Existence and Uniqueness of Solutions for Two Point Second Order Sturm Liouville Boundary Value Problems Via Upper and Lower Solution Method

A Thesis Submitted to the Department of Mathematics,College of Natural Sciences, Jimma University in Partial Fulfillment for the Requirements of the Degree of Masters of Science (M.Sc) in Mathematics.

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June, 2022

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Declaration

Here, I submit a thesis entitled 'Existence and Uniqueness of solution for two point second order Sturm Liouvill boundary value problem by upper and lower solution method' for the award of degree of master of science in mathematics. I, the undersigned declare that, this study is the original and it has not been submitted to any institution elsewhere for the like, where other sources of information have been used, they have been acknowledged or for award of any academic degree achievement.

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Acknowledgment

First of all, I am indebted to Allah who gave me long life and helped me to pass through different ups and downs to reach this time. Next, my Special heartfelt thanks go to my advisor, D.r Wesen Legese and co-advisor, M.r Girma Kebede for their unreserved support, advice and guidance throughout the preparation of this thesis. Finally, I would like to thank my family for being so supportive in both morally and financially and also I would like to thank my lovely wife Yenenesh(Nesia) Tadele for her support of every things started time up to final time. and I express my deepest gratitude to my friend Tesema Mukura he strongly stand with me by moral and support many things in preparing this thesis. Lastly, I would like to thank all my friends.

List of Abbreviations

BVP	Boundary Value Problem
IVP	Initial Value Problem
DE	Differential Equation
ODE	Ordinary Differential Equation
PDE	Partial Differential Equation
$C^1[0,1]$	maps of $[0,1]$ to R continuous first derivative
$C^2[0,1]$	maps of $[0,1]$ to R continuous second derivative
$C^2[a,b]$	maps of $[a,b]$ to R continuous second derivative
R	set of real number
R^+	non negative real number

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Abstract

In this paper, the conditions for the existence and uniqueness of solution of lower and upper solutions of second order non- linear ordinary differential equations were considered. New definitions of upper and lower solutions for our problems are presented. Also considered was the technique for constructing lower and upper solutions of the second order non-linear ordinary differential equation for two point boundary value problem. The existence of solution for two point second order Sturm Liouvill problem boundary value problem determined. The uniqueness of solution for two point second order Sturm Liouvill problem boundary value problem was also determined. the examples illustrating the use of lower and upper solutions were given.

Key Words: lower and upper solution, existence and uniqueness, two point second order Sturm Liouvill boundary value problem .

1 INTRODUCTION

1.1 Background of the Study

The study of differential equations date back to the mid-seventeenth century, when Differential and Integral Calculus was discovered independently by Newton, 1665 and Leibnitz 1676. Newton had laid the foundation stone for the study of differential equations (DEs). He was followed by Leibnitz who coined the name of differential equations in 1676 to denote relationships between differentials and two variables x and y. Boundary value problem consists of finding an unknown solution which satisfies an ordinary differential equation and appropriate boundary conditions at two or more points.

When ordinary differential equations are required to satisfy boundary conditions at more than one value of the independent variable, the resulting problem is called a two point boundary value problem (TynMyint-U 1978).

However, constant coefficient linear two point BVP might have unique solution might have no solution or might have infinitely many solutions. Boundary condition is the set of conditions specified for the behavior of the solution to a set of differential equations at the boundary of its domain. Boundary conditions are important in determining the mathematical solutions to many physical problems.

Boundary value problems arise in almost all branches of science and engineering. The methods commonly used in solving two-point boundary value problems are based on the idea of construction of Green's functions Grossinho & Minhós (2001).

Since then, a lot of work has been done on existence and uniqueness of certain BVP's for third order or higher order differential equations, or differential equations on time scales by matching solutions. The theory of nonlinear boundary value problems is an important and interesting area of research in differential equations. Due to the entirely different nature of the underlying physical processes, its study is more difficult than that of initial value problems.

A variety of techniques are employed in the theory of nonlinear boundary value problems for existence results. One of the most powerful tools for proving existence of solutions is the method of upper and lower solutions, Dobkevich et al. (2014).

Boundary value problems associated with linear as well as non-linear ordinary differential equations or finite difference equations have great deal of interest and play an important role in many fields of applied mathematics such as Engineering and Technology, major industries like automobile, aerospace, optimization theory, electromagnetic potential and heat power transmission theory are few on the boundary value problems to simulate complex phenomena at different scales for designing and manufacturing of heat-technological products.

In these applied setting, positive solutions are meaningful. Earlier the existence and uniqueness of positive solutions for Caputo fractional order boundary value problem is studied by using fixed point theorem, and other methodDhaigude et al. (2021).

The existence of positive solutions of boundary value problems was studied by many researchers: L.H and Haiyan Wang 1994, Erbe, Hu and Wang 1994,Lian, Wong and Yah 1996, Henderson and Wang in 1997, Karakostas and Tsamatosin 2002,Hederson, Ntouyas and Purnaras in 2008, Dang Quang and Ngo Thi KimQuy in 2018, Bai et al. (2020). The method of lower and upper solutions deals mainly with existence results for boundary value problems.

In this study, the researcher restrict attention to second order boundary value problems with separated boundary conditions. Although some of the ideas can be traced back to in Picard (1890), the method of lower and upper solutions was firmly established by G. ScorzaDragoni considered upper and lower solutions the same author extended his method .

The first steps in the theory of lower and upper solutions have been given by Picard in 1890 for Partial Differential Equations and, three years after, in Picard (1890), for Ordinary Differential Equations. In both cases, the existence of a solution is guaranteed from a monotone iterative technique.

Dragoni (1931),introduces in 1931 the notion of the method of lower and upper solutions for ordinary differential equations with Dirichlet boundary value conditions. In particular, by assuming stronger conditions than now a days, the author considers the second-order boundary value problem Upper and lower solutions with corners were considered by De Coster & Habets (2006) . Since then a multitude of variants have been introduced.

The existence of a metric projection and the uniqueness of its point images on to closed convex sets in Banach spaces require at least reflexivity of the Banach space and strict convexity of the norm. Furthermore, higher-order boundary value problems (BVPs) have been studied in many authors, such as Graef et al. (2011) for two-point BVP, Du et al. (2007) for multi point BVP, and R. Agarwal & O'Regan (2003) for infinite interval problem. However, most of these works have been done either on finite intervals or for bounded solutions on an infinite interval. The authors, R. Agarwal & O'Regan (2003): Nonlinear boundary value problems on the semi-infinite interval an upper and lower Solution approach.

Assumed one pair of well-ordered upper and lower solutions, and then applied some fixed point theorems or a monotone iterative technique to obtain a solution. Infinite interval problems occur in the study of radially symmetric solutions of nonlinear elliptic equations R. Agarwal & O'Regan (2003).

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Knobloch & Schmitt (1977), nonlinear boundary value problems on semi-infinite intervals. During the last few years, fixed point theorems, shooting methods, upper and lower technique, etc.

Have been used to prove the existence of a single solution or multiple solutions to infinite interval problem .When applying the upper and lower solution method to infinite interval problems, the solutions are always assumed to be bounded. R. P. Agarwal & O'Regan (2001), they employed the technique of lower and upper solutions and the theory of fixed point index to obtain the existence of at least three solutions.

The problems related to global solutions, especially when the boundary data are prescribed asymptotically and the solutions may be unbounded, have been briefly discussed. By using the upper and lower solutions method and a fixed point theorem, they presented sufficient conditions for the existence of unbounded positive solutions however; their results are suitable only to positive solutions. Lian & Ge (2006), existence of unbounded solutions for a third-order boundary value problem on infinite intervals.

To the best of our knowledge, this is the first attempt to find the unbounded solutions to higher-order infinite interval problems by using the upper and lower solution technique. Since, the half-line is non compact, the discussion is rather involved.

We begin with the assumption that there exists a pair of upper and lower solutions for problem. Graef, Graef et al. (2011), and the nonlinear function f satisfy a Nagumo-type condition. Then, by using the truncation technique and the upper and lower solutions, we estimate a-priori bounds of modified problems. Next, the Schäuder fixed point theorem is used which guarantees the existence of solutions.

A drawback of Scorza Dragon's approach is that, assuming the existence of ordered lower and upper solutions alpha and beta, hides the difficulty.

In practical problems there is no clue to finding these functions. This drawback motivated further work to indicate how they can be found. Constructions of lower and upper solutions appear in various proofs (see for exampleEpheser (1954).

The Nagumo condition is to assume some global existence of the solutions, i.e every solution u of

$u'' = f(t, u, u^{'})$,

such that $\alpha(t) \leqslant u(t) \leqslant \beta(t)$,

on its maximal interval of existence exists on the whole interval [a, b]. Such an approach has been used in Heidel (1974). Since 1970, Jackson and several other authors have made a substantial study for the existence and uniqueness of the solutions for the two-point boundary value problems for third order nonlinear ordinary differential equations.

A long history going back to Picard (1890),the existence and uniqueness of solutions of the two point boundary value problem.

$$\begin{aligned} y'' + f(t, y^{'}) &= 0, \\ y(a) &= A, \\ y(b) &= B, \\ f(t, y, y^{'}), \end{aligned}$$

satisfies a Lipschitz conditions. However, we discuss in the present paper the existence and uniqueness of the solutions for two-point second order Sturm Liouville boundary value problems for second order nonlinear ordinary differential equations by applying the method of upper and lower solutions.

And there are some related presented by others as follow. In Knobloch & Schmitt (1977), had considered the general separated boundary value problem .

$$u'' = f(t, u, u'),$$
$$a_1 u(a) - a_2 u'(a) = A,$$
$$b_1 u(b) + b_2 u'(b) = B,$$

assuming $[a_1 + a_2 > 0], \quad a_i \geq 0, \quad b_i \geq 0.$
This problem contains both the Dirichlet and the Neumann problem.
The major breakthrough was due to Dragoni (1931).
In a first, extended and improved, this author considers the Dirichlet boundary value problem.

$$u'' = f(t, u, u'),$$
$$u(a) = A, u(b) = B.$$

He assumes the existence of functions α *and* $\beta \in C^2([a, b])$ such that:

$$\alpha(t) \leqslant \beta(t) \quad on\ [a, b],$$
$$and$$
$$\alpha''(t) + f(t, \alpha(t), y) \geq 0 \quad , if\ t \in [a, b],\ y \leqslant \alpha'(t) \quad (resp. y \geq \alpha'(t),$$
$$\alpha(a) \leqslant A, \alpha(b) \leqslant B.$$
$$\beta''(t) + f(t, \beta(t), y) \leqslant 0, \quad if \quad t \in [a,],\ y \geq \beta'(t) \quad (resp \quad y \leqslant \beta'(t)),$$
$$\beta(a) \geq A,\ \beta(b) \geq B.$$

He then obtains existence of a solution u of together with its localization $\alpha \leqslant u \leqslant \beta$.
The regularity assumptions were that f is continuous and bounded on.
$E = (t, u, v) \in [a, b] \times R^2 \setminus \alpha(t) \leqslant u \leqslant \beta(t).$

Paul, W. EloeB,Johnny Henderson and Jeffrey T. Neugebauer, (2020), studied three point boundary value problems for ordinary differential equations, uniqueness implies existence.

He et al. (2018). Studied existence and asymptotic analysis of positive solutions by constructing suitable upper and lower solutions and employing Schauder's fixed point theorem, for

$$
\begin{aligned}
-D^{\alpha}_{(t)}(t) &= f(t, x(t), D^{\gamma}_{t}x(t)),\\
D^{\gamma}_{t}x(0) &= D^{(\gamma+1)(0)}_{t} = 0,\\
D^{\mu}_{t}(t) &= \int_0^1 D^{\mu}_{t}x(s)dx(s),
\end{aligned}
$$

where $2 < \alpha \leqslant 3$ with $0 < \gamma \leqslant \mu < \alpha - 2$, D^{α}_{t} is defined as the Riemann–Liouville derivatives.

Dong & Yan (2018). Studied the existence and uniqueness of positive solutions for a singular nonlinear three point boundary value problem.

$$
\begin{aligned}
x''(t) + k(t)x^{-q}(t) = \lambda x^{p}(t),\ t \in [0,1],\\
x(0) = 0,\ x(1) = \alpha x(\eta),\ 0 < a < 1,\ 0 < \eta < 1,
\end{aligned}
$$

and x is positive parameters.

Motivated by the above mentioned results, in this paper, we investigate the existence and uniqueness of solutions for two point second order Sturm-Liouville boundary value problem by upper and lower solution method.

$$
-u'' + k^2u(t) = f(t, u(t), u^{'}(t)), \tag{1.1}
$$

$$
\begin{aligned}
au(0) - bu^{'}(0) = 0,\\
cu(1) + du^{'}(1) = 0,
\end{aligned} \tag{1.2}
$$

where $f : [0,1] \times (0,\infty) \times [0,\infty) \to R$, is a continuous function.

$a, c \in (0,\infty)$, $\quad b, d \in [0,\infty)$, $\quad k > 0$, $\quad$ by using upper and lower solution method.

1.2 Statement of the Problem

This study was focus on establishing the existence and uniqueness of solutions for two point second order Sturm Liouville boundary value problem by upper and lower solution method $(1.1) - (1.2)$.

1.3 Objective of the study

1.3.1 General Objective

The main objective of this study is to establish the existence and uniqueness of solutions for two point second order Sturm Liouville boundary value problems by upper and lower solution method $(1.1) - (1.2)$.

1.3.2 Specific Objectives

The specific objectives of the study:

- To define two point second order Sturm Liouville boundary value problem.
- To determine the existence of solution for two point boundary value problem.
- To determine the uniqueness of solution for two point boundary value problem.
- To give related the example

1.4 Significance of the Study

The result of this study may have the following importance:

- It will give a better understanding about research for the researchers.
- It may provide some background information for other researchers who want to conduct a research on related topics.
- Furthermore, this study would be useful for graduate program of the department and built the research skill and scientific of the researchers.

1.5 Delimitation of the Study

This study was delimited to finding the existence and uniqueness of solution for two point second order Sturm Liouville boundary value problem by applying upper and lower method solution.

2 LITERATURE REVIEW

Boundary value problems associated with linear as well as non-linear ordinary differential equations or finite difference equations have great deal of interest and play an important role in many fields of applied mathematics.

Different authors have proved the existence and uniqueness of solutions for boundary value problems by using different methods and conditions.

From those some of them are given below.

De Coster & Habets (2006). Upper and lower solutions with corners. Since then a multitude of variants have been introduced. For the boundary value problem.

$$\begin{aligned} u'' &= f(t, u, u'), \\ a_1 u(a) - a_2 u'(a) &= A, \\ b_1 u(b) + b_2 u'(b) &= B. \end{aligned} \tag{2.1}$$

Cabaniss (1974). Found existence and uniqueness conditions for the boundary value problem.

$$\begin{aligned} y'' &= f(x, y, y'), \\ a_o y(a) + a_1 y'(a) &= \alpha, \\ b_o y(b) + b_1 y'(b) &= \beta, \end{aligned} \tag{2.2}$$

using the sub function concept, they also showed that the unique solution $y(x, \alpha, \beta)$ of this boundary value problem depends continuously on the boundary data α and β.

Lian & Ge (2006). Studied the existence of at least one positive solution and multiple positive solutions for the two-point boundary value problem.

$$\begin{aligned} u''(t) + f(t, u(t)), 0 < t < 1, \\ \alpha u(0) - \beta u'(0) = 0, \\ \delta u(1) + \gamma u'(1) = 0. \end{aligned} \tag{2.3}$$

Cherpion et al. (2001). Studied α and β be lower and upper solutions of

$$\begin{aligned} u'(t) &= f(t, u(t)), \\ u(a) &= u(b), \end{aligned} \tag{2.4}$$

such that $\alpha \leqslant \beta$. Assume f is continuous on,

$(t, u(t)) \in [a, b] \times R;\ \ \alpha(t) \leqslant u(t) \leqslant \beta(t)$ and solution of the Cauchy problems

$u^{'}(t) = f(t, u(t)),\ \ u(a) = u_0,$ with $u_0 \in [\alpha(a), \beta(a)],$ is unique.

The problem $u^{'}(t) = f(t, u(t)),\ u(a = u(b))$ has at least one solution $u \in C^1(A, B)$such that for all $t \in [a, b],\ \ \alpha(t) \leqslant u(t) \leqslant \beta(t)$.

Bai et al. (2004). Studied lower solution a function $\alpha \in C^1([a, b])$ is a lower solution of the periodic problems.

$$\begin{aligned} u^{'}(t) &= f(t, u(t)), \\ u(a) &= u(b), \end{aligned} \tag{2.5}$$

where, $a < b$ and $f : [a, b] \times R$ is a continuous function.

If for all (a) $t \in [a, b],\ \alpha^{'}(t) \leqslant f(t, \alpha(t)),$

(b) $\alpha(a) \leqslant \alpha(b)$.

Bai et al. (2004). Studied upper solutions a function $\beta \in [a, b]$ is an upper solution of the periodic problems.

$$\begin{aligned} u^{'}(t) &= f(t, u(t)), \\ u(a) &= u(b), \end{aligned} \tag{2.6}$$

if (a) For all $t \in [a, b],\ \beta^{'}(t) \geq f(t, \beta(t))$ and

(b) $\beta(a) \geq \beta(b)$.

He et al. (2018). Studied the existence of solutions of the following BVPs.

$$\begin{aligned} y'' &= f(t, y, y^{'}) + e(t),\ 0 \leqslant t \leqslant 1, \\ y(0) &= 0,\ y^{'}(1) = \alpha y(\eta), \end{aligned} \tag{2.7}$$

where $\eta \in [0, 1],\ \alpha \in R,\ f : [0, 1] \times R^2 \to R$ is a continuous function.

Dobkevich et al. (2014). Solutions of the second order nonlinear boundary value problem,

$$\begin{aligned} x'' &= f(t, x, x^{'}), \\ a_1 x(a) + a_2 x^{'}(a) &= A, \\ b_1 x(b) + b_2 x^{'}(b) &= B, \\ L[Y] &= 0, \\ U_1[Y] &= 0, U_2(0) = 0, \end{aligned} \tag{2.8}$$

has non trivial solution.

Heidel (1974). Studied existence of solutions to second order problems with nonlinear boundary conditions.

$$u''(t) = f(t, u, u^{'}), \quad t \in [0, T], \ T \geq 0, \tag{2.9}$$

and

$u''(t) = f(t, u(t), u'(t)), \ t \in I$, Satisfying the conditions

$g(u(0), u^{'}(T)u^{'}(0)) = 0, \ u(T) + h(u(0)) = 0,$

where $f : I \times R \to R$ (or $f : I \times R \times R \to R$), $R^3 \to Rg : R$ and $h : R \to R$ are continuous functions.

Hu & Zhang (2017). Studied "upper and lower solution method for boundary value problems at resonance".

$$y''(t) = f(t, y, y^{'}), \ 0 < t < 1, \quad y^{'}(0) = 0, \ y^{'}(1) = 0, \tag{2.10}$$

where $f : [0, 1] \times R \to R$ is continuous and $y'' = f(t, y, y^{'}), \ 0 \leqslant t \leqslant 1,$

$y(0) = 0, \ y^{'}(1) = 0$. Where $f : [0, 1] \times R^2 \to R$ is continuous.

Alanazi et al. (2021). Studies Quasilinearization and boundary value problems at resonance.Consider the second order boundary value problem for the differential equation.

$$y''(t) = f(t, y(t)), \quad 0 \leqslant t \leqslant 1, \tag{2.11}$$

with homogeneous Neumann boundary condition $y^{'}(0) = 0, \ y^{'}(1) = 0$, where $f : [0, 1] \times R \times R$ is continuous.

The boundary value problem is at resonance, since constant functions are solutions of the homogeneous problem $y'' = 0$ and satisfy the boundary conditions.

We begin with the assumption that f: $y > 0$ on $[0, 1] \times R$, and obtain the result on the uniqueness of solutions.

Knobloch & Schmitt (1977). Studies uniqueness of solutions of boundary value problems at resonance.

Assume $f : [0,1] \times R^2 \to R \quad f:$ is continuous. Consider the boundary value problem

$$\begin{aligned} y''(t) = f(t, y(t), y^{'}(t)),\ 0 \leqslant t \leqslant 1,\ 0 \leqslant t \leqslant 1, \\ y(0) = 0,\ y^{'}(0) = y^{'}(1). \end{aligned} \tag{2.12}$$

The boundary value problem is at resonance since the linear functions,$y = ct, c \in R$, are solutions of the homogeneous problem $y^{''} = 0$ and satisfy the homogeneous boundary conditions.

Aniaku et al. (n.d.). Studies lower and upper solutions of second order non- linear ordinary differential equations.

Consider the boundary value problem.

$$\begin{aligned} u'' = f(t, u, u^{'}), \\ a_1 u(a) - a_2 u^{'}(a) = A, \\ b_1(b) + b_2 u^{'}(b) = B, \end{aligned} \tag{2.13}$$

where $F : [a, b] \times R^2$ is continuous.

Cabada (2011). Studies an overview of the lower and upper solutions method with nonlinear boundary value conditions.

The author considers the second-order boundary value problem

$$\begin{aligned} u''(t) = f(t, u(t), \\ u^{'}(t),\ t \in [a, b] = I,\ u(a) = A,\ u(b) = B, \end{aligned} \tag{2.14}$$

for $f : I \times R^2 \to R$ a continuous function and $A, B \in R$.

3 Methodology of the Study

This chapter contains study period and site, study design, source of information and mathematical procedures.

3.1 Study Site and period

The study was conducted from October 2021 to June 2022 in Jimma University under the department of mathematics.

3.2 Study Design

In order to achieve the objective of the study employed analytical method of design by using lower and upper solution method.

3.3 Source of Information

The relevant sources of information for this study were different mathematics books, published articles, journals and related studies from internet.

3.4 Mathematical Procedures

In this study we have followed the procedures stated below:

- Defining two point second order Sturm Liouville boundary value problem.
- Determining the existence of solution for two point second order Sturm Liouville boundary value problem by using upper and lower solution method.
- Determining the uniqueness of solution for two point second order Sturm Liouville boundary value problem by using upper and lower solution method.
- Give the related example.

4 RESULT AND DISCUSSION

4.1 Preliminaries

Definition 4.1.1 *Gladwell (2008). We say that* α *is a lower solution of* $x'' = f(t, \alpha(t), \alpha'(t))$ *on an interval* I *provided,*

$$\alpha''(t) \geq f(t, \alpha(t,)\alpha'(t)), \; for \; t \in I. \tag{4.1}$$

Similarly,we say that β *is an upper solution of*

$$\begin{aligned} x'' = f(t, x, x'), \quad & on \; an \; interval \; I \; provided, \\ \beta''(t) \leqslant f(t, \beta(t), \beta'(t)), \; & for \; t \in I. \end{aligned} \tag{4.2}$$

Definition 4.1.2 *Coddington & Levinson (1955). We say that* $T : X \to X$ *is a contraction mapping on a normed linear space* X *provided there is a constant* $\alpha \in (0, 1)$. *such that* $\|T_x - T_y\| \leqslant \alpha\|x - y\|$ *For all* $x, y \in X$.
We say $\bar{x} \in X$ *is a fixed point of* T *provided* $T\bar{x} = \bar{x}$.

Theorem 4.1.1 *Coddington & Levinson (1955).(Existence solution) Let* α *and* β *be* C^2 *-lower and upper solutions of the problem , Consider the periodic boundary value problem.*

$$\begin{aligned} u'' &= f(t, u, u'), \\ u(a) &= u(b), \\ u'(a) &= u'(b). \end{aligned} \tag{4.3}$$

The dependence of f in the derivative u' does not really change the definitions of lower and upper solutions. such that $\alpha \leqslant \beta$, E be defined in ,
$\phi : R^+ \to R$ be a positive continuous function and $F : E \to R$ be a continuous function. Then the problem (4.3) has at least one solution $u \in C^2([a, b])$ such that for all $t \in [a, b]$, $\alpha(t) \leqslant u(t) \leqslant \beta(t)$.

Theorem 4.1.2 *Coddington & Levinson (1955). (Contraction Mapping Theorem) If* T *is contraction mapping on a Banach space* x *with contraction constant* α *with* $0 <$

$\alpha < 1$, *then* T *has a unique fixed point* $\bar{x}$ *in* X *if* $x_0 \in X$ *and we set.*

$$\begin{aligned} x_{(n+1)} = Tx_n,\ for\ n \geq 0,\quad then \\ lim_{(n\to\infty)}x_n = \bar{x}, \\ then\ furthermore \\ \|x_n - x\| \leqslant \frac{a^n}{(1-n)}\|x_1 - x_0\|\ for,\quad n \geq 1. \end{aligned} \tag{4.4}$$

Theorem 4.1.3 *Cabada (2011). Assume* $f : [a,b] \times R \to R$ *is continuous and satisfies a uniform Lipschitz condition with respect to* x *on* $[a,b] \times R$ *with Lipschitz constant* K *; that is ,*

$$\begin{aligned} |f(t,x) - f(t,y)| \leqslant k|x-y|, \\ for\quad all\quad (t,x),(t,y) \in [a,b] \times R,\ if\ b-a \leqslant \frac{\pi}{\sqrt{k}}, \\ then\quad BVPs\quad x'' = f(t,x), \\ x(a) = A, \\ x(b) = B, \end{aligned} \tag{4.5}$$

has a unique solution.

Theorem 4.1.4 *Coddington & Levinson (1955). Assume* $f : [a,b] \times R \times R$ *is continuous and satisfies a uniform Lipschitz condition with respect to* x *and* $x^{'}$
$|f(t,x,x^{'}) - f(t,y,y^{'})| \leqslant k|x-y| + L|x^{'} - y^{'}|$ *for* $(t,x,x^{'}),\ (t,y,y^{'}) \in [a,b] \times R^2$,
where $L \geq 0,\ K \geq 0$ *are constants,*
if $K\frac{(b-a)^2}{8} + L\frac{(b-a)}{2} < 1$,
then the BVP $f(t,x,x^{'}),\ x(a) = A,\ X(b) = B,$ *has a unique solution.*

Theorem 4.1.5 *Aniaku et al. (n.d.). (Uniqueness Theorem) Assume that* $f : [a,b] \times R^2 \to R$ *is continuous and for each fixed,*
$(t,x^{'}) \in [a,b] \times R,$ $f(t,x,x^{'})$ *is strictly increasing with respect to x, then the BVP* $f(t,x,x^{'}),\ x(a) = A,\ x(b) = B$ *has at most one solution.*

Theorem 4.1.6 *Cherpion et al. (2001). If there exist* $\alpha(t)$ *and* $\beta(t)$ *upper and lower solutions respectively for the problem satisfying:*
$\beta(t) \leqslant \alpha(t)$ *and* $\beta''(t) \geq \alpha''(t)$

and if $f : [0,1] \times R \times R \to R$ *is continuous and satisfies*

$f(t, u_1, v) - f(t, u_2, v) \leqslant 0,$

for $\beta(t) \leqslant u_1 \leqslant \alpha(t), \quad v \in R, \quad t \in [0,1],$

$f(t, u_1, v) - f(t, u_2, v) \geq 0,$

for $\alpha'' \leqslant v_1 \leqslant v_2 \leqslant \beta"(t), \quad u \in R, \quad t \in [0,1].$

Then there exist two function sequences $\alpha_n(t)$ *and* $\beta_n(t)$ *that converge uniformly to the solutions of the boundary value problem.*

Theorem 4.1.7 *De Coster & Habets (2006). Assume that* α, β *are coupled lower and upper solutions for the problems.*

$$\begin{aligned} u'' &= f(t, u(t, u^{'}(t))),\ t \in I \quad satisfies\ condition, \\ (g(u(0), u^{'}(T), u^{'}(t))) &= 0, \\ u(T) + h(u(0)) &= 0, \end{aligned} \tag{4.6}$$

also assume that f satisfies a Nagumo condition relative to the interval$[\alpha, \beta]$. Suppose that g is no decreasing in the second variable. In addition, suppose that the function h in $[\alpha(0), \beta(0)]$ is monotone (either none increasing or non-decreasing) and that the functions.

$g_{\alpha(x)} = g((\alpha(0)), \alpha^{'}(0), x) \in R,$

$g_{\beta(x)} = g(\beta(0)\beta^{'}(0), x) \in R$, have got the same kind of monotonicity as h. Then there exists at least one solutions,

$u \in [a,b]\ of\ u''(t) = f(t, u(t), u^{'}(t)),\ t \in I$ of satisfies condition,

$g(u(0)), u^{'}(T), u^{'}(t)) = 0,$

$u(T) + h(u(0)) = 0,$

Moreover $-M < u^{'}(t) < M,\ t \in I.$

Theorem 4.1.8 *Coddington & Levinson (1955). Assume that* $f : [a,b] \times R^2 \to R$ *is continuous and bounded. Then the boundary value problem,*

$$\begin{aligned} x'' &= f(t, x, x^{'}), \\ x(a) &= A, \\ x(b) &= B, \end{aligned} \tag{4.7}$$

where A *and* B *constants has a solution .*

Definition 4.1.3 *De Coster & Habets (2006). We say that* $f : [a, b] \times R^2 \to R$ *satisfies a Nagumo condition with respect to the pair* $\alpha(t), \beta(t)$ *on* $[a, b]$ *provided* $\alpha, \beta : [a, b] \to R$ *are continuous* $\alpha(t) \leqslant \beta(t)$ *on* $[a, b]$ *and there is a function.*

$$h : [0, \infty) \to (0, \infty), \quad such \;\; that \; |f(t, x, x^{'})| \leqslant |(x^{'})|,$$
$$for \; all \; t \in [a, b], \quad \alpha(t) \leqslant x \leqslant \beta(t), \; x^{'} \in R$$
$$with \; \int_{\lambda}^{\infty} \frac{sds}{h(s)} > max_{a \leqslant t \leqslant b}\beta(t) \geq min_{a \leqslant t \leqslant b,}\alpha(t), \quad where$$
$$\lambda = max \left\{ \frac{|\beta(b) - \alpha(a)|}{b - a}, \frac{|\alpha(b) - \beta(a)|}{b - a} \right\}.$$

Theorem 4.1.9 *Coddington & Levinson (1955). Assume that* $f : [a, b] \times R^2 \to R$ *continuous and* α, β *are lower and upper solutions, respectively of* $x'' = f(t, x, x^{'})$ *on* $[a, b]$ *with* $\alpha(t) \leqslant \beta(t)$ *on* $[a, b]$ *.*
Further assume that f satisfies a Nagumo condition with respect to α, β *on* $[a, b]$ *assume* A, B *are constants satisfying*

$$\alpha(t) \leqslant A \leqslant \beta(t), \quad \alpha(t) \leqslant B \leqslant \beta(t);$$

then the BVP

$$\begin{aligned} x'' &= f(t, x, x^{'}), \\ x(a) &= A, x(b) = B, \end{aligned} \tag{4.8}$$

has a solution satisfying

$$\alpha(t) \leqslant x(t) \leqslant \beta(t), \quad for \;\; t \in [a, b].$$

Lemma 4.1.1 *Aniaku et al. (n.d.). Let there exist a constant* $M > 0$ *such that* $|f(t, u, u^{'})| \leqslant M$ *for all* $(t, u, u^{'}) \in I \times R^2$*. Then the boundary value problem*
$u'' = f(t, u, u')$,
$a_1u(a) - a_2u(a) = A$,
$b_1u(b) + b_2u(b) = B$.
Where $f : [a, b] \times R^2 \to R,$ *is continuous* $A, B \in R;\; a_1, b_1 \in R;\; a_2, b_2 \in R^+$
has a solution .

Theorem 4.1.10 *Aniaku et al. (n.d.). Assume* f *is continuous on* $[a, b] \times R$ *and* α *and* β *are lower and upper solution of* $x'' = f(t, x)$*, respectively with* $\alpha(t) \leqslant \beta(t)$ *on*

$[a, b]$.

$A\ and\ B\ are\ constants\ such\ that\ \alpha(a) \leqslant A \leqslant \beta(a),\ and\ \alpha(b) \leqslant B \leqslant \beta(b), then\ BVP$

$$\begin{aligned} x'' &= f(t, x), \\ x(a) &= A, \\ x(b) &= B, \end{aligned} \tag{4.9}$$

$$has\ a\ solution\ x\ satisfying\ \alpha(t) \leqslant x(t) \leqslant \beta(t),\ on\ [a, b].$$

4.2 The Existence of Solution

In this section, we study existence results of the BVP $(1.1)-(1.2)$, using the method of upper and lower solutions. We show that in the presence of lower and upper solutions, we recall the concept of lower and upper solution for the BVP.
Consider the two point boundary value problem.

$$
\begin{aligned}
-u'' + k^2 u &= f(t, u, u^{'}), \\
au(0) - bu^{'}(0) &= 0, \\
cu(1) + du^{'}(1) &= 0, \\
where\ f : [0,1] \times (0, \infty) \times [0, \infty) &\to R
\end{aligned}
$$

is continuous. $a, c \in (0, \infty), \quad b, d \in [0, \infty), k > 0.$
The interest of this study is existence of solution of $(1.1)-(1.2)$ by the method of lower and upper solution technique.

Definition 4.2.1 (lower solution) *A function $\alpha \in C^2[0,1]$ Will be called a lower solution of* $(1.1) - (1.2)$*, if*

$$
\begin{aligned}
\alpha'' \geq k^2\alpha - f(t, \alpha, \alpha^{'}),\ on\ [0,1], and \\
a\alpha(0) - b\alpha^{'}(0) \leqslant 0, \\
c\alpha(1) + d\alpha^{'}(1) \leqslant 0.
\end{aligned}
$$

Definition 4.2.2 (upper solution) *A function $\beta \in C^2[0,1]$ will be called an upper solution of* $(1.1) - (1.2)$*, if*

$$
\begin{aligned}
\beta'' \leqslant k^2\beta - f(t, \beta, \beta^{'}),\ on\ [0,1], \quad and \\
a\beta(0) - b\beta^{'}(0) \geq 0, \\
c\beta(1) + d\beta^{'}(1) \geq 0.
\end{aligned}
$$

Some Conditions;

Nagumo condition (Xiping Lim ,2018);let $h; R^+ \to R$ be positive continuous function satisfying

$\int_0^\infty \frac{s}{h(s)} ds = \infty.$

Then function $f : E \to R$ is said to be satisfying a Nagumo condition if

$|f(t,u,u^{'})| \leqslant h(|u^{'}|),\ for\ all\ (t,u,u^{'}) \in E.$

$E := \{(t,u,u^{'}) \in [0,1] \times (0,\infty) \times [0,\infty);\ \alpha(t) \leqslant u(t) \leqslant \beta(t)\}.$

And other conditions:

$(A_1) :$ $f(t,u,u^{'})$ satisfies a Lipschitz condition with respect to u $u^{'}$ on the set of E.

$(A_2) :$ $f(t,u,u^{'})$ satisfies a Nagumo condition on the set of E.

$(A_3) :$ For any $(t_0,u_0,u_0^{'}) \in E$, the solution of $(1.1)-(1.2)$ satisfies the initial condition $u(t_0) = u_0,\ u^{'}(t_0) = u_0^{'}$ is unique.

Theorem 4.2.1 *Suppose there exist a lower solution $\alpha(t)$ and upper solution $\beta(t)$ of* $(1.1)-(1.2)$ *such that $\alpha(t) \leqslant \beta(t)$ for all $t \in [0,1]$, and the condition A_1 andA_2 hold, then there exist at least one solution $u(t)$ of* $(1.1)-(1.2)$ *satisfying,*

$\alpha(t) \leqslant u(t) \leqslant \beta(t).$

Proof: Define the function $k^2u - F(t,u,u^{'})$ on $[0,1] \times (0,\infty) \times [0,\infty) \to R.$ by setting

$$k^2u - F(t,u,u^{'}) = \begin{cases} k^2\beta - f(t,\beta,u^{'}) + \frac{u-\beta(t)}{1+u^2}, & if \quad u > \beta(t), \\ k^2u - f(t,u,u^{'}), & if \quad \alpha(t) \leqslant u(t) \leqslant \beta(t), \\ k^2\alpha - f(t,\alpha,u^{'}) + \frac{u-\alpha(t)}{1+u^2}, & if \quad u < \alpha(t). \end{cases}$$

Since f is bounded, F is also bounded . Hence by lemma 4.1.1, there exist a solution $u(t)$ of $(1.1)-(1.2)$. We now show that $(t,u,u^{'}) \in E$.

Let $E := \{(t,u,u^{'}) \in [0,1] \times (0,\infty) \times [0,\infty)\};\ \alpha(t) \leqslant u(t) \leqslant \beta(t).$

Which of course mean that $u(t)$ is solution of $(1.1)-(1.2)$. Assume $u(t) > \beta(t)$ on $t \in [0,1]$.

Then there exist $0 \leqslant t_1 < t_2 \leqslant 1$ such that $u(t_1) = \beta(t_i),\ i = 1,2$ and $u(t) > \beta(t), t_1 < t < t_2$.

The difference $u(t) - \beta(t)$ therefore will assume a positive maximum at a point $t_0,\quad t_1 < t_0 < t_2$.

We see $u^{'}(t_0) = \beta^{'}(t_0)$ and $u''(t_0) - \beta''(t_0) = 0$.

But a computation, however show that

$$u''(t) - \beta''(t) \geq (k^2 u(t_0) - F(t_0, u(t_0), u^{'}(t_0))) - (k^2\beta(t_0) - F(t_0), \beta(t_0), \beta^{'}(t_0))$$

$$\begin{aligned}
u''(t_0) - \beta''(t_0) &\geq k^2\beta(t_0) - f(t_0, \beta(t_0), u^{'}(t_0)) + \frac{u(t_0) - \beta(t_0)}{1 + u^2(t_0)} - (k^2\beta(t_0) - f(t_0, \beta(t_0), u^{'}(t_0))) \\
&\geq k^2\beta(t_0) - f(t_0, \beta(t_0), u^{'}(t_0)) - k^2\beta(t_0) + f(t_0, \beta(t_0), u^{'}(t_0)) + \frac{u(t_0) - \beta(t_0)}{1 + u^2(t_0)} \\
&> 0 + \frac{u(t_0) - \beta(t_0)}{1 + u^2(t_0)} \\
&= \frac{u(t_0) - \beta(t_0)}{1 + u^2(t_0)} > 0.
\end{aligned}$$

Which is contradiction.

Next we show that $\alpha(t) \leqslant u(t), \quad t \in [0,1]$ we assume by contradiction that $u(t) < \alpha(t), \ t \in [0,1]$, then there exist points $0 \leqslant t_1 < t_2 \leqslant 1$, such that $u(t) = \alpha(t_i), \ i = 1, 2$ and $u(t) < \alpha(t), t_1 < t < t_2$.

The difference $u(t) - \alpha(t)$ therefore we will assume a negative maximum value at a point t_0,

$$\begin{aligned}
t_1 < t_0 < t_2, \quad & and \\
u^{'}(t_0) = \alpha^{'}(t_0)&, \\
u''(t_0) - \alpha''(t_0) \geq 0&.
\end{aligned}$$

A computation, however show that

$$\begin{aligned}
u''(t_0) - \alpha''(t_0) &= (k^2 u - F(t_0, u(t_0), u^{'}(t_0))) - (k^2\alpha(t_0) - F(t_0, \alpha(t_0), \alpha^{'}(t_0))) \\
&\leqslant (k^2\alpha(t_0) - f(t_0, \alpha(t_0), \alpha'(t_0))) + \frac{u(t_0) - \alpha(t_0)}{1 + u^2(t_0)} - (k^2\alpha(t_0,) - f(t_0, \alpha(t_0), \alpha^{'}(t_0))) \\
&\leqslant (k^2\alpha(t_0) - k^2\alpha(t_0) - f(t_0, \alpha(t_0), \alpha^{'}(t_0))) + f(t_0, \alpha(t_0), \alpha^{'}(t_0)) + \frac{u(t_0) - \alpha(t_0)}{1 + u^2(t_0)} \\
&= 0 + \frac{u(t_0) - \alpha(t_0)}{1 + u^2(t_0)} < 0.
\end{aligned}$$

This also contradiction. These show that $\alpha(t) \leqslant u(t) \leqslant \beta(t)$.

So, with regards to the above theorem ,we shall interested in the existence of such lower solution $\alpha(t)$ and upper solution $\beta(t)$. $\square$

4.3 The Uniqueness of Solution

Consider the two point boundary value problem (1.1) – (1.2)

$$-u'' + k^2u = f(t,u,u'),$$
$$au(0) - bu^{'}(0) = 0,$$
$$cu(1) + du^{'}(1) = 0.$$

Definition 4.3.1 *Assume α and β are continuous function on $[0,1]$ with $\alpha(t) \leqslant \beta(t)$ on $[0,1]$, and assume $c > 0$, is a given constant ;then we say that $k^2u - F(t,u,u^{'})$ is modification of $k^2 - f(t,u,u^{'})$ associated with the triple $\alpha(t), \beta(t), c$ provided,*

$$k^2u - F(t,u,u^{'}) = \begin{cases} k^2\beta - g(t,\beta(t),u^{'}) + \frac{u-\beta(t)}{1+|u|}, & if \quad u \geq \beta(t), \\ k^2u - g(t,u,u^{'}), \quad if \quad \alpha(t \leqslant u(t) \leqslant \beta(t)), \\ k^2\alpha - g(t,\alpha(t),u^{'}(t)) + \frac{u-\alpha(t)}{1+|u|}, & if \quad u \leqslant \alpha(t). \end{cases}$$

where

$$k^2u - g(t,u,u^{'}) = \begin{cases} k^2u - f(t,u,c), & if \quad u^{'} \geq c, \\ k^2 - f(t,u,u^{'}), & if \quad |u^{'}| \leqslant c, \\ k^2u - f(t,u,-c), & if \quad u^{'} \leqslant -c. \end{cases}$$

F are continuous on $[0,1] \times (0,\infty) \times [0,\infty)$.
F is bounded on $[0,1] \times (0,\infty) \times [0,\infty)$,
$k^2u - F(t,u,u^{'}) = k^2u - f(t,u,u^{'})$, if $t \in [0,1]$, $\alpha(t) \leqslant u \leqslant \beta(t)$, *and* $|u^{'}| \leqslant c$.
Consider the two point boundary value problem (1.1) – (1.2)

$$-u'' + k^2u = f(t,u,u'),$$
$$au(0) - bu^{'}(0) = 0,$$
$$cu(1) + du^{'}(1) = 0,$$

where a continuous function on $[0,1] \times (0,\infty) \times [0,\infty)$ by upper and lower solution method.

Theorem 4.3.1 *Assume f is continuous on $[0,1] \times (0,\infty) \times [0,\infty)$ and α and β are lower and upper solution of* (1.1) *respectively with $\alpha(t) \leqslant \beta(t)$ on $[0,1]$. Further assume* *that solution of initial value problem for* (1.1) *are unique.*

If there is $t_0 \in [0,1]$, *such that*

$$\alpha(t_0) = \beta(t_0),$$
$$\alpha^{'}(t_0) = \beta^{'}(t_0), \ \textit{then}$$
$$\alpha(t) \equiv \beta(t), \quad \textit{on} \quad [0,1].$$

Proof: Assume $\alpha(t), \beta(t), t_0$ are as in the statement of this theorem and it is not that $\alpha(t) \equiv \beta(t)$ on $[0,1]$.

We will only consider case where there are points t_1, t_2 such that $0 \leqslant t_0 \leqslant t_1 < t_2 \leqslant 1$,

$\alpha(t_1) = \beta(t_1)$,

$\alpha^{'}(t_1) = \beta^{'}(t_1)$, and

$\alpha(t) < \beta(t)$, on t_1, t_2.

Pick $c > 0$, so that $|\alpha^{'}(t)| < c, \quad |\beta^{'}(t)| < c$.

For $t \in [t_1, t_2]$. Let F be the modification of f with respect to the triple α, β, c for the interval $[t_1, t_2]$. By theorem 4.1.8 the boundary value problem

$-u'' + k^2 u = F(t, u, u^{'}), \quad u(t_1) = \alpha(t_1), \quad u(t_2) = u_{2,}$

where $\quad \alpha(t_2) < u_2 < \beta(t_2) \quad$ has a solution u.

We claim that $u(t) \leqslant \beta(t)$ on $[t_1, t_2]$. Assume not, then there is a $d \in (t_1, t_2)$ such that $w(t) := u(t) - \beta(t)$ has a positive maximum on $[t_1, t_2]$ at d. It follow that $w(d) > 0, \ w^{'}(d) = 0, \ w''(d) \leqslant 0$, and so

$$u(d) > \beta(d), u^{'}(d) = \beta^{'}(d), u''(d) \leqslant \beta''(d).$$
$$\textit{But} \quad w''(d) = u''(d) - \beta''(d)$$
$$\geq (k^2 u(d)) - F(d, u(d), u^{'}(d)) - (-f(d, \beta(d), \beta^{'}(d))) + k^2 \beta(d)$$
$$= k^2 \beta(d) - f(d, \beta(d), \beta^{'}(d)) + \frac{u(d) - \beta(d)}{1 + |\beta(d)|} - k^2 u(d) + f(d, \beta(d), \beta^{'}(d))$$
$$= \frac{u(d) - \beta(d)}{1 + |\beta(d)|} > 0.$$

Which is contradiction.

Hence $\quad u(t) \leqslant \beta(t)$ on $[t_1, t_2]$.

Similarly, $\quad \alpha(t) \leqslant u(t), \quad$ on $\quad [t_1, t_2]$.

We claim that $\alpha(t) \leqslant u(t)$ on $[t_1, t_2]$. Assume not; then there is a $d \in (t_1, t_2)$ such that

$w(t) := u(t) - \alpha(t)$ therefore we will assume a negative maximum on a point $[t_1, t_2]$ at d, it follow that

$w(d) > 0,\ w^{'}(d) = 0,\ w''(d) \leqslant 0.$

and so $u(d) > \alpha(d),\ u^{'}(d) = \alpha^{'}(d),\ u''(d) = \alpha''(d).$

But

$$\begin{aligned} w'' &= u''(d) - \alpha''(d) \\ &\leqslant k^2 u(d) - F(d, u(d), u^{'}(d)) - (k^2 u(d) - f(d, \alpha(d), \alpha^{'}(d))) \\ &= k^2 \alpha(d) - f(d, \alpha(d), \alpha^{'}(d)) + \frac{u(d) - \alpha(d)}{1 + |\alpha(d)|} - (k^2 \alpha(d) - f(d, \alpha(d), \alpha^{'}(d))) \\ &= \frac{u(d) - \alpha(d)}{1 + |\alpha(d)|} > 0. \\ & \textit{Which is contradiction. Hence } \alpha(t) \leqslant u(t) \textit{ on } [t_1, t_2], \end{aligned}$$

thus $\alpha(t) \leqslant u(t) \leqslant \beta(t)$, on $[t_1, t_2]$. Pick $t_3 \in [t_1, t_2]$,

so that $u(t_3) = \alpha(t_3),\ u^{'}(t_3) = \alpha^{'}(t_3)$, and $\alpha(t) < u(t)$, $\alpha(t) \leqslant \beta(t)$, on $(t_3, t_2]$.

Since $|u^{'}(t_3)| = |\alpha^{'}(t_3)| < c$, we pick $t_4 \in (t_3, t_2)$, so that $|x^{'}| < c$, on $[t_3, t_4]$.

Then u is a solution of (1.1) on $[t_3, t_4]$ and hence u is upper solution of (1.1) on $[t_3, t_4]$.

Let F_1 be the modification of f with respect the triple α, u, c for the interval $[t_3, t_4]$.

Let $u_4 \in (\alpha(t_4), u(t_4))$.

Then by the theorem (4.1.8) the boundary value problem

$$\begin{aligned} -u + k^2 u &= F_1(t, u, u^{'}), \\ u(t_3) &= \alpha(t_3), \\ u(t_4) &= u_4, \\ \textit{has a solution } \ y & \textit{ on } [t_3, t_4], \end{aligned}$$

by a similar argument we can show that $\alpha(t) \leqslant y(t) \leqslant u(t)$, on $[t_3, t_4]$.

Now we can pick $t_5 \in (t_3, t_4)$. So that u and y differ at some points in $[t_3, t_5]$, and $|y^{'}(t)| < c$, for $t \in [t_3, t_5]$,

then y is a solution of (1.1) on $[t_3, t_5]$. But now we have u and y are distinct solution of the same IVP(same initial condition at t_3) which contradict the uniqueness.

□

4.4 Related Examples

Example 4.4.1 *Consider the problem,*

$$u'' = -u - tu^2;\ t \in [0,1],$$
$$2u(0) - 2u^{'}(0) = 0,$$
$$3u^{'}(1) = 0.$$

We not that in this case ,we can define $g(t) = -k^2u - f(t,u,u^{'}) = -u - tu^2$

$$g(t,u,u^{'}) = -u - tu^2.$$

Since $-u - tu^2$ *is decreasing function. We see that*

$(-u - tu^2) \leqslant 1,\quad for\ all\ t \in [0,1]$. *So for upper solution* $\beta(t)$ *, we look for solution to the boundary value problem.*

$$u'' = -1,$$
$$2u(0) - 2u^{'}(0) = 0,$$
$$3u^{'}(1) = 0.$$

Now we have to find $\beta(t)$ *upper solution .*

To get this by applying integration for $\beta''(t) = -1$ *with respect to* t.

$$\int \beta''(t)dt = \int -1dt$$
$$\Rightarrow \beta^{'}(t) = -t + 1$$

and also integrating both side to get $\beta(t)$ *with respect to* t *.*

$$\int \beta^{'}(t)dt = \int (-t+1)dt,$$
$$\Rightarrow \beta(t) = -\frac{t^2}{2} + t + 1,$$

which satisfies the given boundary value problem .

Next, we have to find the lower solution $\alpha(t)$*, for lower solution* $\alpha(t)$ *we look for solution of the boundary value problem.*

$$
\begin{aligned}
u'' &= 1,\\
2u(0) - 2u^{'}(0) &= 0,\\
3u^{'}(1) &= 0.
\end{aligned}
$$

Hence,

$u'' = 1.$

Now, here apply the integration both side to get $\alpha(t)$ *with respect to* t.

$$
\begin{aligned}
\int \alpha''(t)dt &= \int 1dt,\\
\alpha^{'}(t) &= t - 1.
\end{aligned}
$$

Again integrating both side with respect to t *in order to get* $\alpha(t)$

$$
\begin{aligned}
\int \alpha(t)dt &= \int (t-1)dt\\
\alpha(t) &= \frac{t^2}{2} - t - 1.
\end{aligned}
$$

This also satisfies the given boundary condition so there is a lower and upper solution . Therefore it satisfies the existence for the given boundary condition.

Note that

$$\alpha(t) = \frac{t^2}{2} - t - 1,$$

and

$$\beta(t) = \frac{-t^2}{2} + t + 1,$$

are upper and lower solution respectively for the given problem satisfying $\alpha(t) \leqslant \beta(t)$ *on* $[0,1]$.

Assume that A *and* B *are constants satisfying*

$$
\begin{aligned}
\alpha(t) &\leqslant A \leqslant \beta(t)\\
\Rightarrow \frac{t^2}{2} - t - 1 &\leqslant A \leqslant \frac{-t^2}{2} + t + 1\\
\alpha(0) = -1 &\leqslant A \leqslant 1 = \beta(0).
\end{aligned}
$$

And

$$\begin{aligned}
\alpha(t) &\leqslant B \leqslant \beta(t),\\
\Rightarrow \frac{t^2}{2} - t - 1 &\leqslant B \leqslant \frac{-t^2}{2} + t + 1,\\
\alpha(1) &\leqslant B \leqslant \beta(1)\\
\alpha(1) = \frac{1^2}{2} - 1 - 1 &\leqslant B \leqslant \frac{-1^2}{2} + 1 + 1 = \beta(1)\\
\alpha(1) = \frac{1}{2} - 2 &\leqslant B \leqslant \frac{-1}{2} + 2 = \beta(1)\\
\alpha(1) = \frac{-3}{2} &\leqslant B \leqslant \frac{3}{2} = \beta(1).
\end{aligned}$$

Therefore by using upper and lower solution method there exist a solution for given problem satisfying boundary condition

$$\alpha(t) \leqslant u(t) \leqslant \beta(t), \;\; for \; t \in [0, 1].$$

Finally,

$$\Rightarrow \frac{t^2}{2} - t - 1 \leqslant u(t) \leqslant \frac{-t^2}{2} + t + 1.$$

5 CONCLUSION AND FUTURE WORK

5.1 Conclusion

The use of lower solution $\alpha(t)$ and upper solution $\beta(t)$ in the search of solution $u(t)$ of nonlinear second order ordinary differential equation (ODE) is possible provided such function satisfies the Existence and Uniqueness of solution is also determined by upper and lower method solution.

Lower solution $\alpha(t)$ and upper solution $\beta(t)$ of second order ODE help a lot in the search of the solution $u(t)$ of such equation, provided $\alpha(t) \leqslant u(t) \leqslant \beta(t)$ for all $t \in [0, 1]$.

5.2 Future Work

In this work we gave focus to solving second order Sturm Lioviolle boundary value problems using the method of upper and lower solutions further researchers can be done such problems using other methods as contraction mapping theorem ,by using Greens function, Lipschitz condition. In addition higher order Sturm Lioville boundary value problems can be researched on such method of upper and lower solution.

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