Fixed Point Results for Generalized Rational (α, ψ, ϕ)- Geraghty Contraction Type Mappings in b- Metric Spaces

A Thesis Submitted to the Department of Mathematics in Partial Fulfillment for the Requirements of the Degree of Masters of Science in Mathematics

Prepared by:
Bezuayehu Nigusie

Under the supervision of:
Advisor: Kidane Koyas (Ph.D.)
Co-Advisor: Mustefa Abduletif (M.Sc.)

February, 2022
Jimma, Ethiopia

Declaration

I, the undersigned declare that, the thesis entitled "Fixed Point Results for Generalized Rational (α, ψ, ϕ)- Geraghty Contraction Type Mappings in b -Metric Spaces" is original and it has not been submitted to any institution elsewhere for the award of any academic degree or like, where other sources of information that have been used, they have been acknowledged.
Name: Bezuayehu Nigusie
Signature: ————————
Date: ————————
The work has been done under the supervision of:
Name: Kidane Koyas (Ph.D.)
Signature: ————————
Date: ————————
Name: Mustefa Abduletif (M.Sc.)
Signature: ————————
Date: ————————

Acknowledgment

First of all, I am indebted to the almighty God who gave me long life and helped me to pass through different ups and downs to reach this time. Next, my Special heartfelt thank goes to my advisor, Dr. Kidane Koyas and co-advisor Mr. Mustefa Abduletif for their unreserved support, advice, and guidance throughout the preparation of this research work. Also, I would like to thank all my friends who gave me initiation while I was preparing this research work. Last but not least. I would like to thank my wife Tsegayinesh Adisu, my children Nanat, Moyibon and my sister Ejigayehu Nigusie for their encouragement through out my study period.

Abstract

In this research work we introduced generalized rational (α, ψ, ϕ)-Geraghty contraction type mappings and proved existence and uniqueness of fixed point for the mappings introduced in the setting of complete b-metric spaces. Our results extend and generalize related fixed-point results in the literature. Also, we provided examples in support of our main results. In this research under taking, we followed analytical study design and used secondary sources of data such as published paper and related books.

Contents

Chapter 1

Introduction

1.1 Background of the study

The Banach contraction principle is one of the most fundamental results in fixed point theory. Due to its usefulness and applications in many disciplines, several authors have improved, extended and generalized this basic result by defining new contractive conditions and replacing the metric space by more general abstract spaces. In 1973, Geraghty, generalized the Banach contraction principle by considering an auxiliary function. In 2010, Amini-Harandi and Emami, (2010) characterized the result of Geraghty in the context of a partially ordered complete metric spaces. Caballero et al. (2012) discussed the existence of a best proximity point of Geraghty contraction. Gordjiet al. (2012) defined the notion of ψ-Geraghty type contraction and obtained results extending the results of Amini-Harandi and Emami, (2010). Samet et al. (2012) defined the notion of (α, ψ)-contractive maps and obtained remarkable fixed-point results. Karapinar and Samet, (2012) introduced the concept of generalized (α, ψ)- contractive mappings and obtained fixed point-results for such mappings. In 2013, Cho et al., (2013) defined the concept of generalized α-Geraghty contraction type maps and α-Geraghty contraction type maps in the setting of a metric space and proved some fixed-point results for such maps in the context of a complete metric space. Then in 2014, Karapinar, (2014) introduced the concept of generalized (α, ψ)-Geraghty contraction type maps and proved fixed point results by generalizing the results obtained by Cho et al. (2013). In 2017, Arshad and Hussain, (2017) defined generalized rational α-Geraghty contraction type mappings and proved some fixed-point results. In 2020, Afshari et al., (2020) introduced generalized (α,ψ)- Geraghty contractive type mappings and investigated the existence and uniqueness of a fixed point for mappings introduced in the setting of b-metric space. In 2021, Singh, et al., (2021) studied fixed point results for generalized rational (α,ψ)-Geraghty contraction type mappings in the setting of metric spaces. Inspired and motivated by the developments above, in this research work we defined generalized rational (α,ψ,ϕ)-Geraghty contraction type mappings and proved the existence and uniqueness of fixed-points for the maps introduced in the setting of b-metric spaces.

1.2 Statement of the Problem

In 2013, Cho et al., (2013) defined the concept of generalized α -Geraghty contraction type maps in the setting of a metric space and proved some fixed-point results. In 2014,Karapinar,(2014) introduced the concept of generalized (α,ψ)-Geraghty contraction type maps and proved fixed point results in the setting of metric spaces. In 2020, Afshari et al., (2020) introduced generalized (α,ψ)- Geraghty contractive type mappings and investigated the existence and uniqueness of fixed-points for mappings introduced in the setting of b-metric spaces. In 2021, Singh, et al., (2021) studied fixed-point results for generalized rational (α,ψ)-Geraghty contraction type mappings in the setting of metric spaces. However, fixed-point results for generalized rational (α,ψ,ϕ)-Geraghty contraction type mappings in b-metric spaces are not yet studied. Thus, in this study we concentrated on establishing and proving fixed-point results for generalized rational (α,ψ,ϕ)-Geraghty contraction type mappings in b- metric spaces.

1.3 Objectives of the study

1.3.1 General objective

The general objective of this research was to study fixed-point results for generalized rational (α,ψ,ϕ)-Geraghty contraction type mappings in b- metric spaces.

1.3.2 Specific objectives

This study has the following specific objectives

- To define generalized rational (α, ψ, ϕ) -Geraghty contraction type mappings in b-metric spaces.
- Establish fixed-point results for generalized rational (α, ψ, ϕ) -Geraghty contraction type mappings in b-metric spaces.
- To prove the existence of fixed-point results for generalized rational (α, ψ, ϕ) -Geraghty contraction type mappings in b- metric spaces.
- To verify the uniqueness of fixed-points for generalized rational (α, ψ, ϕ) -Geraghty contraction type mappings in b-metric spaces.
- To provide examples in support of our main findings.

1.4 Significance of the study

The result of this study may have the following importance:

- It may be used as a reference for any researcher who has interest in doing research in the area.
- It may give basic research skill to the researcher.
- It may be applied to solve existence of solution of some integral and differential equations.

1.5 Delimitation of the Study

This study was delimited to prove the existence and uniqueness of fixed-point results for generalized rational (α, ψ, ϕ)-Geraghty contraction type mappings in b-metric spaces.

Chapter 2

Review of Related literatures

The Banach contraction principle is one of the earliest and most important results in fixed point theory. Because of its application in many disciplines such as chemistry, physics, biology, computer science and many branches of mathematics, a lot of authors have improved, generalized and extended this classical result in nonlinear analysis, see, e.g., Rhoades, (1977), Amini-Harandi and Emami, (2010), Agarwal et al., (2008), Huang and Zhang, (2007), Yang et al., (2011) and the references there in. One of the interesting results was given by Geraghty, (1973) in the setting of complete metric spaces by considering an auxiliary function. Later, Amini-Harandi and Emami, (2010) characterized the result of Geraghty in the context of a partially ordered complete metric space, and Caballero et al., (2012) discussed the existence of a best proximity point of Geraghty contraction. Gordji et al., (2012) defined the notion of ψ-Geraghty type contraction and supposedly improved and extended the results of Amini-Harandi and Emami, (2010). Cho et al., (2013) defined the concept of α-Geraghty contraction type maps in the setting of a metric space and proved the existence and uniqueness of a fixed point of such maps in the context of a complete metric space. Karapnar and Samet, (2012) proved that the results of Gordji et al., (2012). A very interesting extension of the notion of a metric, called b-metric, was proposed by (Czerwik ,1993, 1998). In these pioneer papers, Czerwik observed some fixed-point results, including the analog of the Banach contraction principle in the context of complete b-metric spaces. In the sequel, several papers have been reported on the existence (and the uniqueness) of (common) fixed points of various classes of single-valued and multi-valued operators in the setting of b-metric spaces (see, e.g., Aydiet al.,(2012, 2014), Paesano and Vetro, (2015), Roshan et al., (2014), Shahkoohi and Razani, (2014) and the related references therein). Samet et al., (2012) considered the concept of an admissible mapping to get a very general structure that combines several existing fixed-point theorems by introducing (α, ψ)-contractive type mappings in complete metric spaces. Karapnar and Samet, (2012) improved the results in Samet et al., (2012) by defining the notion of generalized (α, ψ)-contractive type mappings.They listed several existing results as consequences of their main results. Following these initial papers, Karapnar, (2014), Karapnar and Samet, (2014) introduced (α, ψ)-Geraghty contraction type mappings that generalize the results of Geraghty, (1973). In 2021, Singh, et al., (2021) studied fixed point results for generalized rational (α, ψ)-Geraghty contraction type mappings in the setting of metric spaces. Inspired and motivated by the developments above, in this paper we defined generalized rational (α, ψ, ϕ) Geraghty contraction type mapping in the setting of b-metric spaces and obtained the existence and uniqueness of fixed points of such mappings. We also gave an example to illustrate our main findings.

Chapter 3

Methodology

3.1 Study period and site

The study was conducted at Jimma University under the department of mathematics from September, 2021 G.C to February, 2022 G.C.

3.2 Study Design

In this research work we employed analytical method of design.

3.3 Source of Information

The relevant sources of information for this study were books, published articles and related studies from internet.

3.4 Mathematical Procedure of the Study

In this research under taking, we followed the standard procedures. The procedures are:

1. Establishing fixed point theorems.

2. Constructing sequences.

3. Showing that constructed sequences are b-Cauchy.
4. Showing the b-convergences of the sequences.
5. Proving the existence and uniqueness of fixed points.
6. Giving examples in support of the main findings.

Chapter 4

Preliminaries and Main Results

4.1 Preliminaries

Now, we state some notation, definitions and theorems which are useful for our work as follows.

Notation Through out this thesis we denotes:

(i) $R^+ = [0,\infty)$;

(ii) N is the set of natural numbers;

(iii) R is the set of all real numbers;

(iv) $\Phi = \{\phi = R^+ \to R^+$ such that ϕ is increasing, continuous and $\phi(t) = 0$ if and only if $t = 0.\}$;

(v) $\Theta = \{\theta = R^+ \to [0,1)$ such that $\lim_{n\to\infty} \theta(t_n) = 1$ implies $\lim_{n\to\infty} (t_n) = 0$ $\}$;

(vi) $\Omega = \{\theta = R^+ \to [0,\frac{1}{s})$ such that $\lim_{n\to\infty} \theta(t_n) = \frac{1}{s}$ implies $\lim_{n\to\infty} (t_n) = 0$ $\}$.

Definition 4.1 *(Bakhtin, 1989) and (Czerwik, 1993). Let X be a non-empty set and $s \geq 1$ be a given real number. A function $d : X \times X \to R^+$ is a b-metric on X if, for all, $x,y,z \in X$, the following conditions hold:*

(i) $d(x,y) = 0$ if and only if $x = y$;

(ii) $d(x,y) = d(y,x)$;

(iii) $d(x,z) \leq s[d(x,y) + d(y,z)]$ (b-triangular inequality).

Then the pair (X,d) is called a b-metric space.

It should be noted that, the class of b-metric spaces is effectively larger than that of metric spaces; every metric is a b-metric with s = 1, while the converse is not true.

Example 4.1 *(Roshan et al., 2014). Let $X = R$ and $d : X \times X \to R^+$ be given by $d(x,y) = (x-y)^2$ for $x,y \in X$, then d is a b-metric on X with $s = 2$, but it is not a metric on X since for all $x,y,z \in R$ where, $x = 1, y = 3$ and $z = 5$, we have $d(1,5) \nleq d(1,3) + d(3,5)$. Thus, the triangle inequality for a metric does not hold.*

Definition 4.2 *(Boriceanu et al., 2010) Let (X,d) be a b-metric space with the coefficient $s \geq 1$ and let $T : X \to X$ be a given mapping. We say that T is b-continuous at $x_0 \in X$ if and only if for every sequence $\{x_n\}$ in X, we have $x_n \to x_0$ as $n \to \infty$, then $Tx_n \to Tx_0$ as $n \to \infty$. If T is b-continuous at each points $x \in X$, then we say that T is b-continuous on X. In general, a b-metric is not necessarily b-continuous.*

Example 4.2 *(Boriceanu, 2009) Let $X = NU\{\infty\}$.*
Define a mapping $d : X \times X \to R^+$ as follows:

$$d(m,n) = \begin{cases} 0 & \textit{if } m = n \\ |\frac{1}{m} - \frac{1}{n}| & \textit{if one of m and n is even and the other even or } \infty \\ 5 & \textit{if one of m and n is odd and the other is odd or } \infty \\ 4 & \textit{if others.} \end{cases}$$

$d(m,p) \leq \frac{5}{4}[d(m,n) + d(n,p)]$ for all $m,n,p \in X$.
Then (X,d) is a b-metric space with $s = \frac{5}{4}$.
Choose $x_n = 2n$ for each $n \in N$. Then $d(x_n,\infty) = d(2n,\infty) = \frac{1}{2n} \to 0$ as $n \to \infty$, that is, $x_n \to \infty$ as $n \to \infty$.
But,

$$\lim_{n\to\infty} d(x_n,1) = 4 \neq 5 = d(\infty,1).$$

Hence, it is not continuous.

Definition 4.3 *(Jovanovic et al., 2010). Let $\{x_n\}$ be a sequence in a b-metric space (X,d). Then*
(i) $\{x_n\}$ is called b-convergent if and only if there is $x \in X$ such that $d(x_n,x) \to 0$ as $n \to \infty$;
(ii) $\{x_n\}$ is a b-Cauchy sequence if and only if $d(x_n,x_m) \to 0$ as $n,m \to \infty$;

(iii) A b-metric space is said to be complete if and only if each b-Cauchy sequence in this space is b-convergent.

Definition 4.4 *(Geraghty, 1973). Let X be a non-empty set. An operator $T: X \to X$ is called a Geraghty contraction if there exists a function $\theta \in \Theta$ which satisfies for all $x,y \in X$ the condition $d(Tx,Ty) \leq \theta(d(x,y))d(x,y)$.*

Theorem 4.1 *(Geraghty, 1973). Let (X,d) be a complete metric space. If $T: X \to X$ is a Geraghty contractive mapping, then T has a unique fixed point.*

Definition 4.5 *(Samet, 2012) Let X be a non-empty set, $T: X \to X$ and $\alpha: X \times X \to R^+$ be a function. Then T is said to be α- admissible if $\alpha(x,y) \geq 1$ implies $\alpha(Tx,Ty) \geq 1$, for all $x,y \in X$.*

Definition 4.6 *(Karpinar et al., 2013). Let X be a non-empty set. A map $T: X \to X$ is said to be triangular α-admissible if*
(i) T is α-admissible;
(ii) $\alpha(x,z) \geq 1$ and $\alpha(z,y) \geq 1$ implies $\alpha(x,y) \geq 1$ for all $x,y,z \in X$.

Lemma 4.2 *(Popescu, 2014). Let X be a non-empty set and $T: X \to X$ be a triangular α-admissible map. Assume that there exists $x_0 \in X$ such that $\alpha(x_0,Tx_0) \geq 1$. Define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$. Then we have $\alpha(x_n,x_m) \geq 1$ for all $m,n \in N$ with $n < m$.*

Definition 4.7 *(Popescu, 2014). Let X be a non-empty set, (X,d) be a b-metric space and let $\alpha: X \times X \to R^+$ be a function. X is said to be α-regular, if for every sequence $\{x_n\}$ in X such that $\alpha(x_n,x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with $\alpha(x_{n_k},x) \geq 1$ for all k.*

4.2 Main Results

In this section we introduced generalized rational (α,ψ,ϕ)-Geraghty contraction type mappings in the setting of b-metric spaces and proved existence and uniqueness of fixed- point for mappings introduced.

Definition 4.8 *Let (X,d) be a b-metric space with the coefficient $s \geq 1$ and $\alpha : X \times X \to R^+$ be a function. Then the mapping $T : X \to X$ is called generalized rational (α,ψ,ϕ)-Geraghty contraction type mapping if there exist $\theta \in \Omega$, $\psi,\phi \in \Phi$ such that for all $x,y \in X$,*

$$\alpha(x,y)\psi(s^3 d(Tx,Ty)) \leq \theta(\psi(N(x,y)))\psi(N(x,y)) + L\phi(M(x,y)), \tag{4.1}$$

where $L \geq 0$, $M(x,y) = min\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}$ and

$$\begin{aligned} N(x,y) &= Max\Bigg\{ d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2s}, \frac{d(x,Tx)d(y,Ty)}{1+d(x,y)}, \frac{d(x,Tx)d(y,Ty)}{1+d(Tx,Ty)}, \\ &\quad \frac{d(x,Tx)[1+d(y,Ty)]}{1+d(Tx,Ty)}, \frac{d(y,Ty)[1+d(Tx,Ty)]}{1+d(y,Ty)} \Bigg\}. \end{aligned}$$

Theorem 4.3 *Let (X,d) be a complete b-metric space with the coefficient $s \geq 1$, $\alpha : X \times X \to R^+$ be a function and $T : X \to X$ be a mapping. Suppose that the following conditions hold:*
(i) T is generalized rational (α,ψ,ϕ)-Geraghty contraction type mapping;
(ii) T is triangular α-admissible mapping;
(iii) There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;
(iv) T is b-continuous.
Then, T has a fixed point $x \in X$ and $\{T^n x_0\}$ converges to x.
Proof: *By (iii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. We construct a sequence $\{x_n\}$ in X by $x_{n+1} = Tx_n$ for each $n \in NU\{0\}$. If $x_{n_0} = x_{n_0+1} = Tx_0$, for some $n \in NU\{0\}$ then x_{n_0} is a fixed point of T which completes the proof. Now, we suppose $x_n \neq x_{n+1}$ for all $n \in NU\{0\}$.*

Now, by using (4.1) *and Lemma 4.2 we have*

$$\begin{aligned}
\psi(d(x_{n+1},x_{n+2})) &= \psi(d(Tx_n,Tx_{n+1})) \\
&\leq \alpha(x_n,x_{n+1})\psi(s^3 d(Tx_n,Tx_{n+1})) \\
&\leq \theta(\psi(N(x_n,x_{n+1})))\psi(N(x_n,x_{n+1})) \\
&\quad + L\phi(M(x_n,x_{n+1})), \qquad (4.2)
\end{aligned}$$

where

$$\begin{aligned}
N(x_n,x_{n+1}) &= Max\Bigg\{ d(x_n,x_{n+1}), d(x_n,Tx_n), d(x_{n+1},Tx_{n+1}), \frac{d(x_n,Tx_{n+1})+d(x_{n+1},Tx_n)}{2s}, \\
&\quad \frac{d(x_n,Tx_n)d(x_{n+1},Tx_{n+1})}{1+d(x_n,x_{n+1})}, \frac{d(x_n,Tx_n)d(x_{n+1},Tx_{n+1})}{1+d(Tx_n,Tx_{n+1})}, \frac{d(x_n,Tx_n)[1+d(x_{n+1},Tx_{n+1})]}{1+d(Tx_n,Tx_{n+1})}, \\
&\quad \frac{d(x_{n+1},Tx_{n+1})[1+d(Tx_n,Tx_{n+1})]}{1+d(x_{n+1},Tx_{n+1})} \Bigg\} \\
&= Max\{d(x_n,x_{n+1}), d(x_{n+1},x_{n+2})\}
\end{aligned}$$

and

$$\begin{aligned}
M(x_n,x_{n+1}) &= Min\Big\{ d(x_n,x_{n+1}), d(x_n,Tx_n), d(x_{n+1},Tx_{n+1}), (d(x_n,Tx_{n+1}), d(x_{n+1},Tx_n) \Big\} \\
&= Min\{d(x_n,x_{n+1}), d(x_{n+1},x_{n+2}), d(x_n,x_{n+2}), d(x_{n+1},x_{n+1})\} = 0.
\end{aligned}$$

If $N(x_n,x_{n+1}) = d(x_{n+1},x_{n+2})$*, for some* $n \in N \cup \{0\}$ *then from* (4.2)*, we have*

$$\begin{aligned}
\psi(d(x_{n+1},x_{n+2})) &= \psi(d(Tx_n,Tx_{n+1})) \\
&\leq \alpha(x_n,x_{n+1})\psi(s^3 d(Tx_n,Tx_{n+1})) \\
&\leq \theta(\psi(N(x_n,x_{n+1})))\psi(N(x_n,x_{n+1})) + L\phi(M(x_n,x_{n+1})) \\
&= \theta(\psi(d(x_{n+1},x_{n+2})))\psi(d(x_{n+1},x_{n+2})) \\
&< \frac{1}{s}\psi(d(x_{n+1},x_{n+2})) \\
&\leq \psi(d(x_{n+1},x_{n+2})),
\end{aligned}$$

which is a contradiction. Therefore, $N(x_n,x_{n+1}) = d(x_n,x_{n+1})$*. Hence, from* (4.2)*, we get*

$$\begin{aligned}
\psi(d(x_{n+1},x_{n+2})) &= \psi(d(Tx_n,Tx_{n+1})) \\
&\leq \alpha(x_n,x_{n+1})\psi(s^3 d(Tx_n,Tx_{n+1})) \\
&\leq \theta(\psi(N(x_n,x_{n+1})))\psi(N(x_n,x_{n+1})) + L\phi(M(x_n,x_{n+1})) \\
&= \theta(\psi(d(x_n,x_{n+1})))\psi(d(x_n,x_{n+1})) \\
&< \frac{1}{s}\psi(d(x_n,x_{n+1})) \\
&\leq \psi(d(x_n,x_{n+1})).
\end{aligned}$$

By the property of ψ *from above inequality we obtain* $d(x_{n+1},x_{n+2}) < d(x_n,x_{n+1})$ *for all* $n \in N \cup \{0\}$*. Thus,* $\{d(x_n,x_{n+1})\}$ *is a decreasing sequence of a nonnegative real numbers and bounded below by* 0*. Hence, there exists* $r \geq 0$ *such that* $\lim_{n\to\infty} d(x_n,x_{n+1}) = r$*. Now, we show that* $r = 0$*. Suppose on contrary that* $r > 0$*.*
By using (4.1), we have

$$\begin{aligned}
\frac{1}{s}\psi(d(x_{n+1},x_{n+2})) &\leq \psi(d(x_{n+1},x_{n+2})) \\
&= \psi(d(Tx_n,Tx_{n+1})) \\
&\leq \alpha(x_n,x_{n+1})\psi(s^3 d(Tx_n,Tx_{n+1})) \\
&\leq \theta(\psi(N(x_n,x_{n+1})))\psi(N(x_n,x_{n+1})) + L\phi(M(x_n,x_{n+1})) \\
&= \theta(\psi(d(x_n,x_{n+1})))\psi(d(x_n,x_{n+1})) \\
&< \frac{1}{s}\psi(d(x_n,x_{n+1})).
\end{aligned}$$

By applying limit in the above inequality as $n \to \infty$*, we obtain*

$$\frac{1}{s}\psi(r) \leq \lim_{n\to\infty} \theta(\psi(d(x_n,x_{n+1})))\psi(r) \leq \frac{1}{s}\psi(r).$$

Therefore,

$$\lim_{n\to\infty} \theta(\psi(d(x_n,x_{n+1}))) = \frac{1}{s}.$$

Finally, by properties of θ *and* ψ*, we get*

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0 = r,$$

which is a contradiction. Hence,

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{4.3}$$

We now prove that $\{x_n\}$ *is a b-Cauchy sequence in* X*. Suppose that* $\{x_n\}$ *is not a b-Cauchy sequence. Thus, there exists* $\varepsilon > 0$ *for which we can find subsequences* $\{x_{m_k}\}$ *and* $\{x_{n_k}\}$ *of* $\{x_n\}$ *with* $n_k > m_k > k$ *such that*

$$d(x_{m_k}, x_{n_k}) \geq \varepsilon \tag{4.4}$$

and $d(x_{m_k}, x_{n_{k-1}}) < \varepsilon$*.*

From (4.4) *and the triangle inequality, we obtain*

$$\begin{aligned} \varepsilon \leq d(x_{n_k}, x_{m_k}) &\leq sd(x_{n_k}, x_{n_{k+1}}) + sd(x_{n_{k+1}}, x_{m_k}) \\ &\leq sd(x_{n_k}, x_{n_{k+1}}) + s^2 d(x_{n_{k+1}}, x_{m_{k+1}}) + s^2 d(x_{m_{k+1}}, x_{m_k}) \end{aligned} \tag{4.5}$$

Letting $k \to \infty$ *and taking* (4.3) *into account, inequality* (4.5) *yields*

$$\frac{\varepsilon}{s^2} \leq \limsup_{k\to\infty} d(x_{n_{k+1}}, x_{m_{k+1}}). \tag{4.6}$$

By Lemma 4.2, recall that $\alpha(x_{m_k}, x_{n_k}) \geq 1$*. Consequently, by* (4.1)*, we have*

$$\begin{aligned} \psi(d(x_{n_{k+1}}, x_{m_{k+1}})) &= \psi(d(Tx_{n_k}, Tx_{m_k})) \\ &\leq \psi(s^3 d(Tx_{n_k}, Tx_{m_k})) \\ &\leq \alpha(x_{m_k}, x_{n_k}) \psi(s^3 d(Tx_{n_k}, Tx_{m_k})) \\ &\leq \theta(\psi(N(x_{n_k}, x_{m_k}))) \psi(N(x_{n_k}, x_{m_k})) \\ &\quad + L\phi(M(x_{n_k}, x_{m_k})), \end{aligned} \tag{4.7}$$

where

$$
\begin{aligned}
N(x_{n_k},x_{m_k}) &= Max\left\{ d(x_{n_k},x_{m_k}), d(x_{n_k},Tx_{n_k}), d(x_{m_k},Tx_{m_k}), \frac{d(x_{n_k},Tx_{m_k})+d(x_{m_k},Tx_{n_k})}{2s}, \right. \\
&\quad \frac{d(x_{n_k},Tx_{n_k})d(x_{m_k},Tx_{m_k})}{1+d(x_{n_k},x_{m_k})}, \frac{d(x_{n_k},Tx_{n_k})d(x_{m_k},Tx_{m_k})}{1+d(Tx_{n_k},Tx_{m_k})}, \frac{d(x_{n_k},Tx_{n_k})[1+d(x_{m_k},Tx_{m_k})]}{1+d(Tx_{n_k},Tx_{m_k})}, \\
&\quad \left. \frac{d(x_{m_k},Tx_{m_k})[1+d(Tx_{n_k},Tx_{m_k})]}{1+d(x_{m_k},Tx_{m_k})} \right\} \\
&= Max\left\{ d(x_{n_k},x_{m_k}), d(x_{n_k},x_{n_{k+1}}), d(x_{m_k},x_{m_{k+1}}), \frac{d(x_{n_k},x_{m_{k+1}})+d(x_{m_k},x_{n_{k+1}})}{2s}, \right. \\
&\quad \frac{d(x_{n_k},x_{n_{k+1}})d(x_{m_k},x_{m_{k+1}})}{1+d(x_{n_k},x_{m_k})}, \frac{d(x_{n_k},x_{n_{k+1}})d(x_{m_k},x_{m_{k+1}})}{1+d(x_{n_{k+1}},x_{m_{k+1}})}, \frac{d(x_{n_k},x_{n_{k+1}})[1+d(x_{m_k},x_{m_{k+1}})]}{1+d(x_{n_{k+1}},x_{m_{k+1}})}, \\
&\quad \left. \frac{d(x_{m_k},x_{m_{k+1}})[1+d(x_{n_{k+1}},x_{m_{k+1}})]}{1+d(x_{m_k},x_{m_{k+1}})} \right\}
\end{aligned}
$$

and

$$
\begin{aligned}
M(x_{n_k},x_{m_k}) &= Min\left\{ d(x_{n_k},x_{m_k}), d(x_{n_k},Tx_{n_k}), d(x_{m_k},Tx_{m_k}), d(x_{n_k},Tx_{m_k}), d(x_{m_k},Tx_{n_k}) \right\} \\
&= Min\left\{ d(x_{n_k},x_{m_k}), d(x_{n_k},x_{n_{k+1}}), d(x_{m_k},x_{m_{k+1}}), d(x_{n_k},x_{m_{k+1}}), d(x_{m_k},x_{n_{k+1}}) \right\}.
\end{aligned}
$$

Notice that

$$
\frac{d(x_{n_k},x_{m_{k+1}})+d(x_{m_k},x_{n_{k+1}})}{2s} \leq \frac{s[d(x_{n_k},x_{m_k})+d(x_{m_k},x_{m_{k+1}})]+s[d(x_{m_k},x_{n_k})+d(x_{n_k},x_{n_{k+1}})]}{2s} \tag{4.8}
$$

and

$$
\begin{aligned}
d(x_{n_k},x_{m_k}) &\leq sd(x_{n_k},x_{n_{k-1}})+sd(x_{n_{k-1}},x_{m_k}) \\
&< sd(x_{n_k},x_{n_{k-1}})+s\varepsilon.
\end{aligned} \tag{4.9}
$$

By using (4.8) *and* (4.9)*, we get*

$$
\limsup_{k\to\infty} N(x_{n_k},x_{m_k}) \leq s\varepsilon, \tag{4.10}
$$

$$\lim_{k\to\infty} M(x_{n_k}, x_{m_k}) = 0. \tag{4.11}$$

By taking the upper limit as $k \to \infty$ and using (4.6), (4.10) *and* (4.11), *inequality* (4.7) *becomes*

$$\begin{aligned}
\frac{1}{s}\psi(s\varepsilon) \le \psi(s\varepsilon) \le \limsup_{k\to\infty} \psi(s^3 d(x_{n_{k+1}}, x_{m_{k+1}})) &\le \limsup_{k\to\infty} \alpha(x_{m_k}, x_{n_k})\psi(s^3 d(x_{n_{k+1}}, x_{m_{k+1}})) \\
&= \limsup_{k\to\infty} \alpha(x_{m_k}, x_{n_k})\psi(s^3 d(Tx_{n_k}, Tx_{m_k})) \\
&\le \limsup_{k\to\infty} [\theta(\psi(N(x_{n_k}, x_{m_k})))\psi(N(x_{n_k}, x_{m_k})) + \\
&\quad L\phi(M(x_{n_k}, x_{m_k}))] \\
&\le \psi(s\varepsilon) \limsup_{k\to\infty} \theta(\psi(N(x_{n_k}, x_{m_k}))) \\
&\le \frac{1}{s}\psi(s\varepsilon).
\end{aligned}$$

Then $\limsup_{k\to\infty} \theta(\psi(N(x_{n_k}, x_{m_k}))) = \frac{1}{s}$. *Due to the fact that $\theta \in \Omega$, we have*

$$\limsup_{k\to\infty} \psi(N(x_{n_k}, x_{m_k})) = 0.$$

Thus, we conclude that

$$\lim_{k\to\infty} \psi(d(x_{n_k}, x_{m_k})) = 0.$$

By continuity property of ψ we gate

$$\lim_{k\to\infty} d(x_{n_k}, x_{m_k}) = 0,$$

which contradicts (4.4). *Therefore, the sequence $\{x_n\}$ is a b-Cauchy sequence in X. Since X is a complete b-metric space, there exists $x \in X$ such that $\lim_{n\to\infty} x_n = x$. By the b-continuity of T we have*

$$x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} T(x_n) = T(\lim_{n\to\infty} x_n) = Tx.$$

Therefore, x is a fixed point of T. □

In the following by replacing the continuity assumption in the Theorem 4.3 by α-regularity condition we get the following theorem.

Theorem 4.4 *Let (X,d) be a complete b-metric space with the coefficient $s \geq 1$, $\alpha : X \times X \to R^+$ be a function and $T : X \to X$ be a mapping. Suppose that the following conditions hold:*
(i) T is generalized rational (α, ψ, ϕ)-Geraghty contraction type mapping;
(ii) T is triangular α-admissible mapping;
(iii) There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;
(iv) X is α-regular.
Then, T has a fixed point $x \in X$.

Proof: Following the lines in the proof of Theorem 4.3, by (iii) there exist $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. We construct a sequence $\{x_n\}$ in X by $x_{n+1} = Tx_n$ for each $n \in NU\{0\}$ such that $\lim_{n\to\infty} x_n = x$. By α-regularity of X, $\alpha(x_n, x_{n+1}) \geq 1$ and there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that, for all k

$$\alpha(x_{n_k}, x) \geq 1. \tag{4.12}$$

Now, we show $Tx = x$. Assume to the contrary, that $Tx \neq x$, then $d(x, Tx) > 0$.
By the triangular inequality, we have

$$d(x,Tx) \leq sd(x,x_{n_{k+1}}) + sd(x_{n_{k+1}},Tx) = sd(x,x_{n_{k+1}}) + sd(Tx_{n_k},Tx).$$

Letting k tends to infinity in the above inequality we get

$$d(x,Tx) \leq \liminf_{k\to\infty} sd(Tx_{n_k},Tx). \tag{4.13}$$

By using the property of ψ, by (4.12) and (4.13), we get

$$\begin{aligned} \psi(s^2 d(x,Tx)) &\leq \lim_{k\to\infty} \psi(s^3 d(Tx_{n_k},Tx)) \\ &\leq \lim_{k\to\infty} \alpha(x_{n_k},x)\psi(s^3 d(Tx_{n_k},Tx)) \\ &\leq \lim_{k\to\infty} [\theta(\psi(N(x_{n_k},x)))\psi(N(x_{n_k},x)) + L\phi(M(x_{n_k},x))]. \end{aligned} \tag{4.14}$$

where

$$\begin{aligned} N(x_{n_k},x) &= Max\left\{ d(x_{n_k},x), d(x_{n_k},Tx_{n_k}), d(x,Tx), \frac{d(x_{n_k},Tx)+d(x,Tx_{n_k})}{2s}, \right. \\ & \frac{d(x_{n_k},Tx_{n_k})d(x,Tx)}{1+d(x_{n_k},x)}, \frac{d(x_{n_k},Tx_{n_k})d(x,Tx)}{1+d(Tx_{n_k},Tx)}, \frac{d(x_{n_k},Tx_{n_k})[1+d(x,Tx)]}{1+d(Tx_{n_k},Tx)}, \\ & \left. \frac{d(x,Tx)[1+d(Tx_{n_k},Tx)]}{1+d(x,Tx)} \right\} \\ &= Max\left\{ d(x_{n_k},x), d(x_{n_k},x_{n_{k+1}}), d(x,Tx), \frac{d(x_{n_k},Tx)+d(x,x_{n_{k+1}})}{2s}, \right. \\ & \frac{d(x_{n_k},x_{n_{k+1}})d(x,Tx)}{1+d(x_{n_k},x)}, \frac{d(x_{n_k},x_{n_{k+1}})d(x,Tx)}{1+d(x_{n_{k+1}},Tx)}, \frac{d(x_{n_k},x_{n_{k+1}})[1+d(x,Tx)]}{1+d(x_{n_{k+1}},Tx)}, \\ & \left. \frac{d(x,Tx)[1+d(x_{n_{k+1}},Tx)]}{1+d(x,Tx)} \right\} \end{aligned}$$

and

$$\begin{aligned} M(x_{n_k},x) &= Min\left\{ d(x_{n_k},x), d(x_{n_k},Tx_{n_k}), d(x,Tx), d(x_{n_k},Tx), d(x,Tx_{n_k}) \right\} \\ &= Min\left\{ d(x_{n_k},x), d(x_{n_k},x_{n_{k+1}}), d(x,Tx), d(x_{n_k},Tx), d(x,x_{n_{k+1}}) \right\}. \end{aligned}$$

But,

$$\frac{d(x_{n_k},Tx)+d(x,x_{n_{k+1}})}{2s} \leq \frac{sd(x_{n_k},x)+sd(x,Tx)+d(x,x_{n_{k+1}})}{2s}.$$

Then, by (4.3), we get

$$\limsup_{k\to\infty} \frac{d(x_{n_k},Tx)+d(x,x_{n_{k+1}})}{2s} \leq \frac{d(x,Tx)}{2}.$$

When k tends to infinity, we deduce

$$\lim_{k\to\infty} N(x_{n_k},x) = d(x,Tx)$$

and

$$\lim_{k\to\infty} M(x_{n_k},x)=0.$$

Since $\theta(\psi(N(x_{n_k},x)))\leq \frac{1}{s}$ for all $k\in N$, (4.14), we obtain

$$\psi(s^2 d(x,Tx))\leq \frac{1}{s}\psi(d(x,Tx))\leq \psi(d(x,Tx)).$$

Since $\psi\in\Phi$, the above does not hold unless $d(x,Tx)=0$, that is, $Tx=x$ and x is a fixed point of T. □

For the uniqueness of a fixed point we consider the following condition: Condition (G): For all $x,y\in Fix(T)$ either $\alpha(x,y)\geq 1$ or $\alpha(y,x)\geq 1$. Where $Fix(T)$ denotes the set of all fixed point of T.

Theorem 4.5 *Adding condition (G) to the hypotheses of Theorem 4.3 (respectively, Theorem 4.4), we obtain the uniqueness of the fixed point of T.*

Proof: Suppose x and y are two distinct fixed point of T. Now,

$$\begin{aligned} N(x,y) &= Max\Bigg\{d(x,y),d(x,Tx),d(y,Ty),\frac{d(x,Ty)+d(y,Tx)}{2s},\\ &\quad \frac{d(x,Tx)d(y,Ty)}{1+d(x,y)},\frac{d(x,Tx)d(y,Ty)}{1+d(Tx,Ty)},\frac{d(x,Tx)[1+d(y,Ty)]}{1+d(Tx,Ty)},\\ &\quad \frac{d(y,Ty)[1+d(Tx,Ty)]}{1+d(y,Ty)}\Bigg\}\\ &= Max\Bigg\{d(x,y),d(x,x),d(y,y),\frac{d(x,y)+d(y,x)}{2s},\\ &\quad \frac{d(x,x)d(y,y)}{1+d(x,y)},\frac{d(x,x)d(y,y)}{1+d(x,y)},\frac{d(x,x)[1+d(y,y)]}{1+d(x,x)},\\ &\quad \frac{d(y,y)[1+d(x,y)]}{1+d(y,y)}\Bigg\}\\ &= Max\{d(x,y),0\}=d(x,y) \end{aligned}$$

and

$$
\begin{aligned}
M(x,y) &= Min\left\{d(x,y),d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)\right\} \\
&= Min\{d(x,y),d(x,x),d(y,y),d(x,y),d(y,x)\} = 0.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\psi(d(x,y)) &\leq \psi(s^3 d(x,y)) \\
&\leq \alpha(x,y)\psi(s^3 d(x,y)) \\
&\leq \theta(\psi(N(x,y)))\psi(N(x,y)) + L\phi(M(x,y)) \\
&< \frac{1}{s}\psi(d(x,y)) \\
&\leq \psi(d(x,y)),
\end{aligned}
$$

which is a contradiction. Therefore, $x = y$. Hence, T has a unique fixed point. □

Now, we give supportive example to our main result.

Example 4.3 *Let $X = [-1,1]$ be endowed with the b-metric by $d(x,y) = (x-y)^2$. Then, (X,d) is a complete b-metric space with $s = 2$.*
Let $T : X \to X$ be the mapping defined by

$$
T(x) = \begin{cases} 0 & if\ -1 \leq x \leq 0 \\ -x & if\ 0 < x \leq 1\ . \end{cases}
$$

Define $\alpha : X \times X \to R^+$, $\psi : R^+ \to R^+$, $\phi : R^+ \to R^+$ and $\theta : R^+ \to [0, \frac{1}{2})$ as

$$
\alpha(x,y) = \begin{cases} 1 & if\ x,y \in [-1,0] \\ 0 & if\ x,y \in (0,1] \\ \frac{1}{15} & otherwise, \end{cases} ;
$$

$\psi(t) = \frac{t^2}{2}$; $\phi(t) = \frac{t}{3}$ and $\theta(t) = \frac{1}{20}$.
We next illustrate that all conditions in Theorem 4.3 are hold.

Clearly T is continuous and triangular α-admissible mapping.
Taking $x_0 \in [-1,0]$, we have $\alpha(x_0, Tx_0) = \alpha(x_0, 0) = 1 \geq 1$.
We next prove that T is generalized rational (α, ψ, ϕ)-Geraghty contraction type mapping with $L \geq 0$. To show this we need to consider the following three cases:
***Case (i)** For $x, y \in [-1,0]$, we have*
$\alpha(x,y)\psi(s^3 d(Tx,Ty)) = 0 \leq \theta(\psi(N(x,y)))\psi(N(x,y)) + L\phi(M(x,y))$.
***Case (ii)** For $x, y \in (0,1]$, we have*
$\alpha(x,y)\psi(s^3 d(Tx,Ty)) = 0 \leq \theta(\psi(N(x,y)))\psi(N(x,y)) + L\phi(M(x,y))$.
***Case (iii)** For $x \in [-1,0]$ and $y \in (0,1]$, we have*

$$
\begin{aligned}
\alpha(x,y)\psi(s^3 d(Tx,Ty)) \quad &= \quad \frac{1}{15}\psi(2^3 d(0,-y)) \\
&= \frac{1}{15}\psi(8y^2) \\
&= \frac{32}{15}y^4 \\
&\leq \theta(\psi(N(x,y)))\psi(N(x,y)) + L\phi(M(x,y)).
\end{aligned}
$$

Thus, all the assumptions in Theorem 4.3 are satisfied and T has a unique fixed point which is $x = 0$.

Example 4.4 *Let X be a finite set defined as $X = \{4,5,6\}$.*
Defined $d : X \times X \to R^+$ as
$d(x,x) = 0$ *for all* $x \in X$ *and*
$d(x,y) = d(y,x)$ *for all* $x,y \in X$
$d(4,5) = 1$, $d(4,6) = 4$ *and* $d(5,6) = 2$.
Therefore (X,d) is complete b-metric spaces with $s = \frac{4}{3}$, but (X,d) is not a metric space because it lacks the triangular inequality as follow:
$d(4,6) = 4 > 3 = d(4,5) + d(5,6)$
Let $T : X \to X$ be the mapping defined by

$$
T(x) = \begin{cases} 5 & if\, x \neq 4 \\ 6 & if\, x = 4. \end{cases}
$$

Define $\alpha : X \times X \to R^+$, $\psi : R^+ \to R^+$, $\phi : R^+ \to R^+$, $\theta : R^+ \to [0, \frac{3}{4})$ *as*

$$\alpha(x,y) = \begin{cases} 1 & \text{if } x,y \in X \backslash \{4\} \\ \frac{1}{20} & \text{otherwise.} \end{cases};$$

$\psi(t) = t$; $\phi(t) = \frac{t^2}{2}$ *and* $\theta(t) = \frac{5}{8}$.
We next illustrate that all the conditions in Theorem 4.4 hold.
Taking $x_0 = 5$, *we have* $\alpha(x_0, Tx_0) = \alpha(5, T5) = \alpha(5,5) = 1 \geq 1$ *and clearly,* T *is triangular* α*-admissible mapping. Let* $\{x_n\}$ *be a sequence such that* $\alpha(x_n, x_{n+1}) \geq 1$ *for all* $n \in N$ *and* $x_n \to x$ *as* $n \to \infty$*. By the definition of* α*, for each* $n \in N$*, we get that* $x_n \in X \backslash \{4\} = \{5,6\}$*. We obtain that* $x \in \{5,6\}$*. Thus we have* $\alpha(x_n, x) \geq 1$ *for each* $n \in N$.
We next prove T *is generalized rational* (α, ψ, ϕ)*-Geraghty contraction type mapping with* $L \geq 0$*. So we consider the following cases:*
Case(i) *For* $x = y$ *or* $x = 5$ *and* $y = 6$ *trivially* T *is generalized rational* (α, ψ, ϕ)*-Geraghty contraction type mapping.*
Case(ii) *For* $x = 4$ *and* $y = 5$ *or* $y = 6$*, we have*
$N(x,y) = 4$, $M(x,y) = 0$ *and*

$$\begin{aligned} \alpha(x,y)\psi(s^3 d(Tx,Ty)) &= \frac{1}{20}\psi((\frac{4}{3})^3 d(6,5)) \\ &= \frac{32}{135} \\ &\leq \frac{5}{2} \\ &= \theta(\psi(N(x,y)))\psi(N(x,y)) + L\phi(M(x,y)). \end{aligned}$$

Hence, all assumptions in Theorem 4.4 are satisfied and thus T *has a fixed point which is* $x = 5$.
Finally let $x, y \in Fix(T)$*. Clearly* $x = y = 5$*, therefore, by the definition of* α*, we have* $\alpha(x,y) = \alpha(5,5) = 1 \geq 1$*. So, all assumptions in Theorem 4.5 are satisfied and thus* T *has a unique fixed point which is* $x = 5$.

Now, we give some corollaries to our main findings.

Corollary 4.6 *Let (X,d) be a complete b-metric space with the coefficient $s \geq 1$, $\alpha : X \times X \to R^+$, $\theta \in \Omega$, $\psi \in \Phi$ and $T : X \to X$ be a mapping. Suppose that the following conditions hold:*

(i) $\alpha(x,y)\psi(s^3 d(Tx,Ty)) \leq \theta(\psi(N(x,y)))\psi(N(x,y))$, *where*

$$N(x,y) = Max\left\{ d(x,y), \frac{d(x,Tx)d(y,Ty)}{1+d(x,y)}, \frac{d(x,Tx)d(y,Ty)}{1+d(Tx,Ty)} \right\};$$

(ii) T is triangular α-admissible mapping;

(iii) There exists $x_0 \in X$ such that $\alpha(x_0,Tx_0) \geq 1$;

(iv) T is b-continuous.

Then, T has a fixed point $x \in X$.

Proof: The result follows by taking $L = 0$ and $N(x,y) = Max\{d(x,y), \frac{d(x,Tx)d(y,Ty)}{1+d(x,y)}, \frac{d(x,Tx)d(y,Ty)}{1+d(Tx,Ty)}\}$ in Theorem 4.3. □

Remark 4.1 *By taking $s = 1$ in Corollary 4.6 we get the result of Singh, et al. (2021) in complete metric spaces.*

Corollary 4.7 *Let (X,d) be a complete b-metric space with the coefficient $s \geq 1$, $\alpha : X \times X \to R^+$, $\theta \in \Omega$, $\psi \in \Phi$ and $T : X \to X$ be a mapping. Suppose that the following conditions hold:*

(i) $\alpha(x,y)\psi(s^3 d(Tx,Ty)) \leq \theta(\psi(N(x,y)))\psi(N(x,y))$, *where*

$$N(x,y) = Max\left\{ d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2s}, \frac{d(x,Tx)d(y,Ty)}{1+d(x,y)}, \frac{d(x,Tx)d(y,Ty)}{1+d(Tx,Ty)}, \frac{d(x,Tx)[1+d(y,Ty)]}{1+d(Tx,Ty)}, \frac{d(y,Ty)[1+d(Tx,Ty)]}{1+d(y,Ty)} \right\}.$$

(ii) T is triangular α-admissible mapping;

(iii) There exists $x_0 \in X$ such that $\alpha(x_0,Tx_0) \geq 1$;

(iv) T is b-continuous or X is α-regular.

Then, T has a fixed point $x \in X$.

Proof: The result follows by taking $L = 0$ in Theorem 4.3 (or Theorem 4.4). □

Corollary 4.8 *Let (X,d) be a complete b-metric space with the coefficient $s \geq 1$, $\theta \in \Omega$, $\psi, \phi \in \Phi$ and $T : X \to X$ be a mapping. Suppose that the following conditions hold:*
(i) $\psi(s^3 d(Tx,Ty)) \leq \theta(\psi(N(x,y)))\psi(N(x,y)) + L\phi(M(x,y))$,
where $L \geq 0$, $M(x,y) = min\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}$ and

$$N(x,y) = Max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2s}, \frac{d(x,Tx)d(y,Ty)}{1+d(x,y)}, \frac{d(x,Tx)d(y,Ty)}{1+d(Tx,Ty)}, \frac{d(x,Tx)[1+d(y,Ty)]}{1+d(Tx,Ty)}, \frac{d(y,Ty)[1+d(Tx,Ty)]}{1+d(y,Ty)}\right\};$$

(ii) T is b-continuous.
Then, T has a fixed point $x \in X$.

Proof: The result follows by taking $\alpha(x,y) = 1$ for all $x, y \in X$ in Theorem 4.3 .
□

Chapter 5

Conclusion and Future scope

5.1 Conclusion

Singh, et al. (2021) established a fixed point results for generalized rational (α, ψ)-Geraghty contraction type mappings in complete metric spaces and proved the existence and uniqueness of fixed points. In this research work, the authors established fixed point results for generalized rational (α, ψ, ϕ)-Geraghty contraction type mappings in the setting of complete b-metric space and proved the existence and uniqueness of fixed points for the introduced mappings. The results of this research work extend and generalize related fixed point results in the literature particularly that of Singh, et al. (2021) from metric space to b-metric space. We have also supported the main results of this research work by applicable examples.

5.2 Future scope

There are some published results related to the existence of fixed point theorems of mappings defined on b-metric spaces. The researchers believe the search for the existence and uniqueness of fixed points of self-mappings satisfying generalized rational (α, ψ, ϕ)-Geraghty contraction type mappings in b-metric space is an active area of the study. So, any interested researchers can use this opportunity and conduct their research work in this area.

References

Afshari, H., Aydi, H., & Karapinar, E. (2020). On generalized $\alpha - \psi$-Geraghty contractions on b-metric spaces. *Georgian Mathematical Journal*, 27(1), 9-21.

Amini-Harandi A., Emami H., (2010). A fixed point theorem for contraction type maps in partially ordered metric spaces and applications to ordinary differential equation. *Nonlinear Analysis: Theory, Methods and Applications*, 72(5), 2238-2242

Arshad, M., & Hussain, A. (2017). Fixed Point results for generalized rational α-Geraghty contraction. *Miskolc Mathematical Notes*, 18(2), 611-621

Aydi, H., Karapinar, E., Erhan, I. M., & Salimi, P. (2014). Best proximity points of generalized almost ψ-geraghty contractive non-self-mappings. *Fixed Point Theory and Applications*, 2014(1), 1-13.

Bakhtin, I. A. (1989). The contraction principle in quasi metric spaces, *Func. An. Ulian. Gos.Ped.Ins*, 30, 26-37.

Banach, S. (1922). Sur les operations dans les ensembles abs traits etleur application aux equations integrals. *Fund.math*, 3(1), 133-181.

Boriceanu, M., Bota, M. and Petruel, A. (2010). Multivalued fractals in b-metric spaces. Central European Journal of Mathematics, 8(2), 367-377.

Boriceanu, M. (2009). Strict fixed point theorems for multivalued operators in b-metric spaces. *Int. J. Mod. Math*, 4(3), 285-301.

Caballero, J., Harjani, J., & Sadarangani, K. (2012). A best proximity point theorem for Geraghty contractions. *Fixed Point Theory and Applications*, 2012(1), 1-9.

Cho, S. H., Bae, J. S., & Karapinar, E. (2013). Fixed point theorems for α-Geraghty contraction type maps in metric spaces. *Fixed Point Theory and Applications*, 2013(1), 1-11.

Czerwik, S. (1993).Contraction mappings in b-metric spaces. *Acta Mathematica et informatica universit at iso straviensis*, 1(1), 5-11.

Geraghty, M. A. (1973). On contractive mappings. *Proceedings of the American Mathematical Society*, 40(2), 604-608.

Gordji, M. E., Ramezani, M., Cho, Y. J., & Pirbavafa, S. (2012). A generalization of Geraghty's theorem in partially ordered metric spaces and applications to ordinary differential equations. *Fixed Point Theory and Applications*, 2012(1), 1-9.

Huang, L. G., & Zhang, X. (2007). Cone metric spaces and fixed point theorems of contractive mappings. *Journal of mathematical Analysis and Applications*, 332(2), 1468-1476.

Jovanovic, M., Kadelburg, Z., & Radenovic, S. (2010). Common fixed point results in metric-type spaces. *Fixed point theory and applications*, 2010, 1-15.

Karapinar, E. (2013). On best proximity point of ψ-Geraghty contractions. *Fixed Point Theory and Applications*, 2013(1), 1-9.

Karapinar, E., & Samet, B. (2012, September). Generalized $\alpha-\psi$ contractive type mappings and related fixed point theorems with applications. *In Abstract and Applied Analysis. Hindawi Limited.*

Karapinar, E. (2014). $\alpha-\psi$-Geraghty contraction type mappings and some related fixed point results. *Filomat*, 28(1), 37-48.

Kutbi M.A, M. Arshad and A. Hussain, (2015) On Modiied $\alpha-\eta$-Contractive mappings, *Abstr. Appl. Anal.* Article ID 657858, 7 pages.

O'Regan, D., & Petrusel, A. (2008). Fixed point theorems for generalized contractions in ordered metric spaces. *Journal of Mathematical Analysis and Applications*, 341(2), 1241-1252.

Paesano, D., & Vetro, P. (2015). Fixed point theorems for α-set-valued quasi-contractions in b-metric spaces.

Popescu, O. (2014). Some new fixed point theorems for α-Geraghty contraction type maps in metric spaces, *Fixed Point Theory Appl*. Paper No. 190.

Rhoades, B.E. (1977). A comparison of various definitions of contractive mappings. *Transactions of the American Mathematical Society*, 226, pp.257-290.

Roshan, J. R., Parvaneh, V., & Kadelburg, Z. (2014). Common fixed point theorems for weakly isotone increasing mappings in ordered b-metric spaces. *J. Nonlinear Sci. Appl*, 7(4), 229-245.

Samet, B., Vetro, C., & Vetro, P. (2012). Fixed point theorems for $\alpha-\psi$-contractive type mappings. *Nonlinear analysis: theory, methods and applications*, 75(4), 2154-2165.

Shahkoohi, R. J., & Razani, A. (2014). Some fixed point theorems for rational Geraghty contractive mappings in ordered b-metric spaces. *Journal of Inequalities and Applications*, 2014(1), 1-23.

Singh, K. A., Singh, M., & Singh, T. C. (2021). Fixed point theorems for generalized rational $\alpha - \psi$-Geraghty contraction type mappings in metric space. *Electronic Journal of Mathematical Analysis and Applications*, 9(1), 52-58.

Yang, S. K., Bae, J. S., & Cho, S. H. (2011).Coincidence and common fixed and periodic point theorems in cone metric spaces. *Computers and Mathematics with Applications*, 61(2), 170-177.