TIME FRACTIONAL ORDER FORNBERG-WHITHAM EQUATIONS VIA FRACTIONAL REDUCED DIFFERENTIAL TRANSFORM METHOD

A THESIS SUBMITTED TO THE DEPARTMENT OF MATHEMATICS, COLLEGE OF NATURAL SCIENCES, JIMMA UNIVERSITY IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF MASTER OF SCIENCE IN MATHEMATICS

BY:

YAREGAL MUCHE

ADVISOR: **- YESUF OBSIE (PhD)**

CO-ADVISOR: - ADEME KEBEDE (M.Sc.)

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DECLARATION

I, the undersigned declare that, the research entitled "**Time Fractional order Fornberg-Whitham Equation Via Fractional Reduced Differential Transform Method** " is original and it has not been submitted to any institution elsewhere for the award of any degree or like, where other sources of information that have been used, they have been acknowledge.

Name: **YAREGAL MUCHE**

Signature ________________________

Date _____________________________

The work has been done under the supervision and approval of Advisers:

Principal Advisor: **YESUF OBSIE (Ph.D.)**

Signature ________________________

Date ___________________________

Co-advisor**: ADEME KEBEDE (M.Sc.)**

Signature ________________________

Date ___________________________

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ACRONYM

ADM- Adomian Decomposition Method

DEs- Differential Equations

FPDEs- Fractional Partial Differential Equations

GDTM- Generalized Differential Transform Method

HAM- Homotopy Analysis Method

HPM- Homotopy Perturbation Method

ODEs- Ordinary Differential Equations

PDEs- Partial Differential Equations

FWEs- Fornberg –Whitham equations

RDTM- Reduced Differential Transform Method

FRDTM- Fractional Reduced Differential Transform Method

VIM- Variational Iteration Method

IC-Initial condition

FC- Fractional Calculus

ABSTRACT

The fractional reduced differential transform method (FRDTM) was a well-known method for finding an approximate analytic solution for linear and nonlinear fractional partial differential equations. FRDTM is an effective tool to solve partial differential equations analytically. This method provides the solution in the form of a convergent series with easily calculable terms. In this study FRDTM has been successfully applied on one dimensional time fractional Fornberg-Whitham equation subjected to the given initial condition. The efficacy and accuracy of FRDTM is demonstrated by two examples, which indicate that the presented method is very effective, accurate and easy to emplement.The plotted graphs illustrate the behavior of the solution for different values of order α.

Keywords: Fractional Reduced Differential Transform Method (FRDTM), Time Fractional order Fornberg-Whitham Equation, convergence.

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CHAPTER ONE

INTRODUCTION

1.1 BACKGROUND OF THE STUDY

An equation involving one or more dependent variable and its derivatives with respect to one or more independent variables is called Differential Equation (DE). Differential equations occur in connection with numerous problems that are encountered in the various branches of science and engineering. Some of these are: the problem of determining the motion of projectile, rocket, satellite, or planet, the problem of determining the charge or current in an electric circuit etc. Partial differential equations (PDEs) have broad applications in various branches of sciences and engineering such as fluid mechanics, thermodynamics, heat transfer as well as many other areas of physics (Debtnath. L, 1997). PDEs have an enormous applications compared to Ordinary Differential Equations (ODEs), to mention some of these: dynamics, electricity, heat transfer, electromagnetic theory, quantum mechanics and so on (Erwin, 2006). For many nonlinear PDEs, it is rather challenging to manage the nonlinear terms of these equations. Despite the fact that most researchers utilized numerical methods to obtain the approximate solution of the equations, being able to solve such equations analytically is significant due to the fact that manipulation are easier if the approximation is analytical in nature. A fractional partial differential equation (FPDE) is a general form of a partial differential equation by replacing the integer order derivatives with the fractional order. Due to the extensive application of fractional differential equations in various fields of engineering and science, many researchers have paid attention to find the solutions of fractional Fornberg-Whitham equation. Fractional Fornberg-Whitham equation is an example of fractional partial differential equation. The Fornberg-Whitham equation is a type of traveling wave solutions called kink-like or ant kink like wave solutions variety of application in physics and engineering arise, for example, in the propagation of electrical signals and optimization of guided communication systems. To obtain the approximate analytical solution of fractional Foremberg-Witham equation, many effective methods have been developed, such as Homotopy perturbation method (HPM) by (Gupta, P.K., 2011), Variational iteration method (VIM) by (Saka, M.G., Erdogan, 2012), combination Laplace transform and HPM (Singh, D. Kumar,S., 2013), Homotopy analysis method (HAM) and (Saberinik, H., Buzhabadi,R., 2011), differential transform method (DTM), (Merdan, M. et al, 2012) and fractional Homotopy analysis transform method (FHATM)by (Kumar,S, 2014). One of the most known methods to solve partial differential equation is the integral transform method by (Hazewinkel, M. 2001)

Fractional calculus used in many phenomena in engineering, physics, biology, fluid mechanics, and other sciences (Kumar, S;A 68a,1-7 2013) can be described very successfully by models using mathematical tools from fractional calculus. Fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes (kilbas, A.A., 2006). The fractional derivative has been occurring in many physical and engineering problems such as frequency-dependent damping behavior materials, signal processing and system identification, diffusion and reaction processes, creeping and relaxation for viscoelastic materials.

In modeling of many chemical processes, mathematical biology and many other problems of engineering (Mohsen and Seyedeh, 2018). Due to its extensive applications in various fields of engineering and science, many researchers have paid attention to find their solutions. Physical phenomena are best described by partial differential equations (PDEs), which were first developed in the 18th century for describing heat and wave phenomena by Fourier theory (Fourier, J.B.J., 1878). Since then, PDEs have found applications almost in every fields of: chemistry, fluid dynamics, quantum mechanics, classical mechanics, biology, electrostatics and electrodynamics (Jagdev, S., 2016) and many more. PDEs are described in one, two or three dimensions depending on independent variables. Since PDEs describe the complex situations mathematically, it is quite tricky to find the exact solution of every PDE.

Nonlinear partial differential equations are widely used to describe many important phenomena and dynamic processes in physics, mechanics, chemistry, biology, etc. the study of nonlinear partial differential equations plays an important role in physical sciences and Engineering fields. The investigation of exact solutions of nonlinear partial differential equations plays an important role in the study of nonlinear physical phenomena. Many methods, exact, approximate, and purely numerical are available in the literature for the solution of nonlinear partial differentials (Mahmoud Rawashden, 2013). Solving partial differential equations (PDEs) is completely important in the context of Applied Mathematics, Theoretical Physics and Engineering Sciences. It is expected that PDEs will appear while conducting research in these areas (Jafari, S. Sadeghi et al., 2012). (Murat et al., 2016) solved combined KdV- mKdV numerically by cubic B-Spline collocation method.

In recent years, fractional differential equations have received considerable attention owing to their applicability in different fields of sciences such as chemistry, biology, diffusion, control theory, rheology, viscoelasticity, and so on (Chatibi et al., 2019). Consequently, the solution of FPDEs represents nowadays a vigorous research area for scientists, and finding approximate and exact solutions to FPDEs is an important task (Bishehniasar et al., 2017). However, PDEs are commonly hard to tackle, and their fractional order types are more complicated (Chatibi et al., 2020). Therefore, several methods such as the Homotopy perturbation method (He et al., 2003), sub-ODE method (Tchier et al., 2019 and Yusuf et al., 2019), residual power series method and generalized tanh method (Aliyu et al., 2018), and so on (Akgul et al., 2018) are developed to obtain solutions of some nonlinear fractional differential equations.

Most of these methods sometimes require complex and huge calculation in order to obtain approximate solutions. To overcame such difficulties and drawbacks, an alternative method, the so called the fractional reduced differential transform method (FRDTM), has been developed by (Keskin and Oturanc, 2010). FRDTM plays a vital role among all the listed methods because it takes small size computation, easy to implement as compared to other techniques (Srivastava et al., 2013). Using this method, it is possible to find both exact and approximate solutions in a rapidly convergent power series form. FRDTM is a very reliable, efficient, and effective powerful computational technique for solving physical problems (Srivastava et al., 2014).

Recently, by (Mohamed S.M., 2018) the nonlinear Fornberg-Whitham equation is solved numerically to analyze its behavior by Residual power series method and results have been compared with the exact solution. However, the solutions of time fractional nonlinear Fornberg-Whitham equations with the given initial condition by using fractional reduced differential transform method (FRDTM) is not studied yet. Therefore, the main purpose of this study will be to develop a scheme to find approximate analytical solutions of nonlinear time-fractional Fornberg –Whitham equations of the form:

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} - \frac{\partial^3 u}{\partial x^2 \partial t} = 3\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} + u\frac{\partial^3 u}{\partial x^3} - \frac{\partial u}{\partial x} - u\frac{\partial u}{\partial x}, \quad t > 0 \quad 0 < \alpha \leq 1 \tag{1.1}$$

Subjected to the initial condition:

$$u(x,0) = f_1(x)\,, \tag{1.2}$$

where $u(x,t)$ is the fluid velocity, α is constant and lies in the interval $(0,1], t$ the time and x is the spatial coordinate. (Mehmet Merdan et al. 2012).

1.2. Statement of the problem

Recently, an approximate analytical solution of some nonlinear time fractional order Fornberg-Whitham equation was obtained by using the new iteration method (NIM). Even though time fractional Fornberg- Whitham equation can be found in a wide variety of engineering and scientific application, solving nonlinear time fractional order Fornberg-Whitham equation represented by eq. (1.1) by applying FRDTM is not presumably presented in the existing literature. Therefore, the aim of this study is solving time fractional order Fornberg-Whitham equation with the given initial condition by applying fractional reduced differential transform method. As a result, this study mainly focuses on the following problems related with time fractional order Fornberg-Whitham equation given by (1.1)

- Apply FRDTM to obtain analytical solutions of time fractional order nonlinear Fornberg – Whitham equation with the given initial condition.

- Verify the applicability of fractional reduced differential transform method (FRDTM) for solving nonlinear time fractional order Fornberg –Whitham equations to obtain analytical solutions by using specific examples.

1.3 Objective of the study

1.3.1 General objective

The general objective of this study is to find analytical solutions for nonlinear time fractional order Fornberg-Whitham equations represented by (1.1) under the given initial condition represented by (1.2) using fractional reduced differential transform method (FRDTM).

1.3.2 Specific Objectives.

The specific objectives of the study are:-

- To apply FRDTM for obtaining analytical solutions of time fractional order nonlinear Fornberg – Whitham equation with the given initial condition.
- To show the convergence of the solution obtained by FRDTM.
- To verify the applicability of FRDTM by considering supportive examples.

1.4. Significance of the Study

This research is considered to have vital importance for the following reasons:-

- It provides technique of solving time fractional order Fornberg-Whitham equations with the given initial conditions by using FRDTM.
- It familiarizes the researcher with the scientific communication in mathematics and develops the skills of mathematical research.
- It can be used as reference material for other researchers in the same area.

1.5. Delimitation of the study

The FRDTM is a powerful mathematical technique for solving wide range of problems arising in sciences and engineering fields. This study is delimited to study analytical solutions of time fractional order one dimensional nonlinear Fornberg-Whitham equations with the given initial condition by using FRDTM.

CHAPTER -TWO

REVEIW LITERATURE

Differential equations are mathematical expressions that are used to model real life problems arise in different fields of science and engineering on the world. Due to these reasons many equations are represented by using Differential Equations. Differential Equations DEs can be classified as ordinary differential equations, when the unknown function depends on one independent variable, and Partial Differential Equations, when the unknown function depends on two and more than two independent variables. The most important PDE are the wave equations that can model the vibrating string and the vibrating membrane, the heat equation for temperature in a bar or wire and the Laplace equation for electro static potentials. Partial differential equations are very important in dynamics, electricity, heat transfer, electromagnetic theory and quantum mechanics. A variety of numerical and analytical methods have been developed to obtain accurate approximate and analytic solutions for the problems. In order to understand the physical behavior of these problems it is necessary to have some knowledge about the mathematical character, properties and the solutions of the governing partial differential equations. Since the investigation of exact solution of fractional differential equations is interesting and important, in the past several decades many authors mainly had paid attention to study the solution of such equations by using various developed methods. The Vibrational Iteration Method (VIM) has been applied to handle various kinds of nonlinear problems, for example, fractional differential equations, nonlinear differential equations, nonlinear thermos elasticity and nonlinear wave equations. Adomain's Decomposition Method (ADM), Homotopy Perturbation Method (HPM), Homotopy Analysis Method (HAM) and Variation of Parameter Method (VPM) are successfully applied to obtain the exact solution of differential equations Muhammad S. & Syed T. M., (2012).

Fractional differential equations arise in almost all areas of physics, applied and engineering sciences. In order to better understand these physical phenomena, as well as further applications in practical scientific research, it is important to find the analytical solutions. The investigation of analytical solution to these equations is interesting and important. In the past, many authors had studied the solutions of such equations. Recently, several analytical and numerical techniques were successfully applied to deal with differential equations and fractional differential equations. Studies shows that the Adomian decomposition method (ADM) (Cherruauit.Y, 1993), Homotopy perturbation method (HPM) (He.J.H, 2004), Homotopy analysis method (HAM), (Matinfar.M, and Saeidy.M, 2010) and variation of parameter method (VPM), (Biazar.J, and Ghazvini.H, 2009) are successfully applied to obtain the exact solutions of differential equations.

Fractional calculus theory is more than 200 years old to be presented in the literature. Several definitions of fractional integrals and derivatives have been proposed but the first contribution to give proper and most meaning full definition is due to Riemann-Liouville. Application of fractional calculus to dynamics of particles, fields and media, (Luo.AC, et al, 2006). Reduced differential transform method has been applied to solve many physical problems; in particular it is applied in solving the heat and wave like equations. The method is applied in direct way without using linearization, transformation, discretization or restrictive assumptions.

The reduced differential transform technique is an iterative procedure for obtaining Taylor series solution of differential equations. This method reduces the size of computational work and is easily applicable to many physical problems. Nonlinear Fornberg-Whitham equation is solved numerically to analyze its behavior by Residual Power Series Method and results have been compared with the exact solution. The Foremberg-Whitham equation (FWE) has been found to require peak on results as simulation for limiting wave heights as well as the frequency of wave breaks. In fractional calculus (FC) has gained considerable significance and popularity, primarily because of its well-shown applications in a wide range of apparently disparate areas of engineering and science (Purohit, S.D., 2013). Many scholars, such as (Singh et al., 2013), (Kumar et al., 2018), (Gupta and Singh, 2011) etc., have therefore researched the fractional extensions of the Foremberg-Whitham model for the Caputo fractional-order derivative (Abidi and Omrani, 2011). FDEs are widely utilized to model in a variety of fields of study, including an analysis of fractional random walking, kinetic control schemes theory, signal processing, electrical networks, reaction and diffusion procedure (Senol et al., 2019). Fractional derivative provides a splendid method for characterizing the memories and genetic properties of different procedures (Akinyemi et al., 2020). Many scholars have recently solved different types of fractional-order PDEs, for example heat and wave equations by (Khan et al., 2019), coupled Burger equations by (Rawashdeh et al., 2014), hyperbolic telegraph equations by (Baleanu et al., 2019), Harry Dym equations by (Rawashdeh et al., 2017) and diffusion equations by (shah et al., 2019).

Recently, the new iteration method is applied to solve the time fractional Fornberg-Whitham equation. However, solutions of time fractional Fornberg-Whitham equation.by using FRDTM have not been discussed so far. Therefore, the main objective of this study is to apply FRDTM to find the analytical solution of time fractional Fornberg-Whitham equation subject to the given initial condition**.**

CHAPTER THREE

METHODOLOGY

3.1 Study Area and Period

The study is conducted on the topic from PDE which will deal with how to solve time fractional order Fornberg-Whitham equations by using FRDTM in Jimma University, under Mathematics department from August 2021 to February 2022.

3.2 Study Design

Mixed design was used for this study.

3.3. Sources of Information

The source of data for this study is secondary data which can be collected through reference books, internet, reading online books and different published research articles (or journals).

3.4. Mathematical procedures

In order to achieve the objective of the study the following mathematical procedures are applied.

Step 1. Applying fractional reduced differential transform to both sides of equations(1.1)and (1.2) to obtain a recursion relation.

Step 2. Using step 1, obtaining the values of unknown functions $U_1(x), U_2(x), U_3(x), \dots$

Step 3. Applying inverse fractional reduced differential transform on the sequence $\{U_k(x)\}_{k=0}^{n}$ to determine the solution of (1.1).

Step 4.Testing convergence of approximated solution.

Step 5.Mathematica version 7.0 software is used to sketch the solution curves.

CHAPTER FOUR
RESULT AND DISCUSSION

4.1. Preliminaries

This section presents basic definitions and operations or properties related to fractional calculus theory.

4.1.1 Gamma function

Definition 4.1.1(Gamma function, Zahid et.al, 2015): The Gamma function is defined via a convergent improper integral

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \tag{4.1}$$

Provided that $Re(z) > 0$.

(Batir**,** 2008**):** For nonnegative integer n and complex number z such that $Re(z) > 0$, one can derive the following useful properties**.**

a) $\Gamma(z+1) = z\Gamma(z)$

b) $\Gamma(n+1) = \text{n}!, n \in \text{Z}^+$

c) $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

4.1.2. Basic definitions and notations of Fractional Calculus theory

Some essential definitions of fractional order integrals and derivatives that are presented in this study are respectively given by Riemann-Liouville and Caputo.

Definition 4.1.2 let $\mu \in \Re, m \in \mathbb{N}$. A function$f: \Re^+ \to \Re$ belongs the space C_μ if there exists a real number$k \in \Re$ with $k > \mu$ such that$f(t) = t^p g(t)$, where$g(t) \in C[0, \infty)$.

Moreover, $C_\alpha \subset C_\beta$whenever$\beta \leq \alpha$and$f \in \mathbb{C}_\mu^m$ if $f^{(m)} \in \mathbb{C}_\mu$

Definition 4.1.3let J_x^α be Riemann**-**Liouville fractional integral operator and$f \in \mathbb{C}_\mu$then

I. $J_t{}^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \alpha > 0$ (4.2)

II. $J_t{}^0 f(t) = f(t)$ (4.3)

For $f \in C_\mu, \mu \geq -1, \alpha, \beta \geq 0, and\ \gamma > -1$ the operator J_x^αsatisfy the following properties:

I. $J_x{}^\alpha J_x{}^\beta f(x) = J_x{}^{\alpha+\beta} f(x) = J_x{}^\beta J_x{}^\alpha f(x)$ (4.4)

II. $J_x{}^\alpha X^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} X^{\alpha+\gamma}, X > 0$ (4.5)

Remark : The Riemann-Liouville derivative has certain limitations when someone tries to model some real physical problems. In their work, Caputo & Mainaridi proposed a modified fractional differential operator D_x^α to the theory of viscoelasticity to overcome the inconsistency of Riemann-Liouville derivative. The proposed Caputo fractional derivative permits us to use initial and boundary conditions involving integer order derivatives, which have clear physical interpretations in formulation of problem.

Definition4.1.4. If$m-1<\alpha\le m, m\in\mathbb{N}, t>0$,then Caputo fractional derivative of $f\in C_\mu$**(**Carpinteri and Mainaridi, 1997) is defined as

$$D_x{}^\alpha f(x)=J_x{}^{m-\alpha}D_x{}^m f(x)=\frac{1}{\Gamma(m-\alpha)}\int_0^x (x-t)^{m-\alpha-1}f^{(m)}(t)dt \qquad (4.6)$$

The basic properties of the Caputo fractional derivative D_x^α are presented in the following Lemma.

Lemma 1: - If $m-1<\alpha\le m, m\in\mathbb{N}\ and\ f(x)\in C_\mu^m, \mu\ge -1$,then

1. $D_t^\alpha D_t^\beta f(t)=D_t^{\alpha+\beta}f(t)=D_t^\beta D_t^\alpha f(t)$ (4.7)
2. $D_t^\alpha t^\gamma=\frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma-\alpha)}t^{\gamma-\alpha}, t>0$ (4.8)

3. $$\begin{cases} D^\alpha J^\alpha f(x)=f(x) & , \ if\ x>0 \\ J^\alpha D^\alpha f(x)=f(x)-\sum_{k=0}^{m-1}f^{\ (m)}(0^+)\frac{x^k}{k!}, & if\ x>0\end{cases} \qquad (4.9)$$

4.2 Fractional Reduced Differential Transform Method (FRDTM)

In this section, the basic properties of the fractional reduced differential transform method are described.The FRDTM is the most easily implemented analytical method which provides both approximate and the exact solution for both linear and nonlinear fractional differential equations, is very effective, reliable and efficient, and very powerful analytical approach, refer (Gupta, 2011); (Srivastava*et al*., 2013); (Srivastava, *et al*., 2014) ;(Singh *et al*., 2013) and (Singh and Kumar, 2016).

Therefore, this study presents the solution of time fractional Fornberg –Whitham equation by using FRDTM. Consider a function of two variables $u(x,t)$ and suppose that it can be represented as a product of two single-valued functions, i.e. $u(x,t)=f(x)g(t)$.

Based on the properties of one-dimensional differential transform method, the function $u(x,t)$ can be represented as:

$$u(x,t)=\left(\sum_{i=0}^{\infty}F(i)x^i\right)\left(\sum_{i=0}^{\infty}G(j)t^j\right)=\sum_{k=0}^{\infty}U_k(x)t^{k\alpha}, \qquad (4.10)$$

where $U_k(x)$ is called t-dimensional spectrum function of $u(x,t)$ which is also called the reduced transformed function of $u(x,t)$.

In fact, the above definition shows that, the concept of fractional reduced differential transform is derived from the power series expansion (Keskin and Oturanc, 2010).

The basic definition and operation of FRDTM as introduced in (Srivastava*et al*., 2013); (Babaei and pour, 2015) and (Miller and Ross, 1993) were given bellow:-

Definition 4.2.1 If $u(x,t)$ is analytic and continuously differentiable with respect to space variable **x** and time variable **t** in the domain of interest, then the t-dimensional spectrum function or the fractional reduced transformed function of $u(x,t)$ is given by

$$R_D[u(x,t)] = U_k(x) = \frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial \mathrm{t}^{k\alpha}}u(x,t)\right]_{t=t_0} \tag{4.11}$$

whereα is a parameter which describes the order of time fractional derivative in a Caputo sense an $U_k(x)$ is the transformed function of the $u(x,t)$.

Definition 4.2.2 The inverse FRDT of $U_k(x)$ is defined as

$$R_{D^{-1}}[U_k(x)] = u(x,t) = \sum_{k=0}^{\infty} U_k(x)(t-t_0)^{k\alpha} \tag{4.12}$$

Now combining Eq. (4.11**)** and (4.12**)**, we obtain:

$$u(x,t) = \sum_{k=0}^{\infty}\frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial \mathrm{t}^{k\alpha}}u(x,t)\right]_{t=t_0}(t-t_0)^{k\alpha}$$

Which in practical application can be approximated by a finite series.

$\mathrm{U_n(x\,,t)}= \sum_{\mathrm{k=0}}^{\mathrm{n}} \mathrm{U_k(x)(t-t_0)^{k\alpha}}$

Where n is the order of this approximate solution, Therefore, the exact solution can be obtained as:-

$$u(x,t) = \lim_{n\to\infty} u_n(x,t) = \sum_{k=0}^{\infty} U_k(x)(t-t_0)^{k\alpha} \tag{4.13}$$

If $t_0 = 0$ equation (4.13) reduces the form

$$u(x,t) = \lim_{n\to\infty} u_n(x,t) = \sum_{k=0}^{\infty} U_k(x)(t)^{k\alpha} \tag{4.14}$$

Moreover, if α =0 the FRDTM of (4, 14) reduce to classical RDTM.

Applying the fractional reduced differential transformed operator on both sides of equation $u(x,0) = f(x)$ and $U_t(x,0) = g(x)$,we get respectively $U_0(x,0) = f(x)$ and $U_1(x,0) = g(x)$

Hence using equation (4, 12) the function $u(x,t)$ can therefore be written in a finite series as

$u_n(x,t) = \sum_{k=0}^{\infty} U_k(x)(t-t_0)^{k\alpha} + R_n(x,t)$,

where n represents order of estimated solution. Here the tail function $R_n(x,t)$ is negligibly small.

In particular, if $t_0 = 0$this equation takes the form

$$u_n(x,t) = \sum_{k=0}^{\infty} U_k(x)t^{k\alpha}.$$

Finally, the accurate solution is found by taking limit of the function, i.e.

$\lim_{n\to\infty} u_n(x,t) = u(x,t) = \sum_{k=0}^{\infty} U_k(x) t^{k\alpha}$ $=U_0(x) + U_1(x)t^{\alpha} + U_2(x)t^{2\alpha} + U_3(x)t^{3\alpha}\ldots$ (4.15)

Based on the definition and properties of time fractional reduced differential transform of one dimensional Fornberg -Whitham equations we have the following results (Theorems).

Table 4. 1 Basic properties of one dimensional fractional reduced differential transform, (Srivastavaet *al*., 2013) and (Abuteen *et al*., 2016)

No	Original Function	Transformed function (FRDTM)
1	$w(x,t) = \alpha u(x,t) \pm \beta v(x,t)$	$W_k(x) = \alpha U_K(x) \pm \beta V_K(x)$
2	$w(x,t) = au(x,t)$	$W_k(x) = aU_k(x)$, for arbitrary constant a
3	$w(x,t) = sin(\eta x + \theta t)$	$W_k(x) = \left(\frac{\theta^k}{k!}\right)\sin\left(\eta x + (\frac{\pi k}{2})\right), where\ \eta and\ \theta\ are\ constants$
4	$w(x,t) = cos(\eta x + \theta t)$	$W_{k(x)} = \left(\frac{\theta^k}{k!}\right) cos\left(\eta x + (\frac{\pi k}{2})\right)$,where $\eta and\ \theta\ are\ constants$
5	$w(x,t) = \frac{\partial^r}{\partial t^r} u(x,t)$	$W_k(x) = (k+1)(k+2)\ldots(k+r)U_{k+1}(x) = \frac{(k+r)!}{k!} U_{k+r}(x)$
6	$w(x,t) = \frac{\partial}{\partial x} u(x,t)$	$W_k(x) = \frac{\partial}{\partial x} U_k(x)$
7	$w(x,t) = \frac{\partial^r}{\partial x^r} u(x,t)$	$W_k(x) = \frac{\partial^r}{\partial x^r} U_k(x)$
8	$w(x,t) = \frac{\partial^{N\alpha}}{\partial t^{N\alpha}} u(x,t)$	$W_{k(x)} = \frac{\Gamma(k\alpha + N\alpha + 1)}{\Gamma(k\alpha + 1)} U_{k+N(x)}$
9	w(x, t)=u(x, t)v(x, t)	$W_k(x) = \sum_{r=0}^{k} u_n v_{k-r}(x)$

Here we have the detail of some of the theorems with their proofs from table 4.1.

Theorem.4.1. If $w(x,t) = \frac{\partial^r}{\partial x^r} u(x,t)$ then, $W_k(x) = \frac{\partial^r}{\partial x^r} U_k(x)$

Proof: let $W_k(x)$ and $U_k(x)$ t-dimensional spectrum functions of $w(x,t)$ and $u(x,t)$ respectively and is analytic and k-time continuous differentiable function with respect to time t and x in the domain of our interest. Now applying FRDT operator to the left side of the equation $w(x,t) = \frac{\partial^r}{\partial x^r} u(x,t)$ we get

$$W_k(x) = \frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}} w(x,t)\right]_{t=t_0} = \frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}} \frac{\partial^r}{\partial x^r} u(x,t)\right]_{t=t_0}$$

$$= \frac{\partial^r}{\partial x^r}\left(\frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}} u(x,t)\right]_{t=t_0}\right) = \frac{\partial^r}{\partial x^r} U_k(x)$$

Hence $W_k(x) = \frac{\partial^r}{\partial x^r} U_k(x)$.This completes the proofs of the theorem.

Theorem 4.2. If $w(x,t) = \frac{\partial^{N\alpha}}{\partial t^{N\alpha}} u(x,t)$, then $W_k(x) = \frac{\Gamma(k\alpha+N\alpha+1)}{\Gamma(k\alpha+1)} U_{k+N}(x)$

Proof: let $W_k(x)$ and $U_k(x)$ be **t** -dimensional spectrum functions of $w(x,t)$ and $u(x,t)$ respectively and is analytic and **K**-time continuous differentiable function with respect to time **t** and **x** in the domain of our interest. Applying FRDTM operator to the left side equation $w(x,t) = \frac{\partial^{N\alpha}}{\partial t^{N\alpha}} u(x,t)$, we obtain

$$W_k(x) = \frac{1}{\Gamma(k\alpha+1)} \left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}} w(x,t)\right]_{t=t_0},$$

$$W_k(x) = \frac{1}{\Gamma(k\alpha+1)} \left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}} \left(\frac{\partial^{N\alpha}}{\partial t^{N\alpha}} u(x,t)\right)\right],$$

$$W_k(x) = \frac{1}{\Gamma(k\alpha+1)} \left[\frac{\partial^{k\alpha+N\alpha}}{\partial t^{k\alpha+N\alpha}} u(x,t)\right]_{t=t_o},$$

$$W_k(x) = \frac{\Gamma(k\alpha+N\alpha+1)}{\Gamma(k\alpha+N\alpha+1)} \left\{\frac{1}{\Gamma(k\alpha+1)} \left[\frac{\partial^{k\alpha+N\alpha}}{\partial t^{k\alpha+N\alpha}} U(x,t)\right]_{t=t_o}\right\},$$

$$W_k(x) = \frac{\Gamma(k\alpha+N\alpha+1)}{\Gamma(k\alpha+1)} \left\{\frac{1}{\Gamma(k\alpha+N\alpha+1)} \left[\frac{\partial^{\alpha(K+N)}}{\partial t^{\alpha(K+N)}} U(x,t)\right]_{t=t_0}\right\},$$

$$W_k(x) = \frac{\Gamma(k\alpha+N\alpha+1)}{\Gamma(k\alpha+1)} U_{k+N}(x).$$

Theorem 4.3. If $g(x,t) = x^n \sin(\eta x + \theta t)$, then the fractional reduced differential transform of g is $G_k(x) = (\frac{\theta^k}{k!}) x^n \sin(\eta x + \frac{\pi k}{2})$Where$\eta$ and θ are constants.

Proof**: -** Using definition (4.2.1) and properties of fractional reduced differential transform method, we have

$$G_k(x) = \frac{1}{\Gamma(k\alpha+1)} \left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}} g(x,t)\right]_{t=t_0},$$

$$G_k(x) = \frac{1}{\Gamma(k\alpha+1)} \left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}} x^n \sin(\eta x + \theta t)\right]_{t=t_0},$$

$$G_k(x) = x^n \frac{1}{\Gamma(k\alpha+1)} \left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}} \sin(\eta x + \theta t)\right]_{t=t_0},$$

$$G_k(x) = \frac{\theta^k}{k!} x^n \sin(\eta x + \frac{\pi k}{2}),\ \text{for k=0, 1, 2…}$$

Theorem 4.4. If$f(x,t) = x^n \cos(\eta x + \theta t)$, then the fractional reduced differential transform of f is $F_k(x) = (\frac{\theta^k}{k!}) x^n \cos(\eta x + \frac{\pi k}{2})$,where$\eta$ andθ are constants.

Proof**: -** Using definition (4.2.1) and properties of fractional reduced differential transform method, we have

$$F_k(x) = \frac{1}{\Gamma(k\alpha+1)} \left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}} f(x,t)\right]_{t=t_0},$$

$$F_k(x) = \frac{1}{\Gamma(k\alpha+1)} \left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}} x^n \cos(\eta x + \theta t)\right]_{t=t_0},$$

$F_k(x) = x^n \frac{1}{\Gamma(k\alpha+1)} \left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}} \cos(\eta x + \theta t)\right]_{t=t_0},$

$F_k(x) = \frac{\theta^k}{k!} x^n \cos(\eta x + \frac{\pi k}{2})$Where k=0, 1, 2…

Lemma 1: If$w(x,t) = \frac{\partial^3 u(x,t)}{\partial x^2 \partial t}$, then$W_k(x) = \frac{\Gamma(k\alpha+2)}{\Gamma(k\alpha+1)} \frac{\partial^2}{\partial t^2} U_{k+1}(x)$,where $k = 0,1,2,\cdots$

Proof**:**By definition 4.2.1, we have

$W_k(x) = \frac{1}{\Gamma(k\alpha+1)} \left[\frac{\partial^{k\alpha}}{\partial \mathrm{t}^{k\alpha}} w(x,t)\right]_{t=t_0},$

$W_k(x) = \frac{1}{\Gamma(k\alpha+1)} \left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}} w(x,t)\right]_{t=t_0},$

$W_k(x) = \frac{1}{\Gamma(k\alpha+1)} \left[\frac{\partial^{k\alpha}}{\partial \mathrm{t}^{k\alpha}} \left\{\frac{\partial^3 u(x,t)}{\partial x^2 \partial t}\right\}\right]_{t=t_0},$

$W_k(x) = \frac{1}{\Gamma(k\alpha+1)} \left[\frac{\partial^2}{\partial \mathrm{x}^2} \left\{\frac{\partial^{k\alpha+1} u(x,t)}{\partial t^{k\alpha+1}}\right\}\right]_{t=t_0},$

$W_k(x) = \frac{\Gamma(k\alpha+2)}{\Gamma(k\alpha+1)} \frac{\partial^2}{\partial \mathrm{x}^2} \left[\frac{1}{\Gamma(k\alpha+2)} \left\{\frac{\partial^{k\alpha+1} u(x,t)}{\partial t^{k\alpha+1}}\right\}\right]_{t=t_0},$

$W_k(x) = \frac{\Gamma(k\alpha+2)}{\Gamma(k\alpha+1)} \frac{\partial^2 \mathrm{U_{k+1}(x)}}{\partial \mathrm{x}^2}$,where $k = 0,1,2,\cdots$

Lemma 3: If $w(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2} \frac{\partial^3 \mathrm{v(x,t)}}{\partial x^3}$,then $W_k(x) = \sum_{r=0}^{k} \frac{\partial^2 u_r}{\partial x^2} \frac{\partial^3 v_{k-r}}{\partial x^3}$,where $k = 0,1,2,\cdots$

Proof:By using equation (4.14),weget

$W_k(x) = (\sum_{k=0}^{\infty} \frac{\partial^2 u_k}{\partial x^2} (t)^{k\alpha}) \, (\sum_{k=0}^{\infty} \frac{\partial^3 v_k}{\partial x^3} (t)^{k\alpha}) \quad ,$

$= (\frac{\partial^2 u_0}{\partial x^2} + \frac{\partial^2 u_1}{\partial x^2} t^{\alpha} + \frac{\partial^2 u_2}{\partial x^2} t^{2\alpha} + \frac{\partial^2 u_3}{\partial x^2} t^{3\alpha} + \frac{\partial^2 u_4}{\partial x^2} t^{4\alpha} + \cdots)$

$(\frac{\partial^3 v_0}{\partial x^3} + \frac{\partial^3 v_1}{\partial x^3} t^{\alpha} + \frac{\partial^3 v_2}{\partial x^3} t^{2\alpha} + \frac{\partial^3 v_3}{\partial x^3} t^{3\alpha} + \frac{\partial^3 v_4}{\partial x^3} t^{4\alpha} + \cdots),$

$= (\frac{\partial^2 u_0}{\partial x^2} \frac{\partial^3 v_0}{\partial x^3} + (\frac{\partial^2 u_0}{\partial x^2} \frac{\partial^3 v_1}{\partial x^3} + \frac{\partial^2 u_1}{\partial x^2} \frac{\partial^3 v_0}{\partial x^3}) t^{\alpha} + (\frac{\partial^2 u_2}{\partial x^2} \frac{\partial^3 v_0}{\partial x^3} + \frac{\partial^2 u_1}{\partial x^2} \frac{\partial^3 v_1}{\partial x^3} + \frac{\partial^2 u_0}{\partial x^2} \frac{\partial^3 v_2}{\partial x^3}) t^{2\alpha})$

$+ \cdots + (\frac{\partial^2 u_0}{\partial x^2} \frac{\partial^3 v_k}{\partial x^3} + \frac{\partial^2 u_1}{\partial x^2} \frac{\partial^3 v_{k-1}}{\partial x^3} + \cdots + \frac{\partial^2 u_{k-1}}{\partial x^2} \frac{\partial^3 v_1}{\partial x^3} + \frac{\partial^2 u_k}{\partial x^2} \frac{\partial^3 v_0}{\partial x^3}) t^{k\alpha},$

$= \sum_{r=0}^{k} \frac{\partial^2 u_r}{\partial x^2} \frac{\partial^3 v_{k-r}}{\partial x^3} t^{k\alpha},$

Therefore, we obtain

$$W_k(x) = \sum_{r=0}^{k} \frac{\partial^2 u_r}{\partial x^2} \frac{\partial^3 v_{k-r}}{\partial x^3} t^{k\alpha}.$$

4.3. Description of the Method

The aim of this research is to obtain analytical solution of one dimensional time fractional Fornberg - Whitham equation by using fractional reduced differential transform method. This is done based on the works of (Keskin and Oturanc, 2010) that was used to solve fractional partial differential equations. So, the definitions, theorems and some derivations related to FRDTM mentioned in the preceding section were applied here.

I. Consider the one dimensional time fractional homogeneous Fornberg –Whitham equation in Caputo sense:

$$\frac{\partial^\alpha u}{\partial t^\alpha}-\frac{\partial^3 u}{\partial x^2\,\partial t}=3\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}+u\frac{\partial^3 u}{\partial x^3}-\frac{\partial u}{\partial x}-u\frac{\partial u}{\partial x},\quad t>0\quad 0<\alpha\le 1, \tag{4.16}$$

Subjected to the initial condition:

$$u(x,0)=\ f_1(x) \tag{4.17}$$

Applying properties of FRDTM on equation (4.16**)**, we get the following recurrence relation:

$$\frac{\Gamma(\alpha(k+1)+1)}{\Gamma(\alpha k+1)}U_{k+1}-\frac{\Gamma(\alpha k+2)}{\Gamma(\alpha k+1)}\frac{\partial^2}{\partial x^2}U_{k+1}+\frac{\partial}{\partial x}U_k=\sum_{r=0}^{k}U_r\frac{\partial^3}{\partial x^3}U_{k-r}-\sum_{r=0}^{k}U_r\frac{\partial}{\partial x}U_{k-r}+$$
$$3\sum_{r=0}^{k}\frac{\partial}{\partial x}U_r\frac{\partial^2}{\partial x^2}U_{k-r}$$

where $U=U(x)$.

$$U_{k+1}=\frac{\Gamma(\alpha k+1)}{\Gamma(\alpha(k+1)+1)}\left\{\begin{matrix}\frac{\Gamma(\alpha k+2)}{\Gamma(\alpha k+1)}\frac{\partial^2}{\partial x^2}U_{k+1}-\frac{\partial}{\partial x}U_k+\sum_{r=0}^{k}U_r\frac{\partial^3}{\partial x^3}U_{k-r}-\ \sum_{r=0}^{k}U_r\frac{\partial}{\partial x}U_{k-r}\\ +3\sum_{r=0}^{k}\frac{\partial}{\partial x}U_r\frac{\partial^2}{\partial x^2}U_{k-r}\end{matrix}\right\} \tag{4.18}$$

Again applying FRDTM on both sides equation (4.17), we obtain

$$U_0(x)=f_1(x) \tag{4.19}$$

Using equations (4.18) and (4.19), we get the values $U_k(x)$for different values of k = 0, 1, 2, 3, 4 … recursively. i.e.

For k=0,

$$U_1=\frac{\Gamma(1)}{\Gamma(\alpha+1)}\left\{\frac{\Gamma(2)}{\Gamma(1)}\frac{\partial^2}{\partial x^2}U_1-\frac{\partial}{\partial x}U_0+U_0\frac{\partial^3}{\partial x^3}U_0-U_0\frac{\partial}{\partial x}U_0+3\frac{\partial}{\partial x}U_0\frac{\partial^2}{\partial x^2}U_0\right\},$$

$$U_1=\frac{\Gamma(1)}{\Gamma(\alpha+1)}\left\{\frac{\Gamma(2)}{\Gamma(1)}\frac{\partial^2}{\partial x^2}U_1-\frac{\partial}{\partial x}f_1(x)+U_0\frac{\partial^3}{\partial x^3}f_1(x)-f_1(x)\frac{\partial}{\partial x}f_1(x)+3\frac{\partial}{\partial x}f_1(x)\frac{\partial^2}{\partial x^2}f_1(x)\right\},$$

For k=1,

$$U_2=\frac{\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}\left\{\begin{matrix}\frac{\Gamma(\alpha+2)}{\Gamma(\alpha+1)}\frac{\partial^2}{\partial x^2}U_2(x)-\frac{\partial U_1(x)}{\partial x}-\sum_{r=0}^{1}U_{k-r}(x)\frac{\partial U_r}{\partial x}+\sum_{r=0}^{1}\ U_{k-r}(x)\frac{\partial^3 U_r(x)}{\partial x^3}\\ +3\sum_{r=0}^{1}\ \frac{\partial U_{k-r}}{\partial x}\frac{\partial^2 U_r(x)}{\partial x^2}\end{matrix}\right\},$$

$$U_2=\frac{\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}\left\{\begin{matrix}\frac{\Gamma(\alpha+2)}{\Gamma(\alpha+1)}\frac{\partial^2}{\partial x^2}U_2-\frac{\partial}{\partial x}U_1+\left(U_0\frac{\partial^3}{\partial x^3}U_1+U_1\frac{\partial^3}{\partial x^3}U_0\right)\\ -\left(U_0\frac{\partial}{\partial x}U_1+U_1\frac{\partial}{\partial x}U_0\right)+3\left(\frac{\partial}{\partial x}U_0\frac{\partial^2}{\partial x^2}U_1+\frac{\partial}{\partial x}U_1\frac{\partial^2}{\partial x^2}U_0\right)\end{matrix}\right\},$$

For k=2,

$$U_3 = \frac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)}\left\{\begin{matrix}\frac{\Gamma(2\alpha+2)}{\Gamma(2\alpha+1)}\frac{\partial^2}{\partial x^2}U_3(x) - \frac{\partial}{\partial x}U_2 - \sum_{r=0}^{2} U_{k-r}(x)\frac{\partial U_r}{\partial x} + \sum_{r=0}^{2} U_{k-r}(x)\frac{\partial^3 U_r(x)}{\partial x^3} \\ +3\left(\sum_{r=0}^{2} \frac{\partial U_{k-r}}{\partial x}\frac{\partial^2 U_r(x)}{\partial x^2}\right)\end{matrix}\right\},$$

$$U_3 = \frac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)}\left\{\begin{matrix}\frac{\Gamma(2\alpha+2)}{\Gamma(2\alpha+1)}\frac{\partial^2}{\partial x^2}U_3 - \frac{\partial}{\partial x}U_2 + \left(U_0\frac{\partial^3}{\partial x^3}U_2 + U_1\frac{\partial^3}{\partial x^3}U_1 + U_2\frac{\partial^3}{\partial x^3}U_0\right) \\ -\left(U_0\frac{\partial}{\partial x}U_2 + U_1\frac{\partial}{\partial x}U_1 + U_2\frac{\partial}{\partial x}U_0\right) + 3\left(\frac{\partial}{\partial x}U_0\frac{\partial^2}{\partial x^2}U_2 + \frac{\partial}{\partial x}U_1\frac{\partial^2}{\partial x^2}U_1 + \frac{\partial}{\partial x}U_2\frac{\partial^2}{\partial x^2}U_0\right)\end{matrix}\right\},$$

For k=3

$$U_4 = \frac{\Gamma(3\alpha+1)}{\Gamma(4\alpha+1)}\left\{\begin{matrix}\frac{\Gamma(3\alpha+2)}{\Gamma(3\alpha+1)}\frac{\partial^2}{\partial x^2}U_4(x) - \frac{\partial}{\partial x}U_3 - \sum_{r=0}^{3} U_{k-r}(x)\frac{\partial U_r}{\partial x} + \sum_{r=0}^{3} U_{k-r}(x)\frac{\partial^3 U_r(x)}{\partial x^3} \\ +3\sum_{r=0}^{3} \frac{\partial U_{k-r}}{\partial x}\frac{\partial^2 U_r(x)}{\partial x^2}\end{matrix}\right\},$$

$$U_4 = \frac{\Gamma(3\alpha+1)}{\Gamma(4\alpha+1)}\left\{\begin{matrix}\frac{\Gamma(3\alpha+2)}{\Gamma(3\alpha+1)}\frac{\partial^2}{\partial x^2}U_4(x) - \frac{\partial U_3(x)}{\partial x} - \left(U_0\frac{\partial^3}{\partial x^3}U_3 + U_1\frac{\partial^3}{\partial x^3}U_2 + U_2\frac{\partial^3}{\partial x^3}U_1 + U_3\frac{\partial^3}{\partial x^3}U_0\right) \\ -\left(U_0\frac{\partial}{\partial x}U_3 + U_1\frac{\partial}{\partial x}U_2 + U_2\frac{\partial}{\partial x}U_1 + U_3\frac{\partial}{\partial x}U_0\right) \\ +3\left(\frac{\partial}{\partial x}U_0\frac{\partial^2}{\partial x^2}U_3 + \frac{\partial}{\partial x}U_1\frac{\partial^2}{\partial x^2}U_2 + \frac{\partial}{\partial x}U_2\frac{\partial^2}{\partial x^2}U_1 + \frac{\partial}{\partial x}U_3\frac{\partial^2}{\partial x^2}U_0\right)\end{matrix}\right\},$$

and so on

Applying inverse FRDTM on $U_k(x)$, we find

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x)t^{k\alpha} = U_0 + U_1t^{\alpha} + U_2t^{2\alpha} + U_3t^{3\alpha} + U_4t^{4\alpha} + U_5t^{5\alpha} + \cdots$$

$$= f_1(x) + U_1t^{\alpha} + U_2t^{2\alpha} + U_3t^{3\alpha} + U_4t^{4\alpha} + U_5t^{5\alpha} + \cdots$$

4.4. Convergence of reduced differential transform method

In this section, we survey the sufficient condition for convergence of reduce differential transform method, and extend this idea for the convergence of the fractional reduced transform method

First, we discuss the fundamental theorem associated with the convergence of RDTM as in (Roodabeh et al., 2021) for the solution of the problem include ascertaining power series expansion with initial time t_0

$$w(x,t) = \sum_{k=0}^{\infty} d_k(x)(t-t_0)^k \quad, t \in l \tag{4.20}$$

$where\ l = (t_0, t_0 + r), r > 0.$

The important results are proposed in the following theorems

Theorem 4.5: If $\varphi_k(x,t) = d_k(x)(t-t_0)^k$, then the series solution $\sum_{k=0}^{\infty} \varphi_k(x,t)$, stated in equation (4.20) $\forall k \in N \cup \{0\}$

1. It is convergent, if $\exists\ 0 < \lambda < 1$ such that $||\varphi_{k+1}(x,t)|| \le \lambda||\varphi_k(x,t)||$,

2. It is divergent, if $\exists\ \lambda > 1$ such that $||\varphi_{k+1}(x,t)|| \ge \lambda||\varphi_k(x,t)||$.

Using the Banach's fixed point theorem a brief description of its proof. We investigate the truncation error of the series solution equation (4.20), as follow:

Proof: Let $(\mathbb{C}[l], ||.||)$ be the Banach space of all continuous function on l with the norm $||\varphi_k(x,t)|| = ||d_k(x)(t-t_0)^k||$. Also, assume that $||d_0(x)|| < \mathrm{N}_0$, where N_0 is a positive number.

Define the sequence of partial sum $\{S_n\}_{n=0}^{\infty}$ as

$$S_n = \varphi_0 + \varphi_1 + \varphi_2 + \ldots + \varphi_n \ . \tag{4.21}$$

We want to show that $\{S_n\}_{n=0}^{\infty}$ is a Cauchy sequence in this Banach space. To reach this goal, we take

$$\|S_{n+1} - S_n\| = ||\varphi_{n+1}|| \leq \lambda ||\varphi_n|| \leq \ldots \leq \lambda^{n+1} ||\varphi_n|| \leq \lambda^{n+1} \mathrm{N}_0 \ . \tag{4.22}$$

Therefore, for any n, m∈ N, n $\geq m$, using equation (4.22) and the triangle inequality successively, we have

$$\begin{aligned} \|S_n - S_m\| &= \|(S_n - S_{n-1}) + (S_{n-1} - S_{n-2}) + \ldots + (S_{m+1} - S_m)\| \\ &\leq \|(S_n - S_{n-1})\| + \|(S_{n-1} - S_{n-2})\| + \ldots + \|(S_{m+1} - S_m)\| \\ &\leq \frac{1-\lambda^{n-m}}{1-\lambda} \lambda^{m+1} ||\varphi_0|| \ , \end{aligned} \tag{4.23}$$

and because $0 < \lambda < 1$, we obtain $\lim_{n,m\to\infty} ||S_n - S_m|| = 0$. (4.24)

Hence, $\{S_n\}_{n=0}^{\infty}$ is a Cauchy sequence in the Banach space$(\mathbb{C}[l], ||.||)$. Thus the series solution $\sum_{k=0}^{\infty} \varphi_k(x,t)$ defined in equation (4.20), is convergent.

Remark 1: According to the assumption in No. 2 and by using the ratio test, we have

$$\left|\left|\frac{\varphi_{n+1}}{\varphi_n}\right|\right| \geq \lambda > 1. \tag{4.25}$$

As a result, the series is divergent.

Remark 2: If the series solution $\sum_{k=0}^{\infty} d_k(x)(t-t_0)^k$of the nonlinear equation (4.16) convergence than it is an exact solution.

Theorem 4.6: Suppose the series solution $\sum_{k=0}^{\infty} \varphi_k(x,t)$where$\varphi_k(x,t) = d_k(x)(t-t_0)^k$, converges to the solution u(x, t). If the truncated series $\sum_{k=0}^{m} \varphi_k(x,t)$ is used as an approximation to the solution w(x, t), then the maximum absolute truncated error is estimated as (Roodabeh et al., 2021),

$$\left\|w(x,t) - \sum_{k=0}^{m} \varphi_{k(x,t)}\right\| \leq \frac{1}{1-\lambda} \lambda^{m+1} \|\varphi_0\| \ . \tag{4.26}$$

Proof: According to Theorem (4.5), we have the inequality equation (4.23) as follows

$$\|s_n - s_m\| \leq \frac{1-\lambda^{n-m}}{1-\lambda} \lambda^{m+1} \|\varphi_0\|, \tag{4.27}$$

For $n \geq m$. Also, since $0 < \lambda < 1$, in the numerator, we have $1 - \lambda^{n-m} < 1$, therefore the inequality equation (4.27) can be reduced to $\|S_n - S_m\| \leq \frac{1}{1-\lambda} \lambda^{m+1} ||\varphi_0||$.

it is clear when $n \to \infty$, $S_n \to u(x,t)$

Thus, inequality equation (4.26) is obtained and the Theorem is proved.

In generally, Theorem (4.5) and (4.6) state that the reduced differential transform solution of equation (4.16), obtained using the iteration formula (4.18), converges to an exact solution under the condition that $\exists\ 0 < \lambda < 1$ such that $||\varphi_{n+1}|| \leq \lambda||\varphi_n||, \forall k \in N \cup \{0\}$. In other words, if we define, for every $l \in N \cup \{0\}$, the parameters,

$$\gamma_i = \begin{cases} \frac{||\varphi_{i+1}||}{||\varphi_i||}, & ||\varphi_i|| \neq 0, \\ 0 \quad , & ||\varphi_i|| = 0 \quad , \end{cases} \tag{4.28}$$

Then the series solution $\sum_{k=0}^{m} \varphi_k(x,t)$ equation (4.16) converges to an exact solution u(x, t), when $0 \leq \gamma_i < 1\ \forall i \in N \cup \{0\}$. In addition, the maximum absolute truncation error, as discussed Theorem (4.6) is estimated to be

$$\left\|w(x,t) - \sum_{k=0}^{m} \varphi_{k(x,t)}\right\| \leq \frac{1}{\gamma-1}\gamma^{j+1}||\varphi_0|| \ , \quad (4.29$$

Where $\gamma = max\{\gamma_i,$ i=0, 1, 2… j},

Remark 3: The first finite terms have no effect on the convergence of the series solution. In other words if the first finite γ_i's, i=0, 1, 2… L are not less than one and $\gamma_i \leq 1\ for\ i > 1$, then, the series solution $\sum_{k=0}^{m} \varphi_k(x,t)$ of equation (4.16) converges to an exact solution. Because according to Theorem (4.5), we have $\left\|s_{n-}s_j\right\| \leq \frac{1-\lambda^{n-j}}{1-\lambda}\lambda^{j+1}\ ||\varphi_{i+1}||$, and since $0 < \lambda < 1$ for $n \geq j$ and fixed l, we get,

$\lim_{n,j\to\infty}\left\|s_{n-}s_j\right\| = 0$. In this case, the convergence of RDTM approach depends on γ_i for all $i > 1$.

4.5. Convergence of the fractional reduced differential transforms method

In this section, we survey the sufficient condition for convergence of fractional reduce differential transform method, according to the approach described by this method for solving equation (4.16), in the previous section. Below are some important theorems for convergences of the method are proved.

The fundamental point views of FRDTM for the solution of the problem include ascertaining power series expansion with initial time t_0

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x)(t-t_0)^{k\alpha} \ \text{t} \in l \tag{4.30}$$

where $l = (t_0, t_0 + r), r > 0$.

The important results are proposed in the theorems below which were modified form of the above theorem 4.5 and 4.6, according to the fractional reduced differential transform method procedure.

Theorem 4.7. If $u_k(x,t) = U_k(x)(t-t_0)^{k\alpha}$, then the series solution $\sum_{k=0}^{\infty} u_k(x,t)$, stated in equation (4.30) $\forall k \in N \cup \{0\}$ (Seyedeh al., 2021).

(i). It is convergent, if $\exists\ 0 < \lambda < 1$such that $\left||u_{k+1}(x,t)|\right| \leq \lambda\left||u_k(x,t)|\right|$, (ii). It is divergent, if $\exists\ \lambda > 1$ such that$||u_{k+1}(x,t)|| \geq \lambda||u_k(x,t)||$.

Using the Banach's fixed point theorem a brief description of its proof. We investigate the truncation error of the series solution equation (4.30), as follow:

Proof: Let $(\mathbb{C}[l], ||.||)$ be the Banach space of all continuous function on l with the norm $||u_k(x,t)|| = ||u_k(x)(t-t_0)^{k\alpha}||$. Also, assume that $||u_0(x)|| < \mathrm{N}_0$,where N_0 is a positive number.

Define the sequence of partial sum $\{S_n\}_{n=0}^{\infty}$ as

$$S_n = u_0 + u_1 + u_2 + \ldots + u_n \ . \tag{4.31}$$

We want to show that $\{S_n\}_{n=0}^{\infty}$ is a Cauchy sequence in this Banach space. To reach this goal, we take

$$\|S_{n+1} - S_n\| = ||u_{n+1}|| \leq \lambda||u_n|| \leq \ldots \leq \lambda^{n+1}||u_n|| \leq \lambda^{n+1}\mathrm{N}_0 \ . \tag{4.32}$$

Therefore, for any n, m $\in N$,n $\geq m$ using equation (4.32) and the triangle inequality successively, we have $\|S_n - S_m\| = \|(S_n - S_{n-1}) + (S_{n-1} - S_{n-2}) + \ldots + (S_{m+1} - S_m)||$

$\leq \|(S_n - S_{n-1})|| + ||(S_{n-1} - S_{n-2})|| + \ldots + ||(S_{m+1} - S_m)||$

$$\leq \frac{1-\lambda^{n-m}}{1-\lambda}\lambda^{m+1}||u_0|| \ , \tag{4.33}$$

and because $0 < \lambda < 1$, we obtain $\lim_{n,m\to\infty} ||S_n - S_m|| = 0$. (4.34)

Hence, $\{S_n\}_{n=0}^{\infty}$ is a Cauchy sequence in the Banach space$(\mathbb{C}[l], ||.||)$.Thus the series solution $\sum_{k=0}^{\infty} u_k(x,t)$ defined in equation (4.30), is convergent.

Remark 4: According to the assumption in (ii) and by using the ration test, we have

$$||\frac{u_{n+1}}{u_n}|| \geq \lambda > 1. \tag{4.35}$$

As a result, the series is divergent.

Remark 5 . If the series solution $\sum_{k=0}^{\infty} u_k(x)(t-t_0)^{k\alpha}$ of the nonlinear equation (4.16) convergence than it is an exact solution.

Theorem 4.8: Suppose the series solution $\sum_{k=0}^{\infty} u_k(x,t)$ where$u_k(x,t) = u_k(x)(t-t_0)^{k\alpha}$, converges to the solution u(x, t). If the truncated series $\sum_{k=0}^{m} u_k(x,t)$ is used as an approximation to the solution u(x, t) and then the maximum absolute truncated error is estimated as (Seyedeh al., 2021)

$$\|u(x,t) - \sum_{k=0}^{m} u_k(x,t)\| \leq \frac{1}{1-\lambda}\lambda^{m+1}\|u_0\|. \tag{4.36}$$

Proof: According to Theorem (4.7), we have the inequality equation (4.33) as follows

$$\|s_{n-}s_m\| \leq \frac{1-\lambda^{n-m}}{1-\lambda}\lambda^{m+1}\|u_0\|, \tag{4.37}$$

For $n \geq m$. Also, since $0 < \lambda < 1$, in the numerator, we have$1 - \lambda^{n-m} < 1$, therefore the inequality equation (4.37) can be reduced to $\|S_n - S_m\| \leq \frac{1}{1-\lambda}\lambda^{m+1}||u_0||$, it is clear when$n \to \infty$, $S_n \to u(x,t)$ Thus, inequality equation (4.36) is obtained and the Theorem is proved.

In generally, Theorem (4.7) and (4.8) state that the fractional reduced differential transform solution of equation (4.16), obtained using the iteration formula (4.18), converges to an exact solution under the condition that $\exists\ 0 < \lambda < 1$ such that $||u_{n+1}|| \leq \lambda||u_n||, \forall u \in N \cup \{0\}$. In other words, if we define, for every i$\in N \cup \{0\}$, the parameters,

$$\gamma_i = \begin{cases} \frac{||u_{i+1}||}{||u_i||}, & ||u_i|| \neq 0, \\ 0\ , & ||u_i|| = 0\ , \end{cases} \tag{4.38}$$

Then the series solution $\sum_{k=0}^{m} u_k(x,t)$ equation (4.16) converges to an exact solution u(x, t), when $0 \leq \gamma_i < 1\ \forall i \in N \cup \{0\}$. In addition, the maximum absolute truncation error, as discussed Theorem (4.8) is estimated to be

$$\|u(x,t) - \textstyle\sum_{k=0}^{m} u_k(x,t)\,|| \leq \frac{1}{\gamma-1}\gamma^{j+1}||u_0||\ , \tag{4.39}$$

$where\ \gamma = max\{\gamma_i,$ i=0, 1, 2,…,j},

Remark 6: The first finite terms have no effect on the convergence of the series solution. In other words , if the first finite γ_i's, i=0,1,2,…L are not less than one and $\gamma_i \leq 1\ for\ i > 1$, then, the series solution $\sum_{k=0}^{m} u_k(x,t)$ of equation (4.16) converges to an exact solution. Because according to Theorem (4.7), we have $\|S_n - S_j|| \leq \frac{1-\lambda^{n-j}}{1-\lambda}\lambda^{j+1}||u_0||$, and since $0 < \lambda < 1$ for $n \geq j$ and fixed l, we get

$\lim_{n,j\to\infty} ||S_n - S_j|| = 0$. In this case, the convergence of FRDTM approach depends on γ_i for all $i > 1$.

4.6 Illustrative examples

In this part we deal with some examples to show the efficiency and accuracy of fractional reduced differential transform method (FRDTM) explained in the above sections for time fractional Fornberg-Whitham equation.

Example 4.1. Consider the fractional type Fornberg-Whitham equation stated as follows:

$$\frac{\partial^\alpha \mathrm{u}}{\partial \mathrm{t}^\alpha} - u_{xxt} + u_x - uu_{xxx} + uu_x - 3u_x u_{xx} = 0\,, \tag{4.40}$$

Subjected to the initial condition

$$\mathrm{u}(\mathrm{x}, 0) = \mathrm{e}^{\frac{\mathrm{x}}{2}} \tag{4.41}$$

Solution:

Applying FRDTM operator to both sides of the equations (4.40) we obtain the following iteration as follow.

$$\frac{\Gamma(\alpha(k+1)+1)}{\Gamma(\alpha k+1)}U_{k+1}-\frac{\Gamma(\alpha k+2)}{\Gamma(\alpha k+1)}\frac{\partial^2}{\partial x^2}U_{k+1}+\frac{\partial}{\partial x}U_k = U_k\frac{\partial^3}{\partial x^3}U_k-U_k\frac{\partial}{\partial x}U_k+3\frac{\partial}{\partial x}U_k\frac{\partial^2}{\partial x^2}U_k\ ,$$

where $U=U(x)$

$$U_{k+1}=\frac{\Gamma(\alpha k+1)}{\Gamma(\alpha(k+1)+1)}\left\{\begin{matrix}\frac{\Gamma(\alpha k+2)}{\Gamma(\alpha k+1)}\frac{\partial^2}{\partial x^2}U_{k+1}-\frac{\partial}{\partial x}U_k+\sum_{r=0}^{k}U_r\frac{\partial^3}{\partial x^3}U_{k-r}-\sum_{r=0}^{k}U_r\frac{\partial}{\partial x}U_{k-r}\\ +3\sum_{r=0}^{k}\frac{\partial}{\partial x}U_r\frac{\partial^2}{\partial x^2}U_{k-r}\end{matrix}\right\}. \tag{4.42}$$

Again applying FRDTM operator to both sides of the Eq. (4.41) we obtain the following iteration as follow.

$$R_D\mathrm{u}(\mathrm{x},0)=U_0(x)=\mathrm{e}^{\frac{\mathrm{x}}{2}} \tag{4.43}$$

Using equation (4.40) and (4.41), we get the following recursive relation:-

For k=0,

$$U_1=\frac{\Gamma(1)}{\Gamma(\alpha+1)}\left\{\frac{\Gamma(2)}{\Gamma(1)}\frac{\partial^2}{\partial x^2}U_1-\frac{\partial}{\partial x}U_0+U_0\frac{\partial^3}{\partial x^3}U_0-U_0\frac{\partial}{\partial x}U_0+3\frac{\partial}{\partial x}U_0\frac{\partial^2}{\partial x^2}U_0\right\},$$

$$U_1=\frac{2\Gamma(1)\mathrm{e}^{\frac{\mathrm{x}}{2}}}{\Gamma(2)-4\Gamma(\alpha+1)},$$

$$U_1=\frac{2\mathrm{e}^{\frac{\mathrm{x}}{2}}}{1-4\Gamma(\alpha+1)},$$

For k=1,

$$U_2=\frac{\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}\left\{\begin{matrix}\frac{\Gamma(\alpha+2)}{\Gamma(\alpha+1)}\frac{\partial^2}{\partial x^2}U_2-\frac{\partial}{\partial x}U_1+\left(U_0\frac{\partial^3}{\partial x^3}U_1+U_1\frac{\partial^3}{\partial x^3}U_0\right)-\left(U_0\frac{\partial}{\partial x}U_1+U_1\frac{\partial}{\partial x}U_0\right)\\ +3\left(\frac{\partial}{\partial x}U_0\frac{\partial^2}{\partial x^2}U_1+\frac{\partial}{\partial x}U_1\frac{\partial^2}{\partial x^2}U_0\right)\end{matrix}\right\},$$

$$U_2=\mathrm{e}^{\frac{\mathrm{x}}{2}}\left(\frac{4\Gamma(1)\Gamma(\alpha+1)}{\left(\Gamma(2)-4\Gamma(\alpha+1)\right)[\Gamma(\alpha+2)-4\Gamma(2\alpha+1)]}\right),$$

$$U_2=\mathrm{e}^{\frac{\mathrm{x}}{2}}\left(\frac{4\Gamma(\alpha+1)}{\left(1-4\Gamma(\alpha+1)\right)[\Gamma(\alpha+2)-4\Gamma(2\alpha+1)]}\right),$$

For k=2,

$$U_3=\frac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)}\left\{\begin{matrix}\frac{\Gamma(2\alpha+2)}{\Gamma(2\alpha+1)}\frac{\partial^2}{\partial x^2}U_3-\frac{\partial}{\partial x}U_2+\left(U_0\frac{\partial^3}{\partial x^3}U_2+U_1\frac{\partial^3}{\partial x^3}U_1+U_2\frac{\partial^3}{\partial x^3}U_0\right)\\ -\left(U_0\frac{\partial}{\partial x}U_2+U_1\frac{\partial}{\partial x}U_1+U_2\frac{\partial}{\partial x}U_0\right)+3\left(\frac{\partial}{\partial x}U_0\frac{\partial^2}{\partial x^2}U_2+\frac{\partial}{\partial x}U_1\frac{\partial^2}{\partial x^2}U_1+\frac{\partial}{\partial x}U_2\frac{\partial^2}{\partial x^2}U_0\right)\end{matrix}\right\},$$

$$U_3=\mathrm{e}^{\frac{\mathrm{x}}{2}}\left(\frac{8\Gamma(1)\Gamma(\alpha+1)\Gamma(2\alpha+1)}{\left(\Gamma(2)-4\Gamma(\alpha+1)\right)[\Gamma(\alpha+2)-4\Gamma(2\alpha+1)][\Gamma(2\alpha+2)-4\Gamma(3\alpha+1)]}\right),$$

$$U_3=\mathrm{e}^{\frac{\mathrm{x}}{2}}\left(\frac{8\Gamma(\alpha+1)\Gamma(2\alpha+1)}{\left(1-4\Gamma(\alpha+1)\right)[\Gamma(\alpha+2)-4\Gamma(2\alpha+1)][\Gamma(2\alpha+2)-4\Gamma(3\alpha+1)]}\right),$$

In a similar manner we obtain

$$U_4 = e^{\frac{x}{2}}\left(\frac{16\Gamma(1)\Gamma(\alpha+1)\Gamma(2\alpha+1)\Gamma(3\alpha+1)}{(\Gamma(2)-4\Gamma(\alpha+1))[\Gamma(\alpha+2)-4\Gamma(2\alpha+1)][\Gamma(2\alpha+2)-4\Gamma(3\alpha+1)][\Gamma(3\alpha+2)-4\Gamma(4\alpha+1)]}\right),$$

$$U_4 = e^{\frac{x}{2}}\left(\frac{16\Gamma(\alpha+1)\Gamma(2\alpha+1)\Gamma(3\alpha+1)}{(1-4\Gamma(\alpha+1))[\Gamma(\alpha+2)-4\Gamma(2\alpha+1)][\Gamma(2\alpha+2)-4\Gamma(3\alpha+1)][\Gamma(3\alpha+2)-4\Gamma(4\alpha+1)]}\right),$$

and so on.

Applying inverse FRDTM to $U_k(x)$, it yields

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x)t^{k\alpha} = U_0 + U_1 t^{\alpha} + U_2 t^{2\alpha} + U_3 t^{3\alpha} + U_4 t^{4\alpha} + U_5 t^{5\alpha} + \cdots \quad .$$

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x)t^{k\alpha} = e^{\frac{x}{2}} + \left(\frac{2e^{\frac{x}{2}}}{1-4\Gamma(\alpha+1)}\right)t^{\alpha} + \left(e^{\frac{x}{2}}\left(\frac{4\Gamma(\alpha+1)}{(1-4\Gamma(\alpha+1))[\Gamma(\alpha+2)-4\Gamma(2\alpha+1)]}\right)\right)t^{2\alpha}$$

$$+\left(e^{\frac{x}{2}}\left(\frac{8\Gamma(\alpha+1)\Gamma(2\alpha+1)}{(1-4\Gamma(\alpha+1))[\Gamma(\alpha+2)-4\Gamma(2\alpha+1)][\Gamma(2\alpha+2)-4\Gamma(3\alpha+1)]}\right)\right)t^{3\alpha} +$$

$$\left(e^{\frac{x}{2}}\left(\frac{16\Gamma(\alpha+1)\Gamma(2\alpha+1)\Gamma(3\alpha+1)}{(1-4\Gamma(\alpha+1))[\Gamma(\alpha+2)-4\Gamma(2\alpha+1)][\Gamma(2\alpha+2)-4\Gamma(3\alpha+1)][\Gamma(3\alpha+2)-4\Gamma(4\alpha+1)]}\right)\right)t^{4\alpha} + \ldots \quad (4.44)$$

When $\alpha = 1$, equation (4.44) becomes

$$u(x,t) = e^{\frac{x}{2}}\left(1 - \frac{2}{3}t + \frac{2}{9}t^2 - \frac{4}{81}t^3 + \frac{2}{243}t^4 + \ldots\right).$$

The exact solution in the given problem is $u(x,t) = e^{\frac{x}{2}-\frac{2t}{3}}$ as indicated in (4.40).

It is expected that the solution obtained: $\sum_{k=0}^{\infty} U_k(x)$, converges to the exact solution.

In addition, by computing γ_i using theorem 4.8 for the problem (4.40), where $\alpha = 1$, we obtain:

$\gamma_0 = \frac{\|U_1\|}{\|U_0\|} = 0.6666$

$\gamma_1 = \frac{\|U_2\|}{\|U_1\|} = 0.3333$

$\gamma_2 = \frac{\|U_3\|}{\|U_2\|} = 0.2222$

$\gamma_3 = \frac{\|U_4\|}{\|U_3\|} = 0.1666$

…

Hence, for $i \geq 0$, $0 < t \leq 1$, and $\alpha = 1$, we conclude that $\gamma_i < 1$. This confirms that by theorem 4.8, the solution we made by FRDTM for time fractional Fornberg-Whitham equation converges to the exact solution.

The solution curves of the time fractional Fornberg-Whitham equation given in Examples 4.1 for different values of fractional order α is depicted in Figures 1 and 2.

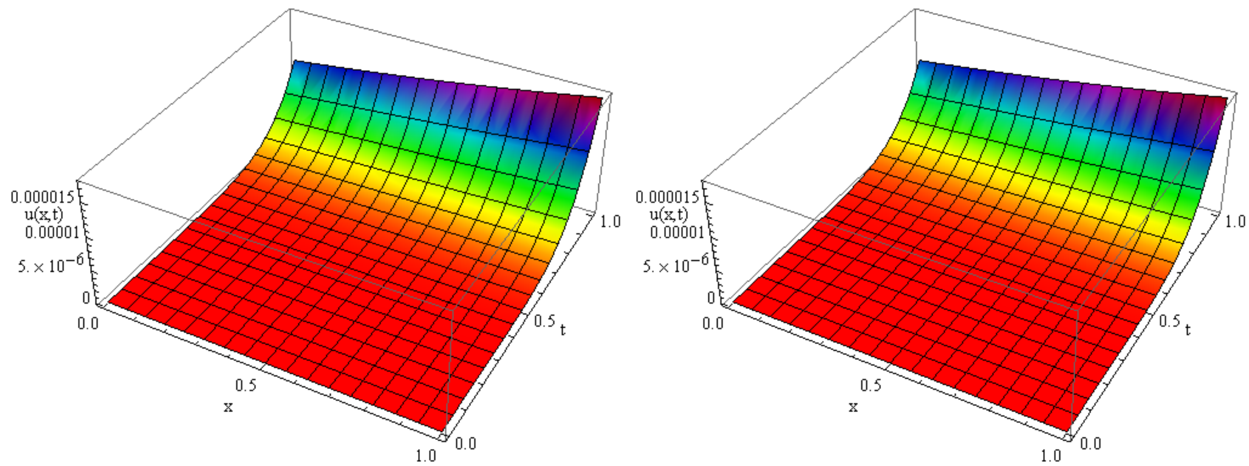

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Figure 1: Solution behavior of Example 4.1: a) $\alpha = \frac{1}{3}$, b) $\alpha = \frac{3}{4}$

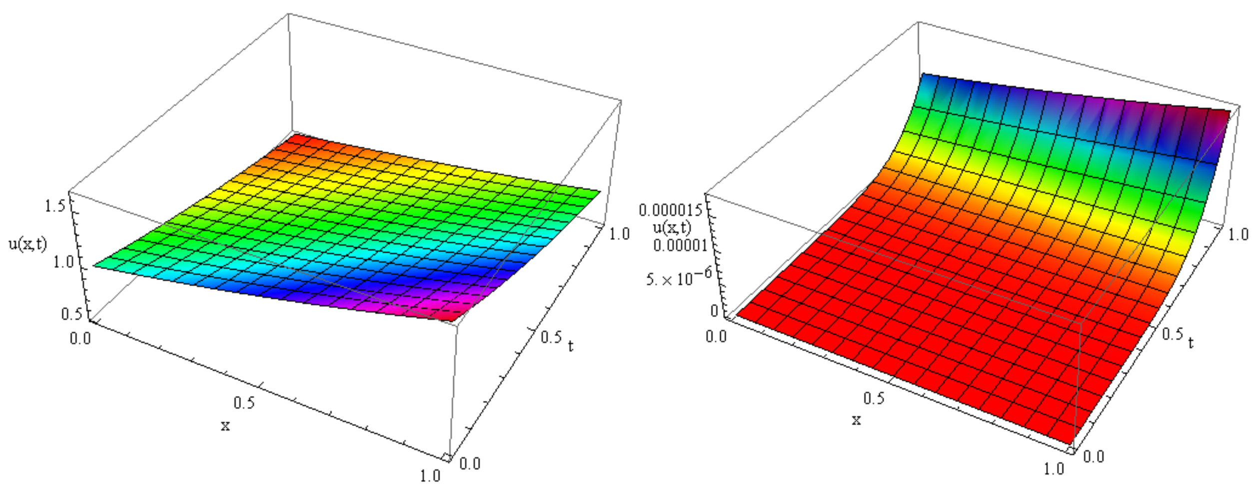

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Figure 2: Solution behavior of Example 4.1: a) $\alpha = 1$, b) Absolute error

Example 4.2. Consider the fractional Fornberg-Whitham equation stated as follows:

$$\frac{\partial^\alpha \mathrm{u}}{\partial \mathrm{t}^\alpha} - u_{xxt} + u_x - uu_{xxx} + uu_x - 3u_x u_{xx} = 0\,, \qquad (4.45)$$

Subjected to the initial condition

$$u(x,0) = \cosh^2\left(\frac{x}{4}\right) \qquad (4.46)$$

Solution:

Applying FRDTM to both sides of the equations (4.45) we obtain

$$\frac{\Gamma(\alpha(k+1)+1)}{\Gamma(\alpha k+1)} U_{k+1} - \frac{\Gamma(\alpha k+2)}{\Gamma(\alpha k+1)} \frac{\partial^2}{\partial x^2} U_{k+1} + \frac{\partial}{\partial x} U_k = U_k \frac{\partial^3}{\partial x^3} U_k - U_k \frac{\partial}{\partial x} U_k + 3 \frac{\partial}{\partial x} U_k \frac{\partial^2}{\partial x^2} U_k$$

where$U = U(x)$

$$U_{k+1} = \frac{\Gamma(\alpha k+1)}{\Gamma(\alpha(k+1)+1)} \begin{Bmatrix} \frac{\Gamma(\alpha k+2)}{\Gamma(\alpha k+1)} \frac{\partial^2}{\partial x^2} U_{k+1} - \frac{\partial}{\partial x} U_k + \sum_{r=0}^{k} U_r \frac{\partial^3}{\partial x^3} U_{k-r} - \sum_{r=0}^{k} U_r \frac{\partial}{\partial x} U_{k-r} \\ + 3 \sum_{r=0}^{k} \frac{\partial}{\partial x} U_r \frac{\partial^2}{\partial x^2} U_{k-r} \end{Bmatrix} \quad (4.47)$$

Again applying FRDTM to both sides of the Eq. (4.46) we obtain

$$R_D \mathrm{u}(\mathrm{x}, 0) = U_0(x) = \cosh^2(\frac{x}{4}) \quad (4.48)$$

Using equation (4.47) and (4.48), we get the following iterated values:

For k=0,

$$U_1 = \frac{\Gamma(1)}{\Gamma(\alpha+1)} \left\{ \frac{\Gamma(2)}{\Gamma(1)} \frac{\partial^2}{\partial x^2} U_1 - \frac{\partial}{\partial x} U_0 + U_0 \frac{\partial^3}{\partial x^3} U_0 - U_0 \frac{\partial}{\partial x} U_0 + 3 \frac{\partial}{\partial x} U_0 \frac{\partial^2}{\partial x^2} U_0 \right\},$$

$$U_1 = \frac{11 sinh\left(\frac{x}{2}\right)}{8-32\Gamma(1+a)} \ ,$$

For k=1,

$$U_2 = \frac{\Gamma(\alpha+1)}{\Gamma(2\alpha+1)} \begin{Bmatrix} \frac{\Gamma(\alpha+2)}{\Gamma(\alpha+1)} \frac{\partial^2}{\partial x^2} U_2 - \frac{\partial}{\partial x} U_1 + \left(U_0 \frac{\partial^3}{\partial x^3} U_1 + U_1 \frac{\partial^3}{\partial x^3} U_0 \right) - \left(U_0 \frac{\partial}{\partial x} U_1 + U_1 \frac{\partial}{\partial x} U_0 \right) \\ + 3 \left(\frac{\partial}{\partial x} U_0 \frac{\partial^2}{\partial x^2} U_1 + \frac{\partial}{\partial x} U_1 \frac{\partial^2}{\partial x^2} U_0 \right) \end{Bmatrix},$$

$$u_2 = -\frac{121 cosh\left(\frac{x}{2}\right) \Gamma(1+a)}{32\big(-1+4\Gamma(1+a)\big)\big(\Gamma(2+a)-4\Gamma(1+2a)\big)} \ ,$$

For k=2,

$$U_3 = \frac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)} \begin{Bmatrix} \frac{\Gamma(2\alpha+2)}{\Gamma(2\alpha+1)} \frac{\partial^2}{\partial x^2} U_3 - \frac{\partial}{\partial x} U_2 + \left(U_0 \frac{\partial^3}{\partial x^3} U_2 + U_1 \frac{\partial^3}{\partial x^3} U_1 + U_2 \frac{\partial^3}{\partial x^3} U_0 \right) \\ - \left(U_0 \frac{\partial}{\partial x} U_2 + U_1 \frac{\partial}{\partial x} U_1 + U_2 \frac{\partial}{\partial x} U_0 \right) + 3 \left(\frac{\partial}{\partial x} U_0 \frac{\partial^2}{\partial x^2} U_2 + \frac{\partial}{\partial x} U_1 \frac{\partial^2}{\partial x^2} U_1 + \frac{\partial}{\partial x} U_2 \frac{\partial^2}{\partial x^2} U_0 \right) \end{Bmatrix},$$

$$U_3 = -\frac{1331 \Gamma(1+a) \Gamma(1+2a) sinh\left(\frac{x}{2}\right)}{128\big(-1+4\Gamma(1+a)\big)\big(\Gamma(2+a)-4\Gamma(1+2a)\big)\big(\Gamma(21+a)-4\Gamma(1+3a)\big)} \ ,$$

In a similar manner we obtain

$$U_4 = -\frac{\left(14641 cosh\left(\frac{x}{2}\right)\Gamma(1+a)\Gamma(1+2a)\Gamma(1+3a)\right)}{\left(512(-1+4\Gamma(1+a))(\Gamma(2+a)-4\Gamma(1+2a))(\Gamma(2+2a)-4\Gamma(1+3a))(\Gamma(2+3a)-4\Gamma(1+4a))\right)} \ ,$$

$$U_5 = -\frac{(161051\Gamma(1+a)\Gamma(1+2a)\Gamma(1+3a)\Gamma(1+4a) sinh\left(\frac{x}{2}\right))}{(2048(-1+4\Gamma(1+a))(\Gamma(2+a)-4\Gamma(1+2a))(\Gamma(2+2a)-4\Gamma(1+3a))(\Gamma(2+3a)-4\Gamma(1+4a))(\Gamma(2+4)-4\Gamma(1+5a)))},$$

$$U_6 = -\frac{(1771561 cosh\left(\frac{x}{2}\right)\Gamma(1+a)\Gamma(1+2a)\Gamma(1+3a)\Gamma(1+4a)\Gamma(1+5a))}{(8192(-1+4\Gamma(1+a))(\Gamma(2+a)-4\Gamma(1+2a))(\Gamma(2+2a)-4\Gamma(1+3a))(\Gamma(2+3a)-4\Gamma(1+4a))(\Gamma(2+4a)-4\Gamma(1+5a))(\Gamma(2+5a)-4\Gamma(1+6a)))},$$

and so on.

Applying inverse FRDTM to$U_k(x)$, it yields

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x) t^{k\alpha} = U_0 + U_1 t^{\alpha} + U_2 t^{2\alpha} + U_3 t^{3\alpha} + U_4 t^{4\alpha} + U_5 t^{5\alpha} + \cdots$$

u(x, t) =

$$\left\{\begin{array}{c} cosh\left(\frac{x}{4}\right)^2+\frac{11sinh\left(\frac{x}{2}\right)}{8-32\Gamma(1+a)}t^{\alpha}-\frac{121cosh\left(\frac{x}{2}\right)\Gamma(1+a)}{32(-1+4\Gamma(1+a))(\Gamma(2+a)-4\Gamma(1+2a))}t^{2\alpha} \\ -\frac{1331\Gamma(1+a)\Gamma(1+2a)sinh\left(\frac{x}{2}\right)}{128(-1+4\Gamma(1+a))(\Gamma(2+a)-4\Gamma(1+2a))(\Gamma(21+a)-4\Gamma(1+3a))}t^{3\alpha} \\ -\frac{\left(14641cosh\left(\frac{x}{2}\right)\Gamma(1+a)\Gamma(1+2a)\Gamma(1+3a)\right)}{\left(512(-1+4\Gamma(1+a))(\Gamma(2+a)-4\Gamma(1+2a))(\Gamma(2+2a)-4\Gamma(1+3a))(\Gamma(2+3a)-4\Gamma(1+4a))\right)} \\ -\frac{(161051\Gamma(1+a)\Gamma(1+2a)\Gamma(1+3a)\Gamma(1+4a)sinh\left(\frac{x}{2}\right))}{\left(2048(-1+4\Gamma(1+a))(\Gamma(2+a)-4\Gamma(1+2a))(\Gamma(2+2a)-4\Gamma(1+3a))(\Gamma(2+3a)-4\Gamma(1+4a))(\Gamma(2+4)-4\Gamma(1+5a))\right)} \\ -\frac{(1771561cosh\left(\frac{x}{2}\right)\Gamma(1+a)\Gamma(1+2a)\Gamma(1+3a)\Gamma(1+4a)\Gamma(1+5a))}{\left(8192(-1+4\Gamma(1+a))(\Gamma(2+a)-4\Gamma(1+2a))(\Gamma(2+2a)-4\Gamma(1+3a))(\Gamma(2+3a)-4\Gamma(1+4a))(\Gamma(2+4a)-4\Gamma(1+5a))(\Gamma(2+5a)-4\Gamma(1+6a))\right)} \end{array}\right. \quad (4.49)$$

When $\alpha = 1$, equation (4.49) becomes

$$u(x,t) = \cosh\left(\frac{x}{4}\right)^2 - \frac{11\,t}{24}\sinh\left(\frac{x}{2}\right) + \left(\frac{11}{24}\,t\right)^2\cosh\left(\frac{x}{2}\right) - \frac{(11\,t)^3\sinh\left(\frac{x}{2}\right)}{20736} + \frac{(11\,t)^4\cosh\left(\frac{x}{2}\right)}{995328} - \frac{(11\,t)^5\sinh\left(\frac{x}{2}\right)}{59719680} + \frac{(11\,t)^6\cosh\left(\frac{x}{2}\right)}{4299816960}\ldots$$

The exact solution in the given problem is $u(x,t) = cosh^2\left(\frac{x}{4} - \frac{11}{24}t\right)$as indicated in (4.45).

It is expected that the solution obtained:$\sum_{k=0}^{\infty} U_k(x)$, converges to the exact solution. For that compute $\gamma_i$using theorem (4.8) for the problem (4.45), i.e.

$\gamma_0 = \frac{\|U_1\|}{\|U_0\|} = \frac{11}{24}\left(\text{sech}\left(\frac{x}{4}\right)\right)^2 sinh\left(\frac{x}{2}\right)$,

For $x = 1$, $\gamma_0 = \frac{\|U_1\|}{\|U_0\|} = 0.254076 < 1$,

For x =2,$\gamma_0 = \frac{\|U_1\|}{\|U_0\|} = 0.6848894 < 1$, similarly for $x > 2$ and 0< $t \leq 1$, $\gamma_0 < 1$.

. . . .

$\gamma_1 = \frac{\|U_2\|}{\|U_1\|} = \frac{11}{24}coth(\frac{x}{2}) < 1$

For x=1, $\gamma_1 = \frac{\|U_2\|}{\|U_1\|}$ =0.991811 < 1,

For x=2, $\gamma_1 = \frac{\|U_2\|}{\|U_1\|}$ =0.6018074 < 1, similarly for $x > 2$ and $0 < t \leq 1$,$\gamma_1 < 1$.

Hence, for $i \geq 0$, $0 < t \leq 1$, for all x, and $\alpha = 1$, we conclude that $\gamma_i < 1$. This confirms that by theorem 4.8, the solution we made by FRDTM for time fractional Fornberg-Whitham equation converges to the exact solution.

The solution curves of the time fractional Fornberg-Whitham equation given in Examples 4.2 for different values of fractional order $\alpha$is depicted in figures 3 and 4.

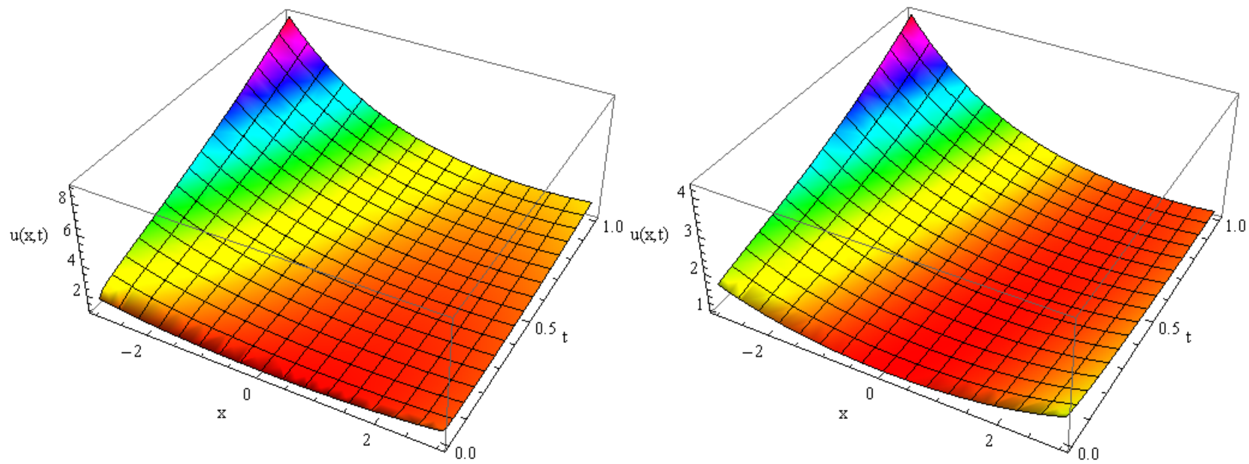

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Figure 3: Solution behavior of Example 4.2: a) $\alpha = \frac{1}{3}$, b) $\alpha = \frac{3}{4}$,

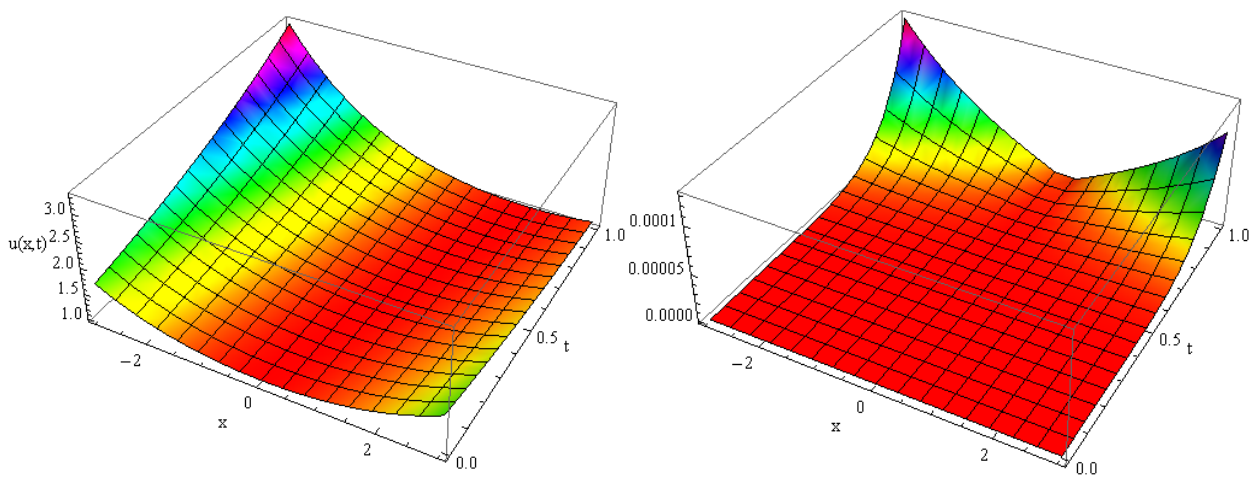

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Figure 4: Solution behavior of Example 4.2: a) $\alpha = 1$, b) Absolute error

4.7. Discussion

In this study the fractional reduced differential transform method (FRDTM) has been successfully applied on one dimensional time fractional Fornberg-Whitham equation subjected to the given initial condition and obtained a rapidly converging series solutions.

The efficiency and capability of the present method have been checked via two examples. As it can be seen in the Figures 1 and 2 of example 1, as the value of the fractional order α approaches 1 the corresponding approximated solution curves are closer and closer to the curve of the exact solution. Further, a comparison of our solution for $\alpha = 1$ (non fractional order) is in excellent agreement with the exact solution. As seen in figures 2 (b) and 4 (b), the errors are very small for both examples. In general, our solutions for Examples 1 and 2 are in excellent agreement with the solution done by (Mehmet Merdan et al. 2012) and (A. A. Alderremy et al.2020).

CHAPTER FIVE

CONCLUSIONS AND FUTURE SCOPE

5.1. Conclusions

In this study, the fractional reduced differential transform method is effectively implemented to find approximate analytics solution of time fractional Fornberg-Whitham equation subject to appropriate initial condition. The fractional derivative used in this study is in the sense of Caputo. The main advantage of this scheme is that it can be used in a direct way without applying techniques of restriction conditions, convincing suppositions, and perturbations. This shows that fractional reduced differential transform method is very simple to utilize and needs brevity of calculation. To check validity and efficiency of the method, two illustrative examples are carried out. The computed results reveal that the fractional reduced differential transform method is accurate and convergent.

5.2. Future Scope

This scheme can be applied to solve linear and non-linear time fractional higher order partial differential equation which arises in various fields of engineering and applied sciences.

REFERENCES

A. A. Alderremy, Hassan Khan, Rasool Shah, Shaban Aly and Dumitru Baleanu (2020). The Analytical Analysis of Time-Fractional Fornberg–Whitham Equations Mathematics 2020, 8, 987; doi: 10.3390/math8060987

Alawad, F.A., Yousif, E.A. and Arbab, A.I. (2013).A new technique of Laplace Variational iteration method for solving space time fractional telegraph equations

Amir, K., and Asaf, K., (2020).Extension of Triple Laplace Transform for Solving Fractional Differential Equations. Discrete and Continuous Dynamical Systems Series, 13(3), 755-768.

Arqub, Omar Abu, & Maayah, Banan. (2018). Numerical solutions of integro differential equations of Fredholm operator type in the sense of the Atangana–Baleanu fractional operator. *Chaos, Solitons & Fractals*, *117*, 117-124.

Basu, S., Gil, M., &Auddy, S. (2015). The MLP distribution: a modified lognormal power-law model for the stellar initial mass function. *Monthly Notices of the Royal Astronomical Society*, *449*(3), 2413-2420.

Batir, Necdet. (2008). On some properties of the gamma function. *Expositiones Mathematicae*, *26*(2), 187-196.

Chen, J., Liu, F., and Anh, V. (2008). Analytical solution for the time fractional telegraph equation by the method of separating variables Journal of Mathematical Analysis and Applications, 338(2), 1364–1377. peregrines**.**UK.

Gupta, V. G., Shrama, B., &Kiliçman, A. (2010).A note on fractional Sumudu transform. *Journal of Applied Mathematics*, *2010*.

Keskin and G. Oturanc, (2010). “Reduced differential transform method: a new approach to +

Fractional partial differential equations,” *Nonlinear Science Letters A*, vol. 1, pp. 61–72.

Kiryakova, Virginia. (2010). The special functions of fractional calculus as generalized fractional calculus operators of some basic functions. *Computers & mathematics with applications, 59*(3), 1128-1141.

Kumar, D.; Singh, J.; Baleanu, D, (2018).A new analysis of the Fornberg-Whitham equation pertaining to a fractional derivative with Mittag-Leffler-type kernel.*Eur. Phys. J. Plus*, *133*, 70.

Mehmet Merdan et al. (2012). Numerical Simulation of Fractional Fornberg-Whitham Equation by Differential Transformation Method. Hindawi Publishing Corporation, Abstract and Applied Analysis. Article ID 965367, 8 pages.

Miller, Kenneth S, & Ross, Bertram. (1993). *An introduction to the fractional calculus and fractional differential equations*: Wiley.

Sahoo, S., & Ray, S. S. (2015). Improved fractional sub-equation method for (3+ 1)-dimensional generalized fractional KdV–Zakharov–Kuznetsov equations. *Computers & Mathematics with Applications*, *70*(2), 158-166.

Seyyedeh Roodabeh Moosavi Noori and Nasir Taghizadeh (2021).Study of Convergence of Reduced Differential Transform Method for Different Classes of Differential Equations International Journal of Differential Equations Volume 2021, Article ID 6696414,16 pages https://doi.org/10.1155/2021/6696414

Srivastava, H. M., & Choi, J. (2001). *Series associated with the zeta and related functions* (Vol. 530). Springer Science & Business Media.

V. K. Srivastava, M. K, (2013).Awasthi, and M. Tamsir, “RDTM solution of Caputo time fractional order Y. hyperbolic Telegraph equation,” *AIP Advances*, vol. 3, Article ID 032142.

Zahid, S. M., & Ray, P. K. (2015).Approximation of Euler Number Using Gamma Function. *American Journal of Undergraduate Research*, *12*(3).