inf_{t\in[0,1]} \frac{f(t,u(t))}{u(t)} = \infty, f_\infty = \lim_{u\to\infty} \max_{t\in[0,1]} \frac{f(t,u(t))}{u(t)} = 0$ in sublinear case.

By a positive solution $u(t)$ of (1.1)-(1.3) we understand a solution $u(t)$ which is positive on $t \in [0,1]$ and satisfies eq. (1.1) for $0 \leq t \leq 1$ and the boundary conditions (1.2) and (1.3).

Theorem 1: Assume that the conditions C_1-C_3 are satisfied. If $f^0 = 0$ and $f_\infty = \infty$ then the boundary value problem has at least one positive solution.

Proof: Now since $f^0 = 0$, there exists $A_1 \in [0,1] \geq 0,$

$\lim_{u \to 0^+} \sup_{t\in[0,1]} \frac{f(t,u(t))}{u(t)} = 0,\ \left| \sup_{t\in[0,1]} \frac{f(t,u(t))}{u(t)} \right| < \eta \ where\ \eta \geq 0,$

$\sup_{t\in[0,1]} \frac{f(t,u(t))}{u(t)} < \eta$, so that $f\big(t,u(t)\big) < \eta,\ \ \eta > 0$ satisfy for every $u \in P$ and $\|u\| = A_1$ for $0 < u < A_1$ where η satisfies

$$\eta \int_0^1 [G(t,s)f(s,u(s))]\, ds. \tag{2.21}$$

$$Tu(t) = \int_0^1 \left[G(t,s)f\big(s,u(s)\big)\right] ds$$

$$\leq \int_0^1 \left[G(s,s)f\big(s,u(s)\big)\right] ds \leq \eta \int_0^1 \left[G(s,s)f\big(s,u(s)\big)\right] ds$$

$\leq \|u\|$by(2.21) .

Consequently,$\|Tu\| \leq \|u\|$ So, if we set $\Omega_1 = \{u \in E: \|u\| < A_1\}$, then

$\|Tu\| \leq \|u\|$for $u \in P \cap \Omega_1$. (2.22)

Next, considering $f_\infty = \infty$, $\lim_{u\to\infty} \min_{t\in[0,1]} \frac{f(t,u(t))}{u(t)} = \infty$ there exists $\eta_\iota > 0$ and $\overline{A_2} > 0$. Let

$A_2 = \min\{2A_1, \pi$ and let $\Omega_2 = \{u \in E: \|u\| < A_2\}$, If $u \in P$ with $\|u\| = A_2$ then $\min_{t\in[0,1]} \frac{f(t,u(t))}{u(t)} > \eta_\iota$, $f(u(t)) > \eta_\iota u(t)$for $u \in \pi_2$.

Where η_ι satisfy $\eta_1\pi_2 \int_{\frac{1}{4}}^{\frac{3}{4}}[G(s,s)f(s,u(s))]\,ds \ \geq 1.$ (2.23)

$$Tu(t) = \int_0^1 [G(s,s)f(s,u(s))]\,ds$$

$$\geq \min_{t\in[\frac{1}{4},\frac{3}{4}]} \int_0^1 [G(t,s)f(s,u(s))]\,ds \geq \eta_1 \int_0^1 [G(t,s)f(s,u(s))]\,ds$$

$$\geq \eta_1\pi_2 \int_{\frac{1}{4}}^{\frac{3}{4}} [G(s,s)f(s,u(s))]\,ds\|u\|$$

$\geq \|u\|$by (4.21).

Hence,$\|Tu\| \geq \|u\|$for $u \in P \cap \alpha\Omega_2$.

Therefore, by first part of fixed-point theorem, it follows that T has in $P \cap (\overline{\Omega_2}\backslash\Omega_1)$ such that $A_1 \leq \|u\| \leq A_2$ by Kransnofelskii fixed-point theorem. Further since, Green's function positive. It follows that $u(t) > 0$ for $t \in [0,1]$ and $u(t)$ is a desired solution for (1.1)-(1.3). The proof is complete.

Theorem (2): Assume that conditions $c_1, c_{2,}$ $and\ c_3$ are satisfied if $f_0 = \infty$ andf^{∞}=0, then the boundary value problem has at least one positive solutions that lies in$P$.

<u>Proof</u>: Let T bethe preserving, completely continuous operator defined by Lemma (2.20) beginning with f_o =∞ , there exits an $A_1 > 0$, ε >0 and satisfy

$\eta_1\,\pi \int_{\frac{1}{4}}^{\frac{3}{4}} G(s,s)\,f(s,u(s))ds \ \geq 1.$(2.24)

$$f_0 = \lim_{u\to 0^+,\, t\in[0,1]} min \frac{f(t,u(t))}{u(t)} = \infty$$

$\frac{f(t,u(t))}{u(t)} \geq \xi_1$for 0< $u \leq A_1$,$f(t,ut)) \geq \xi_1$u.

where$\xi_1 \ \geq \eta_2$ andη_2 is given above. Then for u∈ p and ||u|| = A_1, we have

$$Tu(t) = \int_0^1 G(t,s)f(s,u(s))ds \geq \pi \int_{\frac{1}{4}}^{\frac{3}{4}} G(s,s)f(s,u(s))ds$$

$$\geq \xi_1\,\pi \int_{\frac{1}{4}}^{\frac{3}{4}} G(s,s)f(s.u(s))\,ds\,\|u\|$$

$$\geq ||u||\xi_2\,\pi\int_{\frac{1}{4}}^{\frac{3}{4}} G(s,s)f\big(s,u(s)\big)ds$$

$\geq \|u\|$by (2.24).

Thus$\|u\| \geq \|u\|$,so, if we let $\Omega_1 = \{u \in E: \|u\| < A_1\}$, then

$\|Tu\| \geq \|u\|$, for $u \in p \cap \alpha\Omega_1$.(2.25)

It remains to consider f^{∞}=0, there exists an$\overline{A_2} > 0$ such that$f(t,u(t)) \leq \xi_2 u$ for all u ≥ $\overline{A_2}$, where $\xi_2 > 0$.

Satisfies.

$\xi_2\int_0^1 G(s,s)\,f(s,u(s))ds \leq 1.$ (2.26)

there are twocases:

case (i): f *is bounded and* case (ii): f *is unbounded*

Case (i): Suppose N> 0 is such that $f(t,u(t)) \leq N$ *for* $0 < u < \infty$,

$f^{\infty} = \lim_{u\to\infty} \max_{t\in[0,1]} \frac{f(t,u(t))}{u(t)}$ =0,

$f\big(t,u(t)\big) \leq \xi_2 u(t)$ = N for u(t) $> A_2 > 0$.

Let A_2 = max$\left\{2A_1, N\int_0^1 G(s,s)f\big(s,u(s)\big)ds\right\}$.

Then, for $u(t) \in p$ *with* $\big||u|\big| = A_2$ .We have

$$Tu(t) = \int_0^1 G(t,s)f\big(s,u(s)\big)ds$$

$$\leq \int_0^1 G(s,s)f\big(s,u(s)\big)ds$$

$$\leq \int_0^1 G(s,s)ds\xi_2\,u$$

$= N\int_0^1 G(s,s)ds \leq A_2$ = ||u||

So that $\|u\| \leq \|u\|$. *So , if* Ω_2 =$\{u \in E: ||u|| < A_2\}$then

$\|u\| \leq ||u||$for$u \in p \cap \alpha\Omega_2$.(2.27)

Case(ii): Suppose f is unbounded then,

Let $A_2 > max\{2A_1,\overline{A_2}\}$ be such that $f(t,u(t)) \leq f(t,A_2)$, for 0$< u \leq A_2$.

Choose $u \in p$ with$\|u\| = A_2$, we have

$Tu(t)$ =$\int_0^1 G(t,s)f\big(s,u(s)\big)ds$

$$\leq \int_0^1 G(s,s)f\big(s,u(s)\big)ds$$

$\leq \int_0^1 G(s,s)f(A_2)ds$

$$\leq \int_0^1 G(s,s)u\xi_2 ds$$

$\leq A_2\xi_2 \int_0^1 G(s,s)ds = \|u\|$by(2.27)

and so$\|Tu\| \leq \|u\|$. For this case, if we let$\Omega_2 = \{u \in E: \|u\| < A_2\}$

then ||Tu|| ≤ ||u||, for$u \in p \cap \alpha\Omega_2$.(2.28)

Thus, in either of the cases an application of the second part of the Krensnoselskii's fixed point theorem yields a solution of boundary value problems (1.1)-(1.3) has a positive solution which belongs to p$\cap$ $(\overline{\Omega_2}\backslash\Omega_1)$.

4.3.Uniqueness of positive solution

In this section by applying Banachcontraction principle we verify the uniqueness of the positive solution for boundary value problems (1.1)-(1.3).

Lemma (2.7): Assume $f(t,u(t))$satisfies Lipschitzconditions with respect to the second variable with Lipschiz constant $K$for all$t \in [0,1]$ then the boundary value problem (1,1) – (1.3) has a unique solution when 0$< k\int_0^1 G(t,s)ds < 1$.

Proof: We consider the operator T defined on cone P and given by,

$Tu(t) = \int_0^1 G(t,s)f(s,u(s))ds \, , u \in P$

Since, $T$is self-map on cone$P$, then we prove that the map T is satisfies the Banach contraction principle for all $u, v \in P$ and $t \in [0,1]$,

Where,$Tu(t) = k\int_0^1 G(t,s)f(s,u(s))ds \, , u \in p$. and

$Tv(t) = k\int_0^1 G(t,s)f(s,v(s))ds \, , v \in p$.

So,$|Tu(t) - Tv(t)| = |k\int_0^1 G(t,s)f(s,u(s))ds - \int_0^1 G(t,s)f(s,v(s))ds|$.

$= \left|k\int_0^1 G(t,s)\,[f(s,u(s)) - f(s,v(s))]\,ds\right|$

$$\leq k\int_0^1 \left|G(t,s)\,[f(s,u(s)) - f(s,v(s))]\,ds\right|$$

$$\leq k\int_0^1 G(t,s)ds|u-v| = \alpha|u-v|$$

Therefor,$|Tu(t) - Tv(t)| \leq \alpha|u(t) - v(t)|$.

Notice that $\alpha = k\int_0^1 G(t,s)ds < 1$,the mapping T is a contraction. Hence, by Banachcontraction principle, T has a unique fixed point which is a unique positive solution of boundary value problem (1.1)-(1.3) in a cone.

Chapter 5
Conclusion and Future scope

5.1 Conclusion

In this thesis we construct the Green's function for homogenous boundary value problems and considered existence and uniqueness of a positive solution for four point boundary value problems in a cone by the use of the Karsnoselski'sfixed point theoremandBanach contraction principle respectively for (1.1)-(1.3).

5.2 Future scope

There are some published results related to the existence of positive Solutions for four point boundary value problems.The researcher believes the research for existence and uniqueness of a positive solution for four point boundaryvalue problems in a cone by the use of the Karsnoselski's theorem and Banachfixed-point in a cone for the corresponding non-homogeneous boundary value problems. So,any interested researcher can use this opportunity and conduct their research work in this area.

References

Agarwal, R. P., Wong, F. H., andLian, W. C. (1999). Positive Solutions for Nonlinear Singular Boundary Value Problems. *Applied Mathematics Letters*, *12*(2), 115-120.

Benaicha, S., andHaddouchi, F. (2016). Positive solutions of a nonlinear fourth-order integral boundary value problem. *arXiv preprint arXiv:*1601.03773.

Dong, X., and Bai, Z. (2008). Positive solutions of fourth-order boundary value problem with variable parameters. *The Journal of Nonlinear Sciences and its Applications*, *1*(1), 21-30.

Erbe, L. H., and Wang, H. (1994).On the existence of positive solutions of ordinary differential equations.*Proceedings of the American Mathematical Society*, *120*(3), 743-748.

Krasnoselskii, M.A. (1964). Positive Solutions of Operators Equations, *P. Noordhoff, Groningen, the Netherlands*

Lian, W. C., Wong, F. H. and Yeh, C. C. (1996). On the existence of positive solutions of nonlinear second order differential equations. *Proceedings of the American Mathematical Society*, *124*(4), 1117-1126.

Liu, B. (2004). Positive solutions of a nonlinear four-point boundary value problems. *Applied Mathematics and Computation*, *155*(1), 179-203.

Ma, R., and Wang, H. (2003).Positive solutions of nonlinear three-point boundary-value problems.*Journal of Mathematical Analysis and Applications*, *279*(1), 216-227.

Nieto, J. J. (2013). Existence of a solution for a three-point boundary value problem for a second-order differential equation at resonance. *Boundary value problems*, *2013*(1), 1-7.

Sveikate, N. (2016). On three-point boundary value problem. *Mathematical Modeling and Analysis*, *21*(2), 270-281.

XinDongi and Zhang Bing Bai, (2008). Positive solutions of fourth-order boundary value problem with variable parameters, *J. Nonlinear Sci. Appl. 1, no.*1, 21-30.

Xu, W., and Wang, H. (2017).Positive Solution of a Nonlinear Four-Point Boundary-Value Problem.

Yang, B. (2005). Positive solutions for a fourth order boundary value problem. *Electronic Journal of Qualitative Theory of Differential Equations*, *2005*(3), 1-17.

Zhang, X., and Liu, L. (2007). Positive solutions of fourth-order four-point boundary value problems with p-Laplacian operator. *Journal of Mathematical Analysis and Applications*, *336*(2), 1414-1423.

Z. Bai and Z. Gu(2007). Positive solutions for some second-order four-point boundary value problems, *J. Math. Anal. Appl*. 330 , 34-50.