EXISTENCE OF POSITIVE SOLUTIONS FOR NONLINEAR SECOND ORDER TWO POINT INTEGRAL BOUNDARY VALUE PROBLEM VIA NOVEL FIXED POINT THEOREM

JIMMA UNIVERSITY COLLEGE OF NATURAL SCIENCES

DEPARTMENT OF MATHEMATICS

A thesis submitted to the department of mathematics, college of natural sciences in partial fulfillment for the requirements of the degree of masters of science in mathematics.

(Differential equation)

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Declaration

I, undersigned, declare that ”Existence of positive solutions of nonlinear second order two point integral boundary value problem via novel fixed point theorem” is original and it has not been submitted to any institution elsewhere for the award of any academic degree or like and that all the sources I have used or quoted have been indicated and acknowledged as complete references.

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Contents

Abstract

This thesis is concerned with existence of positive solutions for second order integral boundary value problem. It also presents the construction of Green's function for corrosponding non-trivial homogeneous equation by using its properties. Under the suitable conditions, we established the existence of at least one positive solution by applying Novel fixed point theorem. We provided example to demonstrate for the applicability of our main result.

Chapter 1

Introduction

1.1 Background of the study

Boundary value problems associated with linear as well as non- linear ordinary differential equations or finite difference equations have created a great deal of interest and play an important role in many fields of applied mathematics such as engineering and manufacturing. Major industries like automobile, aerospace, chemical, pharmaceutical, petroleum, electronics and communication as well as engineering technologies like biotechnology and nanotechnology rely on the boundary value problems to simulate complex phenomena at different scales for designing and manufacturing of high technological products. In these applied settings, positive solutions are meaningful (Zill and Cullen, 2001).

A boundary value problem consists in finding an unknown solution which satisfies an ordinary differential equation and appropriate boundary conditions at two or more point. This problem play a very important role for ordinary differential equations in both theory and applications. They are used to describe a large number of physical, biological and chemical phenomena (Tyn Myint - U, 1978).

In the field of differential equations, a boundary value problem is a differential equation together with a set of additional constraints, called the boundary conditions. A solution to a boundary value problem is a solution to the differential equation witch also satisfies the boundary conditions. The theory of boundary value problems with integral boundary conditions for ordinary differential equations arises in different areas of mathematics and physics. For example, a class of boundary value problems with integral boundary conditions arises naturally in thermal conduction problems by Canon(1963), semiconductor problems by Ionkin(1977), hydrodynamics problems by Chegis(1984).

A set C is a cone in real Banach space R if it is closed, nonvoid, the sum of two numbers of C is a number of C, and non- negative scalar multiples of members of C are members of bounded linear functional which are non-negative on C, i.e., $f : f \in R^*$ and $f(c) \geq 0$ for $c \in C$. (Clearly C^1 is a cone in R^* (Kelley and Vaught, 1953).

Fixed point theory is one of the most power full and fruitful tools of modern mathematics and may be considered a cone subject of nonlinear analysis (Guo D.Lakshmikantham, 1988 and AmannH., 1976). Fixed point theory is associated with many celebrated mathematics like Cauchy, Fredholm, Liouville, Lipschitz, Peano and Picard.

A fixed point of a function F is a point p such that $F(p) = p$. One of the most frequently used tools for proving the existence of positive solutions to the integral equations and boundary value problems is the fixed point index theory (AmannH., 1976). The index i has the following properties, let K be a closed convex set in a Banach space E and D be a bounded open set such that :$D_k := D \bigcap K \neq \emptyset$. Let $D_k \longrightarrow K$ be a compact map. Suppose that $X \neq T_x$ for all $x \in \partial D_k$, then

(Existence): If $i(T, D_k, K) \neq 0$, then T has a fixed point in D_k.

(Normalization): If $u \in D_k$, then $i(u, D_k, K) = 1$, where $u(x) = u$ for $x \in D_k$.

(Homotopy): Let $h : [0,1] \times D_k \longrightarrow K$ be a compact map such that $x \neq h(t,x)$ for $x \in \partial D_k$ and $t \in [0,1]$. Then $i(h(0,.), D_k) = i(h(1,.), D_k, K)$.

A large class of boundary value problems that occur often in physical science consists of the second order equation with boundary condition of the type:

$$y'' = f(x, y, y\prime; a < x < b),$$
$$u_1[y] = a_1 y(a) + a_2 y'(a) = \alpha,$$
$$u_2[y] = b_1 y(b) + b_2 y'(b) = \beta,$$

where $a_1, a_2, b_1, b_2, \alpha$ and β are constants. The existence and uniqueness of solutions to this problem are treated by (Tyn Myint, 1978).

Some theories such as the Krasnoselskiis fixed point theorem, the Leggett-Williams fixed point theorem, Avery generalization of the Leggett-Williams fixed point theorem and Avery Henderson fixed point theorem have given decisive impetus for the development of the modern theory of differential equations. The advantage of these techniques lies in that they do not demand the knowledge of solution, but has a great power in application in finding positive solutions, multiple positive solutions, and Eigen-value interval for which there exists one or more positive solutions (Henderson and Wang, 1997).

In analyzing nonlinear phenomena many mathematical models given rise to problems for which only positive solutions make sense. Therefore, since the publication of the monograph positive solutions of operator equations in the year 1964 by academician M.A. Krasnoselskii, hundreds of research articles on the theory of positive solutions of nonlinear problems have appeared (Krasnoselskii, M.A., 1964). The fixed point theorem concern maps f of a set X in to itself that is, a point $x \in X$ such that $f(x) = x$ (Mohammed and Bashir, 2016).

The knowledge of the existence of fixed point has relevant applications in many branches of analysis and topology. Most results so far have been obtained mainly by using other fixed point theorems in cones, such as the Guo-Krasnoselskiis fixed point theorems (Krasnoselskii, M.A., 1964), the LeggettWilliams theorems (1997), Avery and Hendersons theorem (2001), and so on.

The aim of this study is to examine the existence of positive solution for second order two point integral boundary value problems using Novel fixed point theorem.

1.2 Statement of the problem

The study of existence of positive solution for boundary value problems with different types of boundary condition is vital in various fields of study. Due to this it attracts the attention of many researchers. For instance Ruyun Ma (2001): studied the existence of positive solutions second order ordinary differential equations:

$$u''(t)+a(t)f(u(t)) = 0, 0 < t < 1.$$
$$u(0) = 0, u(1) = \int_\alpha^\beta h(t)u(t)dt.$$

Khan (2003): considered the existence of solution using the method of quasi linearization for the nonlinear boundary value problem with integral boundary conditions:

$$x''(t) = f(t, x), t \in [0, 1]$$
$$x(0) - k_1x'(0) = \int_0^1 h_1(x(s))ds$$
$$x(1) + k_2x'(1) = \int_0^1 h_2(x(s))ds.$$

Zhilin Yang (2006): investigated the existence of positive solutions of second order integral boundary value problems for ordinary differential equations:

$$-(au)' + bu = f(t, u), 0 < t < 1$$
$$\cos\gamma_0u(0) - \sin\gamma_0u'(0) = H_1(\int_0^1 u(\tau)d\alpha(\tau)),$$
$$\cos\gamma_1u(1) + \sin\gamma_1u'(1) = H_2(\int_0^1 u(\tau)d\beta(\tau)).$$

Tariboon and Sittiwirattham (2010): studied the existence of positive solutions of a nonlinear three point integral boundary value problem:

$$u''(t) + a(t)f(u(t)) = 0$$
$$u(0) = 0, a \int_0^\eta u(s)ds = u(1), \eta \in (0, 1).$$

Zhang and Fei Sun (2010): studied the existence of solutions for second order integral boundary value problem:

$$-x''(t) = f(t, x(t)) = 0, 0 < t < 1,$$
$$x(0) = 0, x(1) = \int_0^1 a(s)x(s)ds.$$

Mouffak Benchohra et al (2011): studied the existence of solutions for second order boundary value problem:

$$-y''(t) = f(t, y(t)), 0 < t < 1,$$
$$y(0) = 0, y(1) = \int_0^1 g(s)y(s)ds.$$

Yao (2015): investigated the existence of positive solutions on $C[0, \gamma]$, for the boundary value problem:

$$u''(t) + a(t)f(t) = 0, 0 < t < 1,$$
$$u(0) = 0, \alpha \int_0^\eta u(s)ds = u(\gamma), \eta \in (0, \gamma).$$

Qiuyan and Xingqiu (2017): studied the existence of two positive solutions for the nonlinear second order differential equation involving integral boundary value problem:

$$u''(t) + a(t)u'(t) + b(t)u(t) + f(t, u(t)) = 0, 0 < t < 1$$
$$u(0) = \int_0^1 g(s)u(s)ds, u(1) = \int_0^1 h(s)u(s)ds.$$

However, to the best of the researchers knowledge no study has been conducted on the examination of existence of positive solution for two point integral boundary value problem using

Novel fixed point theorem. Hence, the present study is aimed to investigate the existence of positive solution for two point integral boundary value problem:

$$-u''(t) + k^2u(t) = f(t, u(t)), 0 < t < 1 \tag{1.1}$$

$$\begin{aligned} \alpha u(0) - \beta u'(0) &= \int_0^1 h_1(t)u(t)dt, \\ \gamma u(1) + \delta u'(1) &= \int_0^1 h_2(t)u(t)dt, \end{aligned} \tag{1.2}$$

where $\alpha, \gamma \geq 0, \beta, \delta > 0, k > 0, f(t, u(t)) \in [0, 1] \times R^*$ and $h_1(t), h_2(t) \in L^1[0, 1]$ by applying Novel fixed point theorem in a real Banach space. Investigation of appositive solution of equation (1.1) with integral boundary condition equation (1.2), we mean a function $u(t)$ which is positive on $0 \leq t \leq 1$ and satisfies the differential equation (1.1) for $0 \leq t \leq 1$and boundary conditions (1.2).

As result, this study will focus on the following points:

1). Apply the Novel fixed point theorem to investigate the existence of positive solution for second order two point integral boundary value problem.

2). Verify the applicability of the theorem by using specific example.

1.3 Objective of the study

1.3.1 General objective

The general objective of this research is to study the existence of positive solutions for second order two point integral boundary value problems using Novel fixed point theorem.

1.3.2 specific objectives

The specific objects of this study are:

- To establish the existence of at least one positive solution by using Novel fixed point theorem.

- To verify the applicability of the method by using specific example.

1.4 The significance of the study

It is believed that the present study will have the following significances:

- It may help the researcher to achieve basic understanding of conducting scientific research.

- It may provide some background information for future researchers who are interested to conduct further research on the topic or related topics.

1.5 Delimitation of the study

This study will be delimited to finding the existence of at least one positive solution for second order two point integral boundary value problems in equation (1.1) with integral boundary condition equation (1.2) by applying the Novel fixed point theorem.

The rest of this thesis organized as follows:

We first present some definitions which are needed throughout this work and constructs Greens function by using its properties for corresponding homogeneous boundary value problems and state point result by applying the Novel fixed point theorem in a cone Banach space. Finally, we investigate the existence of at least one positive solution for the second order two point integral boundary value problems (1.1-(1.2).

Chapter 2

Review of Related Literature

2.1 Over view of the study

Many problems in science and engineering fields can be described by the partial differential equations. A partial differential equation is an equation that contains one or more partial derivatives of an unknown function that depends on at least two variables. The most important partial differential equation are the wave equations that can model the vibrating string and the vibrating membrane, the heat equation for temperature is a bar or wire and the Laplace equation for electro static potentials. Partial differential equations are very important in dynamics,elasticity, heat transfer, electromagnetic theory and quantum mechanics. A variety of numerical and analytical methods have been developed to obtain accurate approximate and analytic solutions for the problems in the literature (K.Sankara, 2010). In order to understand the physical behavior of this problem, it is necessary to have some knowledge about the mathematical character, properties and the solutions of the governing partial differential equation.

One of the most dynamic areas of research of the last 50 years is fixed point theory. It is one of the most important and indispensable branches of nonlinear analysis due to the proliferation of its applications in many disciplines such as engineering, computer science, physics, economics, biology, chemistry, etc. In mathematics this technique is credited with clarifying and studying the behavior of dynamical systems, statistical methods, game theory models, differential equations, and many others. Specifically, this technique studies the existence and uniqueness of the solution to many integral and fractional equations, which facilitates the way to find numerical solutions to such problems (Rus, IA, 2006).

Positive solution is very important in divers of mathematics and it is one of the most dynamic research subjects in nonlinear analysis. In this area the first important results the existence of positive solution was proved by Erbe and Wang (1994). In recent years, there has been an increased interest in finding the existence of positive solutions for second order two point boundary value problems. In 2012, T.shiffliwirattham and J, Tariboon, studied that the existence of positive solution for second order boundary value problems of difference equations by applying the Krasnoselskiis fixed point theorem (G.Zhang and R.Medina, 2004). X.Lin and W.Lin (2009) studied the existence of positive solutions of the second order boundary valueproblem using the properties of the associate Greens function and Leggett-Williams fixed point theorem. And also Leggett and Williams (1979) developed a fixed point theorem to prove the existence of three positive solutions for Hammerstein integral equations. Since then, this theorem has been reported to be a successful technique for dealing with the existence of three solutions for the two point boundary value problems of differential and difference equations. Keller and Waltham (1968) studied that the existence and uniqueness of solutions for the second order two point boundary value problem and they claimed a boundary value problem may not possess a solution, and if it does, the solution may not be unique. Henderson and Thompson (2002) studied the existence of positive solution for boundary value problems by applying lower and upper solution methods.

Fing and Xie (2009) considered the existence of positive solutions for the second order boundary value problem. Bai and Ge (2004) established a new fixed point theorem by generalizing Leggett-Williams fixed point theorem given in Leggett and Williams (1979) and using this new fixed point theorem they established some new multiplicity results for nonlinear second order two point boundary value problem (Asaduzzaman, M. and Ali, M.Z, 2020). Mohamed, M. , W.Azmi (2013) investigated the positive solutions of to a singular second order boundary value problem with more generalized boundary conditions. Also the existence of positive solutions of singular boundary value problems of ordinary differential equations has been studied by many researchers such as Agarwal and Stanek (2010) established the existence criteria for positive solutions singular boundary value problems for nonlinear second order ordinary and delay differential equations using the Vitalis convergence theorem (Sarma et al, 2017). Gatical et al (2017) proved the existence of positive solution of the second order boundary value problem using the iterative technique and fixed point theorem for cone for decreasing mappings.

Guo etal, (2018) in recent years some boundary value problems with fractional q-Wang (2014), Zhao and Yang (2015), Jiang and Zhong (2016), Zhai and Ren (2017), Sitthiwirattham (2016). They obtained many results as regards the existence and multiplicity of nontrivial solutions, positive solutions, negative solutions and external solutions by applying some well-known tools of fixed point theory such as the Banach contraction principle, the Guo-Krasnoselskii fixed point theorem on cons, monotone iterative methods and Leray-Schauder degree theory. AlsoZhao et al (2011) investigated the existence of positive solutions for the nonlinear fractional boundary value problem. In 2014, Jia and Liu investigated the existence and nonexistence of positive solutions for the integral boundary value problem of fractional differential equations with a disturbance parameter in the boundary conditions and the impact of the disturbance parameter on the existence of positive solutions. In 2007, Zhai Chengbo and Guo Chunmei investigated the existence of positive solutions for three point boundary value problem using the Novel fixed point theorem in Banach space C[0, 1] and only the sup-norm.

2.1.1 Preliminaries

First we recall some known definitions and basic concepts on Greens function that we used in the proof of our main results.

Definition 2.1.1 (Ravi.p and Donal.O (2000)) *: we consider the second order linear differential equation,*

$$p_0(t)y'' + p_1(t)y' + p_2(t)y = r(t), t \in J = [0, 1] \tag{2.1}$$

where the functions $p_0(t), p_1(t), p_2(t)$ *and* $r(t)$ *are continuous in* J *and boundary conditions of the form,*

$$\begin{aligned} L_1(y) &= a_0 y(0) + a_1 y'(0) + b_0 y(1) + b_1 y'(1) = A, \\ L_2(y) &= c_0 y(0) + c_1 y'(0) + d_0 y(1) + d_1 y'(1) = B, \end{aligned} \tag{2.2}$$

where a_i, b_i, c_i *and* d_i *are given constants.*
The boundary value problems (2.1), (2.2) is called a nonhomogeneous two point linear boundary value problem, whereas the homogenous differential equation:

$$p_0(t)y'' + p_1(t)y' + p_2(t)y = 0, t \in J = [0, 1] \tag{2.3}$$

Together with homogeneous boundary conditions,

$$\begin{aligned} L_1(y) &= 0, \\ L_2(y) &= 0, \end{aligned} \tag{2.4}$$

be called a homogeneous two point linear boundary value problems.

The function called a Greens function G(t,s) for the homogeneous boundary value problems (2.3), (2.4) and the solution of the non homogeneous boundary value problems (2.1), (2.2) can be explicitly expressed in terms of G(t,s).
Obviously, for the homogeneous problems (2.3), (2.4) the trivial solution always exist. Greens function G(t,s) for the boundary value problems (2.3), (2.4) is defined in the square $[0,1] \times [0,1]$ and possesses the following fundamental properties:

i. $G(t,s)$ is continuous in$[0,1] \times [0,1]$.

ii. $\frac{\partial G(s^+,s)}{\partial t} - \frac{\partial G(s^-,s)}{\partial t} = \frac{1}{p_0}$, where $\frac{\partial G(s^+,s)}{\partial t} = \lim_{t\to s, t>s} \frac{\partial G(t,s)}{\partial t} and \frac{\partial G(s^-,s)}{\partial t} = \lim_{t\to s, t<s} \frac{\partial G(t,s)}{\partial t}$.

iii. For every fixed $s \in [0,1], z(t) = G(t,s)$ is a solution of the differential equation (2.3) in each of the intervals $[0,s]$ and $[s,1]$.

iv. For every fixed $s \in [0,1], z(t) = G(t,s)$ satisfies the boundary conditions (2.4). These properties completely characterize Greens function G(t,s).

Definition 2.1.2 *: A norm on a (real or complex) vector space X is a real valued function on x whose value at any $x \in X$ is denoted by $\|x\|$ and which has the properties:*

i. $\|x\| \geq 0.$

ii. $\|x\| = 0 \Longleftrightarrow x = 0.$

iii. $\|\alpha x\| = |\alpha|||x||.$

iv. $\|x+y\| \leq \|x\| + \|y||.$ *Here,x and y are arbitrary vectors in X and α scalar.*

Definition ***2.1.3*** *: Let X be a normed linear space with norm denoted by $\|.\|$.*
A sequence of elements X_n of X is a Cauchy sequence, if for every $\varepsilon > 0$ there exists an integer N such that $\|x_n - x_m\| < \varepsilon$, $\forall m, n \in N$.

Definition ***2.1.4*** *: A normed linear space X is said to be complete, if every Cauchy sequence in X converges to a point in X.*

Definition ***2.1.5*** *: A complete normed linear space is called a Banach space.*

Definition ***2.1.6*** *: Let $-\infty < a < b < \infty$. A collection of real valued functions $A = \left\{ f_i | f_i : [a, b] \to R \right\}$ is said to be uniformly bounded, if there exists a constant $M > 0$ with $|f_i(t)| \leq M$, for all $t \in [a, b]$ and $\forall f_i \in A$.*

Definition ***2.1.7*** *: Let E be A real Banach space. A nonempty closed convex set $p \subset E$ is called a cone, if it satisfies the following two conditions:*

i. $y \in p, \alpha \geq 0$ implies $\alpha y \in p$, and

ii. $y \in p$ and $-y \in p$ implies $y = 0$.

Definition ***2.1.8*** *: Let X and Y be Banach spaces and $T : X \to Y$ an operator T is said to be completely continuous, if T is continuous and for each bounded sequence $x_n \subset X, (Tx_n)$ has a convergent subsequence.*

Definition ***2.1.9*** *: Let $T : X \to X$ be self-map. A point $x \in X$ is called a fixed point of T if $T_x = x$.*

Definition ***2.1.10*** *:[Jinxiu, M et al (2010)]. The function $u(t) \in C[0, 1] \bigcap C^2[0, 1]$ is a positive solution of the boundary value problems (1.1)-(1.2). If $u(t)$ is positive on the given interval and satisfies both the differential equation and the boundary conditions.*

Definition ***2.1.11*** *:[Arzela-Ascoli Theorem]. For $A \subset C[0, 1], A$ is compact if and only if A is closed, bounded, and equicontinuous.*

Chapter 3

METHODOLOGY

This chapter contains study period and site, study design, source of information and mathematical procedure of the study.

3.1 Study area and period

The study will be conducted in mathematics department, College of Natural Sciences in Jimma University from August 2021 to January 2022.

3.2 Study Design

In this study analytic method will be employed.

3.3 Source of information

The relevant sources of information for this study will be:

- *different related mathematics books.*
- *published articles.*
- *related studies present on different websites.*

3.4 Mathematical Procedures

To conduct the present study the researcher aimed to follow the following mathematical procedure:

- *Define the problem.*
- *Construct the Greens function for the corresponding homogeneous equation.*
- *Formulate the equivalent operator equation for the boundary value problems (1.1),(1.2).*
- *Determine the fixed point of the operator equation.*
- *Construct illustrative examples and show the applicability of the method.*

Chapter 4

Main Result and Discussion

4.1 Construction of Greens Function

In this section, we construct Greens function for the homogeneous problem corresponding to (1.1),(1.2).

Let $G(t,s)$ be Greens function for the homogeneous problem,

$-u''(t) + k^2u(t) = 0, 0 < t < 1,$

with the boundary condition (1.2).

We can find the Green's function, from the homogeneous differential equation (1.1), thus two linear independent solutions are $u_1(t) = -\sinh kt + \cos hkt$ and $u_2(t) = sinhkt + coshkt$.

Hence the problem (1.1), (1.2) have only the trivial solution if and only if $\Delta \neq 0$.

$$\Delta = \begin{bmatrix} \alpha u_1(0) - \beta u_1'(0) & \alpha u_2(0) - \beta u_2'(0) \\ \gamma u_1(1) + \delta u_2'(1) & \gamma u_2(1) + \delta u_2'(1) \end{bmatrix}$$

$$\Delta = \begin{bmatrix} \alpha + \beta k & \alpha - \beta k \\ \gamma(-\sin hk + \cos hk) + \delta k(-coshk + \sin hk) & \gamma(\sin hk + coshk) + \delta k(\cos hk + sinhk) \end{bmatrix}$$

$$\Delta = 2\sin hk(\alpha\gamma + \beta\delta k^2) + 2k\cos hk(\alpha\delta + \beta\gamma) \neq 0.$$

From the property (iii) there exists four functions, say $\lambda_1(s), \lambda_2(s), \mu_1(s)$ and $\mu_2(s)$ such that

$$G(t,s)=\begin{cases}(-\sin hkt+\cos hkt)\lambda_1(s)+(\sin hkt+\cos hkt)\lambda_2(s), & 0\le t\le s\le 1,\\ (-\sin hkt+\cos hkt)\mu_1(s)+(\sin hkt+\cos hkt)\mu_2(s) & 0\le s\le t\le 1\end{cases}$$

Now, using properties (i) and (ii), we obtain the following two equations:

$u_1(s)\lambda_1(s)+u_2(s)\lambda_2(s)=u_1(s)\mu_1(s)+u_2(s)\mu_2(s),$

$u_1'(s)\mu_(s)+u_2'(s)\mu_2(s)-u_1'(s)\lambda_1(s)-u_2'(s)\lambda_2(s)=-1.$

Let $v_1(s)=\mu_1(s)-\lambda_1(s)$ *and* $v_2(s)=\mu_2(s)-\lambda_2(s)$*, so that (4.4),(4.5) can be written as:*

$(-\sin hks+\cos hks)v_1(s)+(\sin hks+\cos hks)v_2(s),$

$(-\cos hks+\sin hks)v_1(s)-(\cos hks+\sin hks)v_2(s)=\frac{-1}{k}.$

Since $(-\sin hkt+\cos hkt)$ *and* $(\sin hkt+\cos hkt)$ *are linearly independent the Wronskian* $\Delta\neq 0$ *for all* $t\in[0,1]$.

$v_1(s)=\frac{1}{2k(\cos hks-\sin hks)}$ *and*

$v_2(s)=\frac{\sin hks-\cos hks}{2k}.$

Now, using the relations:

$\mu_1(s)=\lambda_1(s)+\frac{1}{2k(\cos hks-\sin hks)}$ *and*

$\mu_2(s)=\lambda_2(s)+\frac{\sin hks-\cos hks}{2k}.$

Green's function can be written as:

$$\begin{cases}(-\sin hkt+\cos hkt)\lambda_1(s)+(\sin hkt+\cos hkt)\lambda_2(s), \quad 0\le t\le s\le 1,\\ \qquad (-\sin hkt+\cosh kt)[\lambda_2(s)+\frac{1}{2k(\cos hks-\sin hks)}]+\\ \qquad\qquad (\sin hkt+\cos hkt)[\lambda_2(s)+\frac{\sin hks-\cos hks}{2k}], \quad 0\le s\le t\le 1\end{cases}$$

Finally, using property (iv) on the boundary condition (1.2) of Green's function with the given interval, we find $\lambda_1(s) and \lambda_2(s)$*, thus*

$$\begin{cases}\alpha[u_1(0)\lambda_1(s)-u_2(0)\lambda_2(s)]-\beta[u_1'(0)\lambda_1(s)+u'(0)\lambda_2(s)]=\\ \qquad \gamma[u_1(1)(\lambda_1(s)+v_1(s))+u_(2)(1)(\lambda_2(s)+v_2(s))]+\\ \qquad\quad \delta[u_1'(\lambda_1(s)+v_1(s))+u_2'(1)(\lambda_2(1)+v_2(s))]=0\end{cases}$$

$$\begin{cases} (\alpha+\beta k)\lambda_1(s)+(\alpha-\beta k)\lambda_2(s)=0, \\ (\gamma+\delta k)(-\sin hk+\cos hk)\lambda_1(s)+(\gamma+\delta k)(\sin hk+\cos hk)\lambda_2(s)= \\ \frac{\gamma\sin hk(1-s)+\delta k\cos hk(1-s)}{2k} \end{cases}$$

Hence,

$$\lambda_1(s)=\frac{1}{\Delta}\begin{bmatrix} 0 & (\alpha-\beta k) \\ \frac{\gamma\sin hk(1-s)+\delta k\cos hk(1-s)}{2k} & (\gamma+\delta k)(\sin hk+coshk) \end{bmatrix}$$

$$\lambda_1(s)=\frac{-(\alpha-\beta k)(\gamma\sin hk(1-s)+\delta k\cos hk(1-s))}{\Delta 2k}.$$

$$\lambda_2(s)=\frac{1}{\Delta}\begin{bmatrix} (\alpha+\beta k) & 0 \\ (\gamma+\delta k)(-\sin hk+\cos hk) & \frac{\gamma\sin hk(1-s)+\delta k\cos hk(1-s)}{2k} \end{bmatrix}$$

$$\lambda_2(s)=\frac{(\alpha+\beta k)(\gamma\sin hk(1-s)+\delta k\cos hk(1-s))}{\Delta 2k}.$$

Substituting the values of $\lambda_1(s)$ *and* $\lambda_2(s)$ *in (4.1) this becomes,*

$$G(t,s)=\begin{cases} \frac{(\alpha\sin hkt+\beta k\cos hkt)(\gamma\sin hk(1-s)+\delta k(1-s))}{(\alpha\gamma+\beta\delta k^2)k\sin hk+(\alpha\delta+\beta\gamma)k^2\cos hk}, & 0\le t\le s\le 1 \\ \frac{(\alpha\sin hks+\beta k\cos hks)(\gamma\sin hk(1-t)+\delta k\cos hk(1-t))}{(\alpha\gamma+\beta\delta k^2)k\sin hk+(\alpha\delta+\beta\gamma)k^2\cos hk}, & 0\le s\le t\le 1 \end{cases}$$

Lemma *4.1.1* *Let* $y(t)\in L^1[0,1]$*, then the Sturm-Liovvile boundary value problem*

$$-u''(t)+k^2u(t)=y(t)$$

with the boundary conditions

$$\begin{cases} \alpha u(0)-\beta u(0)=0 \\ \gamma u(1)+\delta u(1)=0 \end{cases} \tag{4.1}$$

has a unique solution

$$u(t)=\int_0^1 G(t,s)y(s)ds \tag{4.2}$$

where $G(t,s)$ is Green's function for the homogeneous problem

$$-u''(t)+k^2u(t)=0, 0\le t\le 1, \tag{4.3}$$

satisfying the boundary condition (4.1), and given by

$$G(t,s)=\begin{cases}\frac{(\alpha\sin hkt+\beta k\cos hkt)(\gamma\sin hk(1-s)+\delta k\cos hk(1-s))}{(\alpha\gamma+\beta\delta k^2)k\sin hk+(\alpha\delta+\beta\gamma)k^2\cos k}, & t\le s;\\ \frac{(\alpha\sin hks+\beta k\cos hks)(\gamma\sin hk(1-t)+\delta k\cos hk(1-t))}{(\alpha\gamma+\beta\delta k^2)k\sin hk+(\alpha\delta+\beta\gamma)k^2\cos hk}, & s\le t\end{cases}$$

For convenience, we denote
$a_1=1-\int_0^1\frac{p_1(t)}{\Gamma}h_1(t)dt, a_2=\int_0^1\frac{p_2(t)}{\Gamma}h_1(t)dt, a_3=\int_0^1\frac{p_1(t)}{\Gamma}h_2(t)dt,$
$a_4=1-\int_0^1\frac{p_2(t)}{\Gamma}h_2(t)dt, \Gamma=(\alpha\gamma+\beta\delta k^2)\sin hk+(\alpha\delta+\beta\gamma)k\cos hk,$
$p_1(t)=(\gamma\sin hk(1-t))$ *and* $p_2(t)=(\alpha\sin hkt+\beta\gamma)k\cos hkt)$.

Now, we give the following assumptions:

H_1. $h_1,h_2\in L^1[0,1]$ *are nonnegative and* $a_1a_4>a_2a_3, a_1>0, a_4>0$.

H_2. $f(t,u(t))\in C([0,1]\times R_+,R_+)$

H_3. $h_1,h_2\in C([0,1],R_+)$.

Lemma *4.1.2* *Let $\sigma(t)\in L^1([0,1])$. Then the boundary value problem*

$$-u''(t)+k^2u(t)=\sigma(t) \tag{4.4}$$

with boundary condition (1.2), has a unique positive solutions u that can be expressed in the form

$$u(t)=\int_0^1 H(t,s)\sigma(s)ds, t\in[0,1] \tag{4.5}$$

where

$$H(t) = G(t,s) + \frac{p_1(t)a_4 + p_2(t)a_3}{\Gamma(a_1a_4 - a_2a_3)} \int_0^1 G(\tau,s)h_1(\tau)d\tau + \frac{p_1(t)a_2 + p_2(t)a_1}{\Gamma(a_1a_4 - a_2a_3)} \int_0^1 G(\tau,s)h_2(\tau)d\tau \tag{4.6}$$

Moreover, $u(t) > 0$, *provided that* $\sigma(t) > 0$.

Proof: Let $\sigma(t)$ *is continuous on* $[0,1]$ *then the solution of the boundary value problem (4.4),(1.2) is given by*

$$u(t) = \omega(t) + \int_0^1 G(t,s)\sigma(s)ds \tag{4.7}$$

where $\omega(t)$ *is the solution of the boundary value problem (4.3),(1.2). To find* $\omega(t)$, *let* $u_1 = \cos hkt, u_2 = \sin kkt$ *are the two linear independent solutions of (4.3),*

$$\omega(t) = c_1 \cos hkt + c_2 \sin hkt, \tag{4.8}$$

where c_1 *and* c_2 *to be determined.*

$$\omega'(t) = c_1 k \sin hkt + c_2 k \cos hkt \tag{4.9}$$

$\omega(t)$ *satisfies the non homogeneous boundary conditions (1.2). By substituting (4.8) and (4.9) on (1.2) for* $t = 0$ *and* $t = 1$ *we obtain the system of equation*

$$\begin{aligned} \alpha c_1 - \beta k c_2 &= \int_0^1 h_1(s)u(s)ds \\ (\gamma \cos hk + \delta k \sin hk)c_1 + (\gamma \sin hk + \delta k \cos hk)c_2 &= \int_0^1 h_1(s)u(s)ds \end{aligned} \tag{4.10}$$

From (4.10) by simple calculation we found

$$c_1 = \frac{(\gamma \sin hk + \delta k \cos hk)\int_0^1 h_1(s)u(s)ds + \beta k \int_0^1 h_2(s)u(s)ds}{(\alpha\gamma + \beta\delta k^2)\sin hk + (\alpha\delta + \beta\gamma)k \cos hk}$$

and

$$c_2 = \frac{\int_0^1 h_1(s)u(s)ds - (\gamma\cos hk + \delta k \sin hk)\int_0^1 h_1(s)u(s)ds}{(\alpha\gamma+\beta\delta k^2)\sin hk + (\alpha\delta+\beta\gamma)k\cos hk}$$

Replacing the value of c_1 and c_2 on (4.8) gives us

$\omega(t) = \frac{(\gamma\sin hk(1-t)+\delta k\cos hk(1-t))}{\Gamma}\int_0^1 h_1(s)u(s)ds + \frac{(\alpha\sin hkt+\beta k\cos hkt)}{\Gamma}\int_0^1 h_2(s)u(s)ds$

Now u is the solution of the differential equation (4.4) with boundary conditions (1.2) it will take the form

$$u(t) = (\frac{p_1(t)}{\Gamma})\int_0^1 h_1(t)u(t)dt + (\frac{p_2(t)}{\Gamma})\int_0^1 h_2(t)u(t)dt + \int_0^1 G(t,s)\sigma(s)ds. \qquad (4.11)$$

Consequently,

$\int_0^1 h_1(t)u(t)dt = \int_0^1 \frac{p_1(t)}{\Gamma}h_1(t)dt\int_0^1 h_1(t)u(t)dt + \int_0^1 \frac{p_2(t)}{\Gamma}h_1(t)dt\int_0^1 h_2(t)u(t)dt + \int_0^1 h_1(t)dt\int_0^1 G(t,s)\sigma(s)ds$

$\int_0^1 h_2(t)u(t)dt = \int_0^1 \frac{p_1(t)}{\Gamma}h_2(t)dt\int_0^1 h_1(t)u(t)dt + \int_0^1 \frac{p_2(t)}{\Gamma}h_2(t)dt\int_0^1 h_2(t)dt\int_0^1 h_2(t)dt\int_0^1 G(t,s)\sigma(s)ds.$

To solve the above two equations for $\int_0^1 h_1(t)u(t)dt$ and $\int_0^1 h_2(t)u(t)dt$, we have

$(1-\int_0^1 \frac{p_1(t)}{\Gamma}h_1(t)dt)\int_0^1 h_1(t)u(t)dt - \int_0^1 \frac{p_2(t)}{\Gamma}h_1(t)dt\int_0^1 h_2(t)u(t)dt = \int_0^1 h_1(t)dt\int_0^1 G(t,s)\sigma(s)ds$

Thus,

$$\int_0^1 h_1(t)u(t)dt = \frac{a_4\int_0^1 h_1(t)\int_0^1 G(t,s)\sigma(s)dsdt + a_2\int_0^1 h_2(t)\int_0^1 G(t,s)\sigma(s)dsdt}{a_1a_4 - a_2a_3} \qquad (4.12)$$

and

$$\int_0^1 h_2(t)u(t)dt = \frac{a_1\int_0^1 h_2(t)\int_0^1 G(t,s)\sigma(s)dsdt + a_3\int_0^1 h_1(t)\int_0^1 G(t,s)\sigma(s)dsdt}{a_1a_4 - a_2a_3}, \qquad (4.13)$$

Substituting (4.12) and (4.13) on (4.11), we get

$$u(t) = (\frac{p_1(t)}{\Gamma})\frac{a_4\int_0^1 h_1(t)\int_0^1 G(t,s)\sigma(s)dsdt + a_2\int_0^1 h_2(t)\int_0^1 G(t,s)\sigma(s)dsdt}{a_1a_4 - a_2a_3}$$
$$+\frac{p_2(t)}{\Gamma}\frac{a_1\int_0^1 h_2(t)\int_0^1 G(t,s)\sigma(s)dsdt + a_3\int_0^1 h_1(t)\int_0^1 G(t,s)\sigma(s)dsdt}{a_1a_4 - a_2a_3} + \int_0^1 G(t,s)\sigma(s)ds. \qquad (4.14)$$

Rearranging (4.14) we have

$$u(t) = \int_0^1 \left[\; G(t,s) + \tfrac{p_1(t)a_4 + p_2(t)a_3}{\Gamma(a_1a_4 - a_2a_3)} \int_0^1 G(\tau,s)h_1(\tau)d\tau + \tfrac{p_1(t)a_2 + p_2(t)a_1}{\Gamma(a_1a_4 - a_2a_3)} \int_0^1 G(\tau,s)h_2(\tau)d\tau \; \right] \sigma(s)ds. \tag{4.15}$$

The proof ends by comparing (4.15) with (4.5).

Lemma ***4.1.3*** *For* $t, s \in [0,1] \times [0,1]$*, we have*

i. $0 < G(t,s) \leq G(s,s)$,

ii. $G(t,s) \geq MG(s,s)$,

where

$$M = \min \left\{ \; \tfrac{\beta k}{\alpha \sin hk + \beta k \cos hk}, \tfrac{\delta k}{\gamma \sin hk + \delta k \cos hk} \; \right\}.$$

Proof: i the Green's function $H(t,s)$ *is positive for all* $t, s \in [0,1] \times [0,1]$

For $0 \leq s \leq t \leq 1$*, we have*

$\frac{H(t,s)}{H(s)} = \frac{\alpha \sin hkt + \beta k \cos hkt}{\alpha \sin hks + \beta k \cos hks} \leq 1$

$H(t,s) \leq H(s)$

Therefore,

$0 \leq H(t,s) \leq H(s)$ *for all* $t, s \in [0,1] \times [0,1]$

ii let $t \leq s$*, then*

$\frac{H(t,s)}{H(s)} = \frac{\sin hkt + \beta k \cos hkt}{\alpha \sin hks + \beta k \cos hks} \geq \frac{\beta k}{\alpha \sin hks + \beta k \sin hks}$

let $s \leq t$*, then*

$\frac{H(t,s)}{H(s)} = \frac{\gamma \sin hk(1-t) + \delta k \cos hk(1-t)}{\gamma \sin hk(1-s) + \delta k \cos hk(1-s)} \geq \frac{\delta k}{\gamma \sin hk + \delta k \cos hk}.$

Therefore, $H(t,s) \geq MH(s)$ *for all* $t, s \in [0,1] \times [0,1]$

The proof completes.

Lemma ***4.1.4*** *Suppose* (H_1) *and* (H_2) *hold, then for any* $t, s \in [0,1]$*, we have* *i.* $0 \le H(t,s) \le H(s)$,

ii. $H(t,s) \ge MH(s)$,

where

$$M = \min\left\{ \frac{\beta k}{\alpha \sin hk + \beta k \cos hk}, \frac{\delta k}{\gamma \sin hk + \delta k \cos hk}, \frac{\delta k}{\Gamma}, \frac{\beta k}{\Gamma} \right\}$$

$H(s,s) = G(s,s) + \frac{a_4+a_3}{(a_1a_4-a_2a_3)} \int_0^1 G(\tau,s)h_1(\tau)d\tau + \frac{a_2+a_1}{(a_1a_4-a_2a_3)} \int_0^1 G(\tau,s)h_2(\tau)d\tau.$

Proof: The prove of i is immediate by using the nonnegative property of $G(t,s)$ *and the expression of* $H(t,s)$ *for any* $t,s \in [0,1]$. *By i of Lemma 4.1.3*

$H(t,s) \le G(s,s) + [\frac{p_1(t)}{\Gamma}a_4 + \frac{p_2(t)}{\Gamma}a_3]\frac{1}{(a_1a_4-a_2a_3)} \int_0^1 G(\tau,s)h_1(\tau)d\tau$

$+ [\frac{p_1(t)}{\Gamma}a_2 + \frac{p_2(t)}{\Gamma}a_1]\frac{1}{(a_1a_4-a_2a_3)} \int_0^1 G(\tau,s)h_2(\tau)d\tau,$

$\le G(s,s) + \frac{a_4+a_3}{(a_1a_4-a_2a_3)} \int_0^1 G(\tau,s)h_1(\tau)d\tau + \frac{a_2+a_1}{(a_1a_4-a_2a_3)} \int_0^1 G(\tau,s)h_2(s)d\tau$

since $\max_{0\le t\le 1}\left\{ \frac{p_1(t)}{\Gamma}, \frac{p_2(t)}{\Gamma} \right\} \le 1.$

To prove ii by using ii of Lemma 4.1.3

$H(t,s) \ge MG(s,s) + [\frac{p_1(t)}{\Gamma}a_4 + \frac{p_2(t)}{\Gamma}a_3]\frac{1}{(a_1a_4-a_2a_3)} \int_0^1 G(\tau,s)h_1(\tau)d\tau$

$+ [\frac{p_1(t)}{\Gamma}a_2 + \frac{p_2(t)}{\Gamma}a_1]\frac{1}{(a_1a_4-a_2a_3)} \int G(\tau,s)h_2(\tau)d\tau,$

$\ge MG(s,s) + \frac{\delta k a_4 + \beta k a_3}{\Gamma(a_1a_4-a-2a_3)} \int_0^1 G(\tau,s)h_1(\tau)d\tau + \frac{\delta k a_2 + \beta k a_1}{\Gamma(a_1a_4-a_2a_3)} \int_0^1 G(\tau,s)h_2(\tau)d\tau$

$\ge M[G(s,s) + \frac{a_4+a_3}{(a_1a_4-a_2a_3)} \int_0^1 G(\tau,s)h_1(\tau)d\tau + \frac{a_2+a_1}{(a_1a_4-a_2a_3)} \int_0^1 G(\tau,s)h_2(\tau)d\tau]$

$= MH(s)$

The function $u(t) \in C[0,1] \cap C^2(0,1)$ *is said to be a positive solution of (1.1)-(1.2) and (1.2) if it satisfies the differential equation (1.1). Let* $E = C[0,1]$ *be a Banach space with the norm* $||u(t)|| = \max_{t\in[0,1]} |u(t)|$, *and we define a cone* p_1 *in* E *by*

$$p_1 = \left\{ u \in E : u(t) \ge 0, \min_{t\in[0,1]} u(t) \ge M||u|| \right\} \tag{4.16}$$

We define the operator $T : E \longrightarrow E$ *as*

$$(Tu)(t) = \int_0^1 H(t,s)f(s,u(s))ds \tag{4.17}$$

The integral boundary value problem (1.1) has a solution $u = u(t)$ if and only if u is the fixed point of T.

4.2 Existence of positive solution

In this section, we prove the existence of at least one positive solution for the boundary value problem (1.1)-(1.2) by using Novel fixed point theorem.

Let E be a real Banach space and p be a closed cone in E.

Let $\leq$ denots the usual induced partial ordering in E defined by $x \leq y$ if and only if $y - x \in p$. And let k be a closed convex set in a Banach space E and D be a bounded open sets such that $D_k : D \bigcap K \neq 0$. Let $T := D_k \longrightarrow k$ be a compact map. Suppose that $x \neq Tx$ for all $x \in \partial D_k$,

(D_1). (Existence) if $i(T, D_k, k) \neq 0$, then T has a fixed point in D_k where i is index.

(D_2). (Normalization) if $u \in D_k$, then $i(u, D_k, k) = 1$, where $u(x) = u$ for $x \in D_k$ where i is index.

(D_3). (Homotopy) Let $h : [0,1] \times D_k \longrightarrow k$ be a compact map such that $x \neq h(t,x)$ for $x \in \partial D_k$ and $t \in [0,1]$. Then $i(h(0,.), D_k, k) = i(h(1,.), D_k, k)$, where i is index.

To obtaining positive solutions of the boundary value problem (1.1)-(1.2), the following theorem, also known as Novel fixed point theorem will be the fundamental tool.

***Theorem 4.2.1 (Novel fixed point theorem)** : Let $E = C[0,1]$ be endowed with the maximum norm, $||y|| = \max_{t \in [0,1]} |y(t)|$ and $P = \left\{ x \in E | x(t) \geq 0, t \in [0,1] \right\}$ be a cone in E, and the ordering $x \leq y$ if $x(t) \leq y(t)$ for all $t \in [0,1]$. Suppose that $A : p \longrightarrow p$ is a completely continuous operator and there exist constants $b, c > 0, 0 \leq \mu < v \leq 1, r \in (0,1)$ such that $b < rc$ implies* *i.* $\min_{t\in[\mu.v]} Ax(t) \geq r||Ax||$,*for all* $x \in p$

ii. $\min_{t\in[\mu,v]} Ax(t) > b$, *for all* $x \in p$ *with* $b \leq x(t) \leq \frac{b}{r}, t \in [\mu, v]$;

iii. $||Ax|| \leq c$, *for all* $x \in \bar{p}_c = \left\{ \ x \in p : ||x|| \leq c \ \right\}.$

Then, A has at least one fixed point $x^* \in \bar{p}_c$ *with* $\min_{t\in[\mu,v]} x^*(t) > b$.

Lemma *4.2.2* *Let* $H_1 - H_3$ *hold . Then* $T : p_1 \longrightarrow p_1$ *is completely continuous.*

Proof: For all $u \in p_1$, *clearly* $(Tu(t)) \geq 0$. *from Lemma (4.1.4) i, we have*

$(Tu)(t) = \int_0^1 H(t,s)f(s,u(s))ds \leq \int_0^1 H(s)f(s,u(s))ds$

and, taking the max *for* $t \in [0,1]$ *we obtain*

$$||Tu|| \leq \int_0^1 H(s)f(s,u(s))ds.$$

Also by Lemma (4.1.4) ii.

$$(Tu)(t) = \int_0^1 H(t,s)f(s,u(s))ds.$$

$$\geq \min_{t\in[0,1]} \int_0^1 H(t,s)f(s,u(s))ds$$

$$\geq M \int_0^1 H(s)f(s,u(s))ds$$

and we conclude

$$\min_{t\in[0,1]}(Tu)(t) \geq M||Tu||.$$

Therefore, $Tu \in p_1$. *So* T *is an operator for* p_1 *to* p_1. *By Arzela-Ascoli Theorem, we can prove that* T *is completely continuous.*

Theorem 4.2.3 *There exists* $b, c > 0$, *such that* $b < Mc$ *implies*

i. $f(t,u(t)) > \iota b$ *for* $t \in [0,1], b \leq u < \frac{b}{M}$,

ii. $f(t,u(t)) \leq mc$ *for* $t \in [0,1], 0 \leq u \leq c$.

Then, the integral boundary value problem (1.1),(1.2) has at least one positive solution, where

$$m = \left(\int_0^1 H(s)ds \right)^{-1}, \iota = \left(M^2 \int_0^1 H(s)ds \right)^{-1}.$$

Proof: Recall

$$p_1 = \left\{ u \in E : u(t) \geq 0, \min_{t\in[0,1]} u(t) \geq M||u|| \right\},$$

to see that $u \in p_1$ *is a solution of (1.1),(1.2) if and only if*

$u(t) = \int_0^1 \left[G(t,s) + \frac{p_1(t)a_4+p_2a_3}{\Gamma(a_1a_4-a_2a_3)} \int_0^1 G(\tau,s)h_1(\tau)d\tau + \frac{p_1(t)a_2+p_2(t)a_1}{\Gamma(a_1a_4-a_2a_3)} \int_0^1 G(\tau,s)h_2(\tau)d\tau \right] f(s,u(s))ds$

T *is completely continuous and* $Tu(t) \in p_1$ *setting* $\mu = 0$ *and* $\nu = 1$

$$\min_{t\in[0,1]} Tu(t) \geq M||Tu||, \textit{ for all } u \in p_1.$$

Thus, condition i of Theorem 4.2.1 is satisfied. If $u \in \bar{p_c}$*, then* $0 \leq u(t) \leq c, t \in [0,1]$*, from* (ii)*, we have*

$||Tu|| = \max^{t\in[0,1]} \left[\begin{array}{c} \int_0^1 G(t,s)f(s,u(s))ds + \frac{p_1(t)a_4+p_2(t)a_3}{\Gamma(a_1a_4-a_2a_3)} \int_0^1 \int_0^1 G(\tau,s)h_1(\tau)f(s,u(s))d\tau ds \\ +\frac{p_1(ta_2+p_2(t)a_1)}{\Gamma(a_1a_4-a_2a_3\Gamma)} \int_0^1 \int_0^1 G(\tau,s)h_2(\tau)f(s,u(s))d\tau ds \end{array} \right]$

$\leq \int_0^1 G(s,s)f(s,u(s))ds + \frac{a_4+a_3}{(a_1a_4-a_2a_3)} \int_0^1 \int_0^1 G(\tau,s)h_1(\tau)f(s,u(s))d\tau ds$

$+ \frac{a_2+a_1}{(a_1a_4-a_2a_3)} \int_)^1 \int_0^1 G(\tau,s)h_2(\tau)f(s,u(s))d\tau ds = \int_0^1 H(s)f(s,u(s))ds \leq \int_0^1 H(s)mcds$

$= mc\int_0^1 H(s)ds \leq c.$

Hence, condition iii of Theorem 4.2.1 satisfied.

For $f(t,u(t)) > \iota b, b \leq u(t) \leq \frac{b}{M}, 0 \leq t \leq 1$*, it follows from i that*

$$\begin{aligned} Tu(t) &= \int_0^1 H(t,s)f(s,u(s))ds \\ &\geq \int_0^1 MH(s)f(s,u(s))ds > M\iota b\int_0^1 H(s)ds \\ &\geq \frac{b}{M}. \end{aligned}$$

As a result,

$$\min_{t\in[0,1]} Tu(t) \geq M||Tu|| \geq MTu(t) > M\frac{b}{M} = b.$$

Therefore, the condition (ii) of Theorem 4.2.1 satisfied. Thus, T has at least one fixed point. Then the integral boundary value problem (1.1),(1.2) has at least one positive solution.

Example

Let us consider example to illustrate main result for second order two point integral boundary value problems with the application of Novel fixed point theorem.

$$-u''(t) + k^2u(t) = f(t, u(t)), 0 \leq t \leq 1 \tag{4.18}$$

satisfying the boundary condition

$$\begin{aligned} u(0) - u'(0) &= \int_0^1 u(t)dt \\ u(1) + u'(1) &= \int_0^1 \frac{1}{2}u(t)dt \end{aligned} \tag{4.19}$$

Now

$u(t) = (\frac{p_1(t)}{\Gamma}) \int_0^1 u(t)dt + (\frac{p_1(t)}{\Gamma}) \int_0^1 \frac{1}{2}u(t)dt + \int_0^1 G(t, s)f(t, u(t))dt.$

And

$p = \left\{ \begin{array}{c} p = u \in E : u(t) \geq 0, t \in [0, 1] \end{array} \right\}$

Hence the boundary value problem 4.18,4.19 has at least one positive solution u(t).

Chapter 5

Conclusion and Future scope

5.1 Conclusion

Based on the established result the following conclusion can be derived:

- *In this study, we have considered second order integral boundary value problems and used the properties of Greens function to construct corresponding homogeneous equation.*

- *After these we formulated equivalent integral equation for the boundary value problem (1.1)- (1.2) in the given interval and determined the existence of positive fixed point of the integral equation by applying Novel fixed point theorem. Finally, we established the existence of at least one positive solution for the second order integral boundary value problem (1.1)-(1.2).*

5.2 Future Scope

This study focused on existence of positive solutions for second order integral boundary value problems. Any interested researchers may conduct the research on existence of positive solutions by taking different coefficient and considering orders greater than two.

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