Fixed Point Results for Generalized Suzuki (ψ, ϕ) - Contraction and (ψ, ϕ) -Jungck-Suzuki Contraction Type Mappings in Complete b-Metric Spaces

A RESEARCH SUBMITTED TO THE DEPARTMENT OF MATHEMATICS IN PARTIAL FULFILLMENT FOR THE REQUIREMENTS OF THE DEGREE OF MASTERS OF SCIENCE IN MATHEMATICS

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Declaration

I, the undersigned declare that, this research paper entitled ”Fixed Point Results for Generalized Suzuki (ψ,ϕ) - Contraction and (ψ,ϕ)-Jungck-Suzuki Contraction type Mappings in complete b-Metric Spaces” is my own original work and it has not been submitted for the award of any academic degree or the like in any other institution or university, and that all the sources I have used or quoted have been indicated and acknowledged.

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Abstract

In this research, we introduced new class of mappings called generalized (ψ, ϕ)-Suzuki-type mappings and generalized (ψ, ϕ)- Jungck-Suzuki contraction type mappings and established existence and uniqueness of fixed point for generalized (ψ, ϕ)-Suzuki-type mapping and studied coincidence point results for generalized (ψ, ϕ)-Jungck-Suzuki contraction mappings in the frame work of complete b- metric spaces. Our results improve, extend and generalize some known results in the literature. Also, we provided examples in support of our main findings. In this research undertaking, we followed analytical study design and used secondary sources of data, such as published articles and related books.

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Chapter 1

Introduction

1.1 Background of the study

Fixed point theory is an important tool in the study of nonlinear analysis. It is considered to be the key connection between pure and applied mathematics. It is also widely applied in different fields of study such as Economics, Chemistry, Physics and almost all engineering fields. The contraction mapping principle introduced by Banach, (1922) has wide range of applications in a fixed point theory. In 1922, Banach proved the following famous fixed point theorem.
Let (X,d) be a complete metric space, $T : X \to X$ be a contraction map, then there exists a unique fixed point $x_0 \in X$ of T. This theorem, called the Banach contraction principle is a forceful tool in nonlinear analysis.

Another category of contraction which is separate from Banach contraction, and does not imply continuity, was proposed by Kannan, (1968) who also established in the same work that such mappings necessarily have unique fixed points in complete metric spaces. Mappings belonging to this category are known as Kannan type.
In 1972, a new concept which is different from that of Banach, (1922) and Kannan, (1968) contraction type mappings was introduced by Chatterjea, (1972) which gives a new direction to the study of fixed point theory. There are a class of contractive mappings which are different from Banach contraction and have unique fixed point in complete metric spaces. Banach contraction principle has been extended and generalized in different directions by different researchers. For more details we refer Aage and Salunke, (2012), Choudhuny and Bandyopadhyay, (2015), Doric, (2009), Morales and Rojas, (2012), Alsulami *.et al*, (2015), Ameer *.et al* and etc.The family of contractive mappings in metric spaces is a great interest and has already been studied in the literature since long time.
In 2009, Suzuki Introduced the concept of Suzuki-type generalized non-expansive mapping and proved some fixed-point theorems.
In 1984, Khan *et al* introduced the notion of an altering distance function, which is

a control function that alters distance between two points in a metric space. In 1997, Alber *et al* introduced the concept of weak contraction in Hilbert spaces. Later, in 2001 Rhoades proved that the result which Alber *et al*, 1997 is also valid in complete metric spaces.

Using the concept of altering distance function, Dutta and Choundhury, (2008), Doric, (2009), Harjani and Sadarjani, (2009), established some fixed point results for weak contractions and generalized contraction mappings in the frame work of metric spaces. Jleli, *et al* (2014) introduced the notion of θ-contraction and established fixed point theorems and also Liu, *et al*, (2016) proved new fixed point theorems for (ψ,ϕ)-type Suzuki contractions in complete metric spaces. Recently, Eskandar, *et al*, (2019) proved new fixed point theorems for generalized multivalued (ψ,ϕ)-Suzuki type contractions in complete metric spaces. Very recently, Mebawondu and Mebawondu, (2021) introduced a new class of mappings called (ψ,ϕ)-Suzuki-type mapping and (ψ,ϕ)-Jungck-Suzuki contraction type mappings and they established the existence, uniqueness and coincidence results for (ψ,ϕ)-Suzuki-type mapping and (ψ,ϕ)-Jungck-Suzuki contraction mappings in the frame work of complete metric spaces.

Inspire and motivated by the works of Liu, *et al*, (2016), Eskandar, *et al*, (2019), Mebawondu and Mebawondu, (2021) the main purpose of this research work is to study some fixed point results for generalized Suzuki (ψ,ϕ) and (ψ,ϕ)-Juncgck-Suzuki contraction type mappings in the context of complete *b*-metric spaces.

1.2 Statements of the problem

In 2009, Doric (2009) introduced the class of generalized (ψ,ϕ)-weak contractive mappings and established that these mappings necessarily have unique common fixed point in complete metric spaces. In 2016, Liu, *et al,* (2016) proved new fixed-point theorems for (ψ,ϕ)-type Suzuki contractions in complete metric spaces. In 2019, Eskandar, *et al*, (2019) proved new fixed-point theorems for generalized multivalued (ψ,ϕ)-Suzuki type contractions in complete metric spaces. In 2021, Mebawondu and Mebawondu, (2021) introduced a new class of mappings called the generalized Suzuki (ψ,ϕ) contraction mappings in complete metric spaces. However, fixed-point results for generalized Suzuki (ψ,ϕ) and (ψ,ϕ)-Juncgck-Suzuki contraction type mappings in complete b-metric spaces are not yet studied. Thus, in this study we focused on establishing and proving fixed point results for generalized Suzuki (ψ, ϕ) and (ψ, ϕ)-Juncgck-Suzuki contraction type mappings in complete b-metric spaces.

1.3 Objectives of the study

1.3.1 General objective

The main objective of this research work was to study fixed-point results for generalized Suzuki (ψ, ϕ) contraction and coincidence point results for generalized (ψ, ϕ)-Jungck-Suzuki contraction type mappings in the context of complete b-metric spaces.

1.3.2 Specific objectives

This study has the following specific objectives:

- To prove the existence of fixed point results for generalized Suzuki (ψ, ϕ) contraction and coincidence point results for generalized (ψ, ϕ)-Jungck-Suzuki contraction type mappings in the setting of b-metric spaces.
- To show the uniqueness of the fixed point results for generalized Suzuki (ψ, ϕ)- contraction type mappings in the context of complete b-metric spaces.
- To verify the applicability of the main results obtained using specific examples.

1.4 Significance of the study

The study may have the following importance:

- It may provide basic research skills to the researcher.
- The outcome of this study may contribute to research activities on study area.
- The outcome of this study may be applied in solving some problems in applied sciences.

1.5 Delimitation of the Study

The study focuses on proving the existence and uniqueness of fixed points for generalized Suzuki (ψ, ϕ) contraction and coincidence point results for generalized (ψ, ϕ) -Jungck-Suzuki contraction type mappings in the setting of b-metric spaces.

Chapter 2

Review of Related Literatures

The formal form of the metric fixed point theory appeared by the pioneer and art work of Banach results in 1922. In the fixed point theory, contraction is one of the main tools to prove the existence and uniqueness of a fixed point. Banach's contraction principle, which gives an answer on the existence and uniqueness of the solution of an operator, $Tx = x$ is used in all analysis. There are a class of contractive mappings which are different from Banach contraction and have unique fixed point in complete metric spaces. Every contraction in a complete metric space possesses a unique fixed point. By changing the contraction conditions and involving some auxiliary functions, several new results were reported.In 1976, Jungck proved a common fixed point theorem for commuting maps under the condition that $X = Y$. The metric fixed point theory has been extended and generalized in several aspects by a number of authors from all over the worlds. (See e.g., Alber, *et al*, Berinde, (2009), Harjani, (2009), Sadarangani, (2010) and etc.)
A very interesting extension of the notion of a metric, called b-metric, was proposed by Czerwik, (1993) contraction. In these pioneer papers, Czerwik observed some fixed point results, including the analog of the Banach contraction principle in the context of complete b-metric spaces. In 1997, Alber *et al* introduced the concept of weak contraction in Hilbert spaces.

Definition 2.0.1 *(Alber et al, 1997) Let (X,d) be metric space. A self mapping f on X is said to be weakly contractive if, $d(fx, fy) \leq d(x,y) - \phi(d(x,y))$ for all $x, y \in X$, where ϕ is an altering distance function.*

In 2004, Berinde introduced 'weak contractions' as a generalization of contraction maps in continuation to the extensions of contraction maps. Also in 2008, Berinde renamed 'weak contractions' as 'almost contractions' in his later work.In 2008, Suzuki proved two fixed point point theorems, one of which is a new type of generalization of the Banach contraction principle and does characterize the metric completeness. In 2010, Pacurar proved results on sequences of almost contractions and fixed points in b-metric spaces.In 2013, Kir and Kiziltunc established the results in b-metric spaces which generalized the Kannan and Chatterjea type mappings. Roshan, *et al*, 2016 presented Suzuki type fixed point results in b-metric spaces and some common fixed point results for two mappings under generalized contractive condition in b-metric spaces where the b-metric function is not necessarily continuous. In 2015, Latif, *et al* proved the existence and uniqueness of fixed points of a single self map satisfying Suzuki type contraction condition in b-metric spaces. In 2017, Leyew and Abbas proved the existence and uniqueness of fixed points of generalized Suzuki-Geraghty contraction maps in complete b-metric spaces. In 2019, Babu and Babu proved fixed points of generalized contraction maps with rational expressions in b-metric spaces. In the sequel, several papers have been reported on the existence (and the uniqueness) of (common) fixed points of various classes of single-valued and multi-valued operators in the setting of b-metric spaces (see, e.g., Aydi*et al*, (2012), Roshan, (2014), Liu,*et al* , (2016) , Eskandar ,*et al*, (2019) and the related references therein).

Recently, Mebawondu and Mebawondu, (2021) introduced a new class of mappings called (ψ,ϕ)-Suzuki-type mapping and (ψ,ϕ)-Jungck-Suzuki contraction type mappings and they established the existence, uniqueness and coincidence results for (ψ,ϕ)-Suzuki-type mapping and (ψ,ϕ)-Jungck-Suzuki contraction mappings in the frame work of complete metric spaces.

The main purpose of this research work is to study fixed point results for generalized Suzuki (ψ,ϕ)-contraction in the context of complete b-metric spaces.

Chapter 3

Methodology

This chapter contains study design, description of the research methodology, data collection procedures and data analysis process.

3.1 Study area and period

The study was conducted from September 2021 G.C to February 2022 at Jimma University under mathematics department.

3.2 Study Design

In order to achieve the objective of the study we followed analytical method of design.

3.3 Source of Information

In this study secondary source of data such as, different mathematics books related to the study area, published articles related to the research topic and Internet sources was used.

3.4 Mathematical Procedure of the Study

In this study we followed the procedures stated below:

- Establishing fixed point theorems.
- Constructing sequences.
- Showing the constructed sequences are b-Cauchy .
- Showing the b-convergence of the sequences.

- Proving the existence of fixed points and coincidence points.
- Showing uniqueness of the fixed points.
- Giving examples in support of our main findings of the research work.

Chapter 4

Preliminaries and Main Results

4.1 Preliminaries

Notation 1 *We need the following symbols and class of functions to prove certain results of this section:*

- $\Re^+ = [0,\infty)$*;*
- $\Re$ *is the set of all real numbers;*
- $\mathbb{N}$ *is the set of all natural numbers;*
- ϕ *and* ψ *= altering distance function;*
- Ψ *= family of set.*

Definition 4.1.1 *(Czerwik,1993) Let* X *be a nonempty set and* $s \geq 1$ *be a given real number. A function* $d : X \times X \to \Re^+$ *is said to be a b-metric if and only if for all* $x,y,z \in X$*, the following conditions are satisfied:*

a. $d(x,y) = 0$ *if and only if* $x = y$;

b. $d(x,y) = d(y,x)$*;*

c. $d(x,z) \leq s[d(x,y) + d(y,z)]$.
The pair (X,d) *is called a b-metric space.*

It should be noted that, the class of b-metric spaces is effectively larger than that of metric spaces, since a b-metric is a metric when $s = 1$. But, in general, the converse is not true.

Example 4.1.1 *(Roshan,2014) Let* $X = \Re$ *and* $d : X \times X \to \Re^+$ *be given by* $d(x,y) = |x-y|^2$ *for* $x,y \in X$*, then* d *is a b-metric on* X *with* $s = 2$ *but it is not a metric on* X *since for all* $x,y,z \in \Re$ *where,* $x = 1, y = 3$ *and* $z = 7$*, we have*

$$d(1,7) \nleq d(1,3) + d(3,7).$$

Hence the triangle inequality for a metric does not hold.

Definition 4.1.2 *(Boriceanu et al, 2010) Let X be a b-metric space and $\{x_n\}$ be a sequence in X, we say that*

a. *$\{x_n\}$ is b-converges to $x \in X$ if $d(x_n,x) \to 0$ as $n \to \infty$.*

b. *$\{x_n\}$ is a b-Cauchy sequence if $d(x_n,x_m) \to 0$ as $n,m \to \infty$.*

c. *(X,d) is b-complete if every b-Cauchy sequence in X is b-convergent.*

Definition 4.1.3 *(Boriceanu, 2009)Let (X,d) be a b-metric space with the coefficient $s \geq 1$ and let $T: X \to X$ be a given mapping. We say that T is b-continuous at $x_0 \in X$ if and only if for every sequence $\{x_n\}$ in X, we have $x_n \to x_0$ as $n \to \infty$, then $Tx_n \to Tx_0$ as $n \to \infty$. If T is b-continuous at each point $x \in X$, then we say that T is b-continuous on X.*
In general, a b-metric is not necessarily b-continuous.

Definition 4.1.4 *(Khan et al, 1984) A function $\psi : \Re^+ \to \Re^+$ is called an altering distance function if the following properties hold;*
ψ is continuous and nondecreasing function and $\psi(t) = 0$ if and only if $t = 0$.

Definition 4.1.5 *(Alber et al, 1997) A mapping $T : X \to X$ where (X,d) is a metric space, is said to be weakly contractive if $d(Tx,Ty) \leq d(x,y) - \phi(d(x,y))$ where $\phi : R^+ \to R^+$ is altering distance function.*

Definition 4.1.6 *(Jungck and Hussain, 2007) Let X be a nonempty set and $S,T : X \to X$ be any two mapping;*

a. *a point $x \in X$ is called:*
   $i)$ coincidence point of S and T if $Sx = Tx$,
   $ii)$ common fixed point of S and T if $x = Sx = Tx$.

b. *If $y = Sx = Tx$ for some $x \in X$,then y is called a point of coincidence of S and T.*

Lemma 4.1.1 *(Roshan et al,2014) Let (X,d) be a b-metric space with $s \geq 1$ and let $\{x_n\}$ be a sequence in X such that*

$$\lim_{n\to\infty} d(x_n,x_{n+1}) = 0$$

If $\{x_n\}$ is not a b-Cauchy sequence, then there exist $\varepsilon > 0$ and two subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ such that for the following four sequences
$d(x_{m_k}, x_{n_k}), d(x_{m_k+1}, x_{n_k}), d(x_{m_k}, x_{n_k+1}), d(x_{m_k+1}, x_{n_k+1})$
it holds

a. $\varepsilon \leq \liminf_{k\to\infty} d(x_{m_k}, x_{n_k}) \leq \limsup_{k\to\infty} d(x_{m_k}, x_{n_k}) \leq s\varepsilon.$

b. $\frac{\varepsilon}{s} \leq \liminf_{k\to\infty} d(x_{m_k}, x_{n_k+1}) \leq \limsup_{k\to\infty} d(x_{m_k}, x_{n_k+1}) \leq s^2\varepsilon.$

c. $\frac{\varepsilon}{s} \leq \liminf_{k\to\infty} d(x_{m_k+1}, x_{n_k}) \leq \limsup_{k\to\infty} d(x_{m_k+1}, x_{n_k}) \leq s^2\varepsilon.$

d. $\frac{\varepsilon}{s^2} \leq \liminf_{k\to\infty} d(x_{m_k+1}, x_{n_k+1}) \leq \limsup_{k\to\infty} d(x_{m_k+1}, x_{n_k+1}) \leq s^3\varepsilon.$

Theorem 4.1.2 *(suzuki, 2009) Let (X,d) be a complete metric space and $T : X \to X$ be self mapping. Assume that for all $x, y \in X$ with $x \neq y$,*
$$\frac{1}{2}d(x,Tx) < d(x,y) \Rightarrow d(Tx,Ty) < d(x,y).$$

Then, T has a unique fixed point in X.

Recently, Mebawondu and Mebawondu, (2021) introduced the notion of generalized Suzuki (ψ, ϕ)-contraction in complete metric spaces and they proved the existence and uniqueness of fixed point.

Definition 4.1.7 *(Mebawondu and Mebawondu, 2021) Let (X,d) be a metric space. A mapping $T : X \to X$ is said to be (ψ, ϕ)-Suzuki type if for all $x, y \in X$,*
$$\frac{1}{2}d(x,Tx) \leq d(x,y) \Rightarrow \psi(d(Tx,Ty)) \leq \psi(M(x,y)) - k\phi(N(x,y)) + L\phi(N_1(x,y)),$$

where $0 < k \leq 1, L \geq 0$,
$M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty)\}$,
$N(x,y) = \max\{d(x,y), d(y,Ty)\}$,
$N_1(x,y) = \min\{d(x,Ty), d(x,Tx), d(y,Tx)\}$ *and*
ψ, ϕ are altering distance functions.

Theorem 4.1.3 *(Mebawondu and Mebawondu, 2021) Let (X,d) be a complete metric space and $T : X \to X$ be a mapping satisfying definition (4.1.7). Then T has unique fixed point.*

4.2 Main Results

In this section, we introduced generalized Suzuki (ψ,ϕ)-and (ψ,ϕ)-Juncgck-Suzuki contraction type mappings and study fixed point and coincidence point results for the mappings introduced in the setting of b-metric spaces.
Now, we define generalized Suzuki (ψ,ϕ)- contraction type mappings and study existence and uniqueness of fixed points for such mapping.

Definition 4.2.1 *Let (X,d) be a b-metric space and $s \geq 1$. A mapping $T : X \to X$ is said to be generalized (ψ,ϕ)-Suzuki contraction type mapping if for all $x,y \in X$,*
$\frac{1}{2s}d(x,Tx) \leq d(x,y) \Rightarrow$

$$\psi(s^3 d(Tx,T_y)) \leq \psi(M(x,y)) - k\phi(N(x,y)) + L\phi(N_1(x,y)), \tag{4.1}$$

where
$0 < k \leq \frac{1}{s}, L \geq 0, \psi,\phi \in \Psi,$
$M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2s}\},$
$N(x,y) = \max\{d(x,y), d(y,Ty), \frac{d(x,Tx)d(x,Ty)+d(y,Ty)d(y,Tx)}{1+d(x,Ty)+d(y,Tx)},$
$\frac{d(x,Tx)d(x,Ty)+d(y,Ty)d(y,Tx)}{1+s[d(x,y)+d(Tx,Ty)]}\},$
$N_1(x,y) = \min\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}$
and ψ, ϕ are altering distance functions.

Theorem 4.2.1 *Let (X,d) be a b- complete b-metric space, $s \geq 1$ and $T : X \to X$ be generalized (ψ,ϕ)-Suzuki contraction type mapping. Then T has a unique fixed point.*

proof: Let $x_0 \in X$ be arbitrary. We define a sequence $\{x_n\}$ in X by $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$. If we suppose that $x_{n_0} = x_{n_0+1}$ for some $n_0 \in \mathbb{N}$, since $Tx_{n_0} = x_{n_0+1} = x_{n_0}$ the point x_{n_0} forms a fixed point of T that completes the proof. From now on we suppose that $x_{n-1} \neq x_n$ for all $n \in \mathbb{N}$.

Now we observe that by (4.1)

$$\begin{aligned}\frac{1}{2s}d(x_{n-1},Tx_{n-1}) &= \frac{1}{2s}d(x_{n-1},x_n)\\ &\leq d(x_{n-1},x_n), s\geq 1,\end{aligned}$$

Which implies

$$\begin{aligned}\psi(s^3d(x_n,x_{n+1})) &= \psi(s^3d(Tx_{n-1},Tx_n))\\ &\leq \psi(M(x_{n-1},x_n))-k\phi(N(x_{n-1},x_n))\\ &\quad +L\phi(N_1(x_{n-1},x_n)),\end{aligned} \tag{4.2}$$

where

$$\begin{aligned}M(x_{n-1},x_n) &= \max\{d(x_{n-1},x_n),d(x_{n-1},Tx_{n-1}),d(x_n,Tx_n),\frac{d(x_{n-1},Tx_n)+d(x_n,Tx_{n-1})}{2s}\},\\ &= \max\{d(x_{n-1},x_n),d(x_n,x_{n+1}),\frac{d(x_{n-1},x_{n+1})+d(x_n,x_n)}{2s}\}\\ &= \max\{d(x_{n-1},x_n),d(x_n,x_{n+1})\},\end{aligned}$$

$$\begin{aligned}N(x_{n-1},x_n) &= \max\{d(x_{n-1},x_n),d(x_n,Tx_n),\\ &\quad \frac{d(x_{n-1},Tx_{n-1})d(x_{n-1},Tx_n)+d(x_n,Tx_n)d(x_n,Tx_{n-1})}{1+d(x_{n-1},Tx_n)+d(x_n,Tx_{n-1})},\\ &\quad \frac{d(x_{n-1},Tx_{n-1})d(x_{n-1},Tx_n)+d(x_n,Tx_n)d(x_n,Tx_{n-1})}{1+s[d(x_{n-1},x_n)+d(Tx_{n-1},Tx_n)]}\}\\ &= \max\{d(x_{n-1},x_n),d(x_n,x_{n+1}),\\ &\quad \frac{d(x_{n-1},x_n)d(x_{n-1},x_{n+1})+d(x_n,x_{n+1})d(x_n,x_n)}{1+d(x_{n-1},x_{n+1})+d(x_n,x_n)},\\ &\quad \frac{d(x_{n-1},x_n)d(x_{n-1},x_{n+1})+d(x_n,x_{n+1})d(x_n,x_n)}{1+s[d(x_{n-1},x_n)+d(x_n,x_{n+1})]}\}\\ &= \max\{d(x_{n-1},x_n),d(x_n,x_{n+1})\},\end{aligned}$$

$$
\begin{aligned}
N_1(x_{n-1},x_n) &= \min\{d(x_{n-1},x_n),d(x_{n-1},Tx_{n-1}),d(x_n,Tx_n),d(x_{n-1},Tx_n),d(x_n,Tx_{n-1})\} \\
&= \min\{d(x_{n-1},x_n),d(x_{n-1},x_n),d(x_n,x_{n+1}),d(x_{n-1},x_{n+1}),d(x_n,x_n)\} \\
&= 0.
\end{aligned}
$$

If $\max\{d(x_{n-1},x_n),d(x_n,x_{n+1})\} = d(x_n,x_{n+1})$ for some $n \in N$,
then, (4.2) becomes
$\psi(d(x_n,x_{n+1})) \leq \psi(s^3 d(x_n,x_{n+1})) \leq \psi(d(x_n,x_{n+1})) - k\phi(d(x_n,x_{n+1})) + L\phi(0)$

$$
= \psi(d(x_n,x_{n+1})) - k\phi(d(x_n,x_{n+1})), \tag{4.3}
$$

which implies that

$$
\begin{aligned}
\psi(d(x_n,x_{n+1})) &\leq \psi(d(x_n,x_{n+1})) - k\phi(d(x_n,x_{n+1}) \\
&< \psi(d(x_n,x_{n+1})),
\end{aligned}
$$

which is a contradiction.
Hence, we have

$$
\max\{d(x_{n-1},x_n),d(x_n,x_{n+1})\} = d(x_{n-1},x_n).
$$

Which implies that

$$
\psi(d(x_n,x_{n+1})) \leq \psi(d(x_{n-1},x_n)) - k\phi(d(x_{n-1},x_n)). \tag{4.4}
$$

From (4.4) we have that $\{d(x_n,x_{n+1})\}$ is a nonincreasing sequence of nonnegative real numbers which is bounded below.
Thus, for all $n \geq 1$ there exists $c \geq 0$ such that

$$
\lim_{n\to\infty} d(x_n,x_{n+1}) = c. \tag{4.5}
$$

We show that $c = 0$.
Assume that $c > 0$.
Taking the upper limit as $n \to \infty$ in (4.4) , we have

$$
\psi(c) \leq \psi(c) - k\phi(c) < \psi(c).
$$

Which is a contradiction. Hence,

$$c = 0.$$

It follows that for all $n \geq 1$,

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{4.6}$$

Now, we show that $\{x_n\}$ is a b-Cauchy sequence in X. Assume on contrary that the sequence $\{x_n\}$ is not a b-Cauchy. Then there exists $\varepsilon > 0$ for which we can find two subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ with $n_k > m_k > k$ such that

$$d(x_{m_k}, x_{n_k}) \geq \varepsilon \tag{4.7}$$

which implies that

$$d(x_{m_k}, x_{n_k-1}) < \varepsilon$$

and (a)-(d) of lemma (4.1.1) hold.
By taking $x = x_{m_k}$ and $y = x_{n_k}$ in (4.2) we have,

$$\begin{aligned} \frac{1}{2s} d(x_{m_k}, Tx_{m_k}) &= \frac{1}{2s} d(x_{m_k}, x_{m_k+1}) \\ &\leq d(x_{m_k}, x_{n_k}) \end{aligned}$$

implies,

$$\begin{aligned} \psi(s^3 d(x_{m_k+1}, x_{n_k+1})) &= \psi(s^3 d(Tx_{m_k}, Tx_{n_k})) \\ &\leq \psi(M(x_{m_k}, x_{n_k})) - k\phi(N(x_{m_k}, x_{n_k})) \\ &\quad + L\phi(N_1(x_{m_k}, x_{n_k})), \end{aligned} \tag{4.8}$$

where,

$$\begin{aligned} M(x_{m_k}, x_{n_k}) &= \max\{d(x_{m_k}, x_{n_k}), d(x_{m_k}, Tx_{m_k}), d(x_{n_k}, Tx_{n_k}), \frac{d(x_{m_k}, Tx_{n_k}) + d(x_{n_k}, Tx_{m_k})}{2s}\} \\ &= \max\{d(x_{m_k}, x_{n_k}), d(x_{m_k}, x_{m_k+1}), d(x_{n_k}, x_{n_k+1}), \frac{d(x_{m_k}, x_{n_k+1}) + d(x_{n_k}, x_{m_k+1})}{2s}\}, \end{aligned}$$

$$
\begin{aligned}
N(x_{m_k},x_{n_k}) &= \max\{d(x_{m_k},x_{n_k}),d(x_{n_k},Tx_{n_k}),\\
&\quad \frac{d(x_{m_k},Tx_{m_k})d(x_{m_k},Tx_{n_k})+d(x_{n_k},Tx_{n_k})d(x_{n_k},Tx_{m_k})}{1+d(x_{m_k},Tx_{n_k})+d(x_{n_k},Tx_{m_k})},\\
&\quad \frac{d(x_{m_k},Tx_{m_k})d(x_{m_k},Tx_{n_k})+d(x_{n_k},Tx_{n_k})d(x_{n_k},Tx_{m_k})}{1+s[d(x_{m_k},x_{n_k})+d(Tx_{m_k},Tx_{n_k})]}\}\\
&= \max\{d(x_{m_k},x_{n_k}),d(x_{n_k},x_{n_k+1}),\\
&\quad \frac{d(x_{m_k},x_{m_k+1})d(x_{m_k},x_{n_k+1})+d(x_{n_k},x_{n_k+1})d(x_{n_k},x_{m_k+1})}{1+d(x_{m_k},x_{n_k+1})+d(x_{n_k},x_{m_k+1})},\\
&\quad \frac{d(x_{m_k},x_{m_k+1})d(x_{m_k},x_{n_k+1})+d(x_{n_k},x_{n_k+1})d(x_{n_k},x_{m_k+1})}{1+s[d(x_{m_k},x_{n_k})+d(x_{m_k+1},x_{n_k+1})]}\},
\end{aligned}
$$

$$
\begin{aligned}
N_1(x_{m_k},x_{n_k}) &= \min\{d(x_{m_k},x_{n_k}),d(x_{m_k},Tx_{m_k}),d(x_{n_k},Tx_{n_k}),d(x_{m_k},Tx_{n_k}),d(x_{n_k},Tx_{m_k})\}\\
&= \min\{d(x_{m_k},x_{n_k}),d(x_{m_k},x_{m_k+1}),d(x_{n_k},x_{n_k+1}),d(x_{m_k},x_{n_k+1}),d(x_{n_k},x_{m_k+1})\}.
\end{aligned}
$$

Applying the upper and lower limits as $k \to \infty$, (4.8) becomes

$$
\begin{aligned}
\liminf_{k\to\infty} \psi(s^3 d(x_{m_k+1},x_{n_k+1})) &= \liminf_{k\to\infty} \psi(s^3 d(Tx_{m_k},Tx_{n_k}))\\
&\leq \limsup_{k\to\infty} \psi(M(x_{m_k},x_{n_k})) - k \limsup_{k\to\infty} \phi(N(x_{m_k},x_{n_k}))\\
&\quad + L \limsup_{k\to\infty} \phi(N_1(x_{m_k},x_{n_k})).
\end{aligned}
$$

By applying continuity of ψ and ϕ we have,

$$
\begin{aligned}
\psi(s^3 \liminf_{k\to\infty} d(x_{m_k+1},x_{n_k+1})) &= \psi(s^3 \liminf_{k\to\infty} d(Tx_{m_k},Tx_{n_k}))\\
&\leq \psi(\limsup_{k\to\infty} M(x_{m_k},x_{n_k})) - k\phi(\limsup_{k\to\infty} N(x_{m_k},x_{n_k}))\\
&\quad + L\phi(\limsup_{k\to\infty} N_1(x_{m_k},x_{n_k})).
\end{aligned}
$$

Taking the upper and lower limits as $k \to \infty$ and using (4.6) and lemma (4.1.1),

$$
\begin{aligned}
\limsup_{k\to\infty} M(x_{m_k},x_{n_k}) = \limsup_{k\to\infty} &\max\{d(x_{m_k},x_{n_k}),d(x_{m_k},x_{m_k+1}),d(x_{n_k},x_{n_k+1}),\\
&\frac{d(x_{m_k},x_{n_k+1})+d(x_{n_k},x_{m_k+1})}{2s}\},\\
&= \max\{s\varepsilon,0,0,\frac{s^2\varepsilon+s^2\varepsilon}{2s}\} = s\varepsilon. \qquad (4.9)
\end{aligned}
$$

$$\begin{aligned}\lim_{k\to\infty}\sup N(x_{m_k},x_{n_k})) &= \lim_{k\to\infty}\sup\max\{d(x_{m_k},x_{n_k}),d(x_{n_k},x_{n_k+1}),\\ &\quad \frac{d(x_{m_k},x_{m_k+1})d(x_{m_k},x_{n_k+1})+d(x_{n_k},x_{n_k+1})d(x_{n_k},x_{m_k+1})}{1+d(x_{m_k},x_{n_k+1})+d(x_{n_k},x_{m_k+1})},\\ &\quad \frac{d(x_{m_k},x_{m_k+1})d(x_{m_k},x_{n_k+1})+d(x_{n_k},x_{n_k+1})d(x_{n_k},x_{m_k+1})}{1+s[d(x_{m_k},x_{n_k})+d(x_{m_k+1},x_{n_k+1})]}\}\end{aligned}$$

$$= \max\{s\varepsilon,0,0,0\}=s\varepsilon. \tag{4.10}$$

$$\lim_{k\to\infty}\inf N_1(x_{m_k},x_{n_k}))=\lim_{k\to\infty}\inf\min\{d(x_{m_k},x_{n_k}),d(x_{m_k},x_{m_k+1}),d(x_{n_k},x_{n_k+1}),d(x_{m_k},x_{n_k+1}),$$

$$d(x_{n_k},x_{m_k+1})\},$$

$$= \min\{s\varepsilon,0,0,s^2\varepsilon,s^2\varepsilon\}=0. \tag{4.11}$$

From Lemma (4.1.1) we have,

$\frac{\varepsilon}{s^2}\leq \lim_{n\to\infty}\inf d(x_{m_k+1},x_{n_k+1})\leq \lim_{n\to\infty}\sup d(x_{m_k+1},x_{n_k+1})\leq s^3\varepsilon.$

Which implies

$$\psi(s\varepsilon)=\psi(s^3\frac{\varepsilon}{s^2})\leq \psi(s^3\lim_{k\to\infty}\inf d(x_{m_{k+1}},x_{n_{k+1}})). \tag{4.12}$$

Now using (4.9), (4.10), (4.11) (4.12) and (4.8) becomes;

$$\psi(s\varepsilon)\leq \psi(s\varepsilon)-k\phi(s\varepsilon)<\psi(s\varepsilon),$$

which is a contradiction.
Thus, $\{x_n\}$ is b-Cauchy.

Now completeness of X yields that $\{x_n\}$ *b*-converges to a point say $y\in X$ such that

$$\lim_{n\to\infty}x_n=y.$$

Now suppose that for every $n \geq 0$,we have

$$d(x_n, y) < \frac{1}{2s} d(x_n, x_{n+1}) \text{ and } d(x_{n+1}, y) < \frac{1}{2s} d(x_{n+1}, x_{n+2}).$$

From the triangle inequality we have,

$$\begin{aligned}
d(x_n, x_{n+1}) &\leq sd(x_n, y) + sd(y, x_{n+1}) \\
&< \frac{1}{2s} sd(x_n, x_{n+1}) + \frac{1}{2s} sd(x_{n+1}, x_{n+2}) \\
&\leq \frac{1}{2}[d(x_n, x_{n+1}) + d(x_n, x_{n+1})] \\
&= d(x_n, x_{n+1}),
\end{aligned}$$

which is a contradiction.
Hence, we have

$$\frac{1}{2s} d(x_n, x_{n+1}) \leq d(x_n, y) \text{ and } \frac{1}{2s} d(x_{n+1}, x_{n+2}) \leq d(x_{n+1}, y) \text{ holds for all } n \geq 0.$$

$$\begin{aligned}
\psi(d(x_{n+1}, Ty)) &\leq \psi(s^3 d(x_{n+1}, Ty)) \\
&= \psi(s^3 d(Tx_n, Ty)) \\
&\leq \psi(M(x_n, y)) - k\phi(N(x_n, y)) + L\phi(N_1(x_n, y)), \quad (4.13)
\end{aligned}$$

where

$$\begin{aligned}
M(x_n, y) &= \max\{d(x_n, y), d(x_n, Tx_n), d(y, Ty), \frac{d(x_n, Ty) + d(y, Tx_n)}{2s}\} \\
&= \max\{d(x_n, y), d(x_n, x_{n+1}), d(y, Ty), \frac{d(x_n, Ty) + d(y, x_{n+1})}{2s}\},
\end{aligned}$$

$$
\begin{aligned}
N(x_n,y) &= \max\{d(x_n,y),d(y,Ty),\frac{d(x_n,Tx_n)d(x_n,Ty)+d(y,Ty)d(y,Tx_n)}{1+d(x_n,Ty)+d(y,Tx_n)},\\
&\quad \frac{d(x_n,Tx_n)d(x_n,Ty)+d(y,Ty)d(y,Tx_n)}{1+s[d(x,y)+d(Tx_n,Ty)]}\}\\
&= \max\{d(x_n,y),d(y,Ty),\frac{d(x_n,x_{n+1})d(x_n,Ty)+d(y,Ty)d(y,x_{n+1})}{1+d(x_n,Ty)+d(y,x_{n+1})},\\
&\quad \frac{d(x_n,x_{n+1})d(x_n,Ty)+d(y,Ty)d(y,x_{n+1})}{1+s[d(x,y)+d(x_{n+1},Ty)]}\},
\end{aligned}
$$

$$
\begin{aligned}
N_1(x_n,y) &= \min\{d(x_n,y),d(x_n,Tx_n),d(y,Ty),d(x_n,Ty),d(y,Tx_n)\}\\
&= \min\{d(x_n,y),d(x_n,x_{n+1}),d(y,Ty),d(x_n,Ty),d(y,x_{n+1})\}.
\end{aligned}
$$

Taking upper limits as $n \to \infty$ in (4.13) , we get

$\psi(d(y,Ty)) \leq \psi(d(y,Ty)) - k\phi(d(y,Ty)) + L\phi(0) < \psi(d(y,Ty))$.
Which is a contradiction.
This shows that

$$Ty = y.$$

Hence, we proved that y is a fixed point of T.

Now we verify the uniqueness of the fixed point of T.
Suppose that T has two distinict fixed points $y,z \in X$ such that $Ty = y$ and $Tz = z$ $(y \neq z)$.

$$\frac{1}{2s}d(y,Ty) = 0 < d(y,z),$$

implies that

$$
\begin{aligned}
\psi(d(y,z)) &\leq \psi(s^3 d(y,z))\\
&= \psi(s^3 d(Ty,Tz))\\
&\leq \psi(M(y,z)) - k\phi(N(y,z)) + L\phi(N_1(y,z)),
\end{aligned}
$$

where

$$\begin{aligned} M(y,z) &= \max\{d(y,z), d(y,Ty), d(z,Tz), \frac{d(y,Tz)+d(z,Ty)}{2s}\} \\ &= \max\{d(y,z), d(y,y), d(z,z), \frac{d(y,z)+d(z,y)}{2s}\} \\ &= d(y,z). \end{aligned}$$

$$\begin{aligned} N(y,z) &= \max\{d(y,z), d(z,Tz), \frac{d(y,Ty)d(y,Tz)+d(z,Tz)d(z,Ty)}{1+d(y,Tz)+d(z,Ty)}, \\ &\quad \frac{d(y,Ty)d(y,Tz)+d(z,Tz)d(z,Ty)}{1+s[d(y,z)+d(Ty,Tz)]}\}, \\ &= \max\{d(y,z), d(z,z), \frac{d(y,y)d(y,z)+d(z,z)d(z,y)}{1+d(y,z)+d(z,y)} \\ &\quad \frac{d(y,y)d(y,z)+d(z,z)d(z,y)}{1+s[d(y,z)+d(y,z)]}\} \\ &= d(y,z). \end{aligned}$$

$$\begin{aligned} N_1(x_n,y) &= \min\{d(y,z), d(y,Ty), d(z,Tz), d(y,Tz), d(z,Ty)\} \\ &= \min\{d(y,z), d(y,y), d(z,z), d(y,z), d(z,y)\}, \\ &= 0. \end{aligned}$$

We obtain

$$\psi(d(y,z)) \leq \psi(d(y,z)) - k\phi(d(y,z)) < \psi(d(y,z)),$$

which is a contradiction.

Hence, $y = z$.
Therefore, the fixed point is unique. □

Now we give an example in support of theorem 4.2.1.

Example 4.2.1 *Let $X = [0,1]$ be endowed with the b-metric $d : X \times X \to \Re^+$ defined by;*

$$d(x,y) = \begin{cases} x^2 + y^2 + (x-y)^2 \text{ if } & x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

Then (X,d) is a b− complete b− metric space with s=2.
But(X,d) is not a metric space. For example if $x = 0$, $y = \frac{1}{3}$ andz=1,

d(0,1)$\not\leq d(0,\frac{1}{3}) + d(\frac{1}{3},1)$.
Define T:X→ X and $\phi,\psi : \Re^{+} \to \Re^{+}$ by;

$$T(x) = \begin{cases} 0 & \text{if } x \in [0,\frac{1}{5}], \\ \frac{x}{4} & \text{if } x \in (\frac{1}{5},1]. \end{cases}$$

$$\psi(t) = \frac{t}{8},$$

$$\phi(t) = \frac{t}{2}.$$

Then for any $k \in (0,\frac{1}{2}]$ and $L \geq 0$, T is generalized (ψ,ϕ)-Suzuki contraction type mapping.

Now we verify by the following cases.

case I: *Let $x,y \in [0,\frac{1}{5}]$ and $x \geq y$.*
a) *If $x = y$, then $\frac{1}{2s}d(x,Tx) = \frac{1}{4}d(x,0) = \frac{x^2}{2} \geq 0 = d(x,y)$.*

In this case, we have nothing to show.

b) *If $x > y$, then*

$$\begin{aligned} d(x,y) = x^2+y^2+(x-y)^2 &= 2x^2+2y^2-2xy \\ &\geq x^2+y^2 \\ &\geq \frac{(x^2+y^2)^2}{x^2+y^2} \\ &\geq \frac{x^4+y^4}{x^2+y^2} \\ &\geq \frac{x^4+y^4}{x^2+y^2+(x-y)^2} \\ &\geq \frac{2x^4+2y^4}{1+2[x^2+y^2+(x-y)^2]}. \end{aligned}$$

$$\frac{1}{2s}d(x,Tx) = \frac{1}{4}d(x,0) = \frac{x^2}{2} \leq 2x^2+2y^2-2xy = d(x,y).$$

Then
$d(Tx,Ty) = d(0,0) = 0,$
$d(x,Ty) = d(x,0) = 2x^2,$
$d(y,Tx) = d(y,0) = 2y^2,$
$d(y,Ty) = d(y,0) = 2y^2.$
So, we obtain

$$\begin{aligned} \psi(s^3 d(Tx,Ty)) = \psi(2^3(0)) = 0 &\leq \psi(M(x,y)) - k\phi(N(x,y)) + L\phi(N_1(x,y)) \\ &= \frac{x^2+y^2-xy}{4} - k(x^2+y^2-xy) + Ly^2. \end{aligned}$$

where

$$\begin{aligned} M(x,y) &= \max\{2x^2+2y^2-2xy, 2x^2, 2y^2, \frac{x^2+y^2}{2}\}, \\ &= 2x^2+2y^2-2xy. \end{aligned}$$

$$\begin{aligned} N(x,y) &= \max\{2x^2+2y^2-2xy, 2y^2, \frac{2x^4+2y^4}{1+2x^2+2y^2}, \frac{2x^4+2y^4}{1+2[2x^2+2y^2-2xy]}\}, \\ &= 2x^2+2y^2-2xy. \end{aligned}$$

$$\begin{aligned} N_1(x,y) &= \min\{2x^2+2y^2-2xy, 2x^2, 2y^2\}, \\ &= 2y^2. \end{aligned}$$

Therefore,
$\frac{1}{2s}d(x,Tx) \le d(x,y) \Rightarrow \psi(s^3 d(Tx,Ty)) \le \psi(M(x,y)) - k\phi(N(x,y)) + L\phi(N_1(x,y)).$

case II: *Let $x \in [0, \frac{1}{5}]$ and $y \in (\frac{1}{5}, 1]$. This implies $x < y$.*

$$\frac{1}{2s}d(x,Tx) = \frac{1}{4}d(x,0) = \frac{x^2}{2} \le 2x^2+2y^2-2xy = d(x,y).$$

Then
$d(Tx,Ty) = d(0,\frac{y}{4}) = \frac{y^2}{16} + \frac{y^2}{16} = \frac{y^2}{8},$

$d(x,Ty) = d(x,\frac{y}{4}) = \frac{16x^2+y^2-4xy}{8},$

$d(y,Tx) = d(y,0) = 2y^2,$

$d(y,Ty) = d(y,\frac{y}{4}) = \frac{13y^2}{8}.$

So we obtain,

$$\begin{aligned} \psi(s^3 d(Tx,Ty)) = \psi(2^3(\frac{y^2}{8})) = \frac{y^2}{8} &\le \psi(M(x,y)) - k\phi(N(x,y)) + L\phi(N_1(x,y)) \\ &= \frac{x^2+y^2-xy}{4} - k(x^2+y^2-xy) + L\frac{16x^2+y^2-4xy}{16}, \end{aligned}$$

where

$$\begin{aligned} M(x,y) &= \max\{2x^2+2y^2-2xy, \frac{13y^2}{8}, 2x^2, \frac{16x^2+17y^2-4xy}{32}\}, \\ &= 2x^2+2y^2-2xy. \end{aligned}$$

$$\begin{aligned}
N(x,y) &= \max\{2x^2+2y^2-2xy, \frac{13y^2}{8}, \frac{256x^4+26y^4 16(xy)^2-64x^3y}{16x^2+y^2-4xy+8}, \\
&\quad \frac{256x^4+26y^4+16(xy)^2-64x^3y}{32x^2+34y^2-32xy+8}\}, \\
&= 2x^2+2y^2-2xy.
\end{aligned}$$

$$\begin{aligned}
N_1(x,y) &= \min\{2x^2+2y^2-2xy, 2x^2, 2y^2, \frac{13y^2}{8}, \frac{16x^2+y^2-4xy}{8}\}, \\
&= \frac{16x^2+y^2-4xy}{8}.
\end{aligned}$$

Therefore,

$\frac{1}{2s}d(x,Tx) \leq d(x,y) \Rightarrow \psi(s^3 d(Tx,Ty)) \leq \psi(M(x,y)) - k\phi(N(x,y)) + L\phi(N_1(x,y)).$

case III: *Let* $x, y \in (\frac{1}{5}, 1]$ *and* $x \geq y$.

a) *If* $x = y$, *then* $\frac{1}{2s}d(x,Tx) = \frac{1}{4}d(x,\frac{x}{4}) = \frac{13x^2}{32} \geq 0 = d(x,y).$

In this case, we have nothing to show.

b) *If* $x > y$, *then* $\frac{1}{2s}d(x,Tx) = \frac{1}{4}d(x,\frac{x}{4}) = \frac{13x^2}{32} \leq 2x^2+2y^2-2xy = d(x,y).$

Then

$d(Tx,Ty) = d(\frac{x}{4},\frac{y}{4}) = \frac{x^2+y^2-xy}{8}$

$d(x,Ty) = d(x,\frac{y}{4}) = \frac{16x^2+y^2-4xy}{8}$

$d(y,Tx) = d(y,\frac{x}{4}) = \frac{x^2+16y^2-4xy}{8}$

$d(y,T_y) = d(y,\frac{y}{4}) = \frac{13y^2}{8}$

So we obtain,

$$\begin{aligned}
\psi(s^3 d(Tx,Ty)) = \psi(2^3(\frac{x^2+y^2-xy}{8})) &= \frac{x^2+y^2-xy}{8} \\
&\leq \psi(M(x,y)) - k\phi(N(x,y)) + L\phi(N_1(x,y)) \\
&= \frac{x^2+y^2-xy}{4} - k(x^2+y^2-xy) + L\frac{x^2+16y^2-4xy}{16}.
\end{aligned}$$

where

$$\begin{aligned} M(x,y) &= \max\{2x^2+2y^2-2xy, \frac{13x^2}{8}, \frac{13y^2}{8}, 2x^2, \frac{17x^2+17y^2-8xy}{32}\} \\ &= 2x^2+2y^2-2xy. \end{aligned}$$

$$\begin{aligned} N(x,y) &= \max\{2x^2+2y^2-2xy, \frac{13y^2}{8}, \frac{104x^4+104y^4+13(xy)^2-26x^3y-26y^3x}{68x^2+68y^2-32xy+32}, \\ &\quad \frac{104x^4+104y^4+13(xy)^2-26x^3y-26y^3x}{136x^2+136y^2-136xy+32}\} \\ &= 2x^2+2y^2-2xy. \end{aligned}$$

$$\begin{aligned} N_1(x,y) &= \min\{2x^2+2y^2-2xy, \frac{13x^2}{8}, \frac{13y^2}{8}, \frac{16x^2+y^2-4xy}{8}, \frac{16y^2+x^2-4xy}{8}\} \\ &= \frac{x^2+16y^2-4xy}{8}. \end{aligned}$$

Therefore,
$\frac{1}{2s}d(x,Tx) \leq d(x,y) \Rightarrow \psi(s^3 d(Tx,Ty)) \leq \psi(M(x,y)) - k\phi(N(x,y)) + L\phi(N_1(x,y))$.

case Iv: *Let* $x \in (\frac{1}{5}, 1]$ *and* $y \in [0, \frac{1}{5}]$. *This implies* $x > y$.

$$\frac{1}{2s}d(x,Tx) = \frac{1}{4}d(x,\frac{x}{4}) = \frac{13x^2}{32} \leq 2x^2+2y^2-2xy = d(x,y).$$

Then
$d(Tx,Ty) = d(\frac{x}{4},0) = \frac{x^2}{16} + \frac{x^2}{16} = \frac{x^2}{8}$

$d(x,Ty) = d(x,0) = 2x^2$
$d(y,Tx) = d(y,\frac{x}{4}) = \frac{16y^2+x^2-4xy}{8}$
$d(y,Ty) = d(y,0) = 2y^2$.
So we obtain,

$$\begin{aligned} \psi(s^3 d(Tx,Ty)) = \psi(2^3(\frac{y^2}{8})) = \frac{x^2}{8} &\leq \psi(M(x,y)) - k\phi(N(x,y)) + L\phi(N_1(x,y)) \\ &= \frac{x^2+y^2-xy}{4} - k(x^2+y^2-xy) + L(\frac{16y^2+x^2-4xy}{16}). \end{aligned}$$

where

$$M(x,y) = \max\{2x^2+2y^2-2xy, 2y^2, \frac{13x^2}{8}, \frac{17x^2+16y^2-4xy}{32}\}, = 2x^2+2y^2-2xy.$$

$$N(x,y) = \max\{2x^2+2y^2-2xy, 2y^2, \frac{26x^4+32y^4+2(xy)^2-8xy^3}{17x^2+16y^2-4xy+8}, \frac{13x^4+16y^4+(xy)^2-4xy^3}{17x^2+16y^2-32xy+4}\}, = 2x^2+2y^2-2xy.$$

$$N_1(x,y) = \min\{2x^2+2y^2-2xy, 2x^2, 2y^2, \frac{13x^2}{8}, \frac{16y^2+x^2-4xy}{8}\}, = \frac{16y^2+x^2-4xy}{8}.$$

Therefore,
$\frac{1}{2s}d(x,Tx) \leq d(x,y) \Rightarrow \psi(s^3 d(Tx,Ty)) \leq \psi(M(x,y)) - k\phi(N(x,y)) + L\phi(N_1(x,y)).$

Thus, all the conditions of Theorem (4.2.1) *are satisfied and* 0 *is the unique fixed point of* T.

In the following we give the definition of generalized (ψ,ϕ)-Jungck-Suzuki contraction type for a pair of maps and study coincidence points.

Definition 4.2.2 *Let* (X,d) *be a b-metric space,* Y *an arbitrary nonempty set and* $S,T : Y \to X$ *be two mappings. A mapping* S,T *are said to be* (ψ,ϕ)*-Jungck-Suzuki type mapping if for all* $x,y \in Y$,
$\frac{1}{2s}d(Sx,Tx) \leq d(Sx,Sy) \Rightarrow$

$$\psi(s^3 d(Tx,Ty)) \leq \psi(M(Sx,Sy) - k\phi(N(Sx,Sy)) + L\phi(N_1(Sx,Sy)). \qquad (4.14)$$

where,
$0 < k \leq \frac{1}{s}, L \geq 0,$
$M(Sx,Sy) = \max\{d(Sx,Sy), d(Sy,Ty), d(Sx,Tx), \frac{d(Sx,Ty)+d(Sy,Tx)}{2s}\},$

$$N(Sx,Sy) = \max\{d(Sx,Sy), d(Sy,Ty), \frac{d(Sx,Tx)d(Sx,Ty)+d(Sy,Ty)d(Sy,Tx)}{1+d(Sx,Ty)+d(Sy,Tx)},$$

$$\frac{d(Sx,Tx)d(Sx,Ty)+d(Sy,Ty)d(Sy,Tx)}{1+s[d(Sx,Sy)+d(Tx,Ty)]}\},$$

$N_1(Sx,Sy) = \min\{d(Sx,Sy), d(Sx,Tx), d(Sy,Ty), d(Sx,Ty), d(Sy,Ty)\}$
and ψ,ϕ are altering distance functions.

Theorem 4.2.2 *Let (X,d) be a b- complete b-metric space, $s \geq 1$ and $S,T : Y \to X$ is (ψ,ϕ)-Jungck-Suzuki type mapping such that $T(Y) \subseteq S(Y)$ and $S(Y)$is a b-complete subspace of X, then T and S have a coincidence point.*

proof For every $x_0 \in Y$,there exists $x_1 \in Y$ such that $Sx_1 = Tx_0$, since $T(Y) \subseteq S(Y)$. Using this fact, for any $x_{n-1} \in Y$, there exists x_n such that $Sx_n = Tx_{n-1}$ for all $n \in N$. Now observe that

$$\begin{aligned} \frac{1}{2s}d(Sx_{n-1},Tx_{n-1}) &= \frac{1}{2s}d(Sx_{n-1},Sx_n) \\ &\leq d(Sx_{n-1},Sx_n) \end{aligned}$$

which implies

$$\begin{aligned} \psi(s^3 d(Sx_n,Sx_{n+1})) &= \psi(s^3 d(Tx_{n-1},Tx_n)) \\ &\leq \psi(M(Sx_{n-1},Sx_n)) - k\phi(N(Sx_{n-1},Sx_n)) + L\phi(N_1(Sx_{n-1},Sx_n)). \end{aligned}$$

where

$$\begin{aligned} M(Sx_{n-1},Sx_n) &= \max\{d(Sx_{n-1},Sx_n), d(Sx_{n-1},Tx_{n-1}), d(Sx_n,Tx_n), \frac{d(Sx_{n-1},Tx_n)+d(Sx_n,Tx_{n-1})}{2s}\} \\ &= \max\{d(Sx_{n-1},Sx_n), d(Sx_{n-1},Sx_n), d(Sx_n,Sx_{n+1}), \frac{d(Sx_{n-1},Sx_{n+1})+d(Sx_n,Sx_n)}{2s}\} \\ &= \max\{d(Sx_{n-1},Sx_n), d(Sx_n,Sx_{n+1}),\}. \end{aligned}$$

$$
\begin{aligned}
N(Sx_{n-1},Sx_n) &= \max\{d(Sx_{n-1},Sx_n),d(Sx_n,Tx_n),\\
&\frac{d(Sx_{n-1},Tx_{n-1})d(Sx_{n-1},Tx_n)+d(Sx_n,Tx_n)d(Sx_n,Tx_{n-1})}{1+d(Sx_{n-1},Tx_n)+d(Sx_n,Tx_{n-1})},\\
&\frac{d(Sx_{n-1},Tx_{n-1})d(Sx_{n-1},Tx_n)+d(Sx_n,Tx_n)d(Sx_n,Tx_{n-1})}{1+s[d(Sx_{n-1},Sx_n)+d(Tx_{n-1},Tx_n)]}\},\\
&= \max\{d(Sx_{n-1},Sx_n),d(Sx_n,Sx_{n+1}),\\
&\frac{d(Sx_{n-1},Sx_n)d(Sx_{n-1},Sx_{n+1})+d(Sx_n,Sx_{n+1})d(Sx_n,Sx_n)}{1+d(Sx_{n-1},Sx_{n+1})+d(Sx_n,Sx_n)},\\
&\frac{d(Sx_{n-1},Sx_n)d(Sx_{n-1},Sx_{n+1})+d(Sx_n,Sx_{n+1})d(Sx_n,Sx_n)}{1+s[d(Sx_{n-1},Sx_n)+d(Sx_n,Sx_{n+1})]}\},\\
&= \max\{d(Sx_{n-1},Sx_n),d(Sx_n,Sx_{n+1})\}.
\end{aligned}
$$

$$
\begin{aligned}
N_1(Sx_{n-1},Sx_n) &= \min\{d(Sx_{n-1},Sx_n),d(Sx_{n-1},Tx_{n-1}),d(Sx_n,Tx_n),d(Sx_{n-1},Tx_n),d(Sx_n,Tx_{n-1})\}\\
&= \min\{d(Sx_{n-1},Sx_n),d(Sx_n,Sx_{n+1}),d(Sx_{n-1},Sx_{n+1}),d(Sx_n,Sx_n)\},\\
&= 0.
\end{aligned}
$$

Using similar line of proof as in Theorem(4.2.1), we can show that
$\max\{d(Sx_{n-1},Sx_n),d(Sx_n,Sx_{n+1})\} = d(Sx_{n-1},Sx_n)$
and for all $n \geq 0$,

$$d(Sx_n,Sx_{n+1}) \leq d(Sx_{n-1},Sx_n). \tag{4.15}$$

Hence, (4.14) becomes

$$\psi(s^3 d(Sx_n,Sx_{n+1})) \leq \psi(d(Sx_{n-1},Sx_n)) - k\phi(d(Sx_{n-1},Sx_n)). \tag{4.16}$$

From (4.15), we have that $d(Sx_n,Sx_{n+1})$ is a non increasing sequence.
Thus for all $n \geq 0$ there exists $r \geq 0$ such that

$$\lim_{n\to\infty} d(Sx_n,Sx_{n+1}) = r. \tag{4.17}$$

Assume that $r > 0$,taking the limits as $n \to \infty$ in (4.16), we have

$$\psi(r) \leq \psi(r) - k\phi(r) < \psi(r).$$

Which is a contradiction.
Hence, for all $n \geq 0$

$$\lim_{n\to\infty} d(Sx_n, Sx_{n+1}) = 0. \tag{4.18}$$

We now show that sequence $\{Sx_n\}$ is a b-Cauchy sequence in X.Assume on contrary that the sequence $\{Sx_n\}$ is not a b-Cauchy.By Lemma (4.1.1) there exists $\varepsilon > 0$ for which we can find two subsequences $\{Sx_{n_k}\}$ and $\{Sx_{m_k}\}$ of $\{Sx_n\}$ with $n_k > m_k > k$ such that
$d(Sx_{m_k}, Sx_{n_k}) \geq \varepsilon$ implies $d(Sx_{m_k}, Sx_{n_k-1}) < \varepsilon$,and (a)-(d) of lemma (4.1.1) hold.
By setting $x = Sx_{m_k}$ and $y = Sx_{n_k}$ we have;

$$\begin{aligned} \frac{1}{2s}d(Sx_{m_k}, Tx_{m_k}) &= \frac{1}{2s}d(Sx_{m_k}, Sx_{m_{k+1}}) \\ &\leq d(Sx_{m_k}, Sx_{n_k}), \end{aligned}$$

which implies
$\psi(s^3 d(Sx_{m_{k+1}}, Sx_{n_{k+1}})) = \psi(s^3 d(Tx_{m_k}, Tx_{n_k}))$

$$\leq \psi(M(Sx_{m_k}, Sx_{n_k})) - k\phi(N(Sx_{m_k}, Sx_{n_k})) + L\phi(N_1(Sx_{m_k}, Sx_{n_k})), \tag{4.19}$$

where,

$$\begin{aligned} M(Sx_{m_k}, Sx_{n_k}) &= \max\{d(Sx_{m_k}, Sx_{n_k}), d(Sx_{m_k}, Tx_{m_k}), d(Sx_{n_k}, Tx_{n_k}), \frac{d(Sx_{m_k}, Tx_{n_k}) + d(Sx_{n_k}, Tx_{m_k})}{2s}\} \\ &= \max\{d(Sx_{m_k}, Sx_{n_k}), d(Sx_{m_k}, Sx_{m_{k+1}}), d(Sx_{n_k}, Sx_{n_{k+1}}), \\ &\qquad \frac{d(Sx_{m_k}, Sx_{n_{k+1}}) + d(Sx_{n_k}, Sx_{m_{k+1}})}{2s}\}. \end{aligned}$$

$$\begin{aligned}N(Sx_{m_k},Sx_{n_k}) &= \max\{d(Sx_{m_k},Sx_{n_k}),d(Sx_{n_k},Tx_{n_k}),\\&\frac{d(Sx_{m_k},Tx_{m_k})d(Sx_{m_k},Tx_{n_k})+d(Sx_{n_k},Tx_{n_k})d(Sx_{n_k},Tx_{m_k})}{1+d(Sx_{m_k},Tx_{n_k})+d(Sx_{n_k},Tx_{m_k})},\\&\frac{d(Sx_{m_k},Tx_{m_k})d(Sx_{m_k},Tx_{n_k})+d(Sx_{n_k},Tx_{n_k})d(Sx_{n_k},Tx_{m_k})}{1+s[d(Sx_{m_k},Sx_{n_k})+d(Tx_{m_k},Tx_{n_k})]}\}\\&= \max\{d(Sx_{m_k},Sx_{n_k}),d(Sx_{n_k},Sx_{n_{k+1}}),\\&\frac{d(Sx_{m_k},Sx_{m_{k+1}})d(Sx_{m_k},Sx_{n_{k+1}})+d(Sx_{n_k},Sx_{n_{k+1}})d(Sx_{n_k},Sx_{m_{k+1}})}{1+d(Sx_{m_k},Sx_{n_{k+1}})+d(Sx_{n_k},Sx_{m_{k+1}})},\\&\frac{d(Sx_{m_k},Sx_{m_{k+1}})d(Sx_{m_k},Sx_{n_{k+1}})+d(Sx_{n_k},Sx_{n_{k+1}})d(Sx_{n_k},Sx_{m_{k+1}})}{1+s[d(Sx_{m_k},Sx_{n_k})+d(Sx_{m_{k+1}},Sx_{n_{k+1}})]}\}.\end{aligned}$$

$$\begin{aligned}N_1(Sx_{m_k},Sx_{n_k}) &= \min\{d(Sx_{m_k},Sx_{n_k}),d(Sx_{m_k},Tx_{m_k}),d(Sx_{n_k},Tx_{n_k}),d(Sx_{m_k},Tx_{n_k}),d(Sx_{n_k},Tx_{m_k})\}\\&= \min\{d(Sx_{m_k},Sx_{n_k}),d(Sx_{m_k},Sx_{m_{k+1}}),d(Sx_{n_k},Sx_{n_{k+1}}),d(Sx_{m_k},Sx_{n_{k+1}}),d(Sx_{n_k},Sx_{m_{k+1}})\}.\end{aligned}$$

Applying the upper and lower limits as $k \to \infty$, (4.19) becomes

$$\begin{aligned}\liminf_{k\to\infty} \psi(s^3 d(Sx_{m_{k+1}},Sx_{n_{k+1}})) &= \liminf_{k\to\infty} \psi(s^3 d(Tx_{m_k},Tx_{n_k}))\\&\leq \limsup_{k\to\infty} \psi(M(Sx_{m_k},Sx_{n_k})) - k\limsup_{k\to\infty} \phi(N(Sx_{m_k},Sx_{n_k}))\\&\quad +L\limsup_{k\to\infty} \phi(N_1(Sx_{m_k},Sx_{n_k})).\end{aligned}$$

By taking the upper and lower limits as $k \to \infty$ and using lemma (4.1.1) we get,

$$\begin{aligned}\psi(s\varepsilon) &= \psi(s^3\frac{\varepsilon}{s^2})\\&\leq \psi(s^3\liminf_{k\to\infty} d(Sx_{m_{k+1}},Sx_{n_{k+1}})),\end{aligned}$$

which implies that

$$\psi(s\varepsilon) \leq \psi(s\varepsilon) - k\phi(s\varepsilon) < \psi(s\varepsilon),$$

which is a contradiction.
Hence, $\{Sx_n\}$ is b- Cauchy.
Since $S(Y)$ is a b-complete, then there exists say $x \in$S(Y) such that

$$\lim_{n\to\infty} Sx_n = x.$$

More so, we can find $y \in Y$ such that $Sy = x$.
Now, suppose that for every $n \in N$, we have
$d(Sx_{n-1}, y) < \frac{1}{2s} d(Sx_{n-1}, Sx_n)$ and $d(Sx_n, y) < \frac{1}{2s} d(Sx_n, Sx_{n+1})$.
Now, observe that

$$\begin{aligned} d(Sx_{n-1}, Sx_n) &\leq sd(Sx_{n-1}, y) + sd(y, Sx_n) \\ &< \frac{1}{2s} sd(Sx_{n-1}, Sx_n) + \frac{1}{2s} sd(Sx_n, Sx_{n+1}) \\ &\leq \frac{1}{2}[d(Sx_{n-1}, Sx_n) + d(Sx_{n-1}, Sx_n)] \\ &= d(Sx_{n-1}, Sx_n), \end{aligned}$$

which is a contradiction.
Hence, we have
$\frac{1}{2s} d(Sx_{n-1}, Sx_n) \leq d(Sx_{n-1}, y)$ and $\frac{1}{2s} d(Sx_n, Sx_{n+1}) \leq d(Sx_n, y)$ for all $n \in N$.

It then follows that

$$\begin{aligned} \frac{1}{2s} d(Sx_{n-1}, Tx_{n-1}) &= \frac{1}{2s} d(Sx_{n-1}, Sx_n) \\ &\leq d(Sx_{n-1}, y), \end{aligned}$$

which implies
$\psi(s^3 d(Sx_n, Ty)) = \psi(s^3 d(Tx_{n-1}, Ty))$

$$\leq \psi(M(Sx_{n-1}, S_y)) - k\phi(N(Sx_{n-1}, S_y)) + L\phi(N_1(Sx_{n-1}, S_y)). \tag{4.20}$$

where
$M(Sx_{n-1}, Sy) = \max\{d(Sx_{n-1}, Sy), d(Sx_{n-1}, Tx_{n-1}), d(Sy, Ty), \frac{d(Sx_{n-1}, Ty) + d(Sy, Tx_{n-1})}{2s}\}$,

$$N(Sx_{n-1}, Sy) = \max\{d(Sx_{n-1}, Sy), d(Sy, Ty), \frac{d(Sx_{n-1}, Tx_{n-1}) d(Sx_{n-1}, Ty) + d(Sy, Ty) d(Sy, Tx_{n-1})}{1 + d(Sx_{n-1}, Ty) + d(Sy, Tx_{n-1})},$$

$$\frac{d(Sx_{n-1}, Tx_{n-1}) d(Sx_{n-1}, Ty) + d(Sy, Ty) d(Sy, Tx_{n-1})}{1 + s[d(Sx_{n-1}, Sy) + d(Tx_{n-1}, Ty)]}\},$$

$N_1(Sx_{n-1}, Sy) = \min\{d(Sx_{n-1}, Sy), d(Sx_{n-1}, Tx_{n-1}), d(Sy, Ty), d(Sx_{n-1}, Ty), d(Sy, Tx_{n-1})\}$.

Taking the limits as $n \to \infty$ in (4.20), we get

$$\psi(s^3 d(x,Ty)) \leq \psi(d(x,Ty)) - k\phi(d(x,Ty)),$$

which implies

$$\psi(d(x,Ty)) \leq \psi(d(x,Ty)) - k\phi(d(x,Ty)) < \psi(d(x,Ty)),$$

which is a contradiction.
This implies that $x = Ty$.
It follows that
$\lim_{n\to\infty} d(Sx_n, Ty) = 0$ and $\lim_{n\to\infty} d(Sx_n, Sy) = 0$.
Thus, we have $x = Sy = Ty$,which is a coincidence point of T and S. □

Now we give an example in support of Theorem 4.2.2.

Example 4.2.2 *Let $X = [0,1]$ and defined $d : X \times X \to \Re^+$ as follows*

$$d(x,y) = \begin{cases} 0 & \text{if } x = y, \\ (x+y)^2 & \text{if } x \neq y. \end{cases}$$

Then(X,d) is a b− complete b− metric space with s=2.

LetS, T:X→ X and $\phi, \psi : \Re^+ \to \Re^+$ be defined by ;

$$T(x) = \begin{cases} 0 & \text{if } x \in [0, \frac{1}{2}) \\ \frac{x^2}{4} & \text{if } x \in [\frac{1}{2}, 1], \end{cases}$$

$$Sx = x^2,$$

$$\psi(t) = \frac{t}{2},$$

$$\phi(t) = \frac{t}{4}.$$

Then for any $k \in (0, \frac{1}{2}]$ and $L \geq 0$,S,T is generalized (ψ,ϕ)-Jungck-Suzuki contraction type mappings.
To verify we consider the following cases.

Case I: *Let* $x,y \in [0,\frac{1}{2})$ *and* $x \geq y$.
$\frac{1}{2s}d(Sx,Tx) = \frac{1}{4}d(x^2,0) = \frac{x^4}{4} \leq (x^2+y^2)^2 = d(Sx,Sy)$,

and
$d(Tx,Ty) = d(0,0) = 0$,

$d(Sx,Ty) = d(x^2,0) = x^4$,

$d(Sy,Tx) = d(y^2,0) = y^4$,

$d(Sy,Ty) = d(y^2,0) = y^4$.

Therefore, we have

$M(Sx,Sy) = (x^2+y^2)^2$,

$N(Sx,Sy) = (x^2+y^2)^2$,

$N_1(Sx,Sy) = y^4$,

and

$$\begin{aligned}
\psi(s^3 d(Tx,Ty)) &\leq \psi(M(Sx,Sy)) - k\phi(N(Sx,Sy)) + L\phi(N_1(Sx,Sy)), \\
0 &\leq \frac{(x^2+y^2)^2}{2} - k\frac{(x^2+y^2)^2}{4} + L\frac{y^4}{4}.
\end{aligned}$$

Thus, we have

$$\frac{1}{2s}d(Sx,Tx) \leq d(Sx,Sy) \Rightarrow \psi(s^3 d(Tx,Ty)) \leq \psi(M(Sx,Sy)) - k\phi(N(Sx,Sy)) + L\phi(N_1(Sx,Sy)).$$

Case II: *Let* $x \in [0,\frac{1}{2})$ *and* $y \in [\frac{1}{2},1]$.
$\frac{1}{2s}d(Sx,Tx) = \frac{1}{4}d(x^2,0) = \frac{x^4}{4} \leq (x^2+y^2)^2 = d(Sx,Sy)$,

and

$$d(Tx,Ty) = d(0,\frac{y^2}{4}) = \frac{y^4}{16},$$

$$d(Sx,Ty) = d(x^2,\frac{y^2}{4}) = \frac{(4x^2+y^2)^2}{16},$$

$$d(Sy,Tx) = d(y^2,0) = y^4,$$

$$d(Sy,Ty) = d(y^2,\frac{y^2}{4}) = \frac{25y^4}{16}.$$

Therefore, we have

$$M(Sx,Sy) = \frac{25y^4}{16},$$

$$N(Sx,Sy) = \frac{25y^4}{16},$$

$$N_1(Sx,Sy) = x^4,$$

and

$$\begin{aligned} \psi(s^3 d(Tx,Ty)) &\leq \psi(M(Sx,Sy)) - k\phi(N(Sx,Sy)) + L\phi(N_1(Sx,Sy)), \\ \frac{y^4}{4} &\leq \frac{(25y^4)}{32} - k\frac{(25y^4)}{64} + L\frac{x^4}{4}. \end{aligned}$$

Thus, we have

$$\frac{1}{2s}d(Sx,Tx) \leq d(Sx,Sy) \Rightarrow \psi(s^3 d(Tx,Ty)) \leq \psi(M(Sx,Sy)) - k\phi(N(Sx,Sy)) + L\phi(N_1(Sx,Sy)).$$

Case III: *Let* $x,y \in [\frac{1}{2},1]$ *and* $x \geq y$.

$$\frac{1}{2s}d(Sx,Tx) = \frac{1}{4}d(x^2,\frac{x^2}{4}) = \frac{25x^4}{64} \leq (x^2+y^2)^2 = d(Sx,Sy),$$

and

$$d(Tx,Ty) = d(\frac{x^2}{4},\frac{y^2}{4}) = \frac{(x^2+y^2)^2}{16},$$

$$d(Sx,Ty) = d(x^2,\frac{y^2}{4}) = \frac{(4x^2+y^2)^2}{16},$$

$d(Sy,Tx) = d(y^2, \frac{x^2}{4}) = \frac{(4y^2+x^2)^2}{16},$

$d(Sy,Ty) = d(y^2, \frac{y^2}{4}) = \frac{25y^4}{16}.$

Therefore, we have

$M(Sx,Sy) = (x^2+y^2)^2,$

$N(Sx,Sy) = (x^2+y^2)^2,$

$N_1(Sx,Sy) = \frac{25y^4}{16},$

and

$$\begin{aligned} \psi(s^3 d(Tx,Ty)) &\leq \psi(M(Sx,Sy)) - k\phi(N(Sx,Sy)) + L\phi(N_1(Sx,Sy)), \\ \frac{(x^2+y^2)^2}{4} &\leq \frac{(x^2+y^2)^2}{2} - k\frac{(x^2+y^2)^2}{4} + L\frac{25y^4}{64}. \end{aligned}$$

Thus, we have

$$\frac{1}{2s}d(Sx,Tx) \leq d(Sx,Sy) \Rightarrow \psi(s^3 d(Tx,Ty)) \leq \psi(M(Sx,Sy)) - k\phi(N(Sx,Sy)) + L\phi(N_1(Sx,Sy)).$$

Hence, from all these cases S,T satisfies all hypothesis of Theorem (4.2.2) and 0 *is unique point of coincidence of S and T.*

The following Corollaries follow the main result.

Corollary 4.1: Let (X,d) be a b-metric space , $s \geq 1$ and a mapping $T : X \to X$ is generalized (ψ,ϕ)-Suzuki contraction type mapping. If for all $x,y \in X$,

$$\frac{1}{2s}d(x,Tx) \leq d(x,y) \Rightarrow \psi(s^3 d(Tx,T_y)) \leq \psi(M(x,y)) - k\phi(N(x,y)) + L\phi(N_1(x,y)),$$

where

$0 < k \leq \frac{1}{s}, L \geq 0$ and ψ,ϕ are altering distance functions,

$M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty)\}$,
$N(x,y) = \max\{d(x,y), d(y,Ty)\}$,
$N_1(x,y) = \min\{d(x,Ty), d(x,Tx), d(y,Tx)\}$ and ψ, ϕ altering distance functions.
Then T has unique fixed point.

Remark 4.1 *The proof of Corollary 4.1 is in line with the proof of Theorem 4.2.1 and it is equivalent to the result of (Mebawondu and Mebawondu, 2021) when* $s = 1$.

Corollary 4.2: Let (X,d) be a b-metric space , $s \geq 1$ and a mapping $T : X \to X$ is generalized (ψ, ϕ)-Suzuki contraction type mapping. If for all $x, y \in X$,

$$\frac{1}{2s}d(x,Tx) \leq d(x,y) \Rightarrow \psi(s^3 d(Tx,T_y)) \leq \psi(M(x,y)) - k\phi(N(x,y)).$$

where

$$0 < k \leq \frac{1}{s}, L \geq 0, \psi, \phi \in \Psi,$$

$$M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2s}\},$$

$$N(x,y) = \max\{d(x,y), d(y,Ty), \frac{d(x,Tx)d(x,Ty)+d(y,Ty)d(y,Tx)}{1+d(x,Ty)+d(y,Tx)},$$
$$\frac{d(x,Tx)d(x,Ty)+d(y,Ty)d(y,Tx)}{1+s[d(x,y)+d(Tx,Ty)}]\},$$ and ψ, ϕ are altering distance functions.
Then T has unique fixed point.

Remark 4.2 *Corollary 4.2 is equivalent to Theorem 4.2.1 when* $L = 0$.

Chapter 5

Conclusion and Future scope

5.1 Conclusion

Mebawondu and Mebawondu, (2021) established fixed point theorems for generalized Suzuki (ψ,ϕ)-type mapping and generalized (ψ,ϕ)-Jungck-Suzuki contraction type mappings in complete metric spaces and proved the existence, uniqueness and coincidence results of fixed points.In this research work,we introduced fixed point results for generalized Suzuki (ψ,ϕ)-contraction and coincidence point result for generalized (ψ,ϕ)-Jungck-Suzuki contraction type mappings in the context of complete b-metric spaces and proved the existence and uniqueness of fixed point for the mappings introduced. Our results extend and generalize related fixed point results in the literature in particular that of Mebawondu and Mebawondu, (2021). We have also supported the main results of this research work by applicable examples.

5.2 Future scope

There are some published results related to the existence of fixed point theorems of mappings defined on b-metric spaces. The researcher believe that the search for the existence and uniqueness of fixed points of self-mappings satisfying generalized Suzuki (ψ,ϕ)- type contraction in b-metric spaces is an active area of study. So, any interested researchers can use this opportunity and conduct their research work in this area.

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