Common Fixed Point Theorems for a Pair of Contractive Mappings in Partially Ordered b-Metric Space Involving Simulation Functions

A RESEARCH SUBMITTED TO THE DEPARTMENT OF MATHEMATICS IN PARTIAL FULFILLMENT FOR THE REQUIREMENTS OF THE DEGREE OF MASTERS OF SCIENCE IN MATHEMATICS

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Declaration

I, the undersigned declare that, this research paper entitled common fixed point theorems for a pair of contractive mappings in partially ordered b- metric space involving simulation functions is my own original work and it has not been submitted for the award of any academic degree or the like in any other institution or university, and that all the sources. I have used or quoted have been indicated and acknowledged.

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Abstract

In this research, work we introduced common fixed point theorems for a pair of contractive mappings in partially ordered b-metric spaces involving simulation functions. Our results extend and generalize related fixed point results in the literature, in particular that of Rao et al. (2020). In this research undertaking, we followed analytical study design and used secondary sources of data, such as published articles and related books. Finally, we provided examples in support of our main findings.

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Contents

Chapter 1

Introduction

Notation We need the following symbols and class of functions to prove certain results in this section:

- $\Re^+ = [0, \infty)$;
- $\Re$ is the set of all real numbers;
- $\mathbb{N}$ is the set of all natural numbers;
- Z is the set of all simulation functions;
- $\Re^+ \times \Re^+ \to \Re$ is a simulation function.

1.1 Background of the study

Fixed point theory is a fundamental tool in nonlinear analysis and many other branches of modern mathematics, in particular when we deal with the solvability of a certain functional equation. This theory has many applications, particularly in biology, chemistry, economics, game theory, optimization theory, physics; etc.

Banach contraction principle is the first important result on fixed point theory. The famous Banach contraction principle introduced by Banach (1922) ensures the existence and uniqueness of fixed points for a contraction mapping in a complete metric spaces. Several researchers generalized and extended this principle by introducing various contractions in different ambient spaces. The contraction mapping principle introduced by Banach in 1922 has wide range of application in the fixed point theory. Due to its usefulness Banach contraction principle has been extended and generalized in various spaces using different conditions either by modifying the basic contractive condition or by generalizing the ambient spaces or both.

Khojasteh et al. (2015) introduced the notion of Z-contraction, that is, a nonlinear contraction involving a new class of maps namely simulation functions. They studied the existence and uniqueness of fixed points for $Z-$ contraction type operators. This class of Z-contractions includes large types of nonlinear contractions existing in the literature. That is, they introduced the notion of simulation function in order to express different contractive conditions in a simple, unified way. Thus, it is possible to treat several fixed point problems from a unique, common point of view. Roldan-Lopez-de-Hierro et al. (2015) stretched out fixed point theorems given in Khojasteh et al. (2015) to the coincidence point results. Consequently, they illustrated an iterative technique to find a solution to the equation $Tx = gx$, where T and g are self-maps. As an outcome, the authors explored the existence and uniqueness of coincidence points of two given mappings defined on complete metric spaces employing the preconceived notion of simulation function, consisting of the case of compatible mappings. The authors revised the definition of simulation function to a certain extent. Later, Radenovic and Chandok (2018) obtained some sufficient conditions for the existence and uniqueness of point of coincidence by using simulation functions in the context of metric spaces and prove some results for generalized contractions. In 2018, Babu et al. introduced generalized $Z-$ contraction pair of maps with respect to a simulation function ζ in b-metric spaces and studied the existence of common fixed points of such mappings in complete b-metric spaces. In 2021, Alqahtani et al. combined the notions of simulation functions and Proinov type contraction to get a more general framework to guarantee the existence of a fixed point. They investigated common fixed point results of new types mapping under this construction in the context of complete metric spaces. In 2020, Rao et al. establish some coincidence and common fixed point theorems for monotone f-non decreasing self mappings satisfying certain rational type contraction in the context of a metric spaces endowed with partial order.

The purpose of this study is to prove some coincidence and common fixed point results in the frame work of partially ordered b-metric spaces for a pair of self-mappings satisfying a generalized contractive condition of rational type with simulation function by extending the works of Rao et al. (2020). Our results improved, extended and generalized several related fixed point results in the existing literatures.

1.2 Statements of the problem

In 2013, Chandok established some common fixed point results for $f-$ nondecreasing mappings which satisfy some nonlinear contractions of rational type in the framework of metric spaces endowed with a partial order. In 2020, Rao et al. established some coincidence and common fixed point theorems for T monotone f-non decreasing self mappings satisfying certain rational type contraction in the context of a metric spaces endowed with partial order. In this research work, we concentrated in establishing and proving the existence and uniqueness of common fixed point results for a pair of self-maps satisfying certain rational type contractive condition involving a Simulation function in the setting of partially ordered b-metric space so as to extend and generalize the works of Rao et al. (2020).

1.3 Objectives of the study

1.3.1 General objective

The general objective of this research work is to study common fixed point results for a pair of self-maps satisfying a rational type contractive condition involving simulation function in the setting of partially ordered b-metric spaces.

1.3.2 Specific objectives

This study has the following specific objectives:

- To prove the existence of common fixed points for a pair of contractive mappings involving simulation function in the setting of partially ordered b-metric spaces.

- Showing uniqueness of common fixed points for a pair of contractive mappings involving simulation function in the setting of partially ordered b-metric spaces.

- To Provide examples in support of the results obtained.

1.4 Significance of the study

The study may have the following importance:

- It may give basic research skills to the researcher.
- It may be used as a reference for any researcher who has an interest in doing research in this area of study.
- It may be applied to show existence and uniqueness of solution of some problems involving integral and differential equations.

1.5 Delimitation of the Study

This study focuses only on establishing common fixed point theorems for pair of self-maps satisfying a rational type contractive condition involving simulation function in the setting of partially ordered b-metric spaces.

Chapter 2

Review of Related Literatures

Fixed point theory has fascinated many researchers since 1922 with the celebrated Banach fixed point theorem. According to Banach, every contraction map of a complete metric space into itself has a unique fixed point. In almost all scientific disciplines, most of the problems/mathematical models can be converted into fixed point equations in order to prove the existence and uniqueness of their solutions with the aid of fixed point theorems. It also suggests a numerical algorithm for computing solutions: take a guess at a fixed point, and then repeatedly apply the function. Due to its usefulness Banach contraction principle has been extended and generalized in various spaces using different conditions either by modifying the basic contractive condition or by generalizing the ambient spaces or both. there are so many different types of metric fixed point results.

In 2015 Khojasteh et al. introduced the notion of Z-contraction, that is, a nonlinear contraction involving a new class of maps namely simulation functions. They studied the existence and uniqueness of fixed points for Z-contraction type operators. This class of Z-contractions includes large types of nonlinear contractions existing in the literature. That is, they introduced the notion of simulation function in order to express different contractive conditions in a simple, unified way.The main idea of the simulation function is very simple, but also very useful and effective. By letting $d(x,y) = u$ and $d(Tx,Ty) = r$, the corresponding simulation function for Banachs fixed point theorem $\zeta(u,r) = ku - r$ where $K \in [0,1)$. In other words, simulation function can be considered a generator of different contraction type inequalities Z-contraction generalize the Banach contraction and unify several known type of contraction involving the combination of $d(Tx,Ty)$ and $d(x,y)$ satisfies some particular condition in complete metric space. The advantage of this notion is in providing a unique point of view for several fixed point problems. It is clear that for many other well-known results, one can find a corresponding simulation function. In other words, simulation function can be considered a generator of different contraction type inequalities. After this remarkable result several authors extended and generalized in many directions.

Several research works have done to investigate fixed point results by carrying forward the notion of simulation functions blending with the ideas of $\alpha-$ admissibility, lower semi-continuity, wt-distance mappings, almost contraction, and hybrid contraction. In 2016, Olgun et al. introduced the concept of generalized $Z-$ contraction on metric spaces by modifying the contractive condition of Khojasteh et al. (2015) and proved a fixed point theorem for this contraction. Isik et al. (2018) investigated the existence and uniqueness of a fixed point of almost contractions via simulation functions in metric spaces. Moreover, they provided an application to integral equations. Aydi et al. (2018), unified several fixed point results in the set-up of a quasi-metric space by the help of both simulation functions and admissible mappings. Melliani et al. (2020) introduced a new concept of $\alpha-$ admissible almost type $Z-$ contraction and proved the existence of fixed points for admissible almost type $Z-$ contractions in a metric space. Chifu and Karapinar (2019) introduced a new type of contraction, namely admissible hybrid contraction, in order to unify several linear, nonlinear and interpolative contractions in the set-up of metric and b-metric spaces. Moreover, they unified several existing results in the literature by combining the interesting notions: admissible mappings, simulation functions, and hybrid contractions in the setting of a b-metric space.

Roldn-Lpez-de-Hierro et al. (2015) stretched out fixed point theorems given in Khojasteh et al. (2015) to the coincidence point of view. Consequently, they illustrated an iterative technique to find a solution to the equation Tx = gx, where T and g both are self-maps. As an outcome, the authors explored the existence and uniqueness of coincidence points of two given mappings defined on complete metric spaces employing the preconceived notion of simulation function for a pair of compatible mappings. The authors revised the definition of simulation function to a certain extent. Later, Radenovic and Chandok (2018) obtained some sufficient conditions for the existence and uniqueness of point of coincidence by using simulation functions in the context of metric spaces and prove some results for generalized contractions. In 2021, Alqahtani et al. combined the notions of simulation functions and Proinov type contraction to get a more general framework to guarantee the existence of a fixed point. They investigated the common fixed point of new types mapping under this construction in the context of complete metric space.

Chapter 3

Methodology

In this section, we present the study area and period, mathematical study design, sources of information, and the mathematical procedures.

3.1 Study area and period

The study was conducted from September, 2019 to Apr, 2022 G.C in Jimma University under Research and Post Graduate Coordinating Office, College of Natural Sciences.

3.2 Study Design

In this study, we followed analytical method of design.

3.3 Source of Information

The relevant sources of information are secondary data such as published articles, different mathematics books related to the research topic, and M.Sc thesis works available in department of mathematics.

3.4 Mathematical Procedure of the Study

In this study, we followed the procedures stated below:

- Establishing common fixed point theorems.
- Constructing sequences.
- Showing the constructed sequences are b-Cauchy .
- Showing the b-convergence of the sequences.

- Proving the existence of common fixed points.
- Showing uniqueness of common fixed points.
- providing examples in support of our main findings.

Chapter 4

Preliminaries and Main Result

4.1 Preliminaries

Definition 4.1.1 *(Czerwik, 1993). Let X be a (nonempty) set and $s \geq 1$ be a given real number. A function $d : X \times X \to \Re^{+}$ is said to be a b-metric if for all $x, y, z \in X$, the following conditions are satisfied:*
(a) $d(x,y) = 0$ if and only if $x = y$;
(b) $d(x,y) = d(y,x)$;
(c) $d(x,z) \leq s[d(x,y) + d(y,z)]$.
The pair (X,d) is called a b-metric space.
It should be noted that, the class of b-metric spaces is effectively larger than that of metric spaces, But, in general, the converse is not true.

Example 4.1.1 *Let $X = \Re$ (the set of real numbers) define $d : X \times X \to \mathbb{R}^{+}$by $d(x,y) = |x-y|^2$. Thend is a b-metric with S=2 and the pair $(X,d)$is a b-metric space. but it d is not a metric on X since for $x = 3, y = 5$, and $z = 7$, we get $d(3,7) \nleq d(3,5) + d(5,7)$.*
Hence the triangle inequality for a metric does not hold.

Definition 4.1.2 *(Boriceanu et al., 2010). Let X be a b-metric space and $\{x_n\}$ be a sequence in X, we say that:*

1. *$\{x_n\}$ is b-converges to $x \in X$ if $d(x_n, x) \to 0$ as $n \to \infty$.*
2. *$\{x_n\}$ is a b-Cauchy sequence if $d(x_n, x_m) \to 0$ as $n, m \to \infty$.*
3. *(X,d) is b-complete if every b-Cauchy sequence in X is b-convergent.*

Definition 4.1.3 *A partially ordered set (poset) is a system $(X, \preceq)$ where X is anonempty set and is a binary relation of X satisfying for all $x, y, z \in X$;*
(a) $x \preceq x$ (reflexive);
(b) if $x \preceq y$ and $y \preceq x$ then $x = y$ (ant symmetry);
(c) ifx $\preceq y$ and $y \preceq z$ then $x \preceq z$ (transitive).

Example 4.1.2 *If X is any set $(P(X), \subseteq)$ is a partially ordered set, where $P(X) =$ the power set of X.*

Definition 4.1.4 *Let X is a non-empty set. Then $(X, d, \preceq)$ is called partially ordered $b-$ metric spaces if*
(a) (X, d) is a $b-$ metric space;
(b) $(X, \preceq)$ is partially ordered set.

Definition 4.1.5 *(Rao and Kalyeni, 2020). Let $(X, \preceq)$ be a partially ordered set and $T : X \to X$ is a self- mapping, we say T is monotone f non-decreasing with respect to $\preceq$ if for $x, y \in X$, $fx \preceq fy \Longrightarrow Tx \preceq Ty$.*

Definition 4.1.6 *(Rao et al., 2020). Let $(X \preceq)$ be a partially ordered set and $x, y \in X$ then x and y are said to be comparable elements of X if $x \preceq y$ or $y \preceq x$.*

Definition 4.1.7 *Let A be a non-empty subset of partially ordered set $(X, \preceq)$. If every two elements of A are comparable, then it is called well ordered set.*

Definition 4.1.8 *(Rao et al., 2020). Two self-mapping T and f defined over a subset A of a metric space (X, d) are called commuting if $fTx = Tfx$ for all $x \in A$.*

Example 4.1.3 *Let $X = \Re^+$ and $d : X \times X \to \Re^+$ be defined by*
$d(x, y) = |x - y|$ for all $x, y \in X$.Define $f, T : X \to X$ by
$fx = x + 1$ and $Tx = x + 2$ for all $x, \in X$.
It is easy that to see that the pair (f, T) is committing.

Definition 4.1.9 *(Rao et al., 2020). Two self-mapping T and f defined over $A \subset X$ are said to be compatible,if for any sequence $\{x_n\}$ with $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} Tx_n = \mu$ for some$\mu \in A$,then*

$$\lim_{n\to\infty} d(fTx_n, Tfx_n) = 0.$$

Definition 4.1.10 *Let $f, T : X \to X$be two given mappings. We say that $x \in X$.*

1. *is a coincidence point of f and T if $fx = Tx$;*
2. *is a common fixed point of f and T if $x = fx = Tx$;*
3. *is a fixed point of f if $fx = x$;*

4. *a point u in X is said to be a point of coincidence of f and T if $fx = Tx = u$.*

Definition 4.1.11 *(Rao et al., 2020). Two self-mapping T and f defined over $A \subseteq X$ are said to be weakly compatible, if they commute at their coincidence points.*

Example 4.1.4 *Let $X = \Re$ and $d : X \times X \to \Re^+$ be defined by $d(x,y) = |x-y|$ for all $x, y \in X$. Define $f, T : X \to X$ by $fx = 2x-1$ and $Tx = 3x-2$ for all $x \in X$. Then f and T are weakly compatible.*

Example 4.1.5 *Let $X = \Re$ define $f, T : X \to X$ by $fx = x^3$ and $Tx = -6x^2 + x + 6$ for all $x \in X$, then*

1. *$-6, -1, 1$ are coincidence point of f and T.*
2. *$-1, 1$ are common fixed point of f and T.*

Definition 4.1.12 *(Rao et al., 2020). A partially ordered metric space $(X, d, \preceq)$ is called an ordered complete if for each convergent sequence $\{x_n\}_{n_0}^{\infty} \subset X$, one of the following condition hold.*

1. *If $\{x_n\}$ is non-decreasing sequence in X such that $\{x_n\} \to x$ implies $x_n \preceq x$ for all $n \in N$that is x=sup$\{x_n\}$.*
2. *If $\{x_n\}$ is nonincreasing sequence in X such that $\{x_n\} \to x$ implies $x \preceq x_n$ for all $n \in N$ that is x=inf$\{x_n\}$.*

Definition 4.1.13 *(Khojasteh et al. 2015). A simulation function is a mapping $\zeta : \Re^+ \times \Re^+ \to \Re$ satisfying the following conditions.*
$\zeta_1 : \zeta(t,s) < s-t$ for all $t, s > 0$.
ζ_2 : if $\{t_n\}$ and $\{s_n\}$ are sequence in $(0,\infty)$ such that $\lim\limits_{n\to\infty} t_n = \lim\limits_{n\to\infty} s_n = l \in (0,\infty)$ then $\limsup_{n\to\infty} \zeta(t_n, s_n) < 0$.

Definition 4.1.14 *(Harjani and Sadarangani, 2009). Let (X,d) be a metric space and $T : X \to X$ is called $Z-$ contraction with respect to certain simulation function ζ satisfying the following condition*
$\zeta(d(Tx,Ty), d(x,y)) \geqslant 0$ for all $x, y \in x$.
Then T has a unique fixed point.
Moreover, for every $x_0 \in X$ the Picard sequence $\{T^n x_0\}$ converges to this fixed point.

Definition 4.1.15 *(Demma et al. 2016). We say that $\zeta : \Re^+ \times \Re^+ \to \Re$ is a $b-$simulation function, if there exits such that:*
$\zeta_1 : \zeta(t,s) < s-t$ *for all* $t,s > 0$.
ζ_2 : *if* $\{t_n\}$ *and* $\{s_n\}$ *are sequence*
$0 < \lim\limits_{n\to\infty} t_n \leqslant \lim\limits_{n\to\infty} s_n \leqslant b \lim\limits_{n\to\infty} t_n < \infty$, *then* $\limsup_{n\to\infty} \zeta(bt_n, s_n) < 0$.

Lemma 4.1.1 *(Roshan et al. 2014). Suppose (X,d) is a b-metric space with coefficient $s \geq 1$ and $\{x_n\}$ is a sequence in X such that $d(x_n,x_{n+1}) \longrightarrow 0$ as $n \longrightarrow \infty$. If $\{x_n\}$ is not a Cauchy sequence, then there exist an $\varepsilon > 0$ and sequences of positive integers $\{m_k\}$ and $\{n_k\}$ with $n_k > m_k > k$ such that $d(x_{m_k},x_{n_k}) \geq \varepsilon$,$d(x_{m_k},x_{n_k-1}) < \varepsilon$ and the following results hold:*

(i) $\varepsilon \leq \underline{\lim}\, d(x_{m_k},x_{n_k}) \leq \overline{\lim}\, d(x_{m_k},x_{n_k}) < s\varepsilon$,

(ii) $\frac{\varepsilon}{s} \leq \underline{\lim}\, d(x_{m_k+1},x_{n_k}) \leq \overline{\lim}\, d(x_{m_k+1},x_{n_k}) < s^2\varepsilon$,

(iii) $\frac{\varepsilon}{s} \leq \underline{\lim}\, d(x_{m_k},x_{n_k+1}) \leq \overline{\lim}\, d(x_{m_k},x_{n_k+1}) < s^2\varepsilon$,

(iv) $\frac{\varepsilon}{s^2} \leq \underline{\lim}\, d(x_{m_k+1},x_{n_k+1}) \leq \overline{\lim}\, d(x_{m_k+1},x_{n_k+1}) < s^3\varepsilon$.

Theorem 4.1.2 *(Harjani and Sadarangani, 2009). Let(X,d,S) be a complete $b-$metric space and let $f : X \to X$ be a mapping. Suppose that there exists a $b-$simulation function ζ suchthat Holds, that is,*

$$\zeta(sd(fx,fy),d(x,y)) \geqslant 0$$

for all $x,y \in x$. Then f has a unique point.

Theorem 4.1.3 *(Rao et al. 2020). Let$(X,d,\preceq)$ be a complete partially ordered metric space. Suppose that the self-mappings f and T on X are continuous, T is a monotone f non-decreasing, $T(X) \subseteq f(X)$and satisfying the following condition;*

$$d(Tx,Ty) \leqslant \alpha \frac{d(fx,Tx)d(fy,Ty)}{d(fx,fy)} + \beta[d(fx,Tx)+d(fy,Ty)] + \gamma d(fx,fy)$$

for all $x,y \in X$ With $fx \neq fy$ are comparable where $\alpha,\beta,\gamma \in [0,1)$with $0 < \alpha + 2\beta + \gamma < 1$.If there exists a point $x_0 \in X$such that $f(x_0) \preceq T(x_0)$ and $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to U$, then $x_n \preceq u$ for all $n \in N$.

Then T and f have a coincidence point in X.
Further, if T and f are weakly compatible, then T and f have a common fixed point in X. Moreover, the set of common fixed points of T and f is well ordered if and only if $T$and f have one and only one common fixed point in$X$.

4.2 Main Results

Theorem 4.2.1 *Let $(X,d,\preceq)$ be a complete partially ordered b- metric space and $T,f : X \to X$ be mappings on X satisfying;*

$$\zeta(sd(Tx,Ty), \alpha\frac{d(fx,Tx)d(fy,Ty)}{d(fx,fy)} + \beta[d(fx,Tx)+d(fy,Ty)] + \gamma d(fx,fy)) \geq 0 \ (1)$$

for all $x,y \in X$, with $fx \neq fy$ are comparable, where $\zeta \in Z$ and $\alpha,\beta,\gamma \geq 0$ with $0 \leq \alpha+2\beta+\gamma < \frac{1}{s}$ such that

(a) $T(X) \subseteq f(X)$;

(b) there exists a point $x_0 \in X$ with $fx_0 \preceq Tx_0$;

(c) T is a monotone f non-decreasing;

(d) T and f are $b-$continuous;

(e) T and f are compatible.
Then T and f have a coincidence point in X.

(f) If T and f are weakly compatible.
Then T and f have a common point in X.
Moreover, the set of common fixed points of T and f is well ordered if and only if T and f have one and only one common fixed point in X.

Proof: By (b), there exists $x_0 \in X$ such that $fx_0 \preceq Tx_0$. Since by the hypothesis of the theorem, we have $T(X) \subseteq f(X)$; there exists $x_1 \in X$ such that $fx_1 = Tx_0$. Since $T(X) \subseteq f(X)$ we have $Tx_1 \in f(X)$. Hence there exists point $x_2 \in X$ such that $fx_2 = Tx_1$. By continuing the same way, we can construct a sequence $\{x_n\}$ in X, such that

$$fx_{n+1} = Tx_n,$$

for all $n \geq 0$.
Since T is monotone f non-decreasing and $fx_0 \preceq Tx_0 = fx_1$, we have $Tx_0 \preceq Tx_1$.
By continuing the same procedure, we obtain that

$$Tx_0 \preceq Tx_1 \preceq Tx_2 \preceq \cdots \preceq Tx_n \preceq Tx_{n+1} \preceq \cdots.$$

The equality $Tx_{n+1} = Tx_n$ is impossible because $fx_{n+1} \neq fx_n$ for all $n \in N$.
Thus $d(Tx_{n+1}, Tx_n) > 0$ for all n.
Using (1) with $x = x_n$ and $y = x_{n+1}$, we get

$$\zeta(sd(Tx,Ty), \alpha\frac{d(fx,Tx)d(fy,Ty)}{d(fx,fy)} + \beta[d(fx,Tx) + d(fy,Ty)] + \gamma d(fx,fy)) \geq 0.$$

It follows that by (ζ_1)

$$\begin{aligned}
sd(Tx_n, Tx_{n+1}) &< \alpha\frac{d(fx_n,Tx_n)d(fx_{n+},Tx_{n+1})}{d(fx_n,fx_{n+1})} + \beta[d(fx_n,Tx_n) + d(fx_{n+1},Tx_{n+1})] + \\
&\quad \gamma d(fx_n, fx_{n+1}) \\
&< \alpha d(Tx_n,Tx_{n+1}) + \beta d(Tx_n,Tx_{n+1}) + \beta d(Tx_n,Tx_{n-1}) + \gamma d(Tx_n,Tx_{n-1}) \\
&= (\alpha+\beta)d(Tx_n,Tx_{n+1}) + (\beta+\gamma)d(Tx_n,Tx_{n-1}).
\end{aligned}$$

That is,

$$(S-(\alpha+\beta))d(Tx_n,Tx_{n+1}) < (\beta+\gamma)d(Tx_n,Tx_{n-1}).$$

Finally, we get

$$\begin{aligned}
d(Tx_n,Tx_{n+1}) &< \frac{(\beta+\gamma)}{(s-(\alpha+\beta))}d(Tx_n,Tx_{n-1}) \\
&< Kd(Tx_n,Tx_{n-1}) < d(Tx_n,Tx_{n-1})
\end{aligned}$$

where $K = \frac{(\beta+\gamma)}{(s-(\alpha+\beta))} \in [0, \frac{1}{s})$.
Therefore, the sequence $\{d(Tx_n, Tx_{n-1})\}$ is non-decreasing sequence of positive

real numbers and it converges to some real number $r \geq 0$ such that

$$\lim_{n\to\infty} d(Tx_n, Tx_{n-1}) = r.$$

We show that $r = 0$. Suppose to the contrary that $r > 0$.
Applying (ζ_2) with $t_n = d(Tx_{n+1}, Tx_n)$ and $s_n = d(Tx_n, Tx_{n-1})$, we get

$$0 < \lim_{n\to\infty} t_n \leqslant \liminf_{n\to\infty} s_n \leq \limsup_{n\to\infty} s_n \leq b \lim_{n\to\infty} t_n.\infty$$

It follow that

$$0 \leq \limsup_{n\to\infty} \zeta(sd(Tx_n, Tx_{n+1}, d(Tx_n, Tx_{n-1})) < 0,$$

which is a contradiction.
Hence, we conclude that

$$\lim_{n\to\infty} d(Tx_n, Tx_{n-1}) = 0.$$

Now, we show that $\{Tx_n\}$ is a b-Cauchy sequence in X. Suppose $\{Tx_n\}$ is not b-Cauchy sequence. Then there exists an $\varepsilon > 0$ and sequence positive integer $\{m_k\}$ and $\{n_k\}$ with $n_k > m_k \geq k$ such that

$$d(Tx_{m_k}, Tx_{n_k}) \geq \varepsilon, d(Tx_{m_k}, Tx_{n_k-1}) < \varepsilon.$$

Now, we consider

$$\zeta(sd(Tx_{m_k}, Tx_{n_k}), \alpha\frac{d(fx_{n_k}, Tx_{n_k})d(fx_{m_k}, Tx_{m_k})}{d(fx_{n_k}, fx_{m_k})} + \beta[d(fx_{n_k}, Tx_{n_k}) + d(fx_{m_k}, Tx_{m_k})] + \gamma d(fx_{n_k}, fx_{m_k})) \geq 0$$

By (ζ_1), we have

$$\begin{aligned} sd(Tx_{m_k}, Tx_{n_k}) \quad < \quad & \alpha\frac{d(fx_{n_k}, Tx_{n_k})d(fx_{m_k}, Tx_{m_k})}{d(fx_{n_k}, fx_{m_k})} + \beta[d(fx_{n_k}, Tx_{n_k}) + d(fx_{m_k}, Tx_{m_k})] \\ & + \gamma d(fx_{n_k}, fx_{m_k}). \end{aligned}$$

Let

$$\begin{aligned} S_n &= \alpha\frac{d(fx_{n_k},Tx_{n_k})d(fx_{m_k},Tx_{m_k})}{d(fx_{n_k},fx_{m_k})}+\beta[d(fx_{n_k},Tx_{n_k})+d(fx_{m_k},Tx_{m_k})]+\gamma d(fx_{n_k},fx_{m_k}) \\ t_n &= sd(Tx_{m_k},Tx_{n_k}). \end{aligned}$$

Letting $k\to\infty$ and using lemma 1,

$$\limsup_{k\to\infty} t_n = \limsup_{k\to\infty} sd(Tx_{m_k},Tx_{n_k}) = s(s\varepsilon).$$

$$\begin{aligned} \limsup_{k\to\infty} S_n &= \limsup_{k\to\infty}\alpha\frac{d(fx_{n_k},Tx_{n_k})d(fx_{m_k},Tx_{m_k})}{d(fx_{n_k},fx_{m_k})}+\beta[d(fx_{n_k},Tx_{n_k})+d(fx_{m_k},Tx_{m_k})]+ \\ &\quad \gamma d(fx_{n_k},fx_{m_k}) \\ &= \gamma s^3\varepsilon. \end{aligned}$$

It follows that

$$\begin{aligned} 0 &\leq \limsup_{k\to\infty}\zeta(st_n,S_n) \\ &< \limsup_{n\to\infty} s_n - \limsup_{n\to\infty} st_n \\ &= \gamma s^3\varepsilon - s^2\varepsilon < 0, \end{aligned}$$

which is a contradiction

Hence $\{Tx_n\}$is a $b-$ Cauchy sequence in X.

Since X is $b-$ complete there exists $u\in X$ such that $Tx_n\to u$ as $n\to\infty$.

By the $b-$ continuity of T, we have

$$\lim_{n\to\infty} T(Tx_n) = T\lim_{n\to\infty} f(Tx_n) = Tu.$$

Also by the $b-$continuity of f, we have

$$\lim_{n\to\infty} f(fx_{n+1}) = fu.$$

Since f and T are compatibility, we have

$$\lim_{n\to\infty} d(Tfx_n,fTx_n) = 0.$$

By the triangular in equality, we have

$$d(Tu, fu) \leq S[d(Tu, Tfx_n) + d(Tfx_n, fTx_n) + d(fTx_n, fu)].$$

On taking limit as $n \to \infty$ on both sides of the above inequality, we get

$$Tu = fu.$$

So that, u is a coincidence point of T and f.
Now, suppose that T and f are weakly compatible. Let w be a coincidence point of T and f then,

$$T(w) = f(w) \Longrightarrow T(fw) = f(Tw).$$

There exists $z \in X$such that $Tz = fz = w$. It follows that $Tw = T(fz) = f(Tz) = fw$.
Now by (1) and (ζ_1) , we have

$$\begin{aligned} sd(T(z), T(w)) &< \alpha \frac{d(fz, Tz)d(fw, Tw)}{d(fz, fw)} + \beta[d(fz, Tz) + d(fw, Tw)] + \gamma d(fz, fw) \\ &< \frac{0 \times 0}{d(fz, fw)} + \beta[0+0] + \gamma d(fz, fw). \end{aligned}$$

Since $fz = Tz$ and $fw = Tw$ we have,

$$\begin{aligned} Sd(T(z), T(w)) &< \gamma d(Tz, Tw) \\ d(T(z), T(w)) &< \frac{\gamma}{s} d(T(z), T(w)) \\ d(T(z), T(w)) &< d(T(z), T(w)), \end{aligned}$$

which is a contradiction
Hencee

$$Tz = Tw.$$

From above, we have

$$Tw = fw = w.$$

Hence w is a common fixed point of T and f in X.
Now, suppose that the set of common fixed points of T and f is well ordered, we have to show that the common fixed point of T and f is unique.
Let u and v be two common fixed points of T and f such that $u \neq v$, then from (1) we have,

$$\zeta(sd(Tu,Tv)), \alpha\frac{d(fu,Tu)d(fv,Tv)}{d(fu,fv)} + [d(fu,Tu)+d(fv,Tv)] + d(fv,fu)) \geq 0.$$

By using (ζ_1), we have,

$$sd(Tu,Tv) < \alpha\frac{d(fu,Tu)d(fv,Tv)}{d(fu,fv)} + [d(fu,Tu)+d(fv,Tv)] + d(fv,fu).$$

That is,

$$sd(u,v) < \alpha\frac{d(u,u)d(v,v)}{d(u,v)} + \beta[d(u,u)+d(v,v)] + \gamma d(v,u) = \gamma d(v,u).$$

It follow that

$$d(u,v) < \frac{\gamma}{s}d(u,v) < d(u,v).$$

which is a contradiction
Thus, we have $u = v$.
Therefore the common fixed point T and f is unique.
Conversely suppose $T$and f have only one common fixed point then the set of common fixed points of T and f is $\{u\}$, which is well ordered being a singleton.
□

Theorem 4.2.2 *Let $(X,d,\preceq)$ be a complete partially ordered b- metric space and $T,f : X \to X$ be mappings on$X$ satisfying:*

$$\zeta(sd(Tx,Ty), \alpha\frac{d(fx,Tx)d(fy,Ty)}{d(fx,fy)} + \beta[d(fx,Tx)+d(fy,Ty)] + \gamma d(fx,fy)) \geq 0 \quad (2)$$

for all$x,y \in X$, with $fx \neq fy$ are comparable, where $\zeta \in Z$ and $\alpha,\beta,\gamma \geq 0$ with $0 \leq \alpha+2\beta+\gamma < \frac{1}{s}$ such that *(a)* $T(X) \subseteq f(X)$;

(b) there exists a point $x_0 \in X$ *with* $fx_0 \preceq Tx_0$ *;*

(c) T *is a monotone* f *non-decreasing;*

(d)$\{x_n\}$ *is a non-decreasing sequence in* X *such that* $x_n \to u$ *,then* $x_n \preceq u$ *for all* $n \in N$.

(e) $f(X)$ *is a* $b-$ *complete subset of* X.
Then T *and* f *have a coincidenpce oint in* X*;*

(f) if T *and* f *are weakly compatible. Then* T *and* f *have a common fixed point in* X.
Moreover, the set of common fixed points of T *and*f *is well ordered if and only if* T *and* f *have one and only one common fixed point in* X *.*

Proof: Following as in the proof of Theorem (2) we construct a sequence $\{Tx_n\}$. The constructed sequence is $b-$ Cauchy sequence and hence $\{fx_n\}$ is also b-Cauchy sequence in $(f(X),d)$ as $fx_{n+1} = Tx_n$ and $T(X) \subseteq f(X)$.
Since $f(X)$ is $b-$ complete subset of X, there exist $f(u) \in f(X)$ such that

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} fx_n = fu$$

$\{fx_n\}$ is non-decreasing,$fx_n \preceq fu$ and $Tx_n \preceq fu$.
But T is monotone f non decreasing $Tx_n \preceq Tu$ for all n,
letting $n \to \infty$,we obtain $fu \preceq Tu$.

Since $fu \preceq Tu$, define a sequence by $u_0 = u$ and $fu_{n+1} = Tu_n$ for all $n \in N$.
$\{fu_n\}$ is a non-decreasing sequence and

$$\lim_{n\to\infty} fu_n = \lim_{n\to\infty} Tu_n = fv.$$

for some $v \in X$.
It is clear that
sup $fu_n \preceq fv$ and sup $Tu_n \preceq fv$, for all $n \in N$.
Notice that

$$fx_n \preceq fu \preceq fu_1 \preceq ... \preceq fu_n \preceq ... \preceq fv.$$

We consider two cases:
Case: 1 Suppose that $fx_{n_0} = fu_{n_0}$, then we have

$$fx_{n_0} = fu = fu_{n_0} = fu_1 = Tu.$$

Hence u isa coincidence point of $T$and f in X.
Case: 2 Suppose that $fx_{n_0} \neq fu_{n_0}$.
Then from (2)

$$\zeta(Sd(Tx_n,Tu_n), \frac{d(fx_n,Tx_n)d(fu_n,Tu_n)}{d(fx_n,fu_n)} + \beta[d(fx_n,Tx_n)+d(fu_n,Tu_n)] + \gamma d(fx_n,fu_n)) \geq 0.$$

By (ζ_1),we get

$$\begin{aligned} sd(fx_{n+1},fu_{n+1}) &= sd(Tx_n,Tu_n) \\ &< \alpha\frac{d(fx_n,fx_{n+1})d(fu_n,fu_{n+1})}{d(fx_n,fu_n)} + \beta[d(fx_n,fx_{n+1})+d(fu_n,fu_{n+1})] + \\ &\quad \gamma d(fx_n,fu_n). \end{aligned}$$

Taking limit as $n \to \infty$ on both sides of the above inequality and $\{fx_n\}$, $\{fu_n\}$ are nondecreasing sequence i.e., $fv = supfu_n$ and $fu = supfx_n$.

$$\limsup_{n\to\infty} sd(fx_{n+1},fu_{n+1}) < \limsup_{n\to\infty} \gamma d(fx_n,fu_n).$$

It follows that

$$\begin{aligned} sd(fu,fv) &< \gamma d(fu,fv) \\ d(fu,fv) &< \frac{\gamma}{S}(fu,fv), \end{aligned}$$

which is a Contradiction
Thus ,we have $fu = fv$
But

$$fu \preceq fv = fu_1 = Tu.$$

From the above, we get

$$fu = fv = fu_1 = Tu.$$

Hence

$$fu = Tu.$$

We conclude that u is a coincidence point of T and f in X.
Now, suppose that T and f are weakly compatible. Let w be a coincidence point then,we have

$$T(w) = f(w) \Longrightarrow T(fw) = f(Tw).$$

There exists $z \in X$such that $Tz = fz = w$ (w is point of coincidence $T and f$).
By condition $Tw = T(fz) = f(Tz) = fw$. Now by (2), we have

$$\begin{aligned} sd(T(z),T(w)) &< \alpha\frac{d(fz,Tz)d(fw,Tw)}{d(fz,fw)} + \beta[d(fz,Tz) + d(fw,Tw)] + \gamma d(fz,fw) \\ &< \frac{0\times 0}{d(fz,fw)} + \beta[0+0] + \gamma d(fz,fw) \end{aligned}$$

Since $fz = Tz$ and $fw = Tw$, we have

$$\begin{aligned} Sd(T(z),T(w)) &< \gamma d(Tz,Tw) \\ d(T(z),T(w)) &< \frac{\gamma}{s} d(T(z),T(w)) \\ d(T(z),T(w)) &< d(T(z),T(w)), \end{aligned}$$

which is a contradiction.
Hence

$$Tz = Tw.$$

From above,we have

$$Tw = fw = w.$$

Hence w is a common fixed point of T and f in X.
Now, suppose that the set of common fixed points of T and f is well ordered, we have to show that the common fixed point of T and f is unique.
Let u and v be two common fixed points of T and$f$ such that $u \neq v$, then from (2) we have.

$$\zeta(sd(Tu,Tv)), \alpha\frac{d(fu,Tu)d(fv,Tv)}{d(fu,fv)} + [d(fu,Tu)+d(fv,Tv)] + d(fv,fu)) \geq 0.$$

By using (ζ_1), we have

$$sd(Tu,Tv) < \alpha\frac{d(fu,Tu)d(fv,Tv)}{d(fu,fv)} + [d(fu,Tu)+d(fv,Tv)] + d(fv,fu).$$

That is,

$$sd(u,v) < \alpha\frac{d(u,u)d(v,v)}{d(u,v)} + \beta[d(u,u)+d(v,v)] + \gamma d(v,u) = \gamma d(v,u).$$

It follows that

$$d(u,v) < \frac{\gamma}{s}d(u,v) < d(u,v),$$

which is a contradiction
Thus, we have $u = v$.
The common fixed point T and f is unique.
Conversely suppose $T$and f have only one common fixed point, then the set of common fixed points of T and f is $\{u\}$, being a singleton it is well ordered.
This completes the proof.

□

4.3 Consequences

In this section, we give some consequences of the main results.

Corollary 4.3.1 *Let $(X,d,\preceq)$ be a complete partially ordered $b-$ metric space and*

$T, f : X \to X$be mappings on X satisfying:

$$\zeta(sd(Tx,Ty), \beta[d(fx,Tx)+d(fy,Ty)] + \gamma d(fx,fy)) \geq 0$$

for all$x,y \in X$, with $fx \neq fy$ are comparable, where $\zeta \in Z$ and $\alpha, \beta, \gamma \geq 0$ with $0 \leq \alpha + 2\beta + \gamma < \frac{1}{s}$ such that

(a) $T(X) \subseteq f(X)$;

(b) there exists a point $x_0 \in X$ with $fx_0 \preceq Tx_0$;

(c) T is a monotone f non-decreasing;

(d) T and f are $b-$continuous;

(e) T and f are compatible. Then T and f have a coincidence point in X.

(f) If T and $f$are weakly compatible, then T and f have a common point in X. Moreover, the set of common fixed points of T and f is well ordered if and only if T and f have one and only one common fixed point in$X$

Proof: *The proof follows from Theorem (1) by setting $\alpha = 0$.*

Corollary 4.3.2 *Let $(X, d, \preceq)$ be a complete partially ordered $b-$ metric space and $T, f : X \to X$ be mappings on X satisfying:*

$$\zeta(sd(Tx,Ty), \frac{d(fx,Tx), d(fy,Ty)}{d(fx,fy)} + \beta[d(fx,Tx)+d(fy,Ty)]) \geq 0$$

for all$x,y \in X$, with $fx \neq fy$ are comparable, where $\zeta \in Z$ and $\alpha, \beta, \gamma \geq 0$ with $0 \leq \alpha + 2\beta + \gamma < \frac{1}{s}$ such that

(a) $T(X) \subseteq f(X)$;

(b) there exists a point $x_0 \in X$ with $fx_0 \preceq Tx_0$;

(c) T is a monotone f non-decreasing;

(c) T and f are $b-$continuous ;

(d) T and f are compatible.
Then T and f have a coincidence point in X.

Proof: *The proof follows from Theorem (1) by setting $\gamma = 0$.*

Corollary 4.3.3 *Let $(X,d,\preceq)$ be a complete partially ordered b- metric space and $T,f : X \to X$ be mappings onX satisfying:*

$$\zeta(sd(Tx,Ty),\beta[d(fx,Tx)+d(fy,Ty)]+\gamma d(fx,fy)) \geq 0$$

for all $x,y \in X$, with $fx \neq fy$ are comparable, where $\zeta \in Z$ and $\alpha,\beta,\gamma \geq 0$ with $0 \leq \alpha+2\beta+\gamma < \frac{1}{s}$ such that

(a) $T(X) \subseteq f(X)$;

(b) there exists a point $x_0 \in X$ with $fx_0 \preceq Tx_0$;

(c) T is a monotone f non-decreasing;

(d) $\{x_n\}$ is a non-decreasing sequence in X such that x_n ,then $x_n \preceq u$ for all $n \in N$;

(e) If $f(X)$ is a $b-$complete subset of X then T and f have a coincidence point in X;

(f) If T and f are weakly compatible, then T and f have a common fixed point in X. Moreover, the set of common fixed points of T and$f$ is well ordered if and only if T and f have one and only one common fixed point in X .
Proof: *The proof follows from Theorem (2) by setting $\alpha = 0$.*

Corollary 4.3.4 *Let $(X,d,\preceq)$ be a complete partially ordered b- metric space and $T,f : X \to X$ be mappings onX satisfying:*

$$\zeta(sd(Tx,Ty),\alpha\frac{d(fx,Tx)d(fy,Ty)}{d(fx,fy)}+\beta[d(fx,Tx)+d(fy,Ty)]) \geq 0$$

for all$x,y \in X$, with $fx \neq fy$ are comparable, where $\zeta \in Z$ and $\alpha,\beta,\gamma \geq 0$ with $0 \leq \alpha+2\beta+\gamma < \frac{1}{s}$ such that

(a) $T(X) \subseteq f(X)$;

(b) there exists a point $x_0 \in X$ with $fx_0 \preceq Tx_0$;

(c) T is a monotone f non-decreasing;

(d) $\{x_n\}$ is a non-decreasing sequence in X such that x_n ,then $x_n \preceq u$ for all $n \in N$;

(e) If $f(X)$ is a$b-$ complete subset of$X$then T and f have a coincidence pointin X;

(f) If T and f are weakly compatible, Then T and f have a common fixed point in X.

Moreover, the set of common fixed points of T and f is well ordered if and only if T and f have one and only one common fixed point in X .

Proof: *The proof follows from Theorem (2) by setting $\gamma = 0$.*

Example 4.3.1 *Let $X = \{5,6,7,8,9\}$ with the usual b-metric space $d(x,y) = |x-y|^2$ for all x,y $\in X$. We define the partial ordered on X.*

$$\preceq = \{(5,5),(6,6),(7,7),(8,8),(9,9),(7,8),(7,9),(8,9)\}$$

Then $(X,\preceq)$ is partially ordered set.
We define the two self-mapping $f : X \to X$ by

$$f(5) = f(6) = f(7) = 7, f(8) = 8 \text{ and } f(9) = 9.$$

and

$$T(x) = 7,$$

for all $x \in X$.
and $\alpha = \frac{1}{36}, \beta \frac{1}{37} =, \gamma = \frac{1}{38}$.
Now we define $\zeta : \Re^+ \times \Re^+ \to \Re^+$ by $\zeta(t,s) = \frac{s}{2} - t$ and take $x_0 = 5$.
Then T and f have coincidence point in X.

Now, let us show that T is monotone $f-$non decreasing.

$$7 = f(5) \preceq 7 = f(5) \Longrightarrow 7 = T(5) \preceq 7 = T(5)$$
$$7 = f(5) \preceq 7 = f(6) \Longrightarrow 7 = T(5) \preceq 7 = T(6)$$
$$7 = f(5) \preceq 7 = f(7) \Longrightarrow 7 = T(5) \preceq 7 = T(7)$$
$$7 = f(5) \preceq 8 = f(8) \Longrightarrow 7 = T(5) \preceq 7 = T(8)$$
$$7 = f(5) \preceq 9 = f(9) \Longrightarrow 7 = T(5) \preceq 7 = T(9)$$
$$7 = f(6) \preceq 7 = f(6) \Longrightarrow 7 = T(6) \preceq 7 = T(6)$$
$$7 = f(6) \preceq 7 = f(7) \Longrightarrow 7 = T(6) \preceq 7 = T(7)$$
$$7 = f(6) \preceq 8 = f(8) \Longrightarrow 7 = T(6) \preceq 7 = T(8)$$
$$7 = f(6) \preceq 9 = f(9) \Longrightarrow 7 = T(6) \preceq 7 = T(9)$$
$$7 = f(7) \preceq 7 = f(7) \Longrightarrow 7 = T(7) \preceq 7 = T(7)$$
$$7 = f(7) \preceq 8 = f(8) \Longrightarrow 7 = T(7) \preceq 7 = T(8)$$
$$7 = f(7) \preceq 9 = f(9) \Longrightarrow 7 = T(7) \preceq 7 = T(9)$$
$$8 = f(8) \preceq 8 = f(8) \Longrightarrow 7 = T(8) \preceq 7 = T(8)$$
$$8 = f(8) \preceq 9 = f(9) \Longrightarrow 7 = T(8) \preceq 7 = T(9)$$
$$9 = f(9) \preceq 9 = f(9) \Longrightarrow 7 = T(9) \preceq 7 = T(9).$$

Thus T is monotone f nondecreasing.
We have the following possible cases.
Case (i) *when $x = 5$ and $y = 8$.*
Then $f(5) = 7, f(8) = 7$, $T(5) = 7, T(6) = 7$.
in this case
$d(f(5), f(8)) = |7-8|^2 = 1.$
$d(f(5), T(5)) = |7-7|^2 = 0.$
$d(f(8),, T(8)) = |8-7|^2 = 1.$
$d(T(5), T(8)) = |7-7|^2 = 0.$

Now, we get

$$\begin{aligned}
s &= \alpha\frac{d(f(5),T(5))d(f(8),T(8))}{d(f(8),f(5)} + \beta[d(f(5),T(5))d(f(8),T(8))] + \gamma d(f(8),f(5)) \\
&= \alpha\frac{|7-7|^2|8-7|^2}{|7-8|^2} + \beta[|7-7|^2 + |8-7|^2] + \gamma|7-8|^2 \\
&= \beta+\gamma \\
&= \frac{1}{37}+\frac{1}{38} \\
&= \frac{75}{1406}. \\
t &= d(T(5),T(8)) = |7-7|^2 = 0.
\end{aligned}$$

Now, we consider

$$\begin{aligned}
&\zeta(sd(Tx,Ty),\alpha\frac{d(fx,Tx)d(fy,Ty)}{d(fx,fy)} + \beta[d(fx,Tx)+d(fy,Ty)] + \gamma d(fx,fy)) \\
&= \frac{1}{2}[\alpha\frac{d(fx,Tx)d(fy,Ty)}{d(fx,fy)} + \beta[d(fx,Tx)+d(fy,Ty)] + \gamma d(fx,fy))] - sd(Tx,Ty) \\
&= \frac{1}{2}[\beta+\gamma] - s\times 0 = \frac{1}{2}[\frac{1}{37}+\frac{1}{38}] = \frac{75}{2812}.
\end{aligned}$$

That is,

$$\zeta(sd(Tx,Ty),\alpha\frac{d(fx,Tx)d(fy,Ty)}{d(fx,fy)} + \beta[d(fx,Tx)+d(fy,Ty)] + \gamma d(fx,fy)) \geq 0.$$

Case (ii) *When $x=5$ and $y=9$*
Then $f(5)=7$, $f(9)=9$ and $T(7)=T(9)=7$.
In this case
$T(9)=7, T(5)=7.$
$d(T(9),f(9) = |7-9|^2 = 4.$
$d(T(5),T(9)) = |7-7|^2 = 0.$
$d(f(5),f(9)) = |7-9|^2 = 4.$

$d(T(5), f(5) = |7-7|^2 = 0.$

$$\begin{aligned}
s &= \alpha \frac{(d(T(5), f(5)d(T(9), f(9))}{d(f(5), f(9))} + \beta[d(T(5), f(5) + d(T(9), f(9)] + \gamma d(f(5), f(9)) \\
&= \alpha \frac{|7-7|^2)|7-9|^2}{|7-9|^2} + \beta[|7-7|^2) + |7-9|^2] + \gamma)|7-9|^2 \\
&= 4\beta + 4\gamma \\
&= \frac{4}{37} + \frac{4}{38} \\
&= \frac{152+148}{5406} \\
&= \frac{300}{5406} \\
t &= d(T(5), T(9)) = |7-7|^2 = 0.
\end{aligned}$$

Now, we consider

$$\begin{aligned}
&\zeta(sd(Tx, Ty), \alpha \frac{d(fx, Tx)d(fy, Ty)}{d(fx, fy)} + \beta[d(fx, Tx) + d(fy, Ty)] + \gamma d(fx, fy)) \\
= &\ \frac{1}{2}[\alpha \frac{d(fx, Tx)d(fy, Ty)}{d(fx, fy)} + \beta[d(fx, Tx) + d(fy, Ty)] + \gamma d(fx, fy)] - sd(Tx, Ty) \\
= &\ \frac{1}{2}[4\beta + 4\gamma] - 2(0) = \frac{1}{2}[\frac{4}{37} + \frac{4}{38}] - 0 \geq 0.
\end{aligned}$$

That is,

$$\zeta(sd(Tx, Ty), \alpha \frac{d(fx, Tx)d(fy, Ty)}{d(fx, fy)} + \beta[d(fx, Tx) + d(fy, Ty)] + \gamma d(fx, fy)) \geq 0.$$

Case (iii) *When $x = 8$ and $y = 9$.*
Then $f(8) = 8$, $f(9) = 9$ and $T(8) = 7$, $T(9) = 7$.
In this case ;
$d(T(8), f(8) = |8-7|^2 = 1.$
$d(T(8), T(9)) = |7-7|^2 = 0.$
$d(f(8), f(9)) = |7-9|^2 = 1.$
$d(T(9), f(9) = |7-9|^2 = 4.$

Now

$$\begin{aligned}
s &= \alpha\frac{(d(T(8),f(8)d(T(9),f(9))}{d(f(8),f(9))}+\beta[d(T(8),f(8)+d(T(9),f(9)]+\gamma d(f(8),f(9))\\
&= \alpha\frac{|8-7|^2|7-9|^2}{|7-9|^2}+\beta[|8-7|^2+|7-9|^2]+\gamma|7-9|^2\\
&= 4\alpha+5\beta+\gamma\\
&= \frac{4}{36}+\frac{5}{37}+\frac{1}{38}=\frac{13796}{50616}\\
t &= sd(Tx,Ty)=s|Tx-Ty|^2=2|7-7|^2=0
\end{aligned}$$

Now, we consider

$$\begin{aligned}
&\zeta(sd(Tx,Ty),\alpha\frac{d(fx,Tx)d(fy,Ty)}{d(fx,fy)}+\beta[d(fx,Tx)+d(fy,Ty)]+\gamma d(fx,fy))\\
&= \frac{1}{2}[\alpha\frac{d(fx,Tx)d(fy,Ty)}{d(fx,fy)}+\beta[d(fx,Tx)+d(fy,Ty)]+\gamma d(fx,fy))]-sd(Tx,Ty\\
&= \frac{1}{2}[4\alpha+5\beta+\gamma]-2(0)=\frac{1}{2}[4\times\frac{1}{36}+\frac{5}{37}+\frac{1}{38}]=\frac{1}{2}[\frac{13796}{50616}]-0\geq 0
\end{aligned}$$

That is,

$$\zeta(sd(Tx,Ty),\alpha\frac{d(fx,Tx)d(fy,Ty)}{d(fx,fy)}+\beta[d(fx,Tx)+d(fy,Ty)]+\gamma d(fx,fy))\geq 0$$

Hence all the hypothesis of the theorem are satisfied. Moreover, $5,6,7$*, are the coincidence poiints of* T *and* f*,* 7 *is the common fixed point of* T *and* f*, and* 7 *is the unique common fixed point of* T *and* f*.*

Example 4.3.2 *Let* $X=[0,20]$ *and* $d:X\times X\to[0,\infty)$ *be defined by* $d(x,y)=|x-y|^2$*for all* $x,y\in X$*,* $s=2$ *and let* $(x,\preceq)$ *is the usual partially ordered set.*
Define $f,T:X\to X$ *by*

$$f(x)=\begin{cases}0 & \text{if } x=0\\ x+16 & \text{if } 0<x\leqslant 4\\ x-4 & \text{if } 4<x\leqslant 20.\end{cases}$$

and

$$T(x) = \begin{cases} 0 & \text{if } x = \{0\} \cup (4, 20] \\ 3 & \text{if } 0 < x \leqslant 4. \end{cases}$$

T and f are $b-$ continuous functions.

We now define $\zeta : \Re^+ \times \Re^+ \to \Re^+$ by $\zeta(t,s) = \dfrac{3}{4}s - t$ for all $t, s > 0$.

and with the usual b-metric space

$d(x,y) = |x-y|^2 forallx, y \in X.where$ $\alpha = \dfrac{1}{7}, \beta = \frac{1}{9}, \gamma = \frac{1}{8}$ *and* $x_o = 0$.

We have the following possible cases.

***Case (i)** : When $x = 0$ and $y \in (0,4]$*

Then $fx = 0, fy = y + 16 \in (0,4], Tx = 0$ and $Ty = 3$

In this case;

$d(fx,Tx) = |0-0|^2 = 0.$

$d(fy,fx) = |y+16|^2.$

$d(fy,Ty) = |y+13|^2.$

$d(Tx,Ty) = |3-0|^2 = 9.$

Now

$$\begin{aligned} s &= \alpha\frac{(d(fx,Tx)d(fy,Ty)}{(d(fx,fy)} + \beta[d(fx,Tx) + d(fy,Ty)] + \gamma d(fx,fy) \\ &= \alpha\frac{|0-0|^2|y+13|^2}{|y+16|^2} + \beta[|0-0|^2 + |y+13|^2] + \gamma|y+16|^2 \\ &= \beta|y+13|^2 + \gamma|y+16|^2 \\ &= \frac{1}{9}|y+13|^2 + \frac{1}{8}|y+16|^2 \\ t &= sd(Tx,Ty) = s|Ty-Tx|^2 = 2|3-0|^2 = 18. \end{aligned}$$

Now, we consider

$$\begin{aligned}
& \zeta(sd(Tx,Ty),\alpha\frac{d(fx,Tx)d(fy,Ty)}{d(fx,fy)}+\beta[d(fx,Tx)+d(fy,Ty)]+\gamma d(fx,fy)) \\
= & \frac{3}{4}(\alpha\frac{d(fx,Tx)d(fy,Ty)}{d(fx,fy)}+\beta[d(fx,Tx)+d(fy,Ty)]+\gamma d(fx,fy))-sd(Tx,Ty) \\
= & \frac{3}{4}(\beta|y+13|^2+\gamma|y+16|^2)-s|3-0|^2 \\
= & \frac{3}{4}(\frac{1}{9}|y+13|^2+\frac{1}{8}|y+16|^2)-2|3-0|^2 \geq 0.
\end{aligned}$$

That is,

$$\zeta(sd(Tx,Ty),\alpha\frac{d(fx,Tx)d(fy,Ty)}{d(fx,fy)}+\beta[d(fx,Tx)+d(fy,Ty)]+\gamma d(fx,fy)) \geq 0.$$

***Caes (ii)**: When $x \in (0,4]$ and $y = 0$.*
Then $fx = x+16 \in (0,4]$ and $fy = 0$, $Tx = 3$ and $Ty = 0$.
In thise case;
$d(fx,Tx) = |x+13|^2$.
$d(fy,fx) = |x+16|^2$.
$d(fy,Ty) = |0-0|^2 = 0$.
$d(Tx,Ty) = |3-0|^2 = 9$.
Now

$$\begin{aligned}
s &= \alpha\frac{d(fx,Tx)d(fy,Ty)}{d(fx,fy)}+\beta[d(fx,Tx)+d(fy,Ty)]+\gamma d(fx,fy) \\
&= \alpha\frac{|x+13|^2|0-0|^2}{|x+16|^2}+\beta[|x+13|^2+|0-0|^2]+\gamma||x+16|^2 \\
&= \beta|x+13|^2+\gamma|x+16|^2 \\
&= \frac{1}{9}|x+13|^2+\frac{1}{8}|x+16|^2 \\
t &= sd(Tx,Ty) = 2|3-0|^2 = 18.
\end{aligned}$$

now, we consider

$$\begin{aligned}
& \zeta(sd(Tx,Ty), \alpha\frac{d(fx,Tx)d(fy,Ty)}{d(fx,fy)} + \beta[d(fx,Tx)+d(fy,Ty)] + \gamma d(fx,fy)) \\
= \;& \frac{3}{4}(\alpha\frac{d(fx,Tx)d(fy,Ty)}{d(fx,fy)} + \beta[d(fx,Tx)+d(fy,Ty)] + \gamma d(fx,fy)) - sd(Tx,Ty) \\
= \;& \frac{3}{4}(\beta|x+13|^2 + \gamma|x+16|^2 - s|3-0|^2] \\
= \;& \frac{3}{4}(\frac{1}{9}|x+13|^2 + \frac{1}{8}|x+16|^2 - 2\times 9 \geq 0.
\end{aligned}$$

That is ,

$$\zeta(sd(Tx,Ty), \alpha\frac{d(fx,Tx)d(fy,Ty)}{d(fx,fy)} + \beta[d(fx,Tx)+d(fy,Ty)] + \gamma d(fx,fy)) \geq 0.$$

***Case (iii)**: When $x = 0$ and $y \in (4,20]$.*
Then$fx = 0, Tx = 0, fy = y-4, Ty = 0$.
In thise case;
$d(fx,Tx) = |0-0|^2 = 0$
$d(fy,fx) = |y-4|^2$
$d(fy,Ty) = |y-4|^2$
$d(Tx,Ty) = |0-0|^2 = 0$
Now,

$$\begin{aligned}
s &= \alpha\frac{d(fx,Tx)d(fy,Ty)}{d(fx,fy)} + \beta[d(fx,Tx)+d(fy,Ty)] + \gamma d(fx,fy) \\
&= \alpha\frac{|0-0|^2|y-4|^2}{|y-4|^2} + \beta[|0-0|^2 + |y-4|^2] + \gamma|y-4|^2 \\
&= \beta|y-4|^2 + \gamma|y-4|^2 \\
&= \frac{1}{9}|y-4|^2 + \frac{1}{8}|y-4|^2 \\
t &= s|Tx-Ty|^2 = 2|0-0|^2 = 0
\end{aligned}$$

now consider

$$
\begin{aligned}
& \zeta(sd(Tx,Ty), \alpha\frac{d(fx,Tx)d(fy,Ty)}{d(fx,fy)} + \beta[d(fx,Tx)+d(fy,Ty)] + \gamma d(fx,fy)) \\
= \;& \frac{3}{4}(\alpha\frac{d(fx,Tx)d(fy,Ty)}{d(fx,fy)} + \beta[d(fx,Tx)+d(fy,Ty)] + \gamma d(fx,fy)) - sd(Tx,Ty) \\
= \;& \frac{3}{4}(\alpha\frac{|0-0|^2|y-4|^2}{|y-4|^2} + \beta[|0-0|^2+|y-4|^2] + \gamma|y-4|^2) - 2|0-0|^2 \\
= \;& \frac{3}{4}(\beta|y-4|^2 + \gamma|y-4|^2) - 2|0-0|^2 \geq 0. \\
= \;& \frac{3}{4}(\frac{1}{9}|y-4|^2 + \frac{1}{8}|y-4|^2) - 2|0-0|^2 \geq 0
\end{aligned}
$$

That is,

$$
\zeta(sd(Tx,Ty), \alpha\frac{d(fx,Tx)d(fy,Ty)}{d(fx,fy)} + \beta[d(fx,Tx)+d(fy,Ty)] + \gamma d(fx,fy)) \geq 0.
$$

Case (iv)*: When* $x \in (4,20]$ *and* $y = 0$.
Then $fx = x-4, fy = 0, Tx = 0 = Ty$.
In thise case;
$d(fx,Tx) = |x-4|^2$.
$d(fy,fx) = |x-4|^2$.
$d(fy,Ty) = |0-0|^2 = 0$.
$d(Tx,Ty) = |0-0|^2 = 0$.
Now,

$$
\begin{aligned}
s &= \alpha\frac{d(fx,Tx)d(fy,Ty)}{d(fx,fy)} + \beta[d(fx,Tx)+d(fy,Ty)] + \gamma d(fx,fy) \\
&= \alpha\frac{|x-4|^2|0-0|^2}{|x-4|^2} + \beta[|x-4|^2+|0-0|^2] + \gamma|x-4|^2 \\
&= \beta|x-4|^2 + \gamma|x-4|^2 \\
&= \frac{1}{9}|x-4|^2 + \frac{1}{8}|x-4|^2 \\
t &= s|Tx-Ty|^2 = 2|0-0|^2 = 0.
\end{aligned}
$$

now consider

$$\begin{aligned}
& \zeta(sd(Tx,Ty), \alpha\frac{d(fx,Tx)d(fy,Ty)}{d(fx,fy)} + \beta[d(fx,Tx)+d(fy,Ty)] + \gamma d(fx,fy)) \\
= & \frac{3}{4}(\alpha\frac{d(fx,Tx)d(fy,Ty)}{d(fx,fy)} + \beta[d(fx,Tx)+d(fy,Ty)] + \gamma d(fx,fy)) - sd(Tx,Ty) \\
= & \frac{3}{4}(\alpha\frac{|x-4|^2|0-0|^2}{|x-4|^2} + \beta[|x-4|^2+|0-0|^2] + \gamma|x-4|^2) - s|0-0|^2 \\
= & \frac{3}{4}(\beta|x-4|^2 + \gamma|x-4|^2) - 0 \\
= & \frac{3}{4}(\frac{1}{9}|x-4|^2 + \frac{1}{8}|x-4|^2) - 0 \geq 0.
\end{aligned}$$

That is,

$$\zeta(sd(Tx,Ty), \alpha\frac{d(fx,Tx)d(fy,Ty)}{d(fx,fy)} + \beta[d(fx,Tx)+d(fy,Ty)] + \gamma d(fx,fy)) \geq 0.$$

Case (v) *Whenx*, $y \in (0,4]$.
Then $fx = x+16, fy = y+16$
and $Tx = 3, Ty = 3$ *but* $fx \neq fy$.
In this case;
$d(fx,Tx) = |x+13|^2$.
$d(fy,fx) = |x-y|^2$.
$d(fy,Ty) = |y+13|^2$.
$d(Tx,Ty) = |3-3|^2 = 0$.
Now

$$\begin{aligned}
s = & \alpha\frac{d(fx,Tx)d(fy,Ty)}{d(fx,fy)} + \beta[d(fx,Tx)+d(fy,Ty)] + \gamma d(fx,fy) \\
= & \alpha\frac{|x+13|^2,|y+13|^2}{|x-y|^2} + \beta[|x+13|^2+|y+13|^2] + \gamma|x-y|^2 \\
= & \frac{1}{7}\frac{|x+13|^2,|y+13|^2}{|x-y|^2} + \frac{1}{9}[|x+13|^2+|y+13|^2] + \frac{1}{8}|x-y|^2.
\end{aligned}$$

$$t = sd(Tx,Ty) = 2|3-3|^2 = 0.$$

Now we consider

$$
\begin{aligned}
& \zeta(sd(Tx,Ty),\alpha\frac{d(fx,Tx)d(fy,Ty)}{d(fx,fy)}+\beta[d(fx,Tx)+d(fy,Ty)]+\gamma d(fx,fy)) \\
= & \frac{3}{4}(\alpha\frac{d(fx,Tx)d(fy,Ty)}{d(fx,fy)}+\beta[d(fx,Tx)+d(fy,Ty)]+\gamma d(fx,fy))-sd(Tx,Ty) \\
= & \frac{3}{4}(\alpha\frac{|x+13|^2|y+13|^2}{|x-y|^2}+\beta[|x+13|^2+|y+13|^2]+\gamma|x-y|^2)-2|3-3|^2 \\
= & \frac{3}{4}(\frac{1}{7}\frac{|x+13|^2|y+13|^2}{|x-y|^2}+\frac{1}{9}[|x+13|^2+|y+13|^2]+\frac{1}{8}|x-y|^2)-2|3-3|^2\geq 0.
\end{aligned}
$$

That is,

$$\zeta(sd(Tx,Ty),\alpha\frac{d(fx,Tx)d(fy,Ty)}{d(fx,fy)}+\beta[d(fx,Tx)+d(fy,Ty)]+\gamma d(fx,fy))\geq 0.$$

Case (vi) *: Wwhen* $x,\in(0,4]$ *and* $y\in(4,20]$
Then$fx=x+16, fy=y-4, Tx=3, Ty=0$
In this case;
$d(fx,Tx)=|x+13|^2.$
$d(fy,fx)=|x-y+20|^2.$
$d(fy,Ty)=|y-4|^2.$
$d(Tx,Ty)=|3-0|^2=9.$
Now

$$
\begin{aligned}
s &= \alpha\frac{d(fx,Tx)d(fy,Ty)}{d(fx,fy)}+\beta[d(fx,Tx)+d(fy,Ty)]+\gamma d(fx,fy) \\
&= \alpha\frac{(|x+13|^2|y-4|^2}{|x-y+20|^2}+\beta[|x+13|^2+|y-4|^2]+\gamma|x-y+20|^2 \\
&= \frac{1}{7}\frac{(|x+13|^2|y-4|^2}{|x-y+20|^2}+\frac{1}{9}[|x+13|^2+|y-4|^2]+\frac{1}{8}|x-y+20|^2 \\
t &= sd(Tx,Ty)=2|3-0|^2=18.
\end{aligned}
$$

Now consider

$$\begin{aligned}
&\zeta(sd(Tx,Ty),\alpha\frac{d(fx,Tx)d(fy,Ty)}{d(fx,fy)}+\beta[d(fx,Tx)+d(fy,Ty)]+\gamma d(fx,fy))\\
=\ &\frac{3}{4}(\alpha\frac{d(fx,Tx)d(fy,Ty)}{d(fx,fy)}+\beta[d(fx,Tx)+d(fy,Ty)]+\gamma d(fx,fy))-sd(Tx,Ty)\\
=\ &\frac{3}{4}(\alpha\frac{(|x+13|^2|y-4|^2}{|x-y+20|^2}+\beta[|x+13|^2|y-4|^2]+\gamma|x-y+20|^2)-s|3-0|^2\\
=\ &\frac{3}{4}(\frac{1}{7}\frac{(|x+13|^2|y-4|^2}{|x-y|^2}+\frac{1}{9}[|x+13|^2+||y-4|^2]+\frac{1}{8}|x-y+20|^2)-2|3-0|^2\geq 0.
\end{aligned}$$

That is,

$$\zeta(sd(Tx,Ty),\alpha\frac{d(fx,Tx)d(fy,Ty)}{d(fx,fy)}+\beta[d(fx,Tx)+d(fy,Ty)]+\gamma d(fx,fy))\geq 0.$$

***Case (vii)**: When* $x\in(4,20]$ *and* $y\in(0,4]$.
Then $fx=x-4, fy=y+16, Tx=0, Ty=3$.
In this case;
$d(fx,Tx)=|x-4|^2$.
$d(fy,fx)=|x-y-20|^2$.
$d(fy,Ty)=|y+13|^2$.
$d(Tx,Ty)=|3-0|^2=9$.
Now

$$\begin{aligned}
s &= \alpha\frac{d(fx,Tx)d(fy,Ty)}{d(fx,fy)}+\beta[d(fx,Tx)+d(fy,Ty)]+\gamma d(fx,fy)\\
&= \alpha\frac{|x-4|^2,|y+13|^2}{|x-y-20|^2}+\beta[|x-4|^2+|y+13|^2]+\gamma|x-y-20|^2\\
&= \frac{1}{7}\frac{|x-4|^2,|y+13|^2}{|x-y-20|^2}+\frac{1}{9}[|x-4|^2+|y+13|^2]+\frac{1}{8}|x-y-20|^2\\
t &= sd(Tx,Ty)=2|3-0|^2=18.
\end{aligned}$$

Now we consider

$$\begin{aligned}
& \zeta(sd(Tx,Ty),\alpha\frac{d(fx,Tx)d(fy,Ty)}{d(fx,fy)}+\beta[d(fx,Tx)+d(fy,Ty)]+\gamma d(fx,fy)) \\
= & \frac{3}{4}(\alpha\frac{d(fx,Tx)d(fy,Ty)}{d(fx,fy)}+\beta[d(fx,Tx)+d(fy,Ty)]+\gamma d(fx,fy))-sd(Tx,Ty) \\
= & \frac{3}{4}(\alpha\frac{|x-4|^2|y+13|^2}{|x-y-20|^2}+\beta[|x-4|^2+|y+13|^2]+\gamma|x-y-20|^2)-s|3-0|^2 \\
= & \frac{3}{4}(\frac{1}{7}\frac{|x-4|^2|y+13|^2}{|x-y-20|^2}+\frac{1}{9}[|x-4|^2+|y+13|^2]+\frac{1}{8}|x-y-20|^2)-2|3-0|^2\geq 0.
\end{aligned}$$

That is,

$$\zeta(sd(Tx,Ty),\alpha\frac{d(fx,Tx)d(fy,Ty)}{d(fx,fy)}+\beta[d(fx,Tx)+d(fy,Ty)]+\gamma d(fx,fy))\geq 0.$$

Case (viii)*: When* $x,y\in(4,20]$.
Then $fx=x-4, fy=y-4, Tx=Ty=0$.
since $fx\neq fy$
In this case;
$d(fx,Tx)=|x-4|^2$.
$d(fy,fx)=|x-y|^2$.
$d(fy,Ty)=|y-4|^2$.
$d(Tx,Ty)=|0-0|^2=0$.
Now,

$$\begin{aligned}
s &= \alpha\frac{d(fx,Tx)d(fy,Ty)}{d(fx,fy)}+\beta[d(fx,Tx)+d(fy,Ty)]+\gamma d(fx,fy) \\
&= \alpha\frac{|x-4|^2,|y-4|^2}{|y-4|^2}+\beta[|x-4|^2+|y-4|^2]+\gamma|y-4|^2 \\
&= \frac{1}{7}\frac{|x-4|^2,|y-4|^2}{|x-y|^2}+\frac{1}{9}[|x-4|^2+|y-4|^2]+\frac{1}{8}|x-y|^2 \\
t &= sd(Tx,Ty)=2|0-0|^2=0.
\end{aligned}$$

Now consider

$$\begin{aligned}
&\zeta(sd(Tx,Ty),\alpha\frac{d(fx,Tx)d(fy,Ty)}{d(fx,fy)}+\beta[d(fx,Tx)+d(fy,Ty)]+\gamma d(fx,fy))\\
= \ & \frac{3}{4}(\alpha\frac{d(fx,Tx)d(fy,Ty)}{d(fx,fy)}+\beta[d(fx,Tx)+d(fy,Ty)]+\gamma d(fx,fy))-sd(Tx,Ty)\\
= \ & \frac{3}{4}(\alpha\frac{|x-4|^2|y-4|^2}{|x-y|^2}+\beta[|x-4|^2+|y-4|^2]+\gamma|x-y|^2)-s|0-0|^2\\
= \ & \frac{3}{4}(\frac{1}{7}\frac{|x-4|^2|y-4|^2}{|x-y|^2}+\frac{1}{9}[|x-4|^2+|y-4|^2]+\frac{1}{8}|x-y|^2)\geq 0.
\end{aligned}$$

That is,

$$\zeta(sd(Tx,Ty),\alpha\frac{d(fx,Tx)d(fy,Ty)}{d(fx,fy)}+\beta[d(fx,Tx)+d(fy,Ty)]+\gamma d(fx,fy))\geq 0.$$

Hence all the hypothesis of the theorem are satisfied. Moreover, 0 *are coincidence poiint of* T *and* f*,* 0 *is the common fixed point of* T *and* f*, and* 0 *is the unique common fixed point of* T *and* f.

Chapter 5

Conclusion and Future scope

5.1 Conclusion

Rao et al. (2020) established common fixed point theorems for a pair of contractive mappings in partially ordered metric space. In this research work ,we introduced common fixed theorems for a pair of contractive mappings in partially ordered b-metric space involving simulation functions and proved the existence and uniqueness for the mappings introduced. Our results extended and generalized related fixed point results in the literature, in particular that of Rao et al.(2020).we have also supported the main results of this researches work by examples.

5.2 Future scope

The existence and uniqueness of common fixed point for a pair of contractive mappings in partially ordered b-metric space involving simulation functions is an active area of research. Recently there are a some of published research papers related to this area of the study. So any interested researchers can use this opportunity and conduct their research work in this area.

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