FRACTIONAL NAVIER-STOKES EQUATION VIA CONFORMABLE DOUBLE LAPLACE TRANSFORM METHOD

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Declaration

I, the undersigned declare that, the thesis entitled “solution of Fractional Navier-Stokes equation by using conformable double Laplace transform method**”** is original and it has not been submitted to any institution elsewhere for the award of any academic degree or like, where other sources of information that have been used, they have been acknowledged.

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Abstract

This work presents a conformable double Laplace transform method to get analytic approximate solutions for linear one- dimensional Navier-Stokes partial differential equations (PDEs) of fractional-order in conformable fractional derivative sense.

The scheme is tested through two examples, and the results are shown in figures to demonstrate the efficiency and reliability of the proposed method. Furthermore, the outcome of the present method converges rapidly to the given exact solutions when the fractional orders values of α and β are small. Consequently, the proposed method is found to be reliable, efficient and easy to implement for various related problems of science and engineering.

Keywords: PDE, One dimensional Linear Fractional Navier-Stokes Equation, CDLTM, Description of the method

Acronyms

ADM- Adomain Decomposition Method

DTM - Differential Transform Method

CDLTM- Conformable Double Laplace Transform Method

CSLTM -Conformable Single Laplace Transform Method

PDEs - Partial Differential Equations

FPDE- Fractional Partial Deferential Equation

TFNSE- Time Fractional Navier-Stokes Equation

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CHAPTER ONE

INTRODUCTION

1.1 Background of the study

The subject of fractional calculus is one of the most interesting and useful areas of field in applied mathematics. This wide area of field deals with derivatives and integrals of arbitrary order, and their application extensively used in science and engineering (Kilbas AA, M. O, 1993)and (Adem Kilicman and Hassan Eltayeb, 2013). For example electromagnetics, acoustics, viscoelasticity, electro-chemistry, and material science are well described by fractional derivatives (Ahmad, W. E.-K, 2007). That is a result of the fact that a realistic modeling of a physical phenomenon having dependence not only at the time instant but also previous time history can be achieved by fractional calculus (Podlubny I, 1999). Oldham and Spanier (1974) has played a vital role in the development of the subject.

In recent years, different mathematicians have been studying and analyzing one of the fields in the subject called fractional Navier Stokes equation by the linear and nonlinear fractional differential equations which arise in many fields of physical science and engineering. These types of equations play a variety of roles to advance mathematical tools in order to understand fractional modeling (Baleanu D, D. K, 2012 and Hilfer R, 2000)

The solution of fractional partial differential equations has been obtained through a double Laplace decomposition method by authors (Miller, KS, Ross, B, 1993 and Adem Kılıc¸man and Hassan Eltayeb. G, 2010).Cheng and Yao (2018) studied the solution of few time-fractional partial differential equations by simplest equation method. The Navier Stokes equations are fluid dynamics identical to Newton's second law force equals the product of mass and acceleration, and they are of the crucial significance in fluid dynamics. The conformable double Laplace transform method was introduced by Ozkan and Kurt (2015) in the study of fractional partial differential equations. Cenesiz et al. (2017) applied the first integral method to establish the exact solutions for fractional Naiver-Stokes equation. Different powerful methods in a recent time has been developed and used to find out different type solution of fractional Navier-Stokes equation like Adomian decomposition method (Wazwaz, A, 1999), the q-homotopy analysis transform method (He, J, 1999), the modified Laplace transform method (Miller, KS, Ross, B, 1993 & Wazwaz, A, 1999). Kumar et al. (2014) used various methods to study the solutions of linear and nonlinear fractional differential equation combined with a Laplace transform and so on. The fractional

Navier-Stokes equation has been studied by Momani and Odibat (2006). The Laplace transform method was used by Oldham and Spanier (1974) to obtain the solution of the fractional order homogeneous differential equations. Many papers exist in literature, which are related to conformable fractional derivative with its properties and application (Kurt, O. O, 2018 & Abdeljawad, T, 2015).

The one dimensional Navier-Stokes equation with time fractional derivative has been given as,

(Eltayeb et al.2020).

$$D_t^{\alpha} u(x.t) = D_x^2 u(x,t) + \frac{1}{x} D_x u(x,t) + f(x,t), \;\; x,t > 0, \quad 0 < \alpha \leq 1 \qquad (1.1)$$

with initial condition $u(x,0) = f(x)$ (1.2)

where $D_t^{\alpha} = \frac{\partial^{\alpha}}{\partial t^{\alpha}}$ is the Caputo fractional derivative, $D_x^2 = \frac{\partial^2}{\partial x^2}, D_x = \frac{\partial}{\partial x}$ and the right hand side function $f(x,t)$ is the source term. α is parameter which describes the order of time fractional derivative.

Even if different scholars solved time fractional Navier-Stokes equations by using different methods, none of them yet studied it with one dimensional fractional Navier-Stokes equation via conformable double Laplace transform. As a result the main purpose of this study is to apply the conformable double Laplace transform method to find the approximate analytical solutions of the fractional Navier-Stokes equation (1.1) subject to initial condition (1.2), i.e,

$$\frac{\partial^{\beta}}{\partial t^{\beta}}\left(u\left(\frac{x^{\alpha}}{\alpha},\frac{t^{\beta}}{\beta}\right)\right) = \frac{\partial^{2\alpha}}{\partial x^{2\alpha}}\left(u\left(\frac{x^{\alpha}}{\alpha},\frac{t^{\beta}}{\beta}\right)\right) + \frac{a}{x^{\alpha}}\frac{\partial^{\alpha}}{\partial x^{\alpha}} u\left(\frac{x^{\alpha}}{\alpha},\frac{t^{\beta}}{\beta}\right) + f(\frac{x^{\alpha}}{\alpha},\frac{t^{\beta}}{\beta}), \mathrm{x}, t > 0, 0 < \beta \leq 1, \qquad (1, 3)$$

with initial conditions

$$u\left(\frac{x^{\alpha}}{\alpha},0\right) = f(\frac{x^{\alpha}}{\alpha}) \quad , \qquad (1.4)$$

where $\mathrm{x}, t > 0,\;\; 0 < \alpha, \beta \leq 1$, $\frac{\partial^{2\alpha}}{\partial x^{2\alpha}}(\frac{x^{\alpha}}{\alpha},\frac{t^{\beta}}{\beta})$ twice conformable fractional partial derivative of order $\alpha \; and \; \beta$ of a function u $(\frac{x^{\alpha}}{\alpha},\frac{t^{\beta}}{\beta})$.

1.2 Statement of the problem

This study mainly focuses on the basic issues,

- Fractional CDLT of eq.(1.3)
- The fractional CSLT of eq.(1.4)
- Approximate analytical solution of equation (1.3)

1.3 Objective of the study

1.3.1 General Objective

The main objective of the study is to find out the approximate analytical solutions of one dimensional fractional Navier-Stokes equation with appropriate initial condition by using conformable double Laplace transform method.

1.3.2 Specific objectives

The specific objectives of this study are

- To calculate the algebraic equation of (1.3) by applying CDLTM
- To find the algebraic equation of (1.4) by apply SCLTM
- To find the component equation $u_0, u_1, u_2 \ \ . . .$ of equation (1.3) subject to (1.4)

1.4 Significance of the study

The outcome of this study may have the following importance.

- It provides techniques of solving fractional Navier-Stokes equation by using conformable double Laplace transform method.
- It develops skill on mathematical research.
- It can be used as a source of information for those researchers who work on related titles.

1.5 Delimitation of the study

This study is delimited to one-dimensional fractional linear Navier-Stokes equation via conformable double Laplace transform method.

CHAPTER TWO

LITERATURE REVIEW

The applications of differential equations which arise in the field of medicine, engineering, social sciences, physics and other parts of applied sciences is one of the most relevant and interesting area. Although there are various problems including differential equations, there are not any prevalent techniques for the solution of such problems. Many researchers use the integral transforms which is one of the greatest known schemes applied for the solution of ordinary and partial differential equations. After the use of integral transform methods the differential, partial differential, integral, integro-differential equations turn into algebraic equations. So the solution procedure becomes simpler.

Fractional calculus, which has been aroused great interest with respect to extensive area of applications nearly in all disciplines of applied science and engineering became a favorite subject into the last decades as well described by (Podlubny I, 1999). Fractional derivative were not used in physics, engineering and other disciplines although they have along mathematical history. One of the reasons could be that there are many nonequivalent definitions of fractional derivatives and integrals.

The conformable fractional derivative was used to obtain the exact analytical solutions for the time fractional differential equations. The conformable double Laplace transform first initiated by (Abdeljawad, T, 2015) and studied and modified by (F. Jarada and Abdeljawad, T, 2018). Conformable Laplace transform is not only useful for solving local conformable fractional dynamical systems but also it can be employed to solve systems within nonlocal fractional derivatives that were defined by (F. Jarada and Abdeljawad, T, 2018).

Recently, the solution of fractional partial differential equations has been obtained through the use of conformable double Laplace transform method. By the author Hashemi, M. (2018) the conformable fractional Laplace transform was applied to solve the coupled systems of conformable fractional differential equations.

The Navier-Stokes equations are fluid dynamics and they are crucial in fluid dynamics. Also they are vector equations. Now a time fractional double Laplace transform has been used to find out different type solution of fractional Navier-Stokes equations (Momani, S and Odibat, Z, 2006).

The modified reduced differential transform method with the help of the Elzaki transforms (Khalid, M. S, 2015) was successfully applied to the fractional Navier-Stokes equation.

CHAPTER THREE
METHODOLOGY

3.1 Study area and period

This study was conducted under the Department of Mathematics, College of Natural Sciences, Jimma University from September 2020 G.C to February 2022 G.C.

3.2 Study Design

The design of this study is analytical

3.3 Sources of Information

The sources of data for this study were collected from related document materials such as references books, published research articles (or Journals), and studies from internet services.

3.4 Mathematical Procedures:

In order to achieve the stated objective, the following procedures were undertaken.

Step (1**):** multiplying both sides of equation (1.3) by the term $\frac{x^\alpha}{\alpha}$.

Step (2): applying properties of CDLT on both sides' of functions in step (1) and SCLT

on equation (1.4) and substituting the later in to the first.

Step (3): integrating both sides of the result obtained in step (2), from 0 to p with

respect to p to get $U(p,s)$.

Step (4): taking inverse conformable double Laplace transform on the result obtained in step (3),

to compute the solution $u\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)$.

Step (5**):** representing the result of step (4) in the form $u\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)=\sum_{\mathrm{n}=0}^{\infty}\mathrm{u}_{\mathrm{n}}\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)$

and construct recurrence relation, then iterating $u_i\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)$ for $i=0,1,2,\ldots$

CHAPTER FOUR

RESULT AND DISCUSSION

4.1 Mathematical Preliminaries

This section presents the basic definitions of conformable derivative and properties of conformable double Laplace transform methods that are important for this research work.

Definition 4.1(Eltayeb et al. 2019): Given a function $f:(0,\infty)\to R$, the conformable fractional partial derivative of f of order β of the function $f(t)$ is denoted by;

$$\frac{d^{\beta}f(t)}{dt^{\beta}}=\lim_{\in\to 0}\frac{f\left(\frac{t^{\beta}}{\beta}+\in t^{1-\beta}\right)-f\left(\frac{t^{\beta}}{\beta}\right)}{\in},\quad t>0,\ 0<\beta\leq 1 \tag{4.1}$$

Definition 4.2 (Thabet.H, H. and Kendre, S. 2018).Given a function $f\left(\frac{x^{\alpha}}{\alpha},\frac{t^{\beta}}{\beta}\right):R\times(0,\infty)\to R$, the conformable space fractional partial derivative of order α of the function $f(x,t)$ is denoted by

$$\frac{\partial^{\alpha}}{\partial x^{\alpha}}f\left(\frac{x^{\alpha}}{\alpha},\frac{t^{\beta}}{\beta}\right)=\lim_{\in\to 0}\frac{f\left(\frac{x^{\alpha}}{\alpha}+\epsilon x^{1-\alpha},\frac{t^{\beta}}{\beta}\right)-f\left(\frac{x^{\alpha}}{\alpha},\frac{t^{\beta}}{\beta}\right)}{\epsilon},\ x\,,t>0\quad 0<\alpha,\beta\leq 1 \tag{4.2}$$

Definition 4.3: (Thabet.H, and Kendre, S. 2018). Given a function $f\left(\frac{x^{\alpha}}{\alpha},\frac{t^{\beta}}{\beta}\right):R\times(0,\infty)\to R$, the conformable time fractional partial derivative of order β the function $f(x,t)$ is denoted by:

$$\frac{\partial^{\beta}}{\partial t^{\beta}}f\left(\frac{x^{\alpha}}{\alpha},\frac{t^{\beta}}{\beta}\right)=\lim_{\sigma\to 0}\frac{f\left(\frac{x^{\alpha}}{\alpha},\frac{t^{\beta}}{\beta}+\sigma t^{1-\beta}\right)-f\left(\frac{x^{\alpha}}{\alpha},\frac{t^{\beta}}{\beta}\right)}{\sigma}\ \ x,t>0\ \ 0<\alpha,\beta\leq 1 \tag{4.3}$$

Theorem 4.1: Let $\alpha,\beta\in(0,1]$ and $f(x,t)$ continues function α,β differentiable at point $x,t>0$ then

$$\frac{\partial^{\alpha}f(x,t)}{\partial x^{\alpha}}=x^{1-\alpha}\frac{\partial f(x,t)}{\partial x}$$

$$\frac{\partial^{\beta}f(x,t)}{\partial t^{\beta}}=t^{1-\beta}\frac{\partial f(x,t)}{\partial t}$$

Proof: By using definitions 4.2 and 4.3 and $h=\epsilon x^{1-\alpha}$ in equation (4.3) we have

$$\frac{\partial^{\alpha}}{\partial x^{\alpha}} f\left(\frac{x^{\alpha}}{\alpha}, \frac{t^{\beta}}{\beta}\right) = \lim_{\epsilon \to 0} \frac{f\left(\frac{x^{\alpha}}{\alpha}+\epsilon x^{1-\alpha}, \frac{t^{\beta}}{\beta}\right)-f\left(\frac{x^{\alpha}}{\alpha}, \frac{t^{\beta}}{\beta}\right)}{\epsilon}$$

$$= \lim_{h \to 0} \frac{f\left(\frac{x^{\alpha}}{\alpha}+h, \frac{t^{\beta}}{\beta}\right)-f\left(\frac{x^{\alpha}}{\alpha}, \frac{t^{\beta}}{\beta}\right)}{h x^{\alpha-1}}$$

$$= x^{1-\alpha} \lim_{h \to 0} \frac{f\left(\frac{x^{\alpha}}{\alpha}+h, \frac{t^{\beta}}{\beta}\right)-f\left(\frac{x^{\alpha}}{\alpha}, \frac{t^{\beta}}{\beta}\right)}{h}$$

$$= x^{1-\alpha} \frac{\partial^{\alpha}}{\partial x^{\alpha}} f\left(\frac{x^{\alpha}}{\alpha}, \frac{t^{\beta}}{\beta}\right)$$

Similarly, we can prove

$$\frac{\partial^{\beta} f(x,t)}{\partial t^{\beta}} = t^{1-\beta} \frac{\partial f(x,t)}{\partial t}$$

In the next example, we introduce the conformable fractional derivative of specific functions, by using Theorem 4.1 as follows

Example 1. Let $\alpha, \beta \in (0,1]$ and a, b, m, n, τ and $\mu \in R$ then the conformable derivative are as follows:

(1) $\frac{\partial^{\alpha}}{\partial x^{\alpha}}\left(\frac{x^{\alpha}}{\alpha}\right)\left(\frac{t^{\beta}}{\beta}\right) = \frac{t^{\beta}}{\beta}, \frac{\partial^{\alpha}}{\partial x^{\alpha}}\left(\frac{x^{\alpha}}{\alpha}\right)^{n}\left(\frac{t^{\beta}}{\beta}\right) = n\left(\frac{x^{\alpha}}{\alpha}\right)^{n-1}\left(\frac{t^{\beta}}{\beta}\right)$

(2) $\frac{\partial^{\beta}}{\partial t^{\beta}}\left(\frac{x^{\alpha}}{\alpha}\right)\left(\frac{t^{\beta}}{\beta}\right) = \frac{x^{\alpha}}{\alpha}, \frac{\partial^{\beta}}{\partial t^{\beta}}\left(\frac{t^{\beta}}{\beta}\right)^{m}\left(\frac{x^{\alpha}}{\alpha}\right)^{n} = m\left(\frac{t^{\beta}}{\beta}\right)^{m-1}\left(\frac{x^{\alpha}}{\alpha}\right)^{n}$

(3) $\frac{\partial^{\beta}}{\partial t^{\beta}}\left(sin\left(\frac{x^{\alpha}}{\alpha}\right)\mathrm{Sin}\left(\frac{t^{\beta}}{\beta}\right)\right) = \sin\left(\frac{x^{\alpha}}{\alpha}\right)\cos\left(\frac{t^{\beta}}{\beta}\right)$

(4) $\frac{\partial^{\alpha}}{\partial x^{\alpha}} sina\left(\frac{x^{\alpha}}{\alpha}\right) sin\left(\frac{t^{\beta}}{\beta}\right) = a\cos a\left(\frac{x^{\alpha}}{\alpha}\right)\sin\left(\frac{t^{\beta}}{\beta}\right)$

(5) $\frac{\partial^{\alpha}}{\partial x^{\alpha}}\left(e^{\frac{\tau x^{\alpha}}{\alpha}+\mu\frac{t^{\beta}}{\beta}}\right) = \tau e^{\frac{\tau x^{\alpha}}{\alpha}+\mu\frac{t^{\beta}}{\beta}}$

(6) $\frac{\partial^{\beta}}{\partial t^{\beta}}\left(e^{\frac{\tau x^{\alpha}}{\alpha}+\mu\frac{t^{\beta}}{\beta}}\right) = \mu e^{\frac{\tau x^{\alpha}}{\alpha}+\mu\frac{t^{\beta}}{\beta}}$

Definition 4.4 (Ozkan.O and A.Kurt, 2018) Let $u(x,t)$ be a piecewise continuous function on $[0,\infty) \times [0,\infty)$ of exponential order and for some $a, b \in \mathbb{R}$,

$\sup_{x>0,t>0} \frac{|u(x,t)|}{e^{a\frac{x^\alpha}{\alpha}+b\frac{t^\beta}{\beta}}} < \infty$. Under these conditions the conformable double Laplace transform defined by:

$$L_x^\alpha L_t^\beta\big(u(x,t)\big) = U(p,s) = \int_0^\infty \int_0^\infty e^{-p\frac{x^\alpha}{\alpha}-s\frac{t^\beta}{\beta}} u(x,t) t^{\beta-1} x^{\alpha-1} dt dx \tag{4.4}$$

where p, s ∈C, $0 < \alpha, \beta \leq 1$ and integrals by means of conformable integrals with respect to x and t respectively. Similarly, under the same condition:

The conformable single Laplace transform of $u(x,t)$ with respect to x is defined by

$$L_x^\alpha\big(u(x,t)\big) = U(p,t) = \int_0^\infty e^{-p\frac{x^\alpha}{\alpha}} u(x,t) x^{\alpha-1} dx \tag{4.5}$$

where p ∈ C, $0 < \alpha \leq 1$ and integral by means of conformable integrals with respect to x.

Theorem 4.2: (Thabet.H and Kendre, S. 2018) and (Mohamed et al., 2019) Let f be piecewise continuous on $[0,\infty) \times [0,\infty)$ the (CDLT) of the conformable partial derivatives of orders α^{th} *and* $\beta^{th}, \frac{\partial^\alpha u}{\partial x^\alpha}, \frac{\partial^\beta u}{\partial t^\beta}, \frac{\partial^{2\alpha} u}{\partial x^{2\alpha}}$ *and* $\frac{\partial^{2\beta} u}{\partial t^{2\beta}}$ given by

$$\mathrm{L}_x^\alpha \mathrm{L}_t^\beta \left(\frac{\partial^\alpha \mathrm{u}}{\partial \mathrm{x}^\alpha}\right) = \mathrm{pU(p,s) - U(0,s)} \tag{4.6}$$

$$L_x^\alpha L_t^\beta \left(\frac{\partial^\beta u}{\partial t^\beta}\right) = sU(p,s) - U(p,0) \tag{4.7}$$

$$L_x^\alpha L_t^\beta \left(\frac{\partial^{2\alpha} u}{\partial x^{2\alpha}}\right) = p^2 U(p,s) - pU(0,s) - U_x(0,s) \tag{4.8}$$

$$L_x^\alpha L_t^\beta \left(\frac{\partial^{2\beta} u}{\partial t^{2\beta}}\right) = s^2 U(p,s) - sU(p,0) - U_t(p,0) \tag{4.9}$$

Proof: By using definition (CDLT) for $\frac{\partial^\alpha u}{\partial x^\alpha}$ we have

$$L_x^\alpha L_t^\beta \left(\frac{\partial^\alpha u}{\partial x^\alpha}\right) = \int_0^\infty \int_0^\infty e^{-p\frac{x^\alpha}{\alpha}-s\frac{t^\beta}{\beta}} f\left(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta}\right) \left(\frac{\partial^\alpha u}{\partial x^\alpha}\right) t^{\beta-1} x^{\alpha-1} dt dx \tag{4.10}$$

$$= \int_0^\infty e^{-s\left(\frac{t^\beta}{\beta}\right)} t^{\beta-1} \left(\int_0^\infty e^{-p\left(\frac{x^\alpha}{\alpha}\right)} \frac{\partial^\alpha u}{\partial x^\alpha} x^{\alpha-1} dx\right) dt \tag{4.11}$$

By applying Theorem 4.1 $\frac{\partial^\alpha u}{\partial x^\alpha} = x^{1-\alpha}\left(\frac{\partial u(x,t)}{\partial t}\right)$ in equation (4.10) become

$$L_x^{\alpha} L_t^{\beta}\left(\frac{\partial^{\alpha} u}{\partial x^{\alpha}}\right)=\int_0^{\infty} e^{-s\left(\frac{t^{\beta}}{\beta}\right)} t^{\beta-1}\left(\int_0^{\infty} e^{-p\left(\frac{x^{\alpha}}{\alpha}\right)} \frac{\partial^{\alpha} u}{\partial x^{\alpha}} x^{\alpha-1} d x\right) d t \quad (4.12)$$

The integral inside the bracket given by

$$\int_0^{\infty} e^{-p\left(\frac{x^{\alpha}}{\alpha}\right)} \frac{\partial^{\alpha} u}{\partial x^{\alpha}} x^{\alpha-1} d x=p U(p, t)-U(0, t) \quad (4.13)$$

By substituting the equation (4.12) in to (4.11) we obtained

$$L_x^{\alpha} L_t^{\beta}\left(\frac{\partial^{\alpha} u}{\partial x^{\alpha}}\right)= p U(p, s)-U(0, s). \quad (4.14)$$

In same manner the CDLT of $\frac{\partial^{\beta} \mathrm{u}}{\partial \mathrm{t}^{\beta}}, \frac{\partial^{2 \alpha} \mathrm{u}}{\partial \mathrm{x}^{2 \alpha}}$ *and* $\frac{\partial^{2 \beta} \mathrm{u}}{\partial \mathrm{t}^{2 \beta}}$ can be obtained.

Theorem 4.3: (Ozkan.O and. Kurt, A,2018) If the CDLT of the conformable partial derivatives $\frac{\partial^{\beta} u}{\partial t^{\beta}}$ given by equation (4.6), then double Laplace transform of $\left(\frac{\mathrm{x}^{\alpha}}{\alpha}\right)^{\mathrm{n}}\left(\frac{\partial^{\beta}}{\partial \mathrm{t}^{\beta}} \mathrm{f}\left(\frac{\mathrm{x}^{\alpha}}{\alpha}, \frac{\mathrm{t}^{\beta}}{\beta}\right)\right)$ and $\left(\frac{\mathrm{x}^{\alpha}}{\alpha}\right) g\left(\frac{x^{\alpha}}{\alpha}, \frac{t^{\beta}}{\beta}\right)$ are given by

$$(-1)^n \frac{d^n}{d p^n}\left(L_x^{\alpha} L_t^{\beta}\left[g\left(\frac{x^{\alpha}}{\alpha}, \frac{t^{\beta}}{\beta}\right)\right]\right)=L_x^{\alpha} L_t^{\beta}\left[\left(\frac{x^{\alpha}}{\alpha}\right)^n g\left(\frac{x^{\alpha}}{\alpha}, \frac{t^{\beta}}{\beta}\right)\right], \quad (4.15)$$

$$(-1)^n \frac{d^n}{d p^n}\left(L_x^{\alpha} L_t^{\beta}\left[\frac{\partial^{\beta}}{\partial t^{\beta}} f\left(\frac{x^{\alpha}}{\alpha}, \frac{t^{\beta}}{\beta}\right)\right]\right)=L_x^{\alpha} L_t^{\beta}\left[\left(\frac{x^{\alpha}}{\alpha}\right)^n\left(\frac{\partial^{\beta}}{\partial t^{\beta}} f\left(\frac{x^{\alpha}}{\alpha}, \frac{t^{\beta}}{\beta}\right)\right)\right] \quad (4.16)$$

Where $n= 1,2,3 \ldots$

Proof: using the definition of conformable double Laplace transform for e equation (4.15) we get

$$L_x^{\alpha} L_t^{\beta}\left[g\left(\frac{x^{\alpha}}{\alpha}, \frac{t^{\beta}}{\beta}\right)\right]=\int_0^{\infty} \int_0^{\infty} e^{-p \frac{x^{\alpha}}{\alpha}-s \frac{t^{\beta}}{\beta}} g\left(\frac{x^{\alpha}}{\alpha}, \frac{t^{\beta}}{\beta}\right) t^{\beta-1} x^{\alpha-1} d t d x \quad (4.17)$$

By applying the n^{th} derivative with respect to p for both sides of equation (4.17), we have

$$\frac{d^n}{d p^n}\left(L_x^{\alpha} L_t^{\beta} g\left(\frac{x^{\alpha}}{\alpha}, \frac{t^{\beta}}{\beta}\right)\right)=\int_0^{\infty} \int_0^{\infty} \frac{d^n}{d p^n} e^{-p \frac{x^{\alpha}}{\alpha}-s \frac{t^{\beta}}{\beta}} g\left(\frac{x^{\alpha}}{\alpha}, \frac{t^{\beta}}{\beta}\right) t^{\beta-1} x^{\alpha-1} d t d x$$

$$=(-1)^n \int_0^{\infty} \int_0^{\infty}\left(\frac{x^{\alpha}}{\alpha}\right)^n e^{-p \frac{x^{\alpha}}{\alpha}-s \frac{t^{\beta}}{\beta}} g\left(\frac{x^{\alpha}}{\alpha}, \frac{t^{\beta}}{\beta}\right) t^{\beta-1} x^{\alpha-1} d t d x$$

$$= (-1)^n \left(L_x^{\alpha} L_t^{\beta} \left(\frac{x^{\alpha}}{\alpha}\right)^n g\left(\frac{x^{\alpha}}{\alpha}, \frac{t^{\alpha}}{\alpha}\right) \right)$$

We obtain $\quad (-1)^n \frac{d^n}{dp^n}\left(L_x^{\alpha} L_t^{\beta}\left[g\left(\frac{x^{\alpha}}{\alpha}, \frac{t^{\beta}}{\beta}\right)\right]\right) = L_x^{\alpha} L_t^{\beta}\left[\left(\frac{x^{\alpha}}{\alpha}\right)^n g\left(\frac{x^{\alpha}}{\alpha}, \frac{t^{\beta}}{\beta}\right)\right]$

Similarly, we can prove equation (4.16)

Theorem 4.4: (Ozkan.O and. Kurt, A, 2018) Let $f(x,t)$ and $g(x,t)$ be two functions, then the following properties hold.

(a) $L_x^{\alpha} L_t^{\beta}\left[c_1 f\left(\frac{x^{\alpha}}{\alpha}, \frac{t^{\beta}}{\beta}\right) + c_2 g\left(\frac{x^{\alpha}}{\alpha}, \frac{t^{\alpha}}{\alpha}\right)\right] = c_1 L_x^{\alpha} L_t^{\beta}\left[f\left(\frac{x^{\alpha}}{\alpha}, \frac{t^{\beta}}{\beta}\right)\right] + c_2 L_x^{\alpha} L_t^{\beta}\left[g\left(\frac{x^{\alpha}}{\alpha}, \frac{t^{\beta}}{\beta}\right)\right]$ (4.18)

c_1 And c_2 are constant

(b) $L_x^{\alpha} L_t^{\beta}\left[f\left(\gamma\frac{x^{\alpha}}{\alpha}, \mu\frac{t^{\beta}}{\beta}\right)\right] = \frac{1}{\tau} f\left(\frac{p}{\gamma^{\alpha}}, \frac{s}{\mu^{\beta}}\right)$ Where $\tau = \gamma^{\alpha}\mu^{\beta}$ (4.19)

(c) $L_x^{\alpha} L_t^{\beta}\left[\int_0^{\infty} e^{-c\frac{x^{\alpha}}{\alpha} - d\frac{t^{\beta}}{\beta}} f\left(\frac{x^{\alpha}}{\alpha}, \frac{t^{\beta}}{\beta}\right)\right] = F(p+c, s+d)$ (4.20)

Proof:

(a) By using the definition of conformable double Laplace transforms

$$L_x^{\alpha} L_t^{\beta}\left[c_1 f\left(\frac{x^{\alpha}}{\alpha}, \frac{t^{\beta}}{\beta}\right) + c_2 g\left(\frac{x^{\alpha}}{\alpha}, \frac{t^{\beta}}{\beta}\right)\right] = \int_0^{\infty}\int_0^{\infty}\left[c_1 e^{-\left(p\frac{x^{\alpha}}{\alpha} - s\frac{t^{\beta}}{\beta}\right)} f\left(\frac{x^{\alpha}}{\alpha}, \frac{t^{\beta}}{\beta}\right) + c_2 e^{-\left(p\frac{x^{\alpha}}{\alpha} - s\frac{t^{\beta}}{\beta}\right)} g\left(\frac{x^{\alpha}}{\alpha}, \frac{t^{\beta}}{\beta}\right)\right] t^{\beta-1} x^{\alpha-1} dt dx$$

$$= \int_0^{\infty}\int_0^{\infty} c_1 e^{-p\frac{x^{\alpha}}{\alpha} - s\frac{t^{\beta}}{\beta}} f\left(\frac{x^{\alpha}}{\alpha}, \frac{t^{\beta}}{\beta}\right) t^{\beta-1} x^{\alpha-1} dt dx + \int_0^{\infty}\int_0^{\infty} c_2 e^{-p\frac{x^{\alpha}}{\alpha} - s\frac{t^{\beta}}{\beta}} g\left(\frac{x^{\alpha}}{\alpha}, \frac{t^{\beta}}{\beta}\right) t^{\beta-1} x^{\alpha-1} dt dx$$

$$= c_1 \int_0^{\infty}\int_0^{\infty} e^{-p\frac{x^{\alpha}}{\alpha} - s\frac{t^{\beta}}{\beta}} f\left(\frac{x^{\alpha}}{\alpha}, \frac{t^{\beta}}{\beta}\right) t^{\beta-1} x^{\alpha-1} dt dx + c_2 \int_0^{\infty}\int_0^{\infty} e^{-p\frac{x^{\alpha}}{\alpha} - s\frac{t^{\beta}}{\beta}} g\left(\frac{x^{\alpha}}{\alpha}, \frac{t^{\beta}}{\beta}\right) t^{\beta-1} x^{\alpha-1} dt dx$$

$$= c_1 L_x^{\alpha} L_t^{\beta}\left[f\left(\frac{x^{\alpha}}{\alpha}, \frac{t^{\beta}}{\beta}\right)\right] + c_2 L_x^{\alpha} L_t^{\beta}\left[g\left(\frac{x^{\alpha}}{\alpha}, \frac{t^{\beta}}{\beta}\right)\right]$$

(b) Let $\tau = \gamma x$ and $\varphi = \mu t$ so the proof can be expressed as follows

$$L_x^{\alpha} L_t^{\beta}\left[f\left(\gamma\frac{x^{\alpha}}{\alpha}, \mu\frac{t^{\beta}}{\beta}\right)\right] = \int_0^{\infty}\int_0^{\infty} e^{-p\frac{x^{\alpha}}{\alpha} - s\frac{t^{\beta}}{\beta}} f\left(\gamma\frac{x^{\alpha}}{\alpha}, \mu\frac{t^{\beta}}{\beta}\right) t^{\beta-1} x^{\alpha-1} dt dx$$

$$= \int_0^\infty e^{-p\frac{x^\alpha}{\alpha}} \left(\int_0^\infty e^{-s\frac{t^\beta}{\beta}} f\left(\gamma \frac{x^\alpha}{\alpha}, \mu \frac{t^\beta}{\beta}\right) t^{\beta-1} dt \right) x^{\alpha-1} dx$$

$$= \frac{1}{\mu^\beta} \int_0^\infty e^{-p\frac{x^\alpha}{\alpha}} \left(\int_0^\infty e^{-s\frac{\varphi^\beta}{\beta\mu^\beta}} f\left(\gamma \frac{x^\alpha}{\alpha}, \varphi\right) \varphi^{\beta-1} d\varphi \right) x^{\alpha-1} dx$$

$$= \frac{1}{\mu^\beta \gamma^\alpha} \int_0^\infty e^{-p\frac{\tau^\alpha}{\alpha\gamma^\alpha} f\left(\tau, \frac{s}{\sigma\beta}\right)} \tau^{\alpha-1} d\tau$$

$$= \frac{1}{\mu^\beta \gamma^\alpha} F\left(\frac{p}{\gamma^\alpha}, \frac{s}{\mu^\beta}\right)$$

(c) We prove (c) with the aid of conformable Laplace transform definition, i.e.

$$L_x^\alpha L_t^\beta \left[\int_0^\infty e^{-c\frac{x^\alpha}{\alpha} - d\frac{t^\beta}{\beta}} f\left(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta}\right) \right] = \int_0^\infty \int_0^\infty e^{-p\frac{x^\alpha}{\alpha} - s\frac{t^\beta}{\beta}} e^{-c\frac{x^\alpha}{\alpha} - d\frac{t^\beta}{\beta}} f\left(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta}\right) t^{\beta-1} x^{\alpha-1} dt dx$$

$$= \int_0^\infty e^{-p\frac{x^\alpha}{\alpha} - c\frac{x^\alpha}{\alpha}} \left(\int_0^\infty e^{-s\frac{t^\beta}{\beta} - d\frac{t^\beta}{\beta}} f\left(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta}\right) t^{\beta-1} dt \right) x^{\alpha-1} dx$$

$$= \mathrm{F}\left(\frac{x^\alpha}{\alpha}, s + d\right)$$

Now inserting equation (4.20) to equation (4.19) yields

$$\int_0^\infty e^{-(p+c)\frac{x^\alpha}{\alpha}} f\left(\frac{x^\alpha}{\alpha}, s + d\right) x^{\alpha-1} dx = F(p + c, s + d)$$

4.1.1 Existence Condition for the conformable double Laplace transform:

If $f\left(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta}\right)$ is an exponential order a and b as $\frac{x^\alpha}{\alpha} \to \infty, \frac{t^\beta}{\beta} \to \infty$ if there exists positive constant K such that for all $x > X$ and $t > T$

$$\left| f\left(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta}\right) \right| \leq \mathrm{K}e^{\boldsymbol{a\frac{x^\alpha}{\alpha} + b\frac{t^\beta}{\beta}}},$$

It is easy to get,

$$f\left(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta}\right) \leq 0e^{a\frac{x^\alpha}{\alpha} + b\frac{t^\beta}{\beta}} \text{ as } \frac{x^\alpha}{\alpha} \to \infty, \quad \frac{t^\beta}{\beta} \to \infty$$

Or, equivalently,

$$\lim_{\substack{\frac{x^\alpha}{\alpha}\to\infty \\ \frac{t^\beta}{\beta}\to\infty}} e^{-\mu\frac{x^\alpha}{\alpha}-\tau\frac{t^\beta}{\beta}}\left|f\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)\right| = K\lim_{\substack{\frac{x^\alpha}{\alpha}\to\infty \\ \frac{t^\beta}{\beta}\to\infty}} e^{-(\mu-a)\frac{x^\alpha}{\alpha}-(\tau-b)\frac{t^\beta}{\beta}} = 0, \text{ where } \mu > a \text{ and } \tau > b$$

The function $f\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)$ is called an exponential order as $\frac{x^\alpha}{\alpha}\to\infty, \frac{t^\beta}{\beta}\to\infty$ and clearly, it does not grow faster than $Ke^{a\frac{x^\alpha}{\alpha}+b\frac{t^\beta}{\beta}}$ as $\frac{x^\alpha}{\alpha}\to\infty, \frac{t^\beta}{\beta}\to\infty$.

Table 1: conformable double Laplace transform of some function Ozkan.O and Ali.K (2020)

Function $f(x,t)$	Conformable double Laplace transform $f(p,s)$
Ab	$\frac{ab}{ps}$
$\frac{x^\alpha}{\alpha}\frac{t^\beta}{\beta}$	$\frac{1}{p^2 s^2}$
$\frac{x^{m\alpha}}{\alpha}\frac{t^{n\beta}}{\beta}$	$\frac{m!\,n!}{p^{m+1}s^{n+1}}$
$e^{\frac{x^\alpha}{\alpha}+\frac{t^\beta}{\beta}}$	$\frac{1}{(p-1)(s-1)}$
$e^{\frac{x^\alpha}{\alpha}+\frac{t^\beta}{\beta}}\frac{x^{m\alpha}}{\alpha}\frac{t^{n\beta}}{\beta}$	$\frac{m!\,n!}{(p-1)^{m+1}(s-1)^{n+1}}$
$\cos\left(\theta\frac{x^\alpha}{\alpha}\right)\cos\left(\theta\frac{t^\beta}{\beta}\right)$	$\frac{ps}{(p^2+\theta^2)(s^2+\theta^2)}$
$\sin\left(\theta\frac{x^\alpha}{\alpha}\right)\sin\left(\theta\frac{t^\beta}{\beta}\right)$	$\frac{\theta^2}{(p^2+\theta^2)(s^2+\theta^2)}$

4.2 Main result

4.2.1. Description of the method

In this section, we use the definitions, properties and theorems mentioned in the preceding sections to check the applicability and reliability of the proposed method.

Now consider the following fractional Navier-Stokes equation,

$$\frac{\partial^\beta}{\partial t^\beta}\left(u\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)\right) = \frac{\partial^{2\alpha}}{\partial x^{2\alpha}}\left(u\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)\right) + \frac{a}{x^\alpha}\frac{\partial^\alpha}{\partial x^\alpha}u\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right) + f\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right), \text{x}, t > 0, 0 < \beta \le 1, \qquad (4,21)$$

with initial conditions $u\left(\frac{x^\alpha}{\alpha},0\right) = f\left(\frac{x^\alpha}{\alpha}\right)$, (4.22)

where $\text{x}, t > 0, 0 < \alpha, \beta \le 1$, In order to obtain the analytical solution of Eq. (4.21) subject to Eq. (4.22) via CDLTM, we use the following steps.

Step 1**:** Multiplying both sides of Eq. (`4.21) by the term $\frac{x^\alpha}{\alpha}$, we obtain

$$\frac{x^\alpha}{\alpha}\frac{\partial^\beta}{\partial t^\beta}u\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)=\frac{x^\alpha}{\alpha}\left(\frac{\partial^{2\alpha}}{\partial x^{2\alpha}}\mathrm{u}\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)\right)+\frac{x^\alpha}{\alpha}\frac{\alpha}{x^\alpha}\frac{\partial^\alpha}{\partial x^\alpha}u\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)+\frac{x^\alpha}{\alpha}f\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right) \tag{4.23}$$

Step 2**:** Applying (CDLT) on both sides of equation (4.23) and CSLT on both sides of equation (4.22), we get

$$L_x^\alpha L_t^\beta\left(\frac{x^\alpha}{\alpha}\frac{\partial^\beta}{\partial t^\beta}u\right)=L_x^\alpha L_t^\beta\left[\frac{x^\alpha}{\alpha}\frac{\partial^{2\alpha}}{\partial x^{2\alpha}}u+\frac{\partial^\alpha}{\partial x^\alpha}u+\frac{x^\alpha}{\alpha}f(x,t)\right] \tag{4.24}$$

and

$$L_x^\alpha\left(u\left(\frac{x^\alpha}{\alpha},0\right)\right)=L_x^\alpha\left(F\left(\frac{x^\alpha}{\alpha}\right)\right)\text{, respectively} \tag{4.25}$$

Applying Theorem 4.2 and Theorem 4.3 on equation (4.24) and using equation (4.25) we have

$$\frac{d}{dp}\left[U(p,s)-\frac{1}{s}U(p,0)\right]=-\frac{1}{s}L_x^\alpha L_t^\beta\left(\frac{x^\alpha}{\alpha}\frac{\partial^{2\alpha}}{\partial x^{2\alpha}}u+\frac{\partial^\alpha}{\partial x^\alpha}\ u\right)+\frac{1}{s}\frac{d}{dp}L_x^\alpha L_t^\beta\left(f\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)\right) \tag{4.26}$$

Step 3**:** By integrating both sides of equation (4.39) from 0 to p with respect to p and solving for $\mathrm{U}(\mathrm{p},\mathrm{s})$, we have

$$U(p,s)=\frac{1}{s}\int_0^p U(p,0)dp-\frac{1}{s}\int_0^p L_x^\alpha L_t^\beta\left(\frac{x^\alpha}{\alpha}\frac{\partial^{2\alpha}}{\partial x^{2\alpha}}u+\frac{\partial^\alpha}{\partial x^\alpha}\ u\right)dp+\frac{1}{s}\frac{d}{dp}L_x^\alpha L_t^\beta\left(f\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)\right) \tag{4.27}$$

Step 4**:** By taking the inverse conformable double Laplace transform of Eq. (4.27) we obtain

$$u\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)=\frac{1}{s}F\left(\frac{x^\alpha}{\alpha}\right)+L_p^{-1}L_s^{-1}\left(\frac{1}{s}\frac{d}{dp}L_x^\alpha L_t^\beta\left(f\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)\right)\right)$$

$$-L_p^{-1}L_s^{-1}\left(\frac{1}{s}\int_0^p L_x^\alpha L_t^\beta\left(\frac{x^\alpha}{\alpha}\frac{\partial^{2\alpha}}{\partial x^{2\alpha}}u+\frac{\partial^\alpha}{\partial x^\alpha}\ u\right)dp\ \right), \tag{4.28}$$

Step 5**:** Using (CDLT) representing the solution of the system as $u\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)$ by the infinite series

$$u\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)=\sum_{n=0}^\infty u_n\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right). \tag{4.29}$$

Substituting Eq. (4.29) into Eq. (4.28), we get

$$\sum_{n=0}^\infty u_n\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)=\frac{1}{s}F\left(\frac{x^\alpha}{\alpha}\right)+L_p^{-1}L_s^{-1}\left(\frac{1}{s}\frac{d}{dp}L_x^\alpha L_t^\beta\left(f\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)\right)\right)-$$

$$L_p^{-1}L_s^{-1}\left(\frac{1}{s}\int_0^p L_x^{\alpha}L_t^{\beta}\left(\frac{x^{\alpha}}{\alpha}\frac{\partial^{2\alpha}}{\partial x^{2\alpha}}\sum_{n=0}^{\infty}u_n+\frac{\partial^{\alpha}}{\partial x^{\alpha}}\sum_{n=0}^{\infty}u_n\right)dp\right). \tag{4.30}$$

We define the following recursive formula

$$u_0\left(\frac{x^{\alpha}}{\alpha},\frac{t^{\beta}}{\beta}\right)=\frac{1}{s}F\left(\frac{x^{\alpha}}{\alpha}\right) \tag{4.31}$$

and the remaining components can be written as

$$u_{n+1}\left(\frac{x^{\alpha}}{\alpha},\frac{t^{\beta}}{\beta}\right)=L_p^{-1}L_s^{-1}\left(\frac{1}{s}\frac{d}{dp}L_x^{\alpha}L_t^{\beta}\left(f\left(\frac{x^{\alpha}}{\alpha},\frac{t^{\beta}}{\beta}\right)\right)\right)-L_p^{-1}L_s^{-1}\left(\frac{1}{s^{\alpha}}\int_0^p L_x^{\alpha}L_t^{\beta}\left(\frac{x^{\alpha}}{\alpha}\frac{\partial^{2\alpha}}{\partial x^{2\alpha}}u_n+\frac{\partial^{\alpha}}{\partial x^{\alpha}}u_n\right)dp\right) \tag{4.32}$$

where, $L_p^{-1}L_s^{-1}$indicates inverse conformable double Laplace transform with respect to p,s, and $n\geq 0$.

To confirm the applicability of our method we consider the following illustrative examples.

4.3. Illustrative Examples

Example 1. Consider the nonhomogeneous fractional Navier-Stokes equation,

$$\frac{\partial^{\beta}}{\partial t^{\beta}}u\left(\frac{x^{\alpha}}{\alpha},\frac{t^{\beta}}{\beta}\right)=\frac{\partial^{2\alpha}}{\partial x^{2\alpha}}u\left(\frac{x^{\alpha}}{\alpha},\frac{t^{\beta}}{\beta}\right)+\frac{\alpha}{x^{\alpha}}\frac{\partial^{\alpha}}{\partial x^{\alpha}}u\left(\frac{x^{\alpha}}{\alpha},\frac{t^{\beta}}{\beta}\right)+\left(\frac{x^{\alpha}}{\alpha}\right)^2e^{\frac{t^{\beta}}{\beta}}-4e^{\frac{t^{\beta}}{\beta}},\ \ 0<\alpha,\beta\leq 1 \tag{4.33}$$

with initial condition: $u\left(\frac{x^{\alpha}}{\alpha},0\right)=\left(\frac{x^{\alpha}}{\alpha}\right)^2$ (4.34)

To obtain the solution of Equation (4.33) we use the following steps

Step 1**:** Multiplying both sides of Eq. (4.33) by $\frac{x^{\alpha}}{\alpha}$ we obtain

$$\frac{x^{\alpha}}{\alpha}\frac{\partial^{\beta}}{\partial t^{\beta}}u\left(\frac{x^{\alpha}}{\alpha},\frac{t^{\beta}}{\beta}\right)=\left[\frac{x^{\alpha}}{\alpha}\frac{\partial^{2\alpha}}{\partial x^{2\alpha}}u\left(\frac{x^{\alpha}}{\alpha},\frac{t^{\beta}}{\beta}\right)\right]+\frac{\partial^{\alpha}}{\partial x^{\alpha}}u\left(\frac{x^{\alpha}}{\alpha},\frac{t^{\beta}}{\beta}\right)+\frac{x^{\alpha}}{\alpha}\left(\left(\frac{x^{\alpha}}{\alpha}\right)^2e^{\frac{t^{\beta}}{\beta}}-4e^{\frac{t^{\beta}}{\beta}}\right) \tag{4.35}$$

Step 2**:** Applying the conformable double Laplace transform on both sides of Eq. (4.35) and single conformable Laplace transform for initial condition we get

$$L_x^{\alpha}L_t^{\beta}\left(\frac{x^{\alpha}}{\alpha}\frac{\partial^{\beta}}{\partial t^{\beta}}u\left(\frac{x^{\alpha}}{\alpha},\frac{t^{\beta}}{\beta}\right)\right)=L_x^{\alpha}L_t^{\beta}\left(\frac{x^{\alpha}}{\alpha}\frac{\partial^{2\alpha}}{\partial x^{2\alpha}}u\left(\frac{x^{\alpha}}{\alpha},\frac{t^{\beta}}{\beta}\right)+\frac{\partial^{\alpha}}{\partial x^{\alpha}}u\left(\frac{x^{\alpha}}{\alpha},\frac{t^{\beta}}{\beta}\right)+\left(\frac{x^{\alpha}}{\alpha}\right)^3e^{\frac{t^{\beta}}{\beta}}-4\frac{x^{\alpha}}{\alpha}e^{\frac{t^{\beta}}{\beta}}\right) \tag{4.36}$$

and

$$L_x^{\alpha}\left(u\left(\frac{x^{\alpha}}{\alpha},0\right)\right)=L_x^{\alpha}\left(\left(\frac{x^{\alpha}}{\alpha}\right)^2\right)\text{, respectively} \tag{4.37}$$

By applying Theorem 4.2 and Theorem 4.3 on Eq. (4.36) and using SCLT on Eq. (4.37), we obtain

$$-\frac{d}{dp}[sU(p,s)-U(p,0)]=L_x^{\alpha}L_t^{\beta}\left(\frac{x^{\alpha}}{\alpha}\frac{\partial^{2\alpha}}{\partial x^{2\alpha}}u+\frac{\partial^{\alpha}}{\partial x^{\alpha}}\ u\right)+L_x^{\alpha}L_t^{\beta}\left(\left(\frac{x^{\alpha}}{\alpha}\right)^3e^{\frac{t^{\beta}}{\beta}}-4\frac{x^{\alpha}}{\alpha}e^{\frac{t^{\beta}}{\beta}}\right)$$

$$-\frac{d}{dp}\left[sU(p,s)-\frac{2!}{p^3}\right]=\frac{3!}{p^4}\frac{1}{(s-1)}-\frac{4}{p^2(s-1)}+L_x^\alpha L_t^\beta\left(\frac{x^\alpha}{\alpha}\frac{\partial^{2\alpha}}{\partial x^{2\alpha}}u+\frac{\partial^\alpha}{\partial x^\alpha}\,u\right) \tag{4.38}$$

Simplifying further Eq. (4.38) we get

$$\frac{d}{dp}\left[U(p,s)-\frac{1}{s}\frac{2!}{p^3}\right]=-\frac{3!}{p^4}\frac{1}{s(s-1)}+\frac{4}{p^2s(s-1)}-\frac{1}{s}L_x^\alpha L_t^\beta\left(\frac{x^\alpha}{\alpha}\frac{\partial^{2\alpha}}{\partial x^{2\alpha}}u+\frac{\partial^\alpha}{\partial x^\alpha}\,u\right) \tag{4.39}$$

Step 3**:** By integrating both sides of equation (4.39) from 0 to p with respect to p and solving for U(p, s), we have

$$U(p,s)=\frac{2!}{p^3s}+\frac{2!}{p^3s(s-1)}-\frac{4}{ps(s-1)}-\frac{1}{s}\int_0^p\left[L_x^\alpha L_t^\beta\left(\frac{x^\alpha}{\alpha}\frac{\partial^{2\alpha}}{\partial x^{2\alpha}}u+\frac{\partial^\alpha}{\partial x^\alpha}\,u\right)\right]dp \tag{4.40}$$

Step 4**:** By applying double inverse Laplace transform on equation (4.40), we have

$$u\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)=\left(\frac{x^\alpha}{\alpha}\right)^2e^{\frac{t^\beta}{\beta}}-4\left(e^{\frac{t^\beta}{\beta}}-1\right)-L_p^{-1}L_s^{-1}\left(\int_0^p\left[\frac{1}{s}L_x^\alpha L_t^\beta\left(\frac{x^\alpha}{\alpha}\frac{\partial^{2\alpha}}{\partial x^{2\alpha}}u+\frac{\partial^\alpha}{\partial x^\alpha}\,u\right)\right]dp\right) \tag{4.41}$$

Step 5**:** Using (CDLT) representing the solution of the system as $u\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)$ by the infinite series

$$u\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)=\sum_{n=0}^\infty u_n\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right) \tag{4.42}$$

By substituting Eq. (4.42) in to Eq. (4.41) we get

$$\sum_{n=0}^\infty u_n\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)=$$

$$\left(\frac{x^\alpha}{\alpha}\right)^2e^{\frac{t^\beta}{\beta}}-4\left(e^{\frac{t^\beta}{\beta}}-1\right)-L_p^{-1}L_s^{-1}\left(\int_0^p\left[\frac{1}{s}L_x^\alpha L_t^\beta\left(\frac{x^\alpha}{\alpha}\frac{\partial^{2\alpha}}{\partial x^{2\alpha}}\sum_{n=0}^\infty u_n+\frac{\partial^\alpha}{\partial x^\alpha}\sum_{n=0}^\infty u_n\right)\right]dp\right) \tag{4.43}$$

Now using Equation (4.31) and (4.32) the components of the solution of the given problem are given by

$$u_0=\left(\frac{x^\alpha}{\alpha}\right)^2e^{\frac{t^\beta}{\beta}}-4\left(e^{\frac{t^\beta}{\beta}}-1\right) \tag{4.44}$$

$$u_{n+1}=-L_p^{-1}L_s^{-1}\left(\frac{1}{s}\int_0^p L_x^\alpha L_t^\beta\left(\frac{x^\alpha}{\alpha}\frac{\partial^{2\alpha}}{\partial x^{2\alpha}}u_n+\frac{\partial^\alpha}{\partial x^\alpha}u_n\right)dp\right). \tag{4.45}$$

Hence, when $n=0$, we get

$$u_1=-L_p^{-1}L_s^{-1}\left(\frac{1}{s}\int_0^p L_x^\alpha L_t^\beta\left(\frac{x^\alpha}{\alpha}\frac{\partial^{2\alpha}}{\partial x^{2\alpha}}u_0+\frac{\partial^\alpha}{\partial x^\alpha}u_0\right)dp\right)$$

$$=-L_p^{-1}L_s^{-1}\left(\frac{1}{s}\int_0^p L_x^\alpha L_t^\beta\left(\frac{x^\alpha}{\alpha}\frac{\partial^{2\alpha}}{\partial x^{2\alpha}}\left[\left(\frac{x^\alpha}{\alpha}\right)^2e^{\frac{t^\beta}{\beta}}-4\left(e^{\frac{t^\beta}{\beta}}-1\right)\right]+\frac{\partial^\alpha}{\partial x^\alpha}\left[\left(\frac{x^\alpha}{\alpha}\right)^2e^{\frac{t^\beta}{\beta}}-4\left(e^{\frac{t^\beta}{\beta}}-1\right)\right]\right)dp\right)$$

$$=-L_p^{-1}L_s^{-1}\left[\frac{1}{s}\int_0^p L_x^\alpha L_t^\beta\left[4\left(\frac{x^\alpha}{\alpha}\right)\left(e^{\frac{t^\beta}{\beta}}\right)\right]dp\right]$$

$$=-L_p^{-1}L_s^{-1}\left[\frac{1}{s}\int_0^p\left(\frac{4}{p^2(s-1)}\right)dp\right]$$

$$=-L_p^{-1}L_s^{-1}\left(-\frac{4}{ps(s-1)}\right)$$

$$= 4\left(e^{\frac{t^\beta}{\beta}} - 1\right). \tag{4.46}$$

In a similar way, we obtain,

$$u_2 = u_3 = u_4 = \ . \ . \ . = 0,$$

The series solution is therefore given by

$$u\left(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta}\right) = u_0\left(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta}\right) + u_1\left(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta}\right) + u_2\left(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta}\right) + \ . \ . \ .$$

$$= \left(\frac{x^\alpha}{\alpha}\right)^2 e^{\frac{t^\beta}{\beta}} - 4\left(e^{\frac{t^\beta}{\beta}} - 1\right) + 4\left(e^{\frac{t^\beta}{\beta}} - 1\right) + 0 \ + \ . \ . \ . \ . \ ,$$

$$= \left(\frac{x^\alpha}{\alpha}\right)^2 e^{\frac{t^\beta}{\beta}}. \tag{4.47}$$

By taking $\alpha = 1$ and $\beta = 1$, the solution of Example 1 becomes $u(x,t) = x^2 e^t$, which is the same as the exact solution of one dimensional Navier-Stokes equation as in Eltayeb et al., (2020).

The solution curves of the one dimensional fractional Navier-Stokes equation given in Examples 1 for different values of fractional order α and β is depicted in Figures 1 and 2

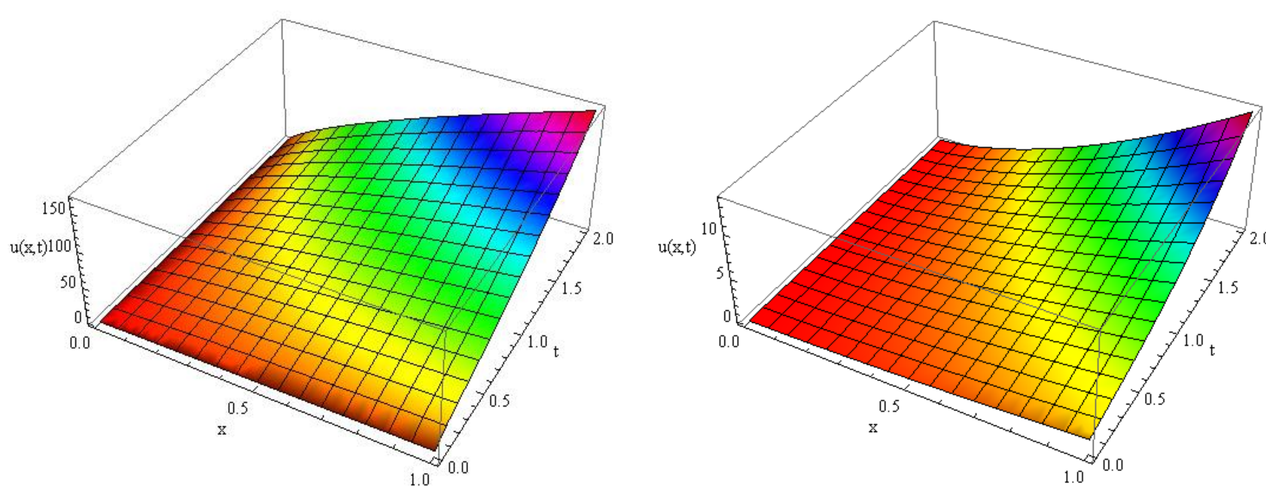
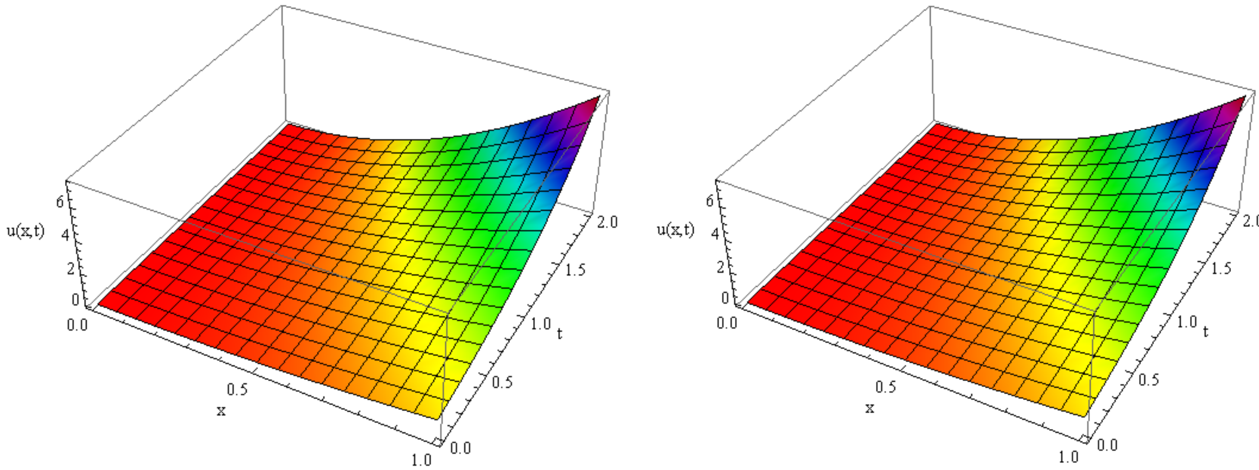


Figure 1: Solution behavior of Example 1: a) $\alpha = \beta = 0.4$, b) $\alpha = \beta = 0.8$,



A B

Figure 2: Solution behavior of Example 1: a) $\alpha = \beta = 1$, b) Exact solution,

Example 2 Consider the fractional Navier-Stokes equation

$$\frac{\partial^{\beta}}{\partial t^{\beta}} u\left(\frac{x^{\alpha}}{\alpha}, \frac{t^{\beta}}{\beta}\right) = k + \frac{\alpha}{x^{\alpha}}\left(\frac{\partial^{\alpha}}{\partial x^{\alpha}} u\left(\frac{x^{\alpha}}{\alpha}, \frac{t^{\beta}}{\beta}\right) + \frac{x^{\alpha}}{\alpha}\frac{\partial^{2\alpha}}{\partial x^{2\alpha}} u\left(\frac{x^{\alpha}}{\alpha}, \frac{t^{\beta}}{\beta}\right)\right), \quad 0 < \alpha, \beta \leq 1 \tag{4.48}$$

with initial conditions $u\left(\frac{x^{\alpha}}{\alpha}, 0\right) = 1 - \left(\frac{x^{\alpha}}{\alpha}\right)^{2}$ (4.49)

Where k = - $\frac{\partial p}{\rho \partial z}$, p is the pressure, u is the fluid velocity and ρ is density.by consider the unsteady flow of a viscous fluid in tube.

To obtain the solution of Equation (4.48) we use the following steps

Step 1**:** Multiplying both side of Eq. (4.48) by $\frac{x^{\alpha}}{\alpha}$ we obtain

$$\frac{x^{\alpha}}{\alpha}\frac{\partial^{\beta}}{\partial t^{\beta}} u\left(\frac{x^{\alpha}}{\alpha}, \frac{t^{\beta}}{\beta}\right) = \frac{x^{\alpha}}{\alpha} k + \left(\frac{\partial^{\alpha}}{\partial x^{\alpha}} u\left(\frac{x^{\alpha}}{\alpha}, \frac{t^{\beta}}{\beta}\right) + \frac{x^{\alpha}}{\alpha}\frac{\partial^{2\alpha}}{\partial x^{2\alpha}} u\left(\frac{x^{\alpha}}{\alpha}, \frac{t^{\beta}}{\beta}\right)\right), \tag{4.50}$$

Step 2**:** Applying the conformable double Laplace transform on both side of Eq. (4.50) and single conformable Laplace transform for initial condition we get

$$L_x^{\alpha} L_t^{\beta}\left(\frac{x^{\alpha}}{\alpha}\frac{\partial^{\beta}}{\partial t^{\beta}} u\left(\frac{x^{\alpha}}{\alpha}, \frac{t^{\beta}}{\beta}\right)\right) = L_x^{\alpha} L_t^{\beta}\left[\frac{x^{\alpha}}{\alpha} k + \left(\frac{\partial^{\alpha}}{\partial x^{\alpha}} u\left(\frac{x^{\alpha}}{\alpha}, \frac{t^{\beta}}{\beta}\right) + \frac{x^{\alpha}}{\alpha}\frac{\partial^{2\alpha}}{\partial x^{2\alpha}} u\left(\frac{x^{\alpha}}{\alpha}, \frac{t^{\beta}}{\beta}\right)\right)\right], \tag{4.51}$$

and

$$L_x^\alpha\left(u\left(\frac{x^\alpha}{\alpha},0\right)\right)=L_x^\alpha\left(1-\left(\frac{x^\alpha}{\alpha}\right)^2\right) \tag{4.52}$$

By applying Theorem 4.2 and Theorem 4.3 on Eq. (4.51) and using SCLT on Eq. (4.52), we obtain

$$-\frac{d}{dp}[sU(p,s)-U(p,0)]=L_x^\alpha L_t^\beta\left[\frac{x^\alpha}{\alpha}k+\left(\frac{\partial^\alpha}{\partial x^\alpha}u\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)+\frac{x^\alpha}{\alpha}\frac{\partial^{2\alpha}}{\partial x^{2\alpha}}u\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)\right)\right]$$

$$-\frac{d}{dp}\left[sU(p,s)-\left[\frac{1}{p}-\frac{2!}{p^3}\right]\right]=\frac{k}{p^2s}+L_x^\alpha L_t^\beta\left(\frac{\partial^\alpha}{\partial x^\alpha}u\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)+\frac{x^\alpha}{\alpha}\frac{\partial^{2\alpha}}{\partial x^{2\alpha}}u\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)\right) \tag{4.53}$$

Simplifying Eq. (4.53) we obtain

$$\frac{d}{dp}\left[U(p,s)-\left[\frac{1}{ps}-\frac{2!}{p^3s}\right]\right]=-\frac{k}{p^2s^2}-\frac{1}{s}L_x^\alpha L_t^\beta\left(\frac{x^\alpha}{\alpha}\frac{\partial^{2\alpha}}{\partial x^{2\alpha}}u\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)+\frac{\partial^\alpha}{\partial x^\alpha}u\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)\right) \tag{4.54}$$

Step 3**:** By applying the integral for both side of Eq. (4.54) from 0 to p with respect to p and solving for $U(p,s)$, we have

$$U(p,s)=\frac{1}{ps}-\frac{2!}{p^3s}+\frac{k}{ps^2}-\frac{1}{s}\int_0^p\left[L_x^\alpha L_t^\beta\left(\frac{x^\alpha}{\alpha}\frac{\partial^{2\alpha}}{\partial x^{2\alpha}}u+\frac{\partial^\alpha}{\partial x^\alpha}u\right)\right]dp \tag{4.55}$$

Step 4**:** By taking the inverse CDLT for Eq. (4.55), we obtain

$$u\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)=1-\left(\frac{x^\alpha}{\alpha}\right)^2+k\frac{t^\beta}{\beta}-L_p^{-1}L_s^{-1}\left(\frac{1}{s}\int_0^p\left[L_x^\alpha L_t^\beta\left(\frac{x^\alpha}{\alpha}\frac{\partial^{2\alpha}}{\partial x^{2\alpha}}u+\frac{\partial^\alpha}{\partial x^\alpha}u\right)\right]dp\right), \tag{4.56}$$

Step 5**:** Using (CDLT) representing the solution of the system as $u\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)$ by the infinite series

$$u\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)=\sum_{n=0}^\infty u_n\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right) \tag{4.57}$$

By substituting Eq. (4.57) in to Eq. (4.56) we get

$$\sum_{n=0}^\infty u_n\left(\frac{x^\alpha}{\alpha},\frac{t^\beta}{\beta}\right)=1-\left(\frac{x^\alpha}{\alpha}\right)^2+k\frac{t^\beta}{\beta}$$

$$-L_p^{-1}L_s^{-1}\left(\frac{1}{s}\int_0^p\left[L_x^\alpha L_t^\beta\left(\frac{x^\alpha}{\alpha}\frac{\partial^{2\alpha}}{\partial x^{2\alpha}}\sum_{n=0}^\infty u_n+\frac{\partial^\alpha}{\partial x^\alpha}\sum_{n=0}^\infty u_n\right)\right]dp\right), \tag{4.58}$$

Now using Equation (4.31) and (4.32) the components of the solution of the given problem are given by

$$u_0=1-\left(\frac{x^\alpha}{\alpha}\right)^2+k\frac{t^\beta}{\beta}.$$

$$u_{n+1}=-L_p^{-1}L_s^{-1}\left(\frac{1}{s}\int_0^p\left[L_x^\alpha L_t^\beta\left(\frac{x^\alpha}{\alpha}\frac{\partial^{2\alpha}}{\partial x^{2\alpha}}u_n+\frac{\partial^\alpha}{\partial x^\alpha}u_n\right)\right]dp\right).$$

Hence, when $n=0$, we get

$$u_1 = -L_p^{-1}L_s^{-1}\left(\frac{1}{s}\int_0^p L_x^{\alpha}L_t^{\beta}\left(\frac{x^{\alpha}}{\alpha}\frac{\partial^{2\alpha}}{\partial x^{2\alpha}}u_0 + \frac{\partial^{\alpha}}{\partial x^{\alpha}}\ u_0\right)dp\ \right)$$

$$= -L_p^{-1}L_s^{-1}\left(\frac{1}{s}\int_0^p L_x^{\alpha}L_t^{\beta}\left(\frac{x^{\alpha}}{\alpha}\frac{\partial^{2\alpha}}{\partial x^{2\alpha}}\left[1-\left(\frac{x^{\alpha}}{\alpha}\right)^2 + k\ \frac{t^{\beta}}{\beta}\right] + \frac{\partial^{\alpha}}{\partial x^{\alpha}}\ \left[1-\left(\frac{x^{\alpha}}{\alpha}\right)^2 + k\ \frac{t^{\beta}}{\beta}\ \right]\right)dp\ \right)$$

$$= -L_p^{-1}L_s^{-1}\left[\frac{1}{s}\int_0^p L_x^{\alpha}L_t^{\beta}\left[-4\,\frac{x^{\alpha}}{\alpha}\right]dp\right]$$

$$= -L_p^{-1}L_s^{-1}\left[\frac{1}{s}\int_0^p\left(-\frac{4}{p^2 s}\right)dp\right]$$

$$= -L_p^{-1}L_s^{-1}\left[\frac{4}{ps^2}\right]$$

$$= -4\,\frac{t^{\beta}}{\beta}$$

In a similar way, we obtain,

$$u_2 = u_3 = .\ .\ . = 0,$$

The series solution is given by

$$u\left(\frac{x^{\alpha}}{\alpha},\frac{t^{\beta}}{\beta}\right) = u_0\left(\frac{x^{\alpha}}{\alpha},\frac{t^{\beta}}{\beta}\right) + u_1\left(\frac{x^{\alpha}}{\alpha},\frac{t^{\beta}}{\beta}\right) + u_2\left(\frac{x^{\alpha}}{\alpha},\frac{t^{\beta}}{\beta}\right) + .\ .\ .$$

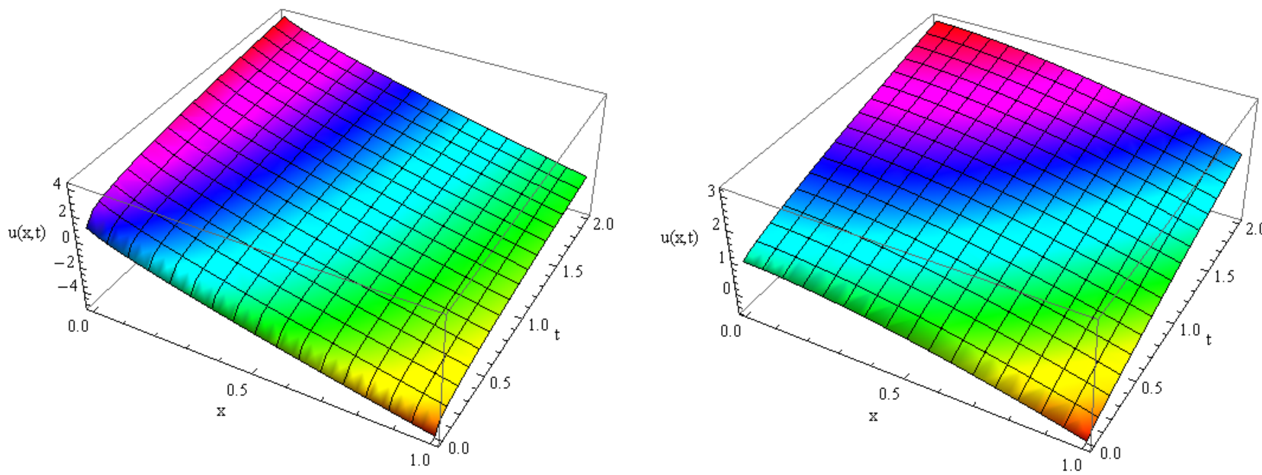
$$= 1-\left(\frac{x^{\alpha}}{\alpha}\right)^2 + k\ \frac{t^{\beta}}{\beta} - 4\,\frac{t^{\beta}}{\beta} + 0 + .\ .\ .,$$

Hence, the solution of Example 2 becomes

$$u\left(\frac{x^{\alpha}}{\alpha},\frac{t^{\beta}}{\beta}\right)\ = 1-\left(\frac{x^{\alpha}}{\alpha}\right)^2 + (k-4)\ \frac{t^{\beta}}{\beta}$$

When $\alpha\ = 1$ and $\beta\ = 1$ the last equation becomes $u(x,t) =\ 1 - x^2 + (k-4)\ t$, which is the same as the exact solution of equation (4.48) as in Eltayeb et al., (2020).

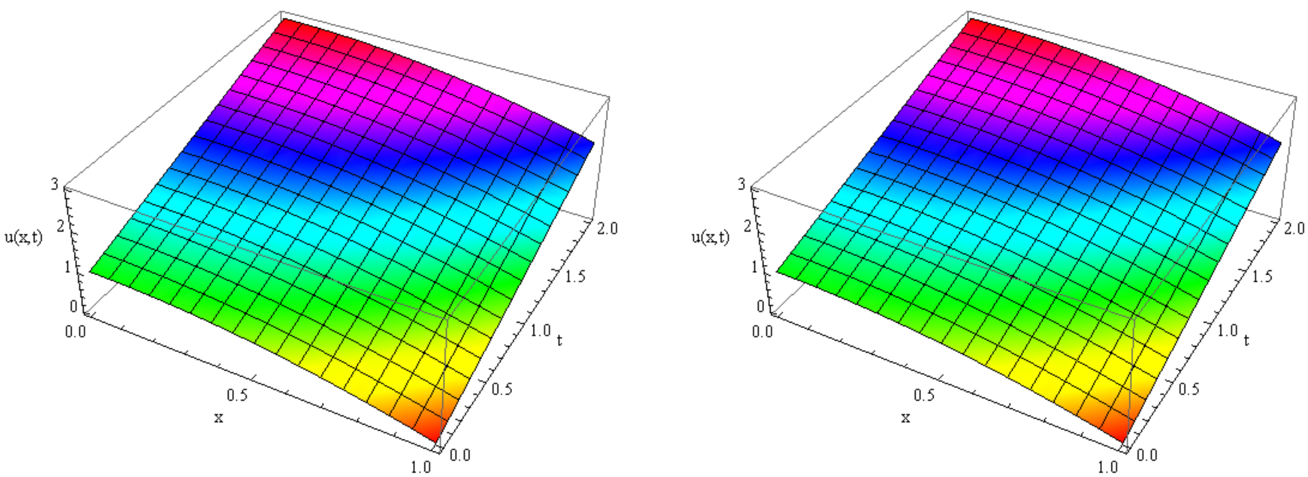
The solution curves of the one dimensional fractional Navier-Stokes equation given in Examples 2 for different values of fractional order α and β is depicted in Figures 3 and 4.



A

B

Figure 3: Solution behavior of Example 2 for $k = 5$: a) $\alpha = \beta = 0.4$, b) $\alpha = \beta = 0.8$,



A

B

Figure 4: Solution behavior of Example 2 for $k = 5$: a) $\alpha = \beta = 1$, b) Exact solution,

4.4. Discussion

The conformable double Laplace transform method (CDLTM) has been successfully applied to one dimensional fractional Navier-Stokes equation (TFNSE) subject to the given initial condition which gives series solution.

The conformable double Laplace transform method was used straightforward. The effectiveness and capability of the present method have been cheeked via two illustrative examples.

The solution curves of the one dimensional fractional Navier-Stokes equation given in example 1 for different values of fractional order alpha and beta was presented in figures 1 - 4. It is shown in the figures 1 and 2 that as the value of the fractional order α approaches 1 the corresponding approximated solution curves are closer and closer to the curve of the exact solution. Also in figures 2 and 4, when the fractional order α equals 1, the approximated solution curve is exactly the same as the corresponding exact solution curve.

In general, the obtained solutions are in excellent agreement when compared with the solution done by Eltayeb et al., (2020).

CHAPTER FIVE
CONCLUSION AND FUTURE SCOPE

5.1 Conclusion

In this study, CDLTM has been applied for solving one dimensional linear TFNSE subject to initial conditions. The CDLTM provides a powerful method for analyzing fractional Navier-Stokes equation. Examples are presented to show the applicability of the method under consideration. In the solution of example 1 and example 2, the same problem has been considered in (Eltayeb et al., 2020). It can be easily seen that theorems that described here can be further generated for other type of functions, relations and equations. This method has been proved to be a powerful tool which enables us to manage fractional Navier-Stokes equation and allow us to reach the desired solution. Therefore, it can be found that the CDLTM is very effective and efficient in the search of exact solution for the fractional Navier-Stokes equation.

5.2 Future Scope

Following the same procedure as discussed in this research work, the CDLTM may be applied to solve linear and non-linear two dimensional fractional Navier-Stokes equation. It may also be used for solving many equations that have the same form.

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