A Common Fixed Point Result for α - Admissible Z-Contraction Mapping via Simulation Function in the Setting of b-Metric Spaces

A Thesis Submitted to the Department of Mathematics in Partial Fulfillment for the Requirements of the Degree of Masters of Science in Mathematics.

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Declaration

I, the undersigned declare that, this research paper entitled ”common fixed point results for α- admissible Z-contraction involving simulation function in the setting of b-metric space ” is my own original work and it has not been submitted for the award of any academic degree or the like in any other institution or university, and that all the sources I have used or quoted have been indicated and acknowledged.
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Abstract

In this thesis, we introduced a new class of maps namely an almost generalized α-admissible Z- contraction pair of maps and establish a common fixed point theorem. Moreover, we prove the existence and uniqueness of common fixed points in the setting of b-metric spaces with the help of simulation function. Our results unify several related results in the existing literature. Finally, we verify the established result by an example.

Contents

Chapter 1

Introduction

1.1 Background of the study

Fixed point theory is an important tool in the study of nonlinear analysis. It is considered to be the key connection between pure and applied mathematics. It is also widely applied in different fields of study such as Economics, Chemistry, Physics and almost all engineering fields. The famous Banach contraction principle introduced by Banach (1922) ensures the existence and uniqueness of fixed points for a contraction mapping in complete metric spaces. Several researchers generalized and extended this principle by introducing various contractions in different ambient spaces. In 1993, Czerwik (1993) introduced the concept of a b-metric space as a generalization of a metric space. In recent years, the theory of fixed points has attracted widespread attention and has been rapidly growing. It was massively studied by many researchers giving new results by using classes of implicit functions defining more general contractive conditions. Khojasteh et al. (2015) presented the notion of contractions involving a new class of simulation functions that has been used and improved by many authors in various spaces. For more works in this line of research, we refer Karapinar (2016), Babu and Dula (2018), Melliani et al. (2020), and Zoto et al. (2020). Recently, Melliani et al. (2020) introduced a new concept of α-admissible almost type Z-contraction and proved the existence of fixed points for admissible almost type Z-contractions in a complete metric space. They generalized the works of Berinde (2004), Ciric (1972), Hardy and Rogers (1973), Kannan (1968) and Karapinar and Samet (2012). The purpose of this study is to prove the existence and uniqueness of common fixed points for an almost generalized α-admissible Z- contraction pair of maps in the setting of b-metric spaces.

Next, we present some notions, definitions and theorems used in the sequel. Throughout the study, we shall use R and R^{+} to represent the set of real numbers and the set of nonnegative real numbers respectively.

Definition 1.1.1 *(Czerwik , 1993). Let X be a (nonempty) set and $s \geq 1$ be a given real number. A function $d : X \times X \to \Re^+$ is said to be a b-metric if and only if for all $x, y, z \in X$, the following conditions are satisfied:*

a. $d(x,y) = 0$ *if and only if* $x = y$;

b. $d(x,y) = d(y,x)$*;*

c. $d(x,z) \leq s[d(x,y) + d(y,z)]$.
The triplet (X,d,s) is called a b-metric space.

Definition 1.1.2 *(Boriceanu et al., 2010). Let X be a b-metric space and $\{x_n\}$ be a sequence in X, we say that:*

a. $\{x_n\}$ *is b-converges to* $x \in X$ *if* $d(x_n, x) \to 0$ *as* $n \to \infty$.

b. $\{x_n\}$ *is a b-Cauchy sequence if* $d(x_n, x_m) \to 0$ *as* $n, m \to \infty$.

c. (X,d) *is b-complete if every b-Cauchy sequence in X is b-convergent.*

Remark (Boriceanu, et al., 2010). In a b-metric space(X,d), the following assertions hold:

(R_1) A convergent sequence has a unique limit;

(R_2) Each convergent sequence is a Cauchy sequence;

(R_3) In general, a b-metric is not continuous;

(R_4) In general, a b-metric does not induce a topology on X.

Example 1.1 (Aghajani text et al.,2014). Let (X,d) be a metric space and $\rho(x,y) = (d(x,y))^p$, where $p > 1$ is a real number. Then ρ is a b-metric with $s = 2^{p-1}$. However, if (X,d) is a metric space, then $\rho(x,y)$ is not necessarily a metric space. For example, if $X = R$ and $d(x,y) = |x-y|$, then $\rho(x,y) = ((d(x,y))^s$ is a b-metric on R with $s = 2$ but it is not a metric on R.

Definition 1.1.3 *(Berinde, 2004). Let (X,d) be a metric space then a map $T : X \longrightarrow X$ is called an almost contraction or (δ, L) contraction if there exist constants $\delta \in (0,1)$ and $L \geq 0$ such that,*

$$d(Tx,Ty) \leq \delta d(x,y) + Ld(y,Tx)$$

for all $x,y \in X$.

Lemma 1.1.1 *(Roshan et al.,2014). Suppose (X,d) is a b-metric space with coefficient $s \geq 1$ and $\{x_n\}$ is a sequence in X such that $d(x_n,x_{n+1}) \longrightarrow 0$ as $n \longrightarrow \infty$. If $\{x_n\}$ is not a Cauchy sequence, then there exist an $\varepsilon > 0$ and sequences of positive integers $\{m_k\}$ and $\{n_k\}$ with $n_k > m_k > k$ such that $d(x_{m_k},x_{n_k}) \geq \varepsilon, d(x_{m_k},x_{n_k-1}) < \varepsilon$ and the following results hold:*

(i) $\varepsilon \leq \underline{\lim} d(x_{m_k},x_{n_k}) \leq \overline{\lim} d(x_{m_k},x_{n_k}) < s\varepsilon$,

(ii) $\frac{\varepsilon}{s} \leq \underline{\lim} d(x_{m_k+1},x_{n_k}) \leq \overline{\lim} d(x_{m_k+1},x_{n_k}) < s^2\varepsilon$,

(iii) $\frac{\varepsilon}{s} \leq \underline{\lim} d(x_{m_k},x_{n_k+1}) \leq \overline{\lim} d(x_{m_k},x_{n_k+1}) < s^2\varepsilon$,

(iv) $\frac{\varepsilon}{s^2} \leq \underline{\lim} d(x_{m_k+1},x_{n_k+1}) \leq \overline{\lim} d(x_{m_k+1},x_{n_k+1}) < s^3\varepsilon$.

Definition 1.1.4 *(Khojasteh et al., 2015). Let $\zeta : R^+ \times R^+ \longrightarrow R$ be a mapping, then $\zeta \in Z$ is called a simulation function if it satisfies the following conditions:*
(ζ_1) $\zeta(0,0) = 0$;
$(\zeta_2)\zeta(t,s) < s - t$ *for all* $t,s > 0$;
(ζ_3) *if* s_n , t_n *are sequences in* $(0,\infty)$ *such that*

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} t_n = \ell > 0.$$

Then $\lim_{n\to\infty} sup\zeta(t_n,s_n) < 0$.

The collection of all simulation functions is denoted by Z. Many different examples of simulation functions can be found in Khojasteh et al. (2015).

Definition 1.1.5 *(Khojasteh et al., 2015). Let X be a complete metric space with metric d, $T : X \to X$ a mapping and $\zeta \in Z$. Then T is called a Z-contraction with respect to ζ if the following condition is satisfied*:

$$\zeta(d(Tx,Ty),d(x,y)) \geq 0,$$

for all $x,y \in X$.

Example 2:(Khojasteh et al., 2015). Let $\zeta_i : R^+ \times R^+ \longrightarrow R$ where $i = 1,2,3$ is defined as follows:

(i) $\zeta_1(t,s) = \lambda s - t$ for all $s,t \in R^+$ where $\lambda \in [0,1)$.

(ii) $\zeta_2(t,s) = \varphi(s) - t$ for all $s,t \in R^+$ where $\varphi : R^+ \longrightarrow R^+$ is an upper semi-continuous function such that $\varphi(t) = 0$ if and only if $t = 0$ and $\varphi(t) < t$ for all $t > 0$.

(iii) $\zeta_3(t,s) = \varphi(s) - \theta(t)$ for all $s,t \in R^+$ where $\varphi, \theta : R^+ \longrightarrow R^+$ are continuous functions such that $\varphi(t) = \theta(t) = 0$ if and only if $t = 0$ and $\varphi(t) < t \leq \theta(t)$ for all $t > 0$.

Definition 1.1.6 *(Samet et al. 2012). Let X be a nonempty set, and $T : X \longrightarrow X$ and $\alpha : X \times X \longrightarrow R^+$ be maps. Then T is called α-admissible if $\alpha(x,y) \geq 1$ implies $\alpha(Tx,Ty) \geq 1$, for each $x,y \in X$.*

Definition 1.1.7 *(Popescu, 2014). Let X be a nonempty set, and $T : X \longrightarrow X$ and $\alpha : X \times X \longrightarrow R^+$ be maps. Then T is said to be α-orbital admissible mapping if $\alpha(x,Tx) \geq 1$ implies $\alpha(Tx,T^2x) \geq 1$, for each $x \in X$.*

Definition 1.1.8 *(Felhi et al., 2016). For a nonempty set X, let $A,B : X \longrightarrow X$ and $\alpha : X \times X \longrightarrow R^+$ be mappings. We say that (A,B) is an α-admissible pair if for all $x,y \in X$, we have if $\alpha(x,y) \geq 1$ implies $\alpha(Ax,By) \geq 1$and $\alpha(By,Ax) \geq 1$.*

Definition 1.1.9 *(Popescu, 2014). Let X be a nonempty set, and $T : X \longrightarrow X$ and $\alpha : X \times X \longrightarrow R^+$ be maps. Then T is said to be a triangular α-orbital admissible if:*

(i) T is α-orbital admissible mapping.

(ii) $\alpha(x,y) \geq 1$ and $\alpha(y,Ty) \geq 1$ implies $\alpha(x,Ty) \geq 1$,for each $x,y \in X$.

Definition 1.1.10 *(Karapinar, 2016). Let X be a nonempty set and $T : X \longrightarrow X$ be a self-map defined on a metric space (X,d). If there exist $\zeta \in Z$ and $\alpha : X \times X \longrightarrow R^+$ and $\alpha : X \times X \longrightarrow R^+$ such that*

$$\zeta(\alpha(x,y)d(Tx,Ty), d(x,y)) \geq 0,$$

for all $x,y \in X$, then T is called an α-admissible Z- contraction with respect to ζ.

Definition 1.1.11 *(Melliani et al., 2020). Let (X,d) be a metric space with simulation function $\zeta \in Z$. We say that $T : X \longrightarrow X$ is an α-admissible almost Z-contraction if there exists $\alpha : X \times X \longrightarrow R^+$ and a constant $L \geq 0$ such that,*

$$\zeta(\alpha(x,y)d(Tx,Ty), M(x,y) + LN(x,y)) \geq 0$$

for all $x,y \in X$, where $M(x,y)$, $N(x,y)$, $m(x,y)$ are given below.

$$M(x,y) = max\{d(x,y), \frac{d(x,Tx)+d(y,Ty)}{2}, \frac{d(x,Ty)+d(y,Tx)}{2}\},$$

$$N(x,y) = min\{d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}.$$

Theorem 1.1.2 *(Melliani et al., 2020). Let (X,d) be a complete metric space and T is an α-admissible almost Z-contraction with respect to ζ. Suppose that,*

(i) T is triangular α-orbital admissible,

(ii) T is continuous,

(iii) there exist $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$,

Then T has a fixed point.

1.2 Statements of the problem

Khojasteh et al. (2015) introduced the notion of Z-contraction and studied existence and uniqueness of fixed points for Z-contraction type operators. This class of Z-contractions unifies large types of nonlinear contractions existing in the literature. Afterwards, Karapinar (2016) originated the concept of α-admissible Z-contraction and presented some fixed point results in the setting of a complete metric spaces by defining a new contractive condition via admissible mapping imbedded in simulation function. Recently, Melliani et al. (2020) introduced a new concept of α-admissible almost type Z-contraction and proved the existence of fixed points for admissible almost type Z-contractions in a complete metric space.
Inspired and motivated by the works of Melliani et al. (2020), in this research we will extend their work to finding common fixed mapping by using a pair of maps, modify their contractive condition, and generalize the space under their consideration by taking to b-metric space.

1.3 Objectives of the study

1.3.1 General objective

The main objective of this research work is to study a common fixed point result for almost generalized α-admissible Z- contraction pair of maps in the setting of b-metric space.

1.3.2 Specific objectives

The specific objectives of this study are:

- To prove the existence of common fixed points for almost generalized α-admissible Z- contraction pair of maps in the setting of b-metric space.
- To prove the uniqueness of the common fixed points for an almost generalized α-admissible Z- contraction pair of maps.
- To provide an example to verify the established result.

1.4 Significance of the study

The study may have the following importance:

- The results obtained in this study may contribute to research activities in this area.
- It may be used to solve some applicable problems in applied mathematics.
- It may help the researcher to develop scientific research writing skills and scientific communication in Mathematics.

1.5 Delimitation of the Study

This study was delimited to prove common fixed point results for an almost generalized α-admissible Z- contraction pair of maps in the setting of b-metric space.

Chapter 2

Review of Related Literatures

The origin of the fixed point theory goes back a century, to the pioneer work of Banach. Since the first study of Banach, researchers have been extended, improved, and generalized this very simple stated but at the same time very powerful theorem. The Banach contraction mapping principle is one of the most versatile elementary results of mathematical analysis. It is widely applied in different branches of mathematics and is regarded as the source of metric fixed point theory. Hence, the terms of the contraction inequality and the abstract structure of Banachs theorem have been investigated.There is a vast literature dealing with technical extensions and generalizations of Banach contraction principle. For instance, in 1993, Czerwik introduced the concept of a b-metric space as a generalization of a metric space. One of the interesting notions, α-admissibility was introduced by Samet et al. (2012). This study, which attracted the attention of many researchers, has been developed and generalized in many directions. For instance, Karapinar and Samet (2012) generalized the results derived in Czerwik (1993) by proposing the concept of generalized $\alpha-\psi$-contractive type. In 2016, Karapinar originated the concept of α-admissible Z-contraction to obtain some interesting fixed point results in the context of complete metric spaces. In 2020 Melliani et al., introduced a new concept of α-admissible almost type Z-contraction with respect to a simulation function ζ, and proved some results about existence and uniqueness of fixed points for such mappings; their results unify several well-known types of contraction and generalize several existing results in the literature. The crucial notion of this research is the simulation function which is defined by Khojasteh et al. (2015).

Inspired and motivated by the works of Melliani et al. (2020), in this research we will extend their work to finding common fixed results for a pair of maps, modify their contractive condition, and generalize the space under their consideration.

Chapter 3

Methodology

3.1 Study Site and period

This study was conducted from September 2021 G.C to February 2022 G.C in Jimma University under the department of Mathematics.

3.2 Study Design

In order to achieve the objectives of the study, we employed analytical method of design.

3.3 Source of Information

This study mostly depended on document materials, so the available sources of information for the study were books, journals, and internet.

3.4 Mathematical Procedure of the Study

In this study we followed the procedures stated below:

- Establishing common fixed point theorem.
- Constructing sequences.
- Showing the constructed sequences is b-Cauchy .
- Showing the b-convergences of the Cauchy sequence.
- Proving the existence of common fixed points.
- Proving the uniqueness of the common fixed points.
- Verifying the main finding of the research by an applicable example.

Chapter 4

Result and Discussion

In this section,we present our main findings.

Definition 4.0.1 *Let (X,d) be a b-metric space with parameter $s \geq 1$. Let $f,g : X \longrightarrow X$ and $\alpha : X \times X \longrightarrow R^+$ be maps. Assume that there exists a simulation function ζ and a constant $L \geq 0$, such that,*

$$\zeta(S^4\alpha(fx,gx)d(fx,gx), M(x,y)+L.m(x,y)) \geq 0, \text{ for all } x,y \in X, \tag{4.1}$$

where
$M(x,y) = max\{d(x,y), d(x,fx), d(y,gy), \frac{d(x,fx)+d(y,gy)}{2}, \frac{d(x,gy)+d(y,fx)}{2s}\}$ *and*
$m(x,y) = min\{d(x,fx), d(y,gy), d(x,gy), d(y,fx)\}$.
Then the pair (f,g) is called an almost generalized α-admissible Z-contraction pair of maps.
Proposition 4.0.1 *Let (X,d) be a b-metric space with coefficient $s \geq 1$ and $f,g : X \longrightarrow X$ be two self maps. Assume that (f,g) is an almost generalized α-admissible Z- contraction pair of maps and suppose there exist $x_0 \in X$ such that*
$min\{\alpha(x_0, fx_0), \alpha(fx_0, x_0)\} \geq 1$. *Then u is a fixed point of f if and only if u is a fixed point of g. Moreover, in that case u is unique.*
Proof: *Let $x_0 \in X$ be arbitrary. Since $fX \subseteq X$ and $gX \subseteq X$, there exist x_1,$x_2 \in X$ such that $fx_0 = x_1$ and $gx_1 = x_2$. Similarly, there exist $x_3, x_4 \in X$ such that $fx_2 = x_3$ and $gx_3 = x_4$.*
In general, we can construct a sequence $\{x_n\}$ by

$$fx_{2n} = x_{2n+1}$$

and

$$gx_{2n+1} = x_{2n+2},$$

for $n = 0, 1, 2, ...$

Since f and g are α-admissible pair of maps and using the hypothesis that exists $x_0 \in X$ such that min$\{\alpha(x_0, fx_0), \alpha(fx_0, x_0)\} \geq 1$, we get

$$\alpha(x_0, x_1) \geq 1 \Rightarrow \begin{cases} \alpha(x_1, x_2) = \alpha(fx_0, gx_1) \geq 1 & \text{and} \\ \alpha(x_2, x_1) = \alpha(gx_1, fx_0) \geq 1 & . \end{cases}$$

continuing in this way, we get
$\alpha(x_n, x_{n+1}) \geq 1$ *and* $\alpha(x_{n+1}, x_n) \geq 1$ *for all* $n \geq 0$.
Now, let u be a fixed point of f, i.e., $fu = u$. Suppose $gu \neq u$. We consider

$$\zeta(s^4 \alpha(u, gu)d(u, gu), M(u, u) + L.m(u, u)) \geq 0, \tag{4.2}$$

where

$$M(u, u) = max\{d(u, u), d(u, gu), \frac{d(u, u) + d(u, gu)}{2}, \frac{d(u, gu) + d(u, u)}{2s}\} = d(u, gu)$$

and

$$m(u, u) = min\{d(u, u), d(u, gu), d(u, gu), d(u, u)\} = 0.$$

Now using $M(u, u)$ and $m(u, u)$ in (4.2), we get

$$\begin{aligned} 0 &\leq \zeta(s^4 \alpha(u, gu)d(u, gu), M(u, u) + L.m(u, u)) \\ &< d(u, gu) - s^4 \alpha(u, gu)d(u, gu) \leq 0, \end{aligned}$$

which is a contradiction.
Hence $gu = u$, so that u is a common fixed point of f and g.
Similarly, let u be a fixed point of g, i.e., $gu = u$. Suppose $fu \neq u$. We consider

$$\zeta(s^4 \alpha(fu, u)d(fu, u), M(u, u) + L.m(u, u)) \geq 0,$$

where

$$M(u, u) = max\{d(u, u), d(u, fu), \frac{d(u, fu) + d(u, u)}{2}, \frac{d(u, u) + d(u, fu)}{2s}\} = d(u, fu)$$

and

$$m(u, u) = min\{d(u, fu), d(u, u), d(u, u), d(u, fu)\} = 0.$$

Now, using $M(u,u)$ and $m(u,u)$ in (4.2), we get

$$\begin{aligned} 0 &\leq \zeta(s^4\alpha(fu,u)d(fu,u), M(u,u)+L.m(u,u)) \\ &< d(u,fu) - s^4\alpha(fu,u)d(fu,u) \leq 0, \end{aligned}$$

which is a contradiction. Hence $fu = u$, so that u is a common fixed point of f and g. Suppose, u and v with $u \neq v$ are two fixed points of f and g, respectively. From inequality (4.1), we have

$$\zeta(s^4\alpha(u,v)d(u,v), M(u,v)+L.m(u,v)) \geq 0, \tag{4.3}$$

where

$$M(u,v) = max\{d(u,v), d(u,u), d(v,v), \frac{d(u,u)+d(v,v)}{2}, \frac{d(u,v)+d(v,u)}{2s}\} = d(u,v)$$

and

$$m(u,v) = min\{d(u,u), d(v,v), d(u,v), d(v,u)\} = 0.$$

Now using $M(u,v)$ and $m(u,v)$ in (4.3), we get

$$\begin{aligned} 0 &\leq \zeta(s^4\alpha(u,v)d(u,v), M(u,v)+L.m(u,v)) \\ &< d(u,v) - s^4\alpha(u,v)d(u,v) < 0, \end{aligned}$$

which is a contradiction. Hence $u = v$. Hence, the proposition follows.

Theorem 4.0.1 *Let (X,d) be a b-metric space with coefficient $s \geq 1$ and $f,g : X \longrightarrow X$ be two selfmaps. Assume that (f,g) is an almost generalized α-admissible Z-contraction pair of maps. Moreover, $\alpha(u,v) \geq 1$ and $\alpha(fu,u) \geq 1, \alpha(u,gu) \geq 1$ for u is a fixed point of either f or g and v is a fixed point of g, and f is triangular α-orbital admissible. Then f,g have a unique common fixed point in X, provided either f or g is b-continuous.*

Proof: *Suppose $x_{2n} = x_{2n+1}$, for some n,then*

$$x_{2n} = fx_{2n},$$

so that x_{2n} is a fixed point of f. Using proposition (4.0.1), we conclude that x_{2n} is *also a fixed point of g, and hence x_{2n} is a common fixed point of f and g. Similarly, suppose $x_{2n+1} = x_{2n+2}$, for some n, then*

$$x_{2n+1} = gx_{2n+1},$$

so that x_{2n+1} is a fixed point of g. Using proposition (4.0.1), we conclude that x_{2n+1} is also a fixed point of f, and hence x_{2n+1} is a common fixed point of f and g. Hence, assume that $x_n \neq x_{n+1}$, for all n.
Now, we consider

$$\zeta(s^4\alpha(fx_{2n}, gx_{2n+1})d(fx_{2n}, gx_{2n+1}), M(x_{2n}, x_{2n+1}) + L.m(x_{2n}, x_{2n+1})) \geq 0, \quad (4.4)$$

for all $x, y \in X$
$M(x_{2n}, x_{2n+1})$

$$= max\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, gx_{2n+1}), \frac{d(x_{2n}, fx_{2n}) + d(x_{2n+1}, gx_{2n+1})}{2}, \frac{d(x_{2n}, gx_{2n+1}) + d(x_{2n+1}, fx_{2n})}{2s}\}$$

$$= max\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), \frac{d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})}{2}, \frac{d(x_{2n}, x_{2n+2}) + d(x_{2n+1}, fx_{2n+1})}{2s}\}$$

$$= max\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\}$$

and

$$m(x_{2n}, x_{2n+1}) = min\{d(x_{2n}, fx_{2n}), d(x_{2n+1}, gx_{2n+1}), d(x_{2n}, gx_{2n+1}), d(x_{2n+1}, fx_{2n})\}$$

$$= min\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), d(x_{2n}, x_{2n+2}), d(x_{2n+1}, x_{2n+1})\} = 0.$$

Suppose $M(x_{2n}, x_{2n+1}) = d(x_{2n+1}, x_{2n+2})$.

Now, using the values of $M(x_{2n},x_{2n+1})$ *and* $m(x_{2n},x_{2n+1})$ *in* (4.4), *we get*

$$\begin{aligned} 0 &\leq \zeta(s^4\alpha(fx_{2n},gx_{2n+1})d(fx_{2n},gx_{2n+1}),M(x_{2n},x_{2n+1})+L.m(x_{2n},x_{2n+1})) \\ &= \zeta(s^4\alpha(x_{2n+1},x_{2n+1})d(x_{2n+1},x_{2n+2}),d(x_{2n+1},x_{2n+2})) \\ &< d(x_{2n+1},x_{2n+2})-s^4\alpha(x_{2n+1},x_{2n+1})d(x_{2n+1},x_{2n+2})<0, \end{aligned}$$

which is a contradiction.
Therefore, $d(x_{2n},x_{2n+1})\geq d(x_{2n+1},x_{2n+2})$.
Now, we consider

$$\zeta(s^4\alpha(fx_{2n+2},x_{2n+1})d(fx_{2n+2},gx_{2n+1}),M(x_{2n+2},x_{2n+1})+L.m(x_{2n+2},x_{2n+1}))\geq 0 \tag{4.5}$$

for all $x,y\in X$,*where*
$M(x_{2n+2},x_{2n+1})$

$$= max\{d(x_{2n+2},x_{2n+1}),d(x_{2n+2},fx_{2n+2}),\frac{d(x_{2n+2},fx_{2n+2})+d(x_{2n+1},gx_{2n+1})}{2}, \frac{d(x_{2n+2},gx_{2n+1})+d(x_{2n+1},fx_{2n+2})}{2s}\}.$$

$$= max\{d(x_{2n+2},x_{2n+1}),d(x_{2n+2},x_{2n+3}),\frac{d(x_{2n+2},x_{2n+3})+d(x_{2n+1},x_{2n+2})}{2}, \frac{d(x_{2n+2},x_{2n+2})+d(x_{2n+1},x_{2n+3})}{2s}\}.$$

And

$$m(x_{2n+2},x_{2n+1}) = min\{d(x_{2n+2},fx_{2n+2}),d(x_{2n+1},gx_{2n+1}),d(x_{2n+2},gx_{2n+1}),d(x_{2n+1},fx_{2n+2})\}$$

$$= min\{d(x_{2n+2},x_{2n+3}),d(x_{2n+1},x_{2n+2}),d(x_{2n+2},x_{2n+2}),d(x_{2n+1},x_{2n+3})\}=0.$$

Suppose, $M(x_{2n},x_{2n+1})=d(x_{2n+2},x_{2n+3})$.

Now, using the values of $M(x_{2n+2},x_{2n+1})$ and $m(x_{2n+2},x_{2n+1})$ in (4.5), we get

$$\begin{aligned}
0 &\leq \zeta(s^4\alpha(fx_{2n+2},x_{2n+1})d(fx_{2n},gx_{2n+1}),M(x_{2n},x_{2n+1})+L.m(x_{2n},x_{2n+1}))\\
&= \zeta(s^4\alpha(x_{2n+3},x_{2n+1})d(x_{2n+2},x_{2n+3}),d(x_{2n+2},x_{2n+3}))\\
&< d(x_{2n+2},x_{2n+3})-s^4\alpha(x_{2n+3},x_{2n+1})d(x_{2n+2},x_{2n+3})<0,
\end{aligned}$$

which is a contradiction.
Therefore,$d(x_{2n+1},x_{2n+2})\geq d(x_{2n+2},x_{2n+3})$.
Hence,$d(x_{n+1},x_{n+2})\leq d(x_n,x_{n+1})$. Consequently, the sequence $\{d(x_n,x_{n+1})\}$ is a non-increasing and bounded below by zero. Accordingly, there exists $r\geq 0$ such that

$$\lim_{n\to\infty} d(x_n,x_{n+1})=r.$$

Assuming $r>0$ and substituting $M(x_{2n+2},x_{2n+1})=d(x_{n+1},x_{n+2})$ and $m(x_{2n+2},x_{2n+1})=0$ in (4.4),we get,

$$\lim_{n\to\infty} s^4\alpha(fx_{2n},x_n)d(x_n,x_{n+1})=r>0.$$

Letting $t_n=\{s^4\alpha(x_n,x_{n+1})d(x_n,x_{n+1})\}$, $s_n=\{d(x_n,x_{n+1})\}$ and using ζ_3 we obtain,$0\leq \limsup_{n\to\infty}(t_n,s_n)<0$,
which is a contradiction. Thus,we have

$$\lim_{n\to\infty} d(x_n,x_{n+1})=0. \tag{4.6}$$

Now, we need to show that $\{x_n\}$ is a b-Cauchy sequence in X. It suffices to show that $\{x_{2n}\}$ is a b-Cauchy sequence in X. Suppose $\{x_{2n}\}$ is not a b-Cauchy sequence in X, then there exists an $\varepsilon>0$ and subsequences $(2m_k)$ and $(2n_k)$ are two sub-sequences of positive integers with $2n_k>2m_k>k$ for all positive integers k. Moreover, $2m_k$ is chosen as the smallest integer satisfying satisfying (4.7).

$$d(x_{2m_k},x_{2n_k})\geq\varepsilon. \tag{4.7}$$

Thus, we have

$$d(x_{2m_k},x_{2n_k-2})<\varepsilon, \tag{4.8}$$

Since f is triangular α-orbital admissible, we have:

$$\alpha(x_{m_{k-1}},x_{n_{k-1}})\geq 1$$

Now, consider

$$\zeta(s^4\alpha(fx_{2n_k},x_{2m_{k-1}})d(fx_{2n_k},gx_{2m_k-1}),M(x_{2n_k},x_{2m_k-1})+L.m(x_{2n_k},x_{2m_k-1}))\geq 0, \tag{4.9}$$

where

$M(x_{2n_k},x_{2m_k-1})$

$$=max\{d(x_{2n_k},x_{2m_k-1}),d(x_{2n_k},fx_{2n_k}),d(x_{2m_k-1},gx_{2m_k-1})\frac{d(x_{2n_k},fx_{2n_k})+d(x_{2m_k-1},gx_{2m_k-1})}{2},\frac{d(x_{2n_k},gx_{2m_k-1})+d(x_{2m_k-1},fx_{2n_k})}{2s}\}$$

$$=max\{d(x_{2n_k},x_{2m_k-1}),d(x_{2n_k},x_{2n_k+1}),d(x_{2m_k-1},x_{2m_k})\frac{d(x_{2n_k},x_{2n_k+1})+d(x_{2m_k-1},x_{2m_k})}{2},\frac{d(x_{2n_k},x_{2m_k})+d(x_{2m_k-1},x_{2n_k+1})}{2s}\}$$

and $m(x_{2n_k},x_{2m_k-1})$

$$=min\{d(x_{2n_k},fx_{2n_k}),d(x_{2m_k-1},gx_{2m_k-1}),d(x_{2n_k},gx_{2m_k-1}),d(x_{2m_k-1},fx_{2n_k})\}$$
$$=min\{d(x_{2n_k},x_{2n_k+1}),d(x_{2m_k-1},x_{2m_k}),d(x_{2n_k},x_{2m_k}),d(x_{2m_k-1},x_{2n_k+1})\}.$$

Taking the upper limit as $k\longrightarrow\infty$ in (4.9) and using Lemma (4.0.1), we get

$$\limsup M(x_{2n_k},x_{2m_k-1})=max\{s^2\varepsilon,0,0,0,\frac{s\varepsilon+s^2\varepsilon}{2s}\}=s^2\varepsilon.$$

From (4.9),we have

$$0\leq\zeta(s^4\alpha(x_{2n_k+1},x_{2m_{k-1}})d(x_{2n_k+1},x_{2m_k}),M(x_{2n_k},x_{2m_k-1})+L.m(x_{2n_k},x_{2m_k-1})).$$

Now, we have

$$0\leq\overline{\lim}\,\zeta(s^4\alpha(x_{2n_k+1},x_{2m_{k-1}})d(x_{2n_k+1},x_{2m_k}),M(x_{2n_k},x_{2m_k-1})+L.m(x_{2n_k},x_{2m_k-1})).$$

$$0\leq\overline{\lim}M(x_{2n_k},x_{2m_k-1})+L.\overline{\lim}\,m(x_{2n_k},x_{2m_k-1})-s^4.\underline{\lim}\,\alpha(x_{2n_k+1},x_{2m_{k-1}})d(x_{2n_k+1},x_{2m_k})$$

$$0\leq s^2\varepsilon-s^4(\frac{\varepsilon}{s})<0,$$

which is a contradiction.
Hence $\{x_{2n}\}$ *is a b- Cauchy sequence in a b-complete b-metric space* X. *Since X a b-complete b-metric space, we have* $\{x_n\}$ *b-converges to some point u (say) in* X.
so,$u = \lim_{n\to\infty} x_{2n+1} = \lim_{n\to\infty} fx_{2n}$ *and* $u = \lim_{n\to\infty} x_{2n+2} = \lim_{n\to\infty} gx_{2n+1}$.
It follows that

$$\lim_{n\to\infty} fx_{2n} = u = \lim_{n\to\infty} gx_{2n+1}.$$

Assuming f *is b- continuous,*

$$u = \lim_{n\to\infty} x_{2n+1} = \lim_{n\to\infty} fx_{2n} = f(\lim_{n\to\infty} x_{2n}) = fu.$$

Thus, u is a fixed point of f.
Hence, by the Proposition 4.0.1, u is a unique common fixed point of f *and g.*
Now, we give corollaries to our main theorem, Theorem 4.0.1.
If we take $L = 0$ *in Theorem 4.0.1, then we have the following result.*

Corollary ***4.0.2*** *Let* (X,d) *be a b-complete b-metric space with parameter s. Let* $f,g : X \longrightarrow X$ *and* $\alpha : X \times X \longrightarrow R^+$ *be maps. Assume that there exists a simulation function* ζ *such that*

$$\zeta(S^4\alpha(fx,gy)d(fx,gy),M(x,y)) \geq 0$$

for all $x,y \in X$ *where* $M(x,y)$ *is the same as in Theorem 4.0.1,*
also assume that the following conditions hold:

(i) *there exists* $x_0 \in X$ *such that min*$\{\alpha(x_0, fx_0), \alpha(fx_0,x_0)\} \geq 1$;

(ii) f *is triangular* α*-orbital admissible;*

(iii) *Either* f *or* g *is b-continuous;*

(iv) $\alpha(u,v) \geq 1, \alpha(fu,u) \geq 1, \alpha(u,gu) \geq 1$ *for u is a fixed point of either* f *or* g *and v is a fixed point of g.*

Then f,g *have a unique common fixed point in* X.

If we take $\alpha(fx,gy) = 1$ *for all* $x,y \in X$ *in Theorem 4.0.1, we have the following result.*

Corollary ***4.0.3*** *Let (X,d) be a b-complete b-metric space with parameter s. Let $f,g: X \longrightarrow X$ be maps. Assume that there exists a simulation function ζ and a constant $L \geq 0$ such that*

$$\zeta(s^4 d(fx,gy), M(x,y) + Lm(x,y)) \geq 0,$$

For all $x,y \in X$ where $M(x,y)$ and $m(x,y)$ are the same as in Theorem 4.0.1.
Also, assume either f or g is b-continuous.Then f and g have a unique common fixed point in X.
Now, we provide an example in support of Theorem 4.0.1

Example 4.0.1 *Let $X = [0,\infty)$ and let $d: X \times X \longrightarrow [0,\infty)$ be defined by*

$$d(x,y) = \begin{cases} 0 & \text{if } x = y, \\ 4 & \text{if } x,y \in [0,1), \\ 5 + \frac{1}{x+z} & \text{if } x,y \in [0,1), \\ \frac{66}{25} & \text{if otherwise .} \end{cases}$$

The function $\alpha: X \times X \longrightarrow [0,\infty)$ defined by

$$\alpha(x,y) = \begin{cases} 0 & \text{if } x = y, \\ 2 & \text{if } x,y \in [0,1), \\ \frac{3}{2} & \text{if } x,y \in [1,\infty), \\ 1 & \text{if otherwise .} \end{cases}$$

$\alpha(fx,gy)$ defined as given for fx and gx below.
Now we define $f,g: X \longrightarrow X$ by

$$fx = \begin{cases} \frac{x}{4} + 2 & \text{if } x \in [0,1), \\ 3x - 2 & \text{if } x \in [1,\infty), \end{cases}$$

and

$$gx = \begin{cases} x & \text{if } x \in [0,1), \\ \frac{1}{x} & \text{if } x \in [1,\infty). \end{cases}$$

Clearly the pair (f,g) is α-admissible, α-orbital admissible and triangular α-orbital admissible. Now we observe that when $x = \frac{3}{2}$, $z = 2 \in [1,\infty)$ and $y \in [0,1)$ We have

$$d(x,z) = 5 + \frac{1}{(x+z)} = 5 + \frac{2}{7} = \frac{37}{7},$$

$$d(x,y) + d(y,z) = \frac{66}{25} + \frac{66}{25} = \frac{132}{25}.$$

So that

$$d(x,z) \nleq d(x,y) + d(y,z).$$

clearly d is a complete b-metric space with coefficient $s = \frac{25}{24}$. Also, f and g are b-continuous.
Hence the given d is a b-metric with $s = \frac{25}{24} > 1$, but not a metric.
Now we define

$$\zeta : (0,\infty) \times (0,\infty) \longrightarrow (0,\infty)$$

by

$$\zeta(s,t) = \frac{4}{5}t - s$$

and choose $L = 3$. We have the following possible cases
***case**(1): $x, y \in [0,1)$.*
In this case $\alpha(fx,gy) = 1, d(fx,gy) = \frac{66}{25}$,

$$\frac{d(x,fx) + d(y,gy)}{2} = \frac{\frac{66}{25} + 4}{2} = \frac{166}{50},$$

$$\frac{d(x,gy) + d(y,fx)}{2s} = \frac{4 + \frac{66}{25}}{2(\frac{25}{24})} = \frac{3984}{1250},$$

$$m(x,y) = min\{d(x,fx), d(y,gy), d(x,gy,d(y,fx))\}$$
$$= min\{\frac{66}{25}, 4, 4, \frac{66}{25}\} = \frac{66}{25}$$

and

$$M(x,y) = max\{d(x,y), d(x,fx), d(y,gy), \frac{d(x,fx) + d(y,fy)}{2}, \frac{d(x,gy) + d(y,fx)}{2s}\}$$

$$= max\{4, \frac{66}{25}, 4, \frac{166}{50}, \frac{3984}{1250}\} = 4.$$

Now, we consider

$$\zeta(\alpha(fx,gy)d(fx,gy), M(x,y) + L.m(x,y))$$

$$= \frac{4}{5}(M(x,y)) + L.m(x,y)) - s^4\alpha(fx,gy)d(fx,gy)$$

$$= \frac{4}{5}(4 + L.\frac{66}{5}) - (\frac{25}{24})^4(1)(\frac{66}{25}) \geq 0.$$

Hence

$$\frac{16}{5} + L.\frac{264}{125} - \frac{25781250}{8294400} \geq 0$$

case(2)*:* $x, y \in [1, \infty)$.

In this case $fx = 3x - 2 \in [1, \infty)$, $gy = \frac{1}{y} \in [0, 1)$ *and* $\alpha(fx, gy) = 1$,

$d(fx, gy) = \frac{66}{25}, \frac{d(x,fx)+d(y,gy)}{2} = \frac{4+\frac{66}{25}}{2} = \frac{166}{50}$,

$$\frac{d(x,gy) + d(y,fx)}{2} = \frac{\frac{66}{25} + 5}{25} = \frac{\frac{191}{25}}{2(\frac{25}{24})} = \frac{4584}{1250}.$$

$$m(x,y) = min\{d(x,fx), d(y,gy), d(x,gy), d(y,fx)\}$$

$$= min\{4, \frac{66}{25}, \frac{66}{25}, 5\}.$$

$$M(x,y) = max\{d(x,y), d(x,fx), d(y,gy), \frac{d(x,fx) + d(y,gy)}{2}, \frac{d(x,gy) + d(y,fx)}{2s}\}$$

$$= max\{5, 4, \frac{66}{25}, \frac{166}{50}, \frac{4584}{1250}\} = 5.$$

Now we consider

$$\zeta(s^4\alpha(fx,gy)d(fx,gy), M(x,y) + L.m(x,y))$$

$$= \frac{4}{5}(M(x,y) + L.m(x,y) - s^4\alpha(fx,gy)d(fx,gy)$$

$$= \frac{4}{5}(5 + L.\frac{66}{25}) - (\frac{25}{24})^4(1)(\frac{66}{25}) \geq 0.$$

Hence $\frac{20}{5} + L.\frac{264}{125} - \frac{25781250}{8294400} \geq 0$.

case(3)*: $x \in [0,1), y \in [1,\infty)$.*
In this case $fx = \frac{x}{4} + 2 \in [1,\infty), gy = \frac{1}{y} \in [0,1)$

$$\alpha(fx,gy) = 1, d(fx,gy) = \frac{66}{25},$$

$$\frac{d(x,fx)+d(y,fx)}{2} = \frac{\frac{66}{25}+\frac{66}{25}}{2} = \frac{132}{50},$$

$$\frac{d(x,gy)+d(y,fx)}{2s} = \frac{4+4}{2(\frac{25}{24})} = \frac{192}{50},$$

$$m(x,y) = min\{d(x,fx),d(y,gy),d(x,gy),d(y,fx)\}$$

$$= min\{\frac{66}{25},\frac{66}{25},4,4\} = \frac{66}{25}$$

and

$$M(x,y) = max\{d(x,y),d(x,fx),d(y,gy),\frac{d(x,fx)+d(y,gy)}{2},\frac{d(x,gy)+d(y,fx)}{2s}\}$$

$$= max\{\frac{66}{25},\frac{66}{25},\frac{66}{25},\frac{132}{50},\frac{192}{50}\} = \frac{192}{50}.$$

Now we consider

$$\zeta(s^4\alpha(fx,gy)d(fx,gy),M(x,y)+L.m(x,y))$$

$$= \frac{4}{5}(M(x,y)+L.m(x,y) - s^4\alpha(fx,gy)d(x,gy)$$

$$= \frac{4}{5}(\frac{192}{50}+L.\frac{66}{25}) - (\frac{25}{24})^4(1)(\frac{66}{25}) \geq 0.$$

Hence

$$\frac{768}{250}+L.\frac{264}{1250} - \frac{25781250}{8294400} \geq 0.$$

case(4)*: $x \in [1,\infty)$, $y \in [0,1)$.*
In this case $fx = 3x-2 \in [1,\infty)$, $gy = y \in [0,1)$, $\alpha(fx,gy) = 1$, $d(fx,gy) = \frac{66}{25}$

$$\frac{d(x,fx)+d(y,gy)}{2} = \frac{5+4}{2} = \frac{9}{2},$$

$$\frac{d(x,gy)+d(y,fx)}{2s} = \frac{\frac{66}{25}+\frac{66}{25}}{2(\frac{25}{24})} = \frac{3168}{1250},$$

$$m(x,y) = min\{d(x,fx),d(y,gy),d(x,gy),d(y,fx)\}$$
$$= min\{5,4,\frac{66}{25},\frac{66}{25}\} = \frac{66}{25}$$

and

$$M(x,y) = max\{d(x,y),d(x,fx),d(y,gy),\frac{d(x,fx)+d(y,gy)}{2},\frac{d(x,gy)+d(y,fx)}{2s}\}$$

$$= max\{\frac{66}{25},5,4,\frac{9}{2},\frac{3168}{1250}\} = 5.$$

Now we consider

$$\zeta(s^4\alpha(fx,gy)d(fx,gy),M(x,y)+L.m(x,y))$$
$$= \frac{4}{5}(M(x,y)+L.m(x,y)-s^4\alpha(fx,gy)d(x,gy)$$
$$= \frac{4}{5}(5+L.\frac{66}{25}-(\frac{25}{24})^4(1)(\frac{66}{25})$$
$$= \frac{20}{5}+L.\frac{264}{125}-\frac{25781250}{8294400} \geq 0.$$

So that the pair (f, g) is α-admissible almost z -contraction pair of maps and satisfy all the hypothesis of Theorem 4.0.1 and $x = 1$ is the unique common fixed point of f and g since $f_1 = g_1 = 1$.

Chapter 5

Conclusion and Future scope

In this thesis, we introduce a new class of maps namely an almost generalized α-admissible Z- contraction pair of maps and established a common fixed point theorem. Moreover, we prove the existence and uniqueness of common fixed points in the setting of b-metric spaces with the help of simulation function. We have also established corollaries. Our results unify several related results in the existing literature. Finally, we verify an established result by an example. It is clear that we can list several consequences of our main results by choosing suitable ζ in Z. We omit the details since they are obvious.

Any interested researcher can extend this work by modifying the contractive condition or the space (or both) we used to a more general one.

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