ACCELERATED NONSTANDARD FINITE DIFFERENCE METHOD FOR SOLVING SINGULARLY PERTURBED PARABOLIC REACTION DIFFUSION PROBLEM.

JIMMA UNIVERSITY
COLLEGE OF NATURAL SCIENCES
DEPARTMENT OF MATHEMATICS

A THESIS SUBMITTED TO THE DEPARTMENT OF MATHEMATICS JIMMA UNIVERSITY IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF MASTER OF SCIENCE IN MATHEMATICS (NUMERICAL ANALYSIS)

BY: ALEMAYEHU FERSHA
ADVISOR: TESFAYE AGA (PhD)
CO-ADVISOR: WORKU TILAHUN(MSc.)

FEBRUARY 2022
JIMMA, ETHIOPIA

TABLE OF CONTENTS PAGE

SCHOOL OF GRADUATE STUDIES

We, the undersigned, number of Board of Examiners of the final open defense by Alemayehu Fersha, have read and evaluated his/her thesis entitled "Accelerated non-standard finite difference method for solving singularly perturbed parabolic reaction-diffusion problem" and examined the candidate. This is therefore to certify that the thesis has been accepted the partial fulfillment of the requirements for the degree master of science in mathematics.

________________________	________________	_______________
Name of the chair person	Signature	Date
Dr.Tesfaye Aga	________________	_______________
Name of Major advisor	Signature	Date
________________________	________________	_______________
Name of the Internal Examiner	Signature	Date
________________________	________________	_______________
Name of the External Examiner	Signature	Date

Title of the thesis: Accelerated nonstandard finite difference method for solving singularly perturbed parabolic reaction diffusion problem.

Degree awarded : MSc

Graduate Thesis Ownership Agreement

This Thesis is a property of Jimma University, an institution that awarded MSc/ PhD degree to the graduate student and funded its research cost fully or partly. The researcher work was accomplished under the closed support and supervision of assigned university's academic staff. It is therefore strictly forbidden to publish, modify, or communicate to put at the disposal of third party the entire document or any part thereof without the common consent the research supervisor(s) and the graduate student. Disregarding this agreement would lead to accountability according to the Jimma University's research and publication misconduct policy.

Article 1.7 of the university's document for "guidelines and procedures for research, JANUARY 2022."

Name of graduate students	Signature	Date
________________________	________________	_______________

Name research supervisor(s)	Signature	Date
________________________	________________	_______________
________________________	________________	_______________
________________________	________________	_______________

Title of the thesis: accelerated nonstandard finite difference method for solving singularly perturbed parabolic reaction diffusion problem.

Degree awarded: MSc

Declaration

I hereby declare that this work which is being presented in the research thesis entitled "Accelerated nonstandard finite difference for singularly perturbed parabolic reaction-diffusion problem" Is an authentic record of my own work it has not been summited elsewhere University for the award of any other degree, and that all the source I have used or quoted have been indicated and acknowledge as complete references.

Name: Alemayehu Fersha

Signature: ________________

Date: ________________

This thesis has been presented with approval of Advisor

Name Dr.Tesfaye Aga

Signature: ________________

Date : _________________

ACKNOWLEDGEMENT

First and for most, I would like to thanks my Almighty God for all his assistance throughout my life as well as accomplishing this project because, nothing can be done without the secret of super Natural God. Secondly, I would like to express deepest gratitude to my Advisor Dr. Tesfaye Aga and Co-Advisor Mr. Worku Tilahun (MSc) for their unreserved support unlimited advice, constructive comments and immediate responses that helped me in the developing this research proposal. Thirdly I would like to thank many people in my upgrade but not list all on this moment. Since they are too many. Finally my thanks reach to my lovely wife Zuriyash Tekle, darling my family spatial my sister Betrakech Fersha for their ideological and financial support.

ACRONYMS

DE: - Differential equation

FDM: - Finite difference method

BVP: - Boundary value problem

SPP: - Singular perturbation problem

PDE: - partial differential equations

SPPPDE: - singularly perturbed parabolic partial differential equation

HODIE: - Higher order via differential identity expression

SPDE:-Singularly perturbed differential equation.

Abstract

In this study, accelerated nonstandard finite difference method is presented for solving singularly perturbed parabolic reaction diffusion initial boundary value problems. Richardson extrapolation technique applied to improve accuracy of the solution and accelerates its rate of convergence from second-order to fourth-order and fourth-order to sixth-order. The consistency and stability of the proposed method have been established very well to guarantee to the convergence of the method. Model examples were considered to illustrate conformation of the theoretical description with experimentation results. The numerical experimentation is carried out some model problems and both the results are presented in tables and graphs. The present method is stable, convergent and gives more accurate solution than some methods existing the literature.

CHAPTER ONE

INTRODUCTION

1.1. Background of the Study

Due to the difficulties in finding the exact solution or analytical solution of a mathematical problems such as, the exact solution of differential equation, the root of non-linear equation, the evaluation of integration involving complex expression and etc., leads to the development of numerical analysis. Numerical analysis is the branch of mathematics that creates analyses and implements algorithms for solving numerically the problems of continuous mathematics (Pal, 2007).

Many physical laws and relations of the real world appear mathematically in the form of a differential equation (DE). A differential equation is an equation that contains the derivative(s) of one/more dependent variable(s) with respect to one/more independent variable(s). If the number of independent variable in the differential variable is one, then the DE is called Ordinary differential equation (ODE) and. If the numbers of independent variables are two/more, then the differential equation is called partial differential equation (Burg&Erwin, 2009).

Many real life problems are modeled by parameter dependent differential equations whose solution behavior depends on the magnitude of the parameter. A PDE in which the highest order derivative is multiplied by a small positive parameter $\varepsilon(0<\varepsilon<<1)$ is called singularly perturbed partial differential equation and the parameter ε is called the perturbation parameter (Gowrisankar &Natesan, 2013). A second order parabolic differential equation is called Convection-diffusion type, if the order of the differential equation is reduced by one whenever the perturbation parameter set zero and it is called reaction-diffusion type .If the order of the differential equation is reduced by two, whenever the perturbation parameter set to zero and it is the main interest of this research.

That is, a singularly perturbed partial differential equation of the form:

$$\varepsilon \frac{\partial^2 u(x,t)}{\partial x^2} - b(x,t)u(x,t) - \frac{\partial u(x,t)}{\partial t} = f(x,t), (x,t) \in D = (0,1)\times(0,1], \qquad (1.1)$$

subjected to the initial-boundary conditions:

$$\begin{aligned} &u(x,0) = s(x), 0 \le x \le 1, \\ &u(0,t) = q_o(t) \\ &u(1,t) = q_1(t), T > 0, \end{aligned} \qquad (1.2)$$

where $\varepsilon(0 < \varepsilon << 1)$ is the perturbation parameter, $b(x,t)$ and $f(x,t)$ are smooth functions such that $b(x,t) > 0$, is a second order singularly perturbed partial differential equation of reaction-diffusion type. A Partial differential problems in which a small parameter multiplied to the highest derivative arise in various field of science and engineering, such as fluid mechanics, elasticity, quantum mechanics, chemical reactor theory, reaction diffusion process hydrodynamics' and etc. (Kumar & Ramos ,2021).

The numerical solution of such problem exhibits a multi-scale character. That is there is (are) a thin layer(s) of the domain where the solution changes rapidly or jumps suddenly forming a boundary layer(s), while away from the layer(s) the solution behaves regularly or changes slowly in the outer region. As a result such problems are called boundary layer problems (Gowrisankar&Natesan, 2014). Due to this multi-scale character of the solution, classical numerical methods which are effective in solving most mathematical problems on uniform mesh, fails to provide reliable numerical result unless the mesh discretization used is extremely refined. Even in this context, careful numerical experiments show that the classical computational methods fail to decrease the maximum point-wise error as the mesh is refined; until the mesh size and the perturbation parameter have the same order of magnitude. Subsequently, the size of the system of algebraic equations was growing more as the dimension of the problem increases. Hence, this results a huge computational cost. This drawback motivates the researcher to develop and analyze different numerical methods (Tesfaye et al., 2019).

The numerical methods for SPPs are broadly classified into fitted operators and fitted mesh methods. In fitted operator methods, exponential fitting factors or artificial viscosity is used to control the rapid growth or decay of the numerical solution in the boundary layer regions. While, fitted mesh methods use non-uniform meshes, which was dense in the boundary layer regions and coarse outside the layer regions (Miller et al., 1996). For the reason that small values of the perturbation parameter, the boundary layer may appear to give rise to difficulties when classical methods are applied on a uniform mesh. Moreover, the error in the approximate solution depends on the variable perturbation parameter. An adapted placement of the nodes or artificial viscosity is needed to ensure that the error is independent of the parameter value and depends only on the number of nodes in the mesh.

The discretization with this property is stated as a uniformly convergent numerical method. Here, both fitted operators and fitted mesh methods help to get uniformly convergent numerical methods. Recently, different scholars outlined in the literature like Clavero and Gracia, (2012), Gowrisanker and Natesan Gowrisankar, (2013) and Munyakazi and Patidar , (2013) formulated a numerical method for singularly perturbed parabolic differential equation of reaction-diffusion type.

More recently, Tesfaye et al., (2021) presented accelerated fitted operator FDM for SPP reaction-diffusion problems. They developed the method by introducing a fitting parameter into the asymptotic solution and applying average finite difference approximation. To improve the accuracy of the developed method they had applied Richardson extrapolation technique. They established the consistent and stability of the method to ensure the convergence of the method. Further they concluded that the method is consistent, stable and produced an accelerated solution for SPPPDE of the reaction type. Mbroh and Munyakazi, (2021) also proposed a parameter-uniform numerical scheme to solve problem under consideration. The continuous problem is first discretized in the space variable using a fitted operator finite difference method. The partial differential equation is thus transformed into a system of initial value problems which are then integrated in time with the Crank–Nicolson finite difference method. They obtained scheme is second-order ε-uniform convergent in space and time. Richardson extrapolation of the space variable results in a fourth order ε-uniform convergence.

However, the obtained accelerated solution is improved. Thus, this study focused with numerical treatment of such problem under consideration via non-standard finite difference method.

1.2. Objectives of Study

1.2.1 General Objective

The general objective of this study is to formulate accelerated nonstandard finite difference method for solving singularly perturbed parabolic Reaction-Diffusion Problems**.**

1.2.2 Specific Objectives

The specific objectives of this study are:

- To construct accelerated nonstandard finite difference method for solving singularly perturbed parabolic reaction diffusion problem type.
- To establish the convergence of the constructed method
- To validate the constructed method by numerical illustration.

1.3.Significance of the Study

The outcome of this study had the following importance:

- Used as a reference material for researchers who works on this area
- Able the graduating students to acquire research skill and scientific procedure.
- Provide a numerical method for the numerical solution of the considered problem.

1.4 Delimitation of the Study

The study is delimited to formulate accelerated nonstandard finite difference method for solving singularly perturbed parabolic reaction-diffusion problems of the form:

$$\left(\varepsilon\frac{\partial^2 u}{\partial x^2}-b(x,t)u-\frac{\partial u}{\partial t}\right)(x,t)=f(x,t),\ (x,t)\in D:=(0,1)\times(0,1].,$$

subjected to the initial - boundary conditions:

$$u(x,0)=s(x),\quad 0\le x\le 1,$$
$$u(0,t)=q_0(t),\ \ u(1,t)=q_1(t),\ \ 0<t\le 1,$$

where ε is a perturbation parameter which satisfies $0<\varepsilon<<1$. Assume that the coefficient function $b(x,t)\ge\beta>0$ and the source function $f(x,t)$ are sufficiently smooth. Under sufficient smoothness and compatibility conditions imposed on the functions $s(x),q_0(t),q_1(t)$ and $f(x,t)$, the problem has a unique solution which exhibits twin boundary layer of width $O(\sqrt{\varepsilon})$ neighboring the boundaries $x=0$ and $x=1$

CHAPTER TWO

REVIEW OF RELATED LITERATURE

2.1 Singular Perturbation Theory

Lundwing Prandtl was the first to introduce the concept of boundary layer in 1904 at the Third International Congruence of Mathematics in Heidelberg Germany. His hypothesis was in the setting of fluid dynamics, fluid adjacent to the boundary sticks to the edge in a thin boundary layer due to friction but this friction has no effect to the flow of the fluid on the interior (Tesfaye et al., 2019).The term singular perturbation appears to have been first coined by (Frendricks and Wasow in,1946) in their paper "Singular Perturbation of Non-linear Oscillation"(Friedrichs & Wasow, 1946). A brief survey for the historical development of perturbation and singular perturbation problems is covered in recent books by (Gowrisankar & Natesan, 2013) and(Wasow, 2018) respectively. More precisely, a perturbation problem is a problem that contains a small parameter ε, called perturbation parameter. If the solution of the problem can be approximated by setting the value of the perturbation parameter equals to zero, then the problem is called regular perturbation problem, otherwise it is called singular perturbation problem. That is, if it is impossible to approximate the solution by an asymptotic expansion as the perturbation parameter tends to zero, then the problem is called singular.

2.2. Finite Difference Method

Among the different classification of numerical methods: like finite difference method, finite element method, finite volume method, spline approximation method and so on; the finite difference method seems to be the simplest approach for the numerical solution of linear differential equation (O'Malley, 1991). Finite difference methods are one of the most widely used numerical methods to solve differential equations. It proceeds by replacing the derivatives appearing in the differential equation by finite difference approximations (Pal, 2007). The replacement of the differential equation into finite difference approximations and incorporating the boundary conditions in the difference equations gives a large algebraic system of equations to be solved by different possibly iterative techniques (Clavero& Gracia,2013). Hence, the solution obtained by solving finite difference approximation indicates that the solution of differential equation at the grid points rather than the continuous solution, so that finite difference method is also called discretization methods.

2.3. Recent Development

Clavero and Gracia,(2012) constructed a high order uniformly convergent finite difference scheme which combines the implicit Euler method to discretize in time, together with the Richardson extrapolation technique, and a higher order via differential identity expression scheme to discretize in space. They splinted the analysis of the uniform convergence completely the contribution to the global error of both the time and the space discretization's. They show numerical results for different test problems confirming in practice the order of uniform convergence proved.

Gowrisankar and Natesan, (2013) presented at a numerical method for the solution of singularly perturbed parabolic reaction–diffusion problems with boundary layers. To solve these problems, they used a modified backward Euler finite difference scheme on layer adapted non-uniform meshes at each time level. The non-uniform meshes are obtained by equidistribution of a positive monitor function, which involves the second-order spatial derivative of the singular component of the solution. The equidistributing monitor function at each time level allows us to use this technique to non-linear parabolic problems. They also analyzed the truncation error and the stability of the method. The proposed scheme is parameter-uniform convergent of order $O(\Delta t + N^{-2})$.

Munyakazi and Patidar, (2013) treat a time-dependent singularly perturbed reaction-diffusion problem. They semi discretize the problem in time by means of the classical backward Euler method. They developed a fitted operator finite difference method (FOFDM) to solve the resulting set of linear problems (one at each time level). The method is shown to be first order convergent in time and second order convergent in space, uniformly with respect to the perturbation parameter.

Gracia andRiordan,(2015) presented a numerical approximation to the solution of a linear singularly perturbed parabolic problem using a classical finite difference operator on a piecewise-uniform Shishkin mesh. First order convergence of these numerical approximations in an appropriately weighted -norm is established.

Tesfaye et al., (2019) studied a fitted operator average finite difference method for solving singularly perturbed parabolic convection-diffusion problems with boundary layer at right side. After discretizing the solution domain uniformly, the differential equation is replaced by average finite difference approximation which gives system of algebraic equation at each time levels. The stability and consistency of the method established very well to guarantee the convergence of the method.

Kumar et al., (2021) developed a parameter-uniform numerical method on equidistributed meshes for solving a class of singularly perturbed parabolic reaction-diffusion problems with Robin boundary conditions. The discretization consists of a modified Euler scheme in time, a central difference scheme in space, and a special finite difference scheme for the Robin boundary conditions. A uniform mesh is used in the time direction while the mesh in the space direction is generated via the equi-distribution of a suitably chosen monitor function and the method is parameter-uniformly convergent of order two in space and order one in time.

More recently, Tesfaye et al., (2021), presented accelerated fitted operator FDM for SPP reaction-diffusion problems. They developed the method by introducing a fitting parameter into the asymptotic solution and applying average finite difference approximation. To improve the accuracy of the developed method they had applied Richardson extrapolation technique. They established the consistent and stability of the method to ensure the convergence of the method. Further they concluded that the method is consistent, stable and produced an accelerated solution for SPPPD of the reaction type. Mbroh and Munyakazi, (2021) also proposed a parameter-uniform numerical scheme to solve problem under consideration. The continuous problem is first discretized in the space variable using a fitted operator finite difference method. The partial differential equation is thus transformed into a system of initial value problems which are then integrated in time with the Crank–Nicolson finite difference method. They obtained scheme is second-order ε-uniform convergent in space and time. Richardson extrapolation of the space variable results in a fourth order ε-uniform convergence.

However, the obtained accelerated solution is improvement. Thus, this study focused with numerical treatment of such problem under consideration via non-standard finite difference method.

2.4 Richardson Extrapolation Method

Richardson extrapolation is a methodology for improving the order of accuracy of numerical solutions that involve the use of a discretization size h. By combining the results from numerical solutions using a sequence of related discretization sizes, the leading order error terms can be methodically removed, resulting in higher order accurate results. Richardson extrapolation is commonly used within the numerical approximation of partial differential equations to improve certain predictive quantities such as the drag or lift of an airfoil, once these quantities are calculated on a sequence of meshes, but it is not widely used to determine the numerical solution of partial differential equations. Within this article, Richardson extrapolation is applied directly to the solution algorithm used within existing numerical solvers of partial differential equations to increase the order of accuracy of the numerical result without referring to the details of the methodology or its implementation within the numerical code. Only the order of accuracy of the existing solver and certain interpolations required to pass information between the mesh levels are needed to improve the order of accuracy and the overall solution accuracy. Using the proposed methodology, Richardson extrapolation is used to increase the order of accuracy of numerical solutions of the linear heat and wave equations and of the nonlinear.

Difference solution of partial differential equation can in certain cases be expanded by even power of a discretization parameter h. If we have n solutions corresponding to different mesh width h_1, h_2 , $h_3 \ldots$, h_N. We can improve the accuracy by Richardson extrapolation and get a solution of order $2n$, yet only on the intersection of all grids used. That is normally on the coarsest grid. To interpolate this high order solution with the same order accuracy in points not be longing to all grids, we need second points in an interval of length$(2n - 1)h$ (Burg & Erwin, 2009).

CHAPTER THREE

METHODOLOGY

3.1 Study Area and Period

This study was conducted at Jimma University, College of Natural Science, Department of Mathematics from August 2020 to January 2022

3.2 Study Design

This study applied both the document at review and numerical experimentation or mixed design

3.3 Source of Information

The source of information required to conduct this study were:

- Related reference books.
- Published articles.
- Journals, etc.

3.4 Mathematical Procedures

To achieve the stated objectives, the researcher followed the following procedures:

- Defining the problem.
- Discretizing the solution domain.
- Constructing the non-standard finite difference scheme for the defined problem.
- Applying Richardson extrapolation technique to accelerate the order of convergence into higher order.
- Establish the stability and consistency the constructed scheme
- Writing MATLAB code for the scheme.
- Providing numerical illustrations and conclusion

CHAPTER FOUR

DESCRIPTION OF THE METHOD AND NUMERICAL RESULT

4.1 Description of the Method

We consider the singularly perturbed parabolic reaction-diffusion initial boundary value problem.

$$\varepsilon\frac{\partial^2 u(x,t)}{\partial x^2}-b(x,t)u(x,t)-\frac{\partial u(x,t)}{\partial t}=f(x,t),(x,t)\in D=(0,1)\times(0,1] \qquad (1),$$

subject to the initial and boundary conditions:

$$u(x,0)=s(x), 0\le x\le 1 \qquad (2)$$
$$u(0,t)=q_0(t), u(1,t)=q_1(t), 0<t\le 1,$$

where ε is perturbation the perturbation parameter that satisfies $0<\varepsilon<<1$ and assume that the coefficient function $b(x,t)\ge\beta>0$ is sufficiently smooth. Under sufficient smoothness and compatibility conditions imposed on the functions $s(x), q_0(x)$, $q_1(t)$ and $f(x,t)$ the initial boundary value problem admit a unique solution $u(x,t)$, which exhibits twin boundary layer of width $O(\sqrt{\varepsilon})$ neighboring the boundaries $x=0$ and $x=1$of Ω.
To formulate the method, let us take the singularly perturbed homogenous differential equation.

$$\varepsilon\frac{\partial^2 u(x,t)}{\partial x^2}-\beta u=0 \qquad (3),$$

subject to the boundary conditions $u(0)=q_0, u(1)=q_1$ and its solution is

$$m^2\varepsilon-\beta=0\Longrightarrow m^2\varepsilon=\beta\Longrightarrow m^2=\frac{\beta}{\varepsilon}\Longrightarrow\rho=\pm\sqrt{\frac{\beta}{\varepsilon}}.$$

$$u(x)=C\exp\left(\pm\sqrt{\frac{\beta}{\varepsilon}}x\right) \qquad (4)$$

for arbitrary constant C.

Here Eq. (3) has two linear independent solutions namely, $e^{\rho x}$ and $e^{-\rho x}$ with $\rho=\sqrt{\frac{\beta}{\varepsilon}}$.

For ordinary differential equation case, representing the approximate solution $u(x)$ at the grid point $x_m$by u_m with the mesh size$h=\frac{1}{M}$and we have

$x_m = mh,\ m = 0,1,2\dots M0 = x_0 < x_1 < x_2 < , \dots < x_{M=}1,$ for M been positive integer. We denote the approximate solution to $u(x)$ at the grid points x_m by U_m. The theory of difference given by Jean et al, (2006) shows that the second order linear difference equation

$$\begin{vmatrix} U_{m-1} & e^{\rho x_{m-1}} & e^{-\rho x_{m-1}} \\ U_m & e^{\rho xm} & e^{-\rho x_m} \\ U_{m+1} & e^{\rho x_{m+1}} & e^{-\rho x_{m+1}} \end{vmatrix} = 0$$

Since $x_{m\pm1} = x_m \pm h$ and we have

$$U_{m-1(e^{-\rho h}-e^{\rho h})} - U_m\left(e^{-2\rho h} - e^{2\rho h}\right) + U_{m+1}\left(e^{-\rho h} - e^{\rho h}\right) = 0$$

Using the relation $\left(e^{-2\rho h} - e^{2\rho h}\right) = \left(e^{-\rho h} - e^{\rho h}\right)\left(e^{-\rho h} + e^{\rho h}\right)$ and

$2cosh(\rho h) = (e^{-\rho h} + e^{\rho h})$ the above equation can be written as

$$U_{m-1} - 2U_m \cosh(\rho h) + U_{m+1} = 0 \qquad (5)$$

Now, Eq. (4) is the exact difference scheme of Eq. (3) and the sense that the difference equation given in Eq. (4) has the same general solution $u(x_m) = c_1 e^{\rho x_m} + c_2 e^{-\rho x_m}$ as the differential equation of Eq. (3) Jean et al., (2006).

Using the identity $\cosh(\rho h) = 1 + 2\left(\sinh\left(\frac{\rho h}{2}\right)\right)^2$, Eq. (5) can be transformed to

$$\varepsilon \frac{U_m - 2U_m + U_{m+1}}{\frac{4}{\rho^2}\left(\sinh\left(\frac{\rho h}{2}\right)\right)^2} - \beta u_m = 0 \qquad (6)$$

This implies that the exact scheme of the non-homogenous equation

$$\varepsilon u''(x) - \beta u = f \qquad (7),$$

where f assumed to be constant, then Eq. (7) is given by

$$\varepsilon \frac{U_{m-1} - 2U_m + U_{m+1}}{\frac{4}{\rho^2}\left(\sinh\left(\frac{\rho h}{2}\right)\right)^2} - \beta U_m = f \qquad (8)$$

The non-standard finite scheme is one the difference equation that used to determine the approximate solution U_m to the solution $u(x)$ of the given differential equation. If the classical denominator h^2 of the discrete second order derivative is replaced by anon-negative function ϕ_m such that

$$\phi_m(h) = h^2 + O(h^3) \text{ as } 0 < h \to 0$$

As Jean et al., (2006) provided important observation that the complex structure of the denominator of discrete derivative in Eq. (8) constitutes a general property of these schemes, is useful designing reliable schemes for such problems. To demonstrate the procedure, let N be a positive integer when working on $\overline{D}$,we custom a rectangular grid D_h^k whose nodes are (x_m, t_n) for $n = 0,1,2, \dots, N$.

Here $0 = t_0 < t_1 < t_2 < \cdots < t_N = 1$ and $t_n = nk, k = \frac{1}{N}, n = 0,1,2, \dots, N$.

.Accordingly, let denote the approximate solution $U_m^{n+\frac{1}{2}} \simeq u(x_m, t_{n+\frac{1}{2}})$ at an arbitrary point $(x_m, t_{n+\frac{1}{2}})$.

Then we can consider Eq. (1) at a fixed node $(x_m, t_{n+\frac{1}{2}})$ and write it

$$\varepsilon \frac{\partial^2 U_m^{n+\frac{1}{2}}}{\partial x^2} - b_m^{n+\frac{1}{2}} U_m^{n+\frac{1}{2}} - \frac{\partial U_m^{n+\frac{1}{2}}}{\partial t} = f_m^{n+\frac{1}{2}}, (x_m, t_{n+\frac{1}{2}}) \epsilon D_h^k \qquad (9)$$

For the derivative concerning t by Taylor series expansion yields:

$$U_m^{n+1} = U_m^{n+\frac{1}{2}} + \frac{k}{2}\frac{\partial U_m^{n+\frac{1}{2}}}{\partial t} + \left(\frac{k}{2}\right)^2 \frac{\partial^2 U_m^{n+\frac{1}{2}}}{2!\,\partial t^2} + \left(\frac{k}{2}\right)^3 \frac{\partial^3 U_m^{n+\frac{1}{2}}}{3!\,\partial t^3} + O(h^4) \qquad (10)$$

$$U_m^{n} = U_m^{n+\frac{1}{2}} - \frac{k}{2}\frac{\partial U_m^{n+\frac{1}{2}}}{\partial t} + \left(\frac{k}{2}\right)^2 \frac{\partial^2 U_m^{n+\frac{1}{2}}}{\partial t^2} - \left(\frac{k}{2}\right)^3 \frac{\partial^3 U_m^{n+\frac{1}{2}}}{\partial t^3} + O(h^4) \qquad (11),$$

subtracting Eq. (11) from Eq. (10)

$$U_m^{n+1} - U_m^{n} = k\frac{\partial U_m^{n+\frac{1}{2}}}{\partial t} + 2\left(\frac{k}{2}\right)^3 \frac{\partial^3 U_m^{n+\frac{1}{2}}}{\partial t^3}$$

$$\frac{U_m^{n+1} - U_m^{n}}{k} + \tau_1 = \frac{\partial U_m^{n+\frac{1}{2}}}{\partial t} \qquad (12),$$

where the truncation term $\tau_1 = -\frac{k^2}{24}\frac{\partial^3 U_m^{n+\frac{1}{2}}}{\partial t^3}$

Again taking the other terms in Eq. (9) related to the points (m, n) and (m, n+1) on the $(n)^{th}$ and$(n+1)^{th}$ time level, average as:

$$\frac{\varepsilon \frac{\partial^2 U_m^{n+1}}{\partial x^2} + \varepsilon \frac{\partial^2 U_m^{n}}{\partial x^2} - b_m^{n+1} U_m^{n+1} - b_m^{n} U_m^{n} - f_m^{n+1} - f_m^{n}}{2} = \frac{\partial U_m^{n+\frac{1}{2}}}{\partial t} \qquad (13)$$

substituting Eq. (12) into Eq. (13) and then it becomes

$$\varepsilon\frac{\partial^2 U_m^{n+1}}{\partial x^2}+\varepsilon\frac{\partial^2 U_m^{n}}{\partial x^2}-b_m^{n+1}U_m^{n+1}-b_m^nU_m^n-f_m^{n+1}-f_m^n=\frac{2}{k}(U_m^{n+1}-U_m^n)$$

$$\varepsilon\frac{\partial^2 U_m^{n+1}}{\partial x^2}-b_m^{n+1}U_m^{n+1}-\frac{2}{k}U_m^{n+1}=f_m^{n+1}+f_m^n-\frac{2}{k}U_m^n-\varepsilon\frac{\partial^2 U_m}{\partial x^2}+b_m^nU_m^n \qquad (14),$$

Now, inserted of Eq. (8), we may approximate Eq. (14) by the non-standard scheme as

$$\varepsilon\frac{U_{m-1}^{n+1}-2U_m^{n+1}+U_{m+1}^{n+1}}{\phi_m^2}-b_m^{n+1}U_m^{n+1}-\frac{2}{k}U_m^{n+1}=f_m^{n+1}+f_m^n$$

$$+b_m^nU_m^n-\frac{2}{k}U_m^n-\varepsilon\frac{U_{m-1}^{n}-2U_m^{n}+U_{m+1}^{n}}{\phi_m^2} \qquad (15),$$

where $\phi_m=\frac{2}{\rho_m}\sinh\left(\frac{h\rho_m}{2}\right)$, $\rho_m=\sqrt{\frac{b_m}{\varepsilon}}$.

For clarity, the obtained scheme can be -written as the three term recurrence relation

$$E_m^{n+1}U_{m-1}^{n+1}-F_m^{n+1}U_m^{n+1}+G_m^{n+1}U_{m+1}^{n+1}=H_m^{n+1} \qquad (16),$$

where $E_m^{n+1}=\frac{\varepsilon}{\phi_m^2}=G_m^{n+1}$, $F_m^{n+1}=\frac{2\varepsilon}{\phi_m^2}+b_m^{n+1}+\frac{2}{k}$ and

$H_m^{n+1}=b_m^n+f_m^{n+1}+f_m^n-\frac{2}{k}U_m^n-\varepsilon\frac{U_{m-1}^{n}-2U_m^{n}+U_{m+1}^{n}}{\phi_m^2}$

4.1.1 **Thomas Algorithm**

In this section, the stability of solving tri-diagonal system concerning to the space direction at each $(n+1)^{th}$- time level is provided. A brief description for solving the system using the discrete invariant imbedding algorism, also called the Thomas algorism, is presented as follows. Consider the scheme above in Eq. (16) for $m=1,2\ldots M-1$ and subject to the boundary condition in Eq. (2) that can be written as:

$u(0,t_{n+1})=q_0(t_{n+1}), u(1,t_{n+1}),=q_1(t_{n+1}), 0<t_{n+1}\le 1$

Assuming that the solution of Eq. (16) is given by

$$U_m^{n+1}=W_m^{n+1}U_{m+1}^{n+1}+T_m^{n+1}, m=M-1, M-2.\ldots,2,1 \qquad (17),$$

where W_m^{n+1} and T_m^{n+1} are to be determined.

Considering Eq. (17) at the nodal point x_{m-1}, we have

$$U_{m-1}^{n+1}=W_{m-1}^{n+1}U_m^{n+1}+T_{m-1}^{n+1} \qquad (18)$$

Substitute Eq. (18) in Eq. (16) gives:

$$E_m^{n+1}(W_{m-1}^{n+1}U_m^{n+1}+T_{m-1}^{n+1})-F_m^{n+1}U_m^{n+1}+G_m^{n+1}U_{m+1}^{n+1}=H_m^{n+1}$$

This leads to obtain the equation

$$U_m^{n+1} = \frac{G_m^{n+1}}{F_m^{n+1} - E_m^{n+1} W_{m-1}^{n+1}} U_{m+1}^{n+1} + \frac{E_m^{n+1} T_{m-1}^{n+1} - H_m^{n+1}}{F_m^{n+1} - E_m^{n+1} W_{m-1}^{n+1}} \tag{19}$$

Comparing Eq. (17) with Eq. (19) the values determine as:

$$\begin{cases} W_m^{n+1} = \frac{G_m^{n+1}}{F_m^{n+1} E_m^{n+1} W_{m-1}^{n+1}}, \\ T_m^{n+1} = \frac{E_m^{n+1} T_{m-1}^{n+1} - H_m^{n+1}}{F_m^{n+1} - E_m^{n+1} W_{m-1}^{n+1}} \end{cases} \tag{20}$$

To solve these recurrence relations form $= 1, 2 \ldots M - 1$, we need the initial condition for $W_0^{n+1} = 0$ and we take$T_0^{n+1} = U_0^{n+1} = s(x_0)$. With these stating points of initial values, we compute W_m^{n+1} and T_m^{n+1} form $= 1, 2 \ldots M - 1$ using Eq. (20) in the forward process, and then obtain U_m^{n+1} in the backward process by Eq. (17) and the boundary condition$u_m^{n+1} = q_1(t_{n+1})$. Further, the conditions for the discrete invariant imbedding algorithm to be stable, if and only if:

$$|E_m^{n+1}| = \frac{\varepsilon}{\phi_m^2} > 0,$$

$$|G_m^{n+1}| = \frac{\varepsilon}{\phi_m^2} > 0, \tag{21}$$

$$|F_m^{n+1}| = \left|\frac{2\varepsilon}{\phi_m^2} + b_m^{n+1} + \frac{2}{k}\right| > 0, \text{and } |F_m| \geq |E_m| + |G_m|$$

Hence, the Thomas Algorithm is stable for the proposed method in Eq. (16).

4.1.2. Stability of the method

The analysis of the proposed method is easily accomplished by the used Fourier analysis. As Zhilin et al., (2018)provides detail reasons, the Von Neumann stability method is applied to investigate the stability of the developed scheme in Eq. (15)or Eq. (16) , by assuming that its solution at the grid point (x_m, t_n) is given by;

$$U_m^n = \xi^n e^{imθ} \tag{22},$$

where $i = \sqrt{-1}$ and θ is the real number with ξ is the amplitude factor.

Now, substituting Eq. (22) into the homogeneous part of Eq. (15) yields the amplitude factor:

$$(\xi^{n+1})(E_m^{n+1} U_{m-1}^{n+1} - F_m^{n+1} U_m^{n+1} + G_m^{n+1} U_{m+1}^{n+1}) = (\xi)^n \left(b_m^n U_m^n - \frac{2}{k} U_n^m - \varepsilon \frac{U_{m-1}^n - 2U_m^n + U_{m+1}^n}{\phi_m^2}\right)$$

$$(\xi^n * \xi)(E_m^{n+1}U_{m-1}^{n+1} - F_m^{n+1}U_m^{n+1} + G_m^{n+1}U_{m+1}^{n+1})$$
$$= (\xi^n)\left(\left(b_m^n U_m^n - \frac{2}{k}U_m^n - \varepsilon\frac{U_{m-1}^n - 2U_m^n + U_{m+1}^n}{\phi_m^2}\right)\right)$$

$$\xi = \frac{b_m^n U_m^n - \frac{2}{k}U_m^n - \varepsilon\frac{U_{m-1}^n - 2U_m^n + U_{m+1}^n}{\phi_m^2}}{E_m^{n+1}U_{m-1}^{n+1} - F_m^{n+1}U_m^{n+1} + G_m^{n+1}U_{m+1}^{n+1}}$$

$$\xi = \frac{k\phi_m^2 b_m^n e^{im\theta} - 2\phi_m^2 e^{im\theta} - k\varepsilon\left(e^{i(m-1)\theta} - 2e^{im\theta} + e^{i(m+1)\theta}\right)}{\frac{\varepsilon}{\phi_m^2}e^{i(m-1)\theta} - \left(\frac{2\varepsilon}{\phi_m^2} + b_m{}^{n+1} + \frac{2}{k}\right)e^{im\theta} + \frac{\varepsilon}{\phi_m^2}e^{i(m+1)\theta}}$$
$$\xi = \frac{k\phi_m^2 b_m^n e^{im\theta} - 2\phi_m^2 e^{im\theta} - k\varepsilon\left(e^{i(m-1)\theta} - 2e^{im\theta} + e^{i(m+1)\theta}\right)}{k\varepsilon e^{i(m-1)\theta} - k\phi_m^2 b_m^{n+1}e^{im\theta} - 2\phi_m^2 e^{im\theta} - 2k\varepsilon e^{im\theta} + k\varepsilon e^{i(m+1)\theta}}$$

For sufficiently small k, the condition of stability is $|\xi| \le 1$ that can be satisfied as:

$$\xi = \left|\frac{k(\ \phi_m^2 bm^n e^{im\theta} - \varepsilon(e^{i(m-1)\theta} - 2e^{im\theta} + e^{i(m+1)\theta})) - 2\phi_m^2 e^{im\theta}}{k(\varepsilon e^{i(m-1)\theta} - \phi_m^2 U_m^{n+1}e^{im\theta} - 2\varepsilon e^{im\theta} + \varepsilon e^{i(m+1)\theta}) - 2\phi_m^2 e^{i(m+1)\theta}}\right|$$
$$\xi = \left|\frac{-2\phi_m^2 e^{im\theta}}{-2\phi_m^2 e^{im\theta}}\right| \le 1.$$

Thus, $|\xi| \le 1$. Hence, the scheme given in Eq. (17) is stable and, we can say the formulated scheme is unconditionally stable.

4.1.2 Truncation Error and consistency of the method

In this section, the truncation error for the described method will be investigated. Truncation error $TE(h,k)$ between the exact solutions (x_m, t_n) the approximation U_m^{n+1} is given by:

$$TE(h,k) = \varepsilon\frac{\partial^2 U_m^{n+\frac{1}{2}}}{\partial x^2} - b_m^{n+\frac{1}{2}}U_m^{n+\frac{1}{2}} - \frac{\partial U_m^{n+\frac{1}{2}}}{\partial t}$$
$$-\{E_m^{n+1}U_{m-1}^{n+1} - F_m^{n+1}U_m^{n+1} + G_m^{n+1}U_{m+1}^{n+1}\}$$
$$-\{b_m^n U_m^n - \frac{2}{k}U_m^n - \varepsilon\frac{U_{m-1}^n - 2U_m^n + U_{m+1}^n}{\phi_m^2}\}$$

$$TE(h,k) = \varepsilon\frac{\partial^2 U_m^{n+\frac{1}{2}}}{\partial x^2} - b_m^{n+\frac{1}{2}}U_m^{n+\frac{1}{2}} - \frac{\partial U_m^{n+\frac{1}{2}}}{\partial t} - \{\frac{\varepsilon}{\phi_m^2}(U_{m-1}^{n+1} - 2U_m^{n+1} + U_{m+1}^{n+1}\} -$$
$$(b_m^{n+1}U_m^{n+1} + b_m^n U_m^n) + \frac{\varepsilon}{\phi_m^2}(U_{m-1}^n - 2U_m^n + U_{m+1}^n) - \frac{2}{k}(U_m^{n+1} - U_m^n)\}. \quad (23)$$

Using Taylor's series expansion to U_m^n around$u(x_m, t_n)$ with respect to the spatial direction, we have the approximation for $U_{m\pm1}^n$ as:

$$U_{m+1}^n = U_m^n + h\frac{\partial U_m^n}{\partial x} + \frac{h^2}{2}\frac{\partial^2 U_m^n}{\partial x^2} + \frac{h^3}{6}\frac{\partial^3 U_m^n}{\partial x^3} + \frac{h^4}{24}\frac{\partial^4 U_m^n}{\partial x^4} + O(h^5)$$

$$U_{m-1}^n = U_m^n - h\frac{\partial U_m^n}{\partial x} + \frac{h^2}{2}\frac{h^2}{2}\frac{\partial^2 U_m^n}{\partial x^2} - \frac{h^3}{6}\frac{\partial^3 U_m^n}{\partial x^3} + \frac{h^4}{24}\frac{\partial^4 U_m^n}{\partial x^4} + O(h^5)$$

From these two basic equations, we obtain

$$U_m^{n+1} - 2U_m^n + U_{m-1}^n = h^2\frac{\partial^2 U_m^n}{\partial x^2} + \frac{h^4}{12}\frac{\partial^4 U_m^n}{\partial x^4} + O(h)^6 \qquad (24)$$

Similarly, we have

$$U_{m+1}^{n+1} - 2U_m^{n+1} + U_{m-1}^{n+1} = h^2\frac{\partial^2 U_m^{n+1}}{\partial x^2} + \frac{h^4}{12}\frac{\partial^4 U_m^n}{\partial x^4} + O(h)^6 \qquad (25)$$

Also, using Taylor's series expansion to U_m^n around $(x_m, t_n)$with respect to the temporal direction, we have the approximation $U_m^{n+\frac{1}{2}}$as:

$$U_m^{n+1} = U_m^{n+\frac{1}{2}} + \frac{k}{2}\frac{\partial U_m^{n+\frac{1}{2}}}{\partial t} + (\frac{k}{2})^2\frac{\partial^2 U_m^{n+\frac{1}{2}}}{2!\,\partial t^2} + (\frac{k}{2})^3\frac{\partial^3 U_m^{n+\frac{1}{2}}}{3!\,\partial t^3} + \frac{(k)^4}{348}\frac{\partial^4 U_m^{n+\frac{1}{2}}}{\partial t^4} + O(k^5)$$

$$U_n^m = U_m^{n+\frac{1}{2}} - \frac{k}{2}\frac{\partial U_m^{n+\frac{1}{2}}}{\partial t} + \left(\frac{k}{2}\right)^2\frac{\partial^2 U_m^{n+\frac{1}{2}}}{2!\,\partial t^2} - \left(\frac{k}{2}\right)^3\frac{\partial^3 U_m^{n+\frac{1}{2}}}{3!\,\partial t^3} + \frac{(k)^4}{348}\frac{\partial^4 U_m^{n+\frac{1}{2}}}{\partial t^4} - O(k^5)$$

From these two basic equations, we get:

$$U_m^{n+1} - U_m^n = k\frac{\partial U_m^{n+\frac{1}{2}}}{\partial t} + \frac{(k)^3}{24}\frac{\partial^3 U_m^{n+\frac{1}{2}}}{3!\,\partial t^3} + O(k^5) \qquad (26)$$

Substitute Eq. (24) - Eq. (26) into Eq. (23)

$$TE(h,k) = \varepsilon\frac{\partial^2 U_m^{n+\frac{1}{2}}}{\partial x^2} - b_m^{n+\frac{1}{2}}U_m^{n+\frac{1}{2}} - \frac{\partial U_m^{n+\frac{1}{2}}}{\partial t} - \Bigg\{\frac{\varepsilon h^2}{\phi_m^2}\left[\frac{\partial^2 U_m^{n+1}}{\partial x^2} + \frac{\partial^2 U_m^n}{\partial x^2}\right] + \frac{\varepsilon h^4}{12\phi_m^2}\left[\frac{\partial^4 U_m^{n+1}}{\partial x^4} + \frac{\partial^4 U_m^n}{\partial x^4}\right] + O(h^6) - (b_m^{n+1}U_m^{n+1} + b_m^nU_m^n) - 2\frac{\partial U_m^{n+\frac{1}{2}}}{\partial t} - \frac{k^2}{12}\frac{\partial^3 U_m^{n+\frac{1}{2}}}{\partial t^3} + O(k^5)\Bigg\} \qquad (27)$$

$$\begin{aligned} TE(h,k) = {} & \varepsilon\frac{\partial^2 u_m^{n+\frac{1}{2}}}{\partial x^2} - b_m^{n+\frac{1}{2}}u_m^{n+\frac{1}{2}} - \frac{\partial u_m^{n+\frac{1}{2}}}{\partial t} \\ & - \frac{\varepsilon}{2\phi_m^2}[(U_{m-1}^{n+1} - 2U_m^{n+1} + U_{m+1}^{n+1}) + U_{m-1}^n - 2U_m^n + U_{m+1}^n] + \cdots \\ & + \frac{1}{2}(b_m^{n+1}U_m^{n+1} + b_m^n bU_m^n) + \frac{1}{2}\left[\frac{2}{k}\left(k\left(\frac{\partial U_m^{n+\frac{1}{2}}}{\partial t} + \frac{k^3\partial^3 U_m^{n+\frac{1}{2}}}{24\partial t^3}\right)\right)\right] \end{aligned}$$

$$TE(h,k) = \varepsilon \frac{\partial^2 u_m^{n+\frac{1}{2}}}{\partial x^2} - b_m^{n+\frac{1}{2}} u_m^{n+\frac{1}{2}} - \frac{\partial u_m^{n+\frac{1}{2}}}{\partial t}$$

$$- \frac{\varepsilon}{2\phi_m^2}\left\{\left[h^2 \frac{\partial^2 U_m^{n+1}}{\partial x^2} + \frac{h^4}{12}\frac{\partial^4 U_m^{n+1}}{\partial x^4}\right] + \left[h^2 \frac{\partial^2 U_m^{n}}{\partial x^2} + \frac{h^4}{12}\frac{\partial^4 U_m^{n+1}}{\partial x^4}\right] + \cdots\right\}$$

$$+ \frac{1}{2}(b_m^{n+1} U_m^{n+1} + b_m^n U_m^n) + \frac{\partial U_m^{n+\frac{1}{2}}}{\partial t} + \frac{k^3}{24}\frac{\partial^3 U_m^{n+\frac{1}{2}}}{\partial t^3}$$

$$TE(h,k) = \varepsilon \frac{\partial^2 u_m^{n+\frac{1}{2}}}{\partial x^2} - b_m^{n+\frac{1}{2}} u_m^{n+\frac{1}{2}} + \frac{1}{2}(b_m^{n+1} U_m^{n+1} + b_m^n U_m^n) - \frac{\partial u_m^{n+\frac{1}{2}}}{\partial t} + \frac{\partial u_m^{n+\frac{1}{2}}}{\partial t}$$

$$- \frac{\varepsilon h^2}{\phi_m^2}\left\{\left[\frac{\frac{\partial^2 U_m^{n+1}}{\partial x^2} + \frac{\partial^2 U_m^{n}}{\partial x^2}}{2}\right] + \frac{h^2}{12}\left[\frac{\frac{\partial^4 U_m^{n+1}}{\partial x^4} + \frac{\partial^4 U_m^{n}}{\partial x^4}}{2}\right] + \cdots\right\} + \frac{k^3}{24}\frac{\partial^3 U_m^{n+\frac{1}{2}}}{\partial t^3},$$

but $b_m^{n+\frac{1}{2}} u_m^{n+\frac{1}{2}} = \frac{1}{2}(b_m^{n+1} U_m^{n+1} + b_m^n U_m^n)$

$$TE(h,k) = \varepsilon \frac{\partial^2 u_m^{n+\frac{1}{2}}}{\partial x^2} - \frac{\varepsilon h^2}{\phi_m^2}\left\{\left[\frac{\frac{\partial^2 U_m^{n+1}}{\partial x^2} + \frac{\partial^2 U_m^{n}}{\partial x^2}}{2}\right] + \left[\frac{h^2}{24}\left(\frac{\partial^4 U_m^{n+1}}{\partial x^4} + \frac{\partial^4 U_m^{n}}{\partial x^4}\right)\right] + \cdots\right\} + \frac{k^2}{24}\frac{\partial^3 U_m^{n+\frac{1}{2}}}{\partial t^3},$$

where $\phi_m = \frac{2sinh\left(\frac{h\rho_m}{2}\right)}{\rho_m}$

$$sinh(\rho h) = \frac{e^{\rho h} - e^{-\rho h}}{2}$$

$2sinh(\rho h) = e^{\rho h} - e^{-\rho h}$, but $e^{\rho h} = 1 + \rho h + \frac{(\rho h)^2}{2} + \frac{(\rho h)^3}{6} + \frac{(\rho h)^4}{24} + \cdots$

$$e^{-\rho h} = 1 - \rho h + \frac{(\rho h)^2}{2} - \frac{(\rho h)^3}{6} + \frac{(\rho h)^4}{24} + \cdots$$

$$e^{\rho h} - e^{-\rho h} = 2\rho h + \frac{(\rho h)^3}{3} + \cdots$$

$$4sinh\left(\frac{\rho h}{2}\right) = e^{\rho h} - e^{-\rho h}$$

$$4sinh\left(\frac{\rho h}{2}\right) = 2\rho h + \frac{(\rho h)^3}{3} + \cdots$$

$$2sinh\left(\frac{\rho h}{2}\right) = \rho h + \frac{(\rho h)^3}{6} + \cdots$$

$$\frac{4sinh^2}{\rho^2}\left(\frac{\rho h}{2}\right) = \frac{\rho^2 h^2}{\rho^2} + \cdots$$

$$\phi_m^2 \sim h^2$$

$$TE(h,k) = \frac{\varepsilon\partial^2 u_m^{n+\frac{1}{2}}}{\partial x^2} - \frac{\varepsilon}{2}\left[\frac{\partial^2 U_m^{n+1}}{\partial x^2} + \frac{\partial^2 U_m^n}{\partial x^2}\right] - \frac{\varepsilon h^2}{24}\left[\frac{\partial^4 U_m^{n+1}}{\partial x^4} + \frac{\partial^4 U_m^n}{\partial x^4}\right] + \cdots + \frac{k^2}{24}\frac{\partial^3 U_m^{n+\frac{1}{2}}}{\partial t^3}.$$

Again $\varepsilon\frac{\partial^2 u_m^{n+1}}{\partial x^2} = \frac{1}{2}\varepsilon\left(\frac{\partial^2 U_m^{n+1}}{\partial x^2} + \frac{\partial^2 U_m^n}{\partial x^2}\right)$

$$TE(h,k) = \frac{-\varepsilon h^2}{24}\left[\frac{\partial^4 U_m^{n+1}}{\partial x^4} + \frac{\partial^4 U_m^n}{\partial x^4}\right] + \frac{k^2}{24}\left(\frac{\partial^3 U_m^{n+\frac{1}{2}}}{\partial t^3}\right) \tag{28}$$

$$TE(h,k) = C_1 h^2 + C_2 k^2 \equiv C(h^2 + k^2),$$

where $C_1 = \frac{\varepsilon}{24}\left\|\frac{\partial^4 U_m^{n+1}}{\partial x^4} + \frac{\partial^4 U_m^n}{\partial x^4}\right\|_\infty$ and $C_2 = \left\|\frac{\partial^3 U_m^{n+\frac{1}{2}}}{24\partial t^3}\right\|$

Since at the nodal point x_m , we have:

$$\frac{d^2 u(x_m)}{dx^2} = u'' \text{, and } b(x_m)u(x_m) = b_m u_m.$$

Therefore the described method is second - order convergent. Truncation error refers to the difference between the original differential equation and the finite difference approximation at the grid points. Thus the developed scheme is second-order accurate. As book of Zhilin et al.,(2008) a finite difference scheme is called consistent if the limit of the truncation error is equal to zero as the mesh size goes to zero. Hence, this definition of the consistency on the described method with the truncation error in Eq. (28)is satisfied as:

$$\lim_{(h,k)\to 0} TE(h,k) = \lim_{(h,k)\to(0,0)} C(h^2 + k^2) = 0$$

Therefore, by Lax equivalence theorem the constructed scheme convergent.

4.1.3. Richardson Extrapolation

The basic idea of i Richardson extrapolation is that whenever the leading term is the error for an approximation formula is known. By combing two or more approximation obtained from those formula using different value of meshes lengths:

$$(h,k), \frac{(h,k)}{2}, \frac{(h,k)}{4}, \frac{(h,k)}{8},$$

to obtain a higher - order approximation and the technique is known as Richardson extrapolation.

This procedure is a convergence acceleration technique which consists of considering a linear combination of two computed approximations of a solution. The linear combination turns out to be a better approximation.

Particularly in our case, the described numerical method is almost second over convergent as verified Eq. (28) So that, from this equation, we have

$$|u(x_m, t_n) - U_m^n| \leq C(h^2 + k^2) \quad (29),$$

where $u(x_m, t_n)$ and U_m^n exact and approximate solutions respectively, C is constant in dependent of mesh sizes inhk and perturbation parameter. Let D_{2M}^{2N} be the mesh obtained by bisecting each mesh interval in D_M^N and denote the approximation of the solution on D_{2M}^{2N} by U_{2M}^{2N}. Consider Eq. (28) works for any $(h, k) \neq 0$ which implies:

$$u(x_m, t_n) - U_m^n \simeq C(h^2 + k^2) + R_M^N \quad (30)$$

So that it works for any $(\frac{h}{2}, \frac{k}{2}) \neq 0$ yields:

$$u(x_m, t_n) - U_m^n \simeq C(\frac{h^2}{4} + \frac{k^2}{4}) + R_{2M}^{2N} \quad (31),$$

where the remainder,$R_M^N$and R_{2M}^{2N} are $O(h^4 + k^{4)})$.

Eliminating the constant from Eq. (30) and Eq. (31)

Leads to $3u(x_m, t_n) - (4U_{2m}^{2n} - U_m^n) \cong O(h^4 + k^4)$, which suggests that

$$(U_m^n)^{ext} = \frac{1}{3}(4U_{2m}^{2n} - U_m^n) \quad (32)$$

is also an approximation $u(x_m)$.

Using this approximation to evaluate the truncation error we obtain:

$$|u(x_m) - (U_m^n)^{ext}| \leq C(h^4 + k^4) \quad (33)$$

Now, using these two different solutions which are obtained by the same scheme given by Eq. (16), we get another third solution in terms the two by Eq. (33).

This is the Richardson exploration technique to accelerate the second –order to fourth -order convergent.

In similarly manner accelerating this fourth order approximation to six orders, we choose the system of equations

$$\begin{cases} u(x_m,t_n) - U_m^n \simeq C_1(h^2+k^2) + C_2(h^4+k^4) + C_3(h^6+k^6) + \cdots \\ u(x_m,t_n) - U_{4m}^{4n} \simeq C_1\left(\frac{h^2+k^2}{16}\right) + C_2\left(\frac{h^4+k^4}{256}\right) + C_3\left(\frac{h^6+k^6}{4096}\right) + \cdots \end{cases}$$

$$15\big(u(x_m,t_n) - (16U_{4m}^{4n} - U_m^n)\big) \equiv O(h^6+k^6) \ ,$$

which leads to

$$(U_{2m}^{2n})^{6ext} = \frac{1}{15}(16(U_{4m}^{4n})^{4ext} - (U_{2m}^{2n})^{4ext}) \qquad (34)$$

Now, using the two different solutions which are obtained the scheme given Eq. (16) we get another third solution in terms of the two by Eq. (34). This is the Richardson extrapolation methods to accelerate the fourth order to six order convergence.

$$\begin{cases} u(x_m,t_n) - U_m^n \simeq C_1(h^2+k^2) + C_2((h^4+k^4) + C_3((h^6+k^6) + \cdots \\ u(x_m,t_m) - U_{2m}^{2n} \simeq C_1(\frac{h^2}{4} + \frac{k^2}{4}) + C_2((\frac{h^4}{16} + \frac{k^4}{16}) + C_3((\frac{h^6}{64} + \frac{k^6}{64}) + \cdots \\ u(x_m,t_n) - U_{4m}^{4n} \simeq C_1(\frac{h^2}{16} + \frac{k^2}{16}) + C_2((\frac{h^4}{256} + \frac{k^4}{256}) + C_3((\frac{h^6}{4096} + \frac{k^6}{4096}) + \cdots \\ u(x_m,t_n) - U_{8n}^{8m} \simeq C_1(\frac{h^2}{64} + \frac{k^2}{64}) + C_2(\frac{h^4}{4096} + \frac{k^4}{4096}) + C_3(\frac{h^6}{262144} + \frac{k^6}{262144}) + \cdots \end{cases}$$

4.2 Numerical illustration

In this section, we provide numerical examples and results to validate the applicability of the describe schemes

Example1: Consider the singularly perturbed parabolic initial value boundary problem:

$$\varepsilon\frac{\partial^2 u}{\partial x^2} - (1 + xe^{-t})u(x,t) - \frac{\partial u}{\partial t} = f(x,t), (x,t)\epsilon(0,1) \times (0,1],$$

subject to the initial boundary conditions:

$$u(x,0) = 0, x\epsilon[0,1], u(0,t) = 0 = u(1,t), t\epsilon[0,1],$$

where the source functions $f(x,t)$ is fitted such that the exact solution is

$$u(x,t) = (1 - \exp(-t)(\frac{\exp\left(\frac{-x}{\sqrt{\varepsilon}}\right) + \exp\left(\frac{-(1-x)}{\sqrt{\varepsilon}}\right)}{1 + \exp\left(\frac{-1}{\varepsilon}\right)} - (\cos(\pi x))^2)$$

Example 2: Consider the singularly perturbed parabolic problem:

$$\varepsilon\frac{\partial^2 u}{\partial x^2} - b(x,t)u(x,t) - \frac{\partial u}{\partial t} = f(x,t), \forall(x,t)\epsilon(0,1) \times (0,1],$$

where $b(x,t) = 1 + x^2 + t^2\exp(t)$ and $f(x,t) = \exp(t) - 1 + \sin(\pi x)$, subject to initial boundary condition

$$u(x,0) = 0, x\epsilon([0,1], u(0,t) = 0 = u(1,t), t\epsilon[0,1]$$

For this example the exact solution is not accessible, so that it's maximum absolute error calculated by double mesh size principle (Tesfaye et al., 2021).

$$E_{\varepsilon}^{M,N} = max(x_m, t_{n+1})\epsilon\overline{D}_M^N|U_{M-}^N U_{2M}^{2N}|$$

4.3 List of tables

Table 4.1: Maximum absolute error for Example 1, when $M = N$.

ε↓ N →	16	32	64	128	256
Present Method					
2^{-5}	6.7162e-09	2.4174e-10	3.9219e-11	3.1199e-12	1.8279e-13
2^{-6}	$1.6260e-08$	$3.0971e-10$	$1.0865e-11$	$1.7436e-12$	$1.3995e-13$
2^{-7}	$5.0027e-08$	$8.7617e-10$	$1.5752e-11$	$5.4736e-13$	$8.7397e-14$
2^{-8}	$1.6164e-07$	$3.0206e-09$	$5.0137e-11$	$8.7010e-13$	$3.0108e-14$
2^{-9}	$4.7537e-07$	$1.0856e-08$	$1.8376e-10$	$2.9633e-12$	$5.0780e-14$
Tesfaye et al., (2021)					
2^{-5}	2.9109e-06	2.1164e-07	1.3395e-08	8.4080e-10	4.5679e-11
2^{-6}	$6.6145e-06$	$4.1973e-07$	$2.6385e-08$	$1.6535e-09$	$1.0346e-10$
2^{-7}	$1.2425e-05$	$8.5139e-07$	$5.3716e-08$	$3.3727e-09$	$2.1095e-10$
2^{-8}	$2.4780e-05$	$1.6801e-06$	$1.0710e-07$	$6.7530e-09$	$4.2260e-10$
2^{-9}	$4.4030e-05$	$3.1975e-06$	$2.1164e-07$	$1.3395e-08$	$8.4080e-10$

Table 4.2:Computed maximum absolute errors for Example 1,when $M = N$

$\varepsilon\downarrow$ N $\rightarrow$	16	32	64	128	256
Sixth order					
2^{-7}	5.0027e-08	8.7617e-10	1.5752e-11	5.4736e-13	8.7397e-14
2^{-8}	$1.6164e-07$	$3.0206e-09$	$5.0137e-13$	$8.7010e-13$	$3.0108e-14$
2^{-9}	$4.7537e-07$	$1.0856e-10$	$1.8376e-10$	$2.0633e-12$	$5.0780e-14$
Fourth order					
2^{-7}	1.2425e-05	8.5139e-07	5.3716e-08	3.3727e-09	2.1094e-10
2^{-8}	$2.4780e-05$	$1.6801e-06$	$1.0710e-07$	$6.7529-09$	$3.0108e-10$
2^{-9}	$4.4030e-05$	$3.1975e-06$	$2.1164e-07$	$1.3395e-08$	$8.4079e-10$
Second order					
2^{-7}	$1.8440e-03$	$4.7001e-04$	$1.1807e-04$	$2.9555e-05$	$7.3017e-06$
2^{-8}	$1.6579e-03$	$4.3054e-04$	$1.0867e-04$	$2.7234e-05$	$6.8138e-06$
2^{-9}	$1.5084e-03$	$4.0599e-04$	$4.0599e-04$	$2.5982e-05$	$6.5044e-06$

The maximum absolute error

$$E_{\varepsilon}^{M,N} = \max_{(x_m, t_{n+1})\epsilon \bar{D}_M^N} |u(x_m, t_{n+1}) - (U_m^{n+1})| \text{ and}$$

$E_{\varepsilon}^{M,N} = \max_{(x_m, t_{n+1})\epsilon \bar{D}_M^N} |u(x_m, t_{n+1}) - (U_m^{n+1})^{ext}|$, where$u(x_m, t_n)$ an exact solution,$U_m^{n+1}$is an approximate solution before extrapolation and$(U_m^{n+1})^{ext}$is also an approximate solution after Richardson extrapolation (Tesfaye et al., 2021).

The corresponding rate of convergence is determined by

$$R_{\varepsilon}^{M,N} = \frac{\log E_{\varepsilon}^{M,N} - \log E_{\varepsilon}^{2M,2N}}{\log 2} = \frac{\log M,N - \log 2M,2N}{\log 2}$$

Table 4.3: Computed Rate of Convergence for Example 1, When $M = N$

$\varepsilon \downarrow N \rightarrow$	16	32	64	128
Sixth order				
2^{-7}	5.8354	5.7976	4.8469	4.6468
2^{-8}	5.7418	5.9128	5.8486	4.8530
2^{-9}	5.4525	5.8845	5.9545	5.8668
Fourth order				
2^{-7}	3.8673	3.9864	3.9934	3.9990
2^{-8}	3.8826	3.9715	3.9873	3.9982
2^{-9}	3.7835	3.9173	3.9818	3.9938
Second order				
2^{-7}	1.9721	1.9930	1.9982	1.9994
2^{-8}	1.9451	1.9862	1.9965	1.9989
2^{-9}	1.8935	1.9728	1.9931	1.9980

Table 4.4: Comparison of Maximum Absolute Errors for Example 2:

$\varepsilon \downarrow N \rightarrow$	$M=128$ $N=4$	$M=256$ $N=8$	$M=512$ $N=16$	$M=1024$ $N=32$
Present Method				
2^{-4}	$3.4191e-05$	$7.9633e-06$	$1.9294e-06$	$4.7462e-07$
2^{-8}	$3.2698e-05$	$7.7775e-06$	$1.9115e-06$	$4.6939e-07$
2^{-12}	$4.4756e-06$	$5.3190e-06$	$1.8283e-06$	$4.1728e-07$
2^{-16}	$3.9622e-05$	$6.7383e-06$	$6.2755e-07$	$8.6046e-08$
Tesfaye et al., (2021)				
2^{-4}	$3.9138e-04$	$9.5343e-05$	$2.3353e-05$	$5.8317e-06$
2^{-8}	$6.5366e-04$	$1.6276e-04$	$4.0499e-05$	$1.0111e-05$
2^{-12}	$6.6576e-04$	$1.6709e-04$	$4.1915e-05$	$1.0552e-05$
2^{-16}	$6.6691e-04$	$3.1446e-04$	$2.7007e-04$	$9.6167e-05$

4.4 List of Figures

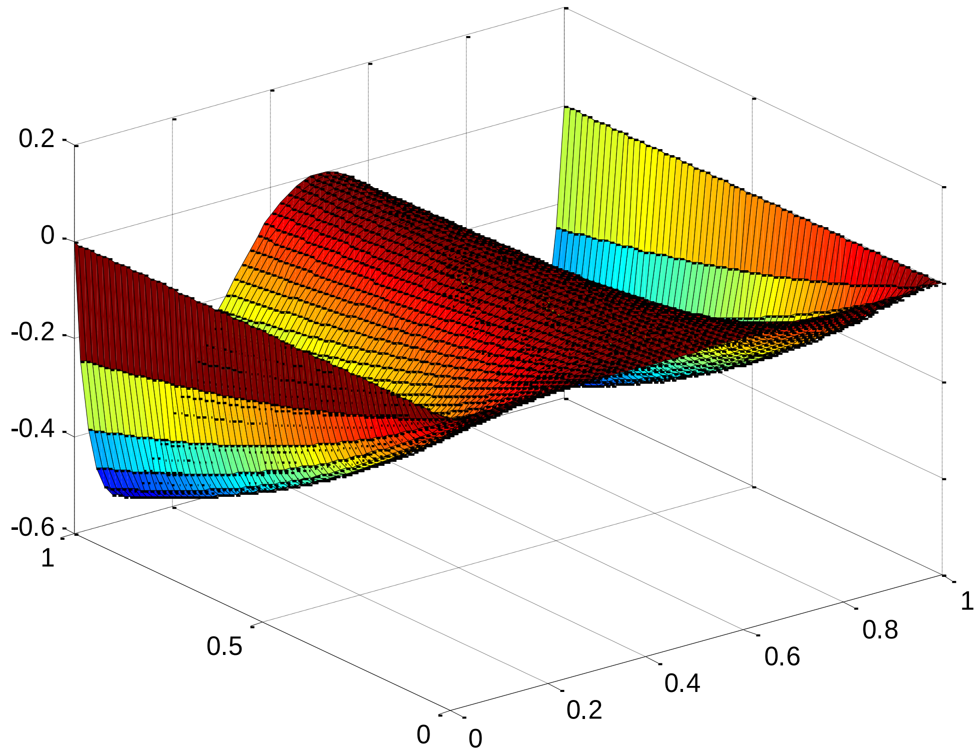

Figure 4.1 Surface plot of the solution behavior for Example 1, where $\varepsilon = 2^{-10}$ and $M = N = 64$.

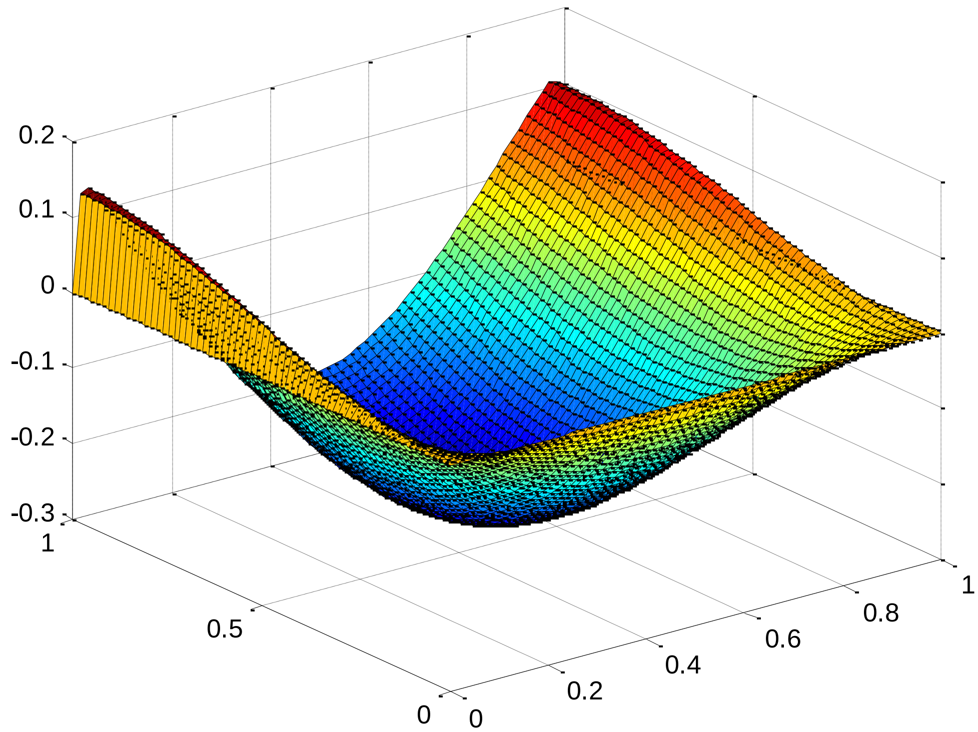

Figure 4.2 Surface plot of the solution behavior for Example 2, where $\varepsilon = 2^{-12}$ and

$M = N = 64$

CHAPTER FIVE

DISCUSSION, CONCLUSION AND SCOPE

5.1 Discussion and Conclusion

In this work, accelerate non-standard finite difference method described and analyzed for solving singularly perturbed parabolic reaction diffusion initial boundary value problem .Richardson extrapolation technique helps to improve accuracy of the solution and accelerate rate of convergence from second order to fourth order and fourth order to sixth order. Consistency and stability of the method established clearly to guaranty the convergence of the method. We consider two model examples to illustrate the numerical results interims of maximum absolute error and rate of convergence for different values of the perturbation parameter and mesh size (see the table 1-4). Specifically 1 and 4 used to verify the betterment of present method by producing more accurate solution the existing methods in the literature. Table 2, shows that the confirmation of fourth and sixth order of convergence in the theoretical analysis with experiment results. Table 3, shows the effect of applying Richardson extrapolation method and improvement of the accuracy of solution. As the number of interval N increases accuracy of solution also increases. Additionally, fig 1 to illustrate the problem has two (left and right) boundary layers. Fig 2 shows that the effect of mesh sizes and perturbation parameter with occurrence of maximum absolute errors in the layer regions. At the end accelerated non-standard finite difference method is formulated for the class of singularly perturbed parabolic reaction-diffusion initial boundary value problem which is stable, convergent and gives more accurate solution than some of the existing method in literature

5.2 Scope of the Future Work

In this thesis, accelerated non-standard finite difference method is introduced for solving singularly perturbed parabolic reaction-diffusion problem. Hence the scheme proposed in this thesis can also be extended to higher order finite difference method to solve singularly perturbed parabolic reaction-diffusion equation.

REFERENCE

Bronson, R., & Costa, G. B. (2014). **Schaum's outline of differential equations**. McGraw-Hill Education.

Bullo, T., Duressa, G. F., & Degla, G. (2021). Accelerated fitted operator finite difference method for singularly perturbed parabolic reaction-diffusion problems. *Comput, Methods Differ. Equ.*, *9*(3), 886-898.

Burg, C., & Erwin, T. (2009). Application of Richardson extrapolation to the numerical solution of partial differential equations. *Numerical Methods for Partial Differential Equations*, *25*(4), 810-83.

Clavero, C., & Gracia, J. L. (2012). A high order HODIE finite difference scheme for 1D parabolic singularly perturbed reaction–diffusion problems. Applied Mathematics and Computation, 218(9), 5067-5080.

Clavero, C., & Gracia, J. L. (2013). A higher order uniformly convergent method with Richardson extrapolation in time for singularly perturbed reaction–diffusion parabolic problems. *Journal of computational and applied mathematics*, *252*, 75-85.

File, G., Gadisa, G., Aga, T., & Reddy, Y. N. (2017). Numerical solution of singularly perturbed delay reaction-diffusion equations with layer or oscillatory behaviour. *American Journal of Numerical Analysis*, *5*(1), 1-10.

Friedrichs, K. O., & Wasow, W. R. (1946). Singular perturbations of non-linear oscillations. *Duke Mathematical Journal*, *13*(3), 367-381.

Gowrisankar, S., & Natesan, S. (2013). The parameter uniform numerical method for singularly perturbed parabolic reaction–diffusion problems on equidistributed grids. *Applied Mathematics Letters*, *26*(11), 1053-1060.

Gowrisankar, S., & Natesan, S. (2014). A robust numerical scheme for singularly perturbed delay parabolic initial-boundary-value problems on equidistributed grids. *Electron. Trans. Numer. Anal*, *41*, 376-395.

Gracia, J. L., & O'Riordan, E. (2015). Numerical approximation of solution derivatives in the case of singularly perturbed time dependent reaction–diffusion problems. *Journal of Computational and Applied Mathematics*, *273*, 13-24.

Kreyszig E,*Advanced engineering mathematics*. Integration. 2008 Mar; 9:4.

Kumar, S., & Ramos, H. (2021). Parameter-uniform approximation on equidistributed meshes for singularly perturbed parabolic reaction-diffusion problems with Robin boundary conditions. *Applied Mathematics and Computation*, *392*, 125677.

Lubuma, J. M. S., & Patidar, K. C. (2007). Solving singularly perturbed advection–reaction equations via non-standard finite difference methods. *Mathematical Methods in the Applied Sciences*, *30*(14), 1627-1637...

Mbroh, N. A., & Munyakazi, J. B. (2021). A robust numerical scheme for singularly perturbed parabolic reaction-diffusion problems via the method of lines. *International Journal of Computer Mathematics*, 1-20.

Miller, J. J., O'riordan, E., & Shishkin, G. I. (1996). **Fitted numerical methods for singular perturbation problems:** *error estimates in the maximum norm for linear problems in one and two dimensions*. World scientific.

Munyakazi, J. B., & Patidar, K. C. (2013). A fitted numerical method for singularly perturbed parabolic reaction-diffusion problems. *Computational and applied mathematics*, *32*(3), 509-519.

Nayfeh, A. H. (2011). **Introduction to perturbation techniques**. John Wiley & Sons..

O'Malley Jr, R. E. (1991). *Singular perturbation methods for ordinary differential equations*. Springer-Verlag..

Pal, M. (2007). Numerical Analysis for Scientists and Engineers: theory and C Programs, Alpha Science.

Smith, G. D., Smith, G. D., & Smith, G. D. S. (1985). Numerical solution of partial differential equations: finite difference methods. Oxford university press.

Tesfaye, A. G. A., Gemechis, F. İ. L. E., & Degla, G. (2019). Fitted Operator Average Finite Difference Method for Singularly Perturbed Parabolic Convection-Diffusion Problem. International Journal of Engineering and Applied Sciences, 11(3), 414-427.

Tesfaye ,Gemechis, Duressa andGuy (2021), presented accelerated fitted operator FDM for SPP reaction-diffusion problems Trefethen LN. The definition of numerical analysis. Cornell University, 1992.

Turuna, D. A., Woldaregay, M. M., & Duressa, G. F. (2020). Uniformly convergent numerical method for singularly perturbed convection-diffusion problems. *Kyungpook Mathematical Journal*, *60*(3), 629-645.

Woldaregay, M. M.,& Duressa, G. F. (2020). Higher-Order Uniformly Convergent Numerical Scheme for Singularly Perturbed Differential Difference Equations with Mixed Small Shifts. International Journal of Differential Equations, 2020.

Wasow, W. (2018). Asymptotic expansions for ordinary differential equations. Courier Dover Publications.

Zill, D. G. (2012). *A first course in differential equations with modeling applications*. Cengage Learning.