ACCELERATED FITTED MESH FINITE DIFFERENCE METHOD FOR SINGULARLY PERTURBED SELF-ADJOINT BOUNDARY VALUE PROBLEM

A thesis submitted to the department of mathematics Jimma university in partial fulfillment of the requirements for the degree of master of science in mathematics.

(Numerical Analysis)

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DECLARATION

I under designed declare that, this research proposal entitled "Accelerated fitted mesh finite difference method for singularly perturbed Self-adjoint boundary value problem" is my own original work and it has not been submitted to any institution elsewhere for the award of any academic degree or the like.

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Contents

Acronyms

* BVP - Boundary Value Problem.
* SP - Singularly Perturbed.
* SPBVP - Singularly Perturbed Boundary Value Problem.
* SPP - Singularly Perturbed Problem.
* RPP -Regular perturbation proble
* 1D -One Dimensional.
* FMFDM -Fited Mesh Finite Difference Method.
* FDM -Finite Difference Method.
* DE-Differential Equation.
* ODE -Ordinary Differential Equation.
* PDE -Partial Differential Equation.

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Abstract

In this thesis, accelerated fitted mesh finite difference method is presented for solving Singularly perturbed Self-adjoint boundary value problems. First, the derivatives of the differential equation are transformed into finite difference approximations that make linear system of algebraic equations in the form of a three-term recurrence relation which can easily be solved by Thomas algorithm. And then, Richardson extrapolation method is applied to accelerate the convergence. Second, establish the convergence of the proposed method very well. Finally, validate results using numerical model examples and compared with other methods listed in the literature and exact solution. Maximum absolute error for each model example shown by tables and behavior of graphs with different perturbation parameters and mesh sizes which shows the betterment of the present method.

Chapter 1

Introduction

1.1 Background of the study

A differential equation(DE) is an equation which contains the derivatives of one or more dependent variables, with respect to one or more independent variables.A DE contains derivatives which are either ordinary or partial derivatives. If an equation contains only partial derivatives of one or more dependent variables with respect to a single independent variable, it is said to be partial differential equation(PDE). where as if an equation contains only ordinary derivatives of one or more dependent variables with respect to a single independent variable, it is said to be ordinary differential equation(ODE). Any ODE obtained from a given DE and having the property that its solution is an integrating factor of the other is known as adjoint DE (Siraj et al., 2019). If the coefficients $a_0(x)$, $a_1(x)$ and $a_2(x)$ in the DE of the form:

$$a_0(x)y''(x) + a_1(x)y'(x) + a_2(x)y(x) = 0, \tag{1.1}$$

are continuous and $a_0(x) \neq 0$ with the given domain, the obtained DE can be transformed into the equivalent self-adjoint equation of $(a(x)y'(x))' + b(x)y(x) = 0$
for the functions $a(x) = e^{\int \frac{a_1(x)}{a_0(x)} d(x)}$ and $b(x) = \frac{a_2(x)}{a_0(x)} a(x)$. A self-adjoint DE, whose highest

order derivative is multiplied by a small positive parameter, and of the form:

$$-\varepsilon(a(x)y'(x))' + b(x)y(x) = f(x) \tag{1.2}$$

is called second order self-adjoint SPP. A SPP is a problem containing a small positive parameter that cannot be approximated by setting the parameter value to zero (Siraj et al., 2019).

In singularly perturbed differential problem, small positive parameter affecting the highest order derivative(s) of the DE which gives rise to large gradients in the solution over narrow regions of the domain. So that, the presence of a small perturbation parameter in the differential equation typically leads to boundary layers in the solution, which makes the convergence analysis very difficult (Suayip and Niyazi, 2013). As Miller (1996), boundary layer is a region of the independent variable over which the dependent variable changes rapidly.

Singularly perturbed second order two-point BVP occur very frequently in fluid motion, chemical reactor theory, elasticity, diffusion in polymer, reaction- diffusion equation, control of chaotic system and so on (Kadalbajoo and Kumar, 2008). If the order of SPDEs of the reduced problem is decreased by one, then the problem called as convection-diffusion type and if the order is reduced by two it is called reaction-diffusion type. Hence, second order SP self-adjoint ODEs are types of convection-diffusion problem.

Due to the importance of these problems in real life situations, the need to develop numerical methods(NMs) for approximating its solution is advantageous. But, numerically solving the singularly perturbed differential equations depends upon the small positive parameters. So that, the solution varies rapidly in some parts of the domain and varies slowly in some other parts of the domain because of the existence of boundary layer (Siraj et al., 2019).

The solution of second order SP self-adjoint two-point BVP exhibits one or two layers. For solving this problem having two layers, the existing numerical methods give good results when the mesh size is smaller than the perturbation parameter. But it is expensive and time-consuming process (Fasika et al., 2017). If we take the mesh size is greater than the perturbation parameter, the classical NMs produce oscillatory solution and pollute the solution in the entire interval, because of boundary layer behavior. As a result, developing NMs for solving SP self-adjoint problems yield consideration of the researches.

Recently, different scholars such as Fasika et al., (2016), and (2017) and Feyisa and Gemchis, (2017) have developed a higher (fourth, sixth, eighth and tenth) order compact FDM to solve SP reaction diffusion problems. These authors developed higher order compact FDMs, by considering the condition for the coefficients of diffusion and reaction terms are constant only. Thus, even if their methods produce more accurate numerical solution, it is restricted to treat the problems with constant coefficients of diffusion and reaction term.

Also, other scholar's, Terefe et al., (2016) and Yitbarek et al., (2017) have presented fourth and sixth order stable central difference method for solving self-adjoint SP two-point BVP. Further (Murad et al., 2020) have presented NM solution of Sp self-adjoint BVP using Galarkin method and also (Siraj et al, 2019) have also presented fourth order stable central difference method with Richardson extrapolation method for second-order singularly perturbed self-adjoint BVPs. So far, some of those recently developed methods works for non-constant coefficient and produce good accurate solution. As well in all of these currently developed NMs, the perturbation parameter is comparable with the mesh size of the solution domain.

However, the numerical solutions are in a good agreement with the exact one for which most classical numerical methods do not give good result. Yet, it needs to improve the accuracy of the solution when the perturbation parameter is smaller than the mesh size of the solution domain with higher order of convergence.

Therefore, the purpose of this study is to develop an accelerated FMFDM that gives a more accurate solution for solving SP self-adjoint BVPs.

1.2 Objectives

1.2.1 General objective

The general objective of this study is to develop an accelerated fitted mesh finite difference method for singularly perturbed self-adjoint boundary value problems.

1.2.2 Specific objectives

The specific objectives of the present study are:

* To formulate accelerated fitted mesh finite difference method for solving singularly perturbed self-adjoint boundary value problems.

* To establish the convergence of the method.

* To investigate the accuracy of the method

1.3 Significance of the study

The outcomes of this study may have the following importance:

* Help the graduate students to acquire research skills and scientific procedures.

* To introduce the application of numerical methods in different field of studies.

* Serve as a reference material for scholars who works on this area.

* Provide a numerical method for solving self-adjoint singularly perturbed boundary value problems.

1.4 Delimitation of the study

This study is delimited to develop and analysis the accelerated fitted mesh finite difference method for singularly perturbed self-adjoint boundary value problem of the form:

$$-\varepsilon(a(x)y'(x))' + b(x)y(x) = g(x), \qquad x \in (0,1),$$

with the boundary conditions:

$$y(0) = \alpha \ and \ y(1) = \beta,$$

where, ε is a perturbation parameter that satisfies, $0 < \varepsilon << 1$ and α, β are arbitrary constants. Functions $a(x) \geq a > 0$, $b(x) \geq b > 0$, and $g(x)$ are assumed to be sufficiently continuous functions on the stated domain.

Chapter 2

Review of Related Literature

2.1 Boundary value problem

In the field of differential equations, a BVP is a DE together with a set of additional restraints, called the boundary conditions. A solution to a BVP is a solution to the differential equation which also satisfies the boundary conditions. It arises in several branches of physics as any physical differential equation will have them. Problems involving the wave equation, such as the determination of normal modes, are often stated as boundary value problems.

A boundary value problem of an ordinary differential equations with solution and derivative values specified at more than one point. Most commonly, the solution and derivatives are specified at just two points (the boundaries) defining a two-point boundary value problem.A boundary value problem for a given differential equation consists of finding a solution of the given differential equation subject to a given set of boundary conditions. A boundary condition is a prescription some combinations of values of the unknown solution and its derivatives at more than one point.

2.2 Singular Perturbation Problem

Singular perturbation problem was first introduced by (Prandtl, 1904) during his talk on fluid motion with small friction in a seven-page report presented at the Third International Congress of Mathematicians in Heidelberg in 1904 in which he demonstrated that fluid flow past over a body can be divide in two regions, a boundary layer and outer region. However, the term singular perturbation was first used by (Friedrichs and Wasow, 1946) in a paper presented at a seminar on non-linear vibrations at New York University. The solutions of singular perturbation problems typically contain layers. Originally (Prandtl, 1904) introduced the term boundary layer, but this term came into more general following the work of (Wasow, 1942).

A brief survey for the historical development of singular perturbation problems is covered in the recent book by (O'Malley, 1991) and (Roos, 2008). More precisely, a perturbation problem is problem that contains a small parameter called perturbation parameter. If the solution of the problem can be approximated by setting the value of the perturbation parameter equal to zero, then the problem is called regular perturbation problem, otherwise it is called singular perturbation problem. That is, if it is impossible to approximate the solution by asymptotic expansion as the perturbation parameter tends to zero, then the problem is called singular.

In real life, we often encounter many problems which are described by parameter dependent differential equations. The behavior of the solution of these types of differential equation depends on the magnitude of the parameter. Any differential equation in which the highest order derivative is multiplied by a small positive parameter is called singular perturbation problem and the parameter is known as the perturbation parameter. Singular perturbation problems (SPPs) have always played prominent role in the theory of differential equation.

If the perturbation parameter is present at other places other than highest derivative, then the problem is called regular perturbation problem (RPPs). In fact, any differential equation whose solution changes rapidly in some parts of the solution domain/interval is generally known as singular perturbation problem and also called boundary layer problem. (Miller, 1996) said that "boundary layer is a region of the independent variable over which the dependent variable changes rapidly".

In numerical analysis, Richardson extrapolation is a sequence acceleration method, used to improve the rate of convergence of a sequence. The basic idea behind extrapolation is that whenever the leading term in the error for an approximation formula is known, we can combine two approximations obtained from that formula using different values of the parameter mesh size h to obtain a higher-order approximation and the technique is known as Richardson extrapolation.

2.3 Fitted Mesh method

A Fitted mesh can be incorporated into both a finite difference and a finite element method. In finite difference method it has two major ingredients: the finite difference operator that is used to approximate the differential operator L and the mesh that replaces the continuous domain Ω.The numerical method with a fitted finite difference scheme on a piecewise uniform mesh with specially chosen transition points separating the coarse and fine mesh are known as fitted mesh methods.

Clearly, the simplest form of fitted mesh is a piecewise uniform mesh with specially chosen transition points separating the coarse and fine meshes. These piecewise uniform fitted meshes were first introduced by and the corresponding numerical methods were further developed, in the book (Shish kin,1992). The first numerical results using a fitted mesh method were presented in (Miller,1991).

2.4 Recent development

Kadalbajoo and Patidar, (2003) are presented, 'Spline approximation method for solving self-adjoint singular perturbation problems on non-uniform grids. In this article, a numerical method based on cubic spline with adaptive grid was given for the self-adjoint singularly perturbed two-point boundary value problems of the form:

$$Ly \equiv -\varepsilon(a(x)y'(x))' + b(x)y(x) = g(x), 0 < x < 1 \tag{2.1}$$

Subject to the boundary conditions:

$$y(0) = \alpha \; and \; y(1) = \beta \tag{2.2}$$

Where α, β are given constants and ε is a small positive parameter. Further, the coefficients of diffusion term $a(x)$ and the coefficient of reaction term $b(x)$ are smooth functions and satisfy the condition $a(x) \geq a > 0$, $a'(x) \geq 0$ and $b(x) \geq b > 0$. The scheme derived in this method is second order accurate and model numerical examples are given to support the predicted theory.

Kailash (2005) developed, 'Higher order fitted operator numerical method for self-adjoint SPPs. Here, authors consider self-adjoint SP two-point BVPs in conservation form. Reducing the original problem into the normal form and then using the theory of inverse monotone matrices, a fitted operator finite difference method is derived via the standard Nemerov's method. The scheme thus derived is fourth order accurate for moderate values of the perturbation parameter ε whereas for very small values of this parameter the method is e-uniformly convergent with order two. Numerical examples are given in support of the theory.

Kadalbajoo and Kumar (2010), Proposed 'Variable mesh finite difference method for self-adjoint singularly perturbed two-point boundary value problems. In this article, a numerical method based on finite difference method with variable mesh is given self-adjoint singularly perturbed two-point boundary value problems. To obtain parameter- uniform convergence, a variable mesh is constructed, which is dense in the boundary region and coarse in the outer region. The uniform convergence analysis of the method discussed. The original problem is reduced to its normal form and the reduced problem solved by finite difference method taking variable mesh. To support the effectiveness of the method, several numerical examples have been considered.

Aruna and Kanth (2012),suggested 'A spline based computational simulations for solving self-adjoint singularly perturbed two-point BVPs. Those proposed a spline based computational simulations for solving self-adjoint SP two-point BVPs. The original problem is reduced to its normal form and the reduced boundary value problem is treated by using difference approximations via cubic splines in tension. The convergence of the method is analyzed. Some numerical examples are given to demonstrate the computational efficiency of the present method.

Khuri and Sayfy (2014), proposed "A patching approach for Self-adjoin SP second-order two-point BVPs. In this article, the basic aim is to introduce and describe a patching approach based on a novel combination of the varational iterative method and adaptive cubic spline collocation scheme for the solution of a class of self-adjoin SP second-order two-point boundary value problems that model various engineering problems. The domain of the problem is decomposed into two subintervals: the varational iterative method is implemented in the area (vicinity) of the boundary layer while in the outer region the resulting problem is tackled by applying an adaptive cubic spline collocation scheme, which comprises the use of mapping/transformation redistribution functions or constructed grading functions.

Numerical results, computational comparisons, appropriate error measures and illustrations are provided to testify the convergence, efficiency and applicability of the method. Performance the method examined through test examples that reveal that the current approach converges to the exact solution rapidly with accurate solution and that the convergence is uniform across the domain. The proposed technique yields numerical solutions in very good agreement with and/or superior to existing exact and approximate solutions.

More recently, Fasika et al., (2016), Fasika et al., (2017) and Feyisa and Gemechis, (2017), have offered the higher (fourth, sixth, eighth and tenth) order compact finite difference method for solving singularly perturbed 1D reaction–diffusion problems. But these methods are developed and applicable only for the coefficient of reaction and diffusion terms are constant in case of second order self-adjoin SPPs.

Besides Terefe et al., (2016), Yitbarek et al., (2017) and Siraj et al, (2019) have proposed, fourth and sixth-order stable central difference methods for solving self-adjoin SPPs. And Murad et al., (2020) have also proposed NM solution of SP self-adjoint BVP usin Galarkin method. Yet, these methods treated the stated problem, for comparable perturbation parameter with the mesh size of the solution domain and till the obtained numerical solution needs to improve accuracy.

Thus, it is necessary to improve the accuracy with higher order of convergence for solving second order self-adjoint singularly perturbed boundary value problems which involves variable coefficient of reaction and diffusion terms. Furthermore, for self-adjoint singularly perturbed boundary value problems with two boundary layers essential to develop numerical method which concerns more accurate numerical solution.

Chapter 3

Methodology

3.1 Study Area and Period

The study was conducted in Jima University under the department of Mathematics from May 2021 to January 2022. Conceptually the study focuses on accelerated fitted mesh finite difference method for singularly perturbed self-adjoint boundary value problems

3.2 Study Design

This study would employ mixed design (documentary review design and numerical experimentation design).

3.3 Source of Information

The relevant sources of information for this study are books, published articles and related studies from Internet services.

3.4 Mathematical Procedure

In order to achieve the stated objectives, the study followed the following steps:

1. Defining the problem.

2. Discretizing the solution domain/interval.

3. Formulating fitted mesh numerical scheme for the defined problem

4. Applying the Richardson extrapolation technique.

5. Establishing the convergence analysis of the formulated problem.

6 Writing MATLAB code for the formulated schemes.

7. Valideting the scheme using numerical illustration.

Chapter 4

Description of the Method and Numerical Results

4.1 Description of the method

In this section, the description of FDM on piecewise uniform mesh has been presented. Consider the sp self-adjoint BVP of the form:

$$-\varepsilon(a(x)y'(x))' + b(x)y(x) = g(x),\ 0 < x < 1, \tag{4.1}$$

with the boundary conditions,

$$y(0) = \alpha \textit{ and } y(1) = \beta, \tag{4.2}$$

where, ε is a perturbation parameter that satisfies, $0 < \varepsilon << 1$ and $a(x)$, $b(x)$ and $f(x)$ are assumed to be sufficiently continuous differentiation functions. By product rule of differentiation, Eq.(4.1) can be re-written as:

$-\varepsilon a(x)y''(x) - \varepsilon a'(x)y'(x) + b(x)y(x) = g(x)$

This can be written as

$$-\varepsilon y''(x) + p(x)y'(x) + q(x)y(x) = f(x) \tag{4.3}$$

where, $p(x) = \dfrac{-\varepsilon a'(x)}{a(x)}$, $q(x) = \dfrac{b(x)}{a(x)}$, and $f(x) = \dfrac{g(x)}{a(x)}$.
Now, to discretized the solution domain for $N \geq 8$ to be an integer such that consider the transition parameter τ is chosen as:

$$\tau = min(\frac{1}{4}, 2\sqrt{\varepsilon}\ln(N)), \tag{4.4}$$

In such discretization,the solution domain is divided into three sub-intervals: $[0,\tau]$, $[\tau, 1-\tau]$ and $[1-\tau, 1]$. Where τ is the width of boundary layer. The intervals $[0,\tau]$ and $[1-\tau,1]$ are each divided into $\dfrac{N}{4}$ equal mesh elements, while the intervals $[\tau, 1-\tau]$ is divided into $\dfrac{N}{2}$ equal mesh elements.Therefore, we have $\dfrac{N}{4}+1$ equidistant grid points in the intervals $[0,\tau]$ and $[1-\tau,1]$ and $\dfrac{N}{2}-1$ equidistant in $[\tau, 1-\tau]$.
We have, $h = h_i + 1 - h_i$ or $h_i = h_i + 1 - h$, where the mesh spacing is given by:

$$h_i = \begin{cases} \frac{4\tau}{N}, & i = 1, 2, ..., \dfrac{N}{4},\ i = \dfrac{3N}{4} + 1, ..., N \\ \dfrac{2(1-2\tau)}{N}, & i = \dfrac{N}{4} + 1, ...3\dfrac{N}{4}. \end{cases} \tag{4.5}$$

For convenience,let $p(x_i) = p_i$, $q(x_i) = q_i$, $y(x_i) = y_i$, $y'(x_i) = y'_i$,....., $y^n(x_i) = y^n_i$.
Assume that $y(x)$ has continuous higher order derivatives on [0,1], and to develop the fitted mesh finite difference (FMFD) scheme, we use Taylor's Series expansion in order to get central difference formula for y''_i and y'_i

$$y_{i+1} = y_i + h_{i+1}y'_i + \frac{h^2_{i+1}}{2}y''_i + \frac{h^3_{i+1}}{6}y'''_i + O(h^4_{i+1}) \tag{4.6}$$

Re-arranging Eq.(4.6) and solving for y_i', we get

$$\delta^+ y_i = y_i' = \frac{y_{i+1} - y_i}{h_{i+1}} - \frac{h_{i+1}}{2} y_i'' + TE_1, \tag{4.7}$$

where, $TE_1 = -\frac{h_{i+1}^2}{6} y_i'''$.

$$y_{i-1} = y_i - h_i y_i' + \frac{h_i^2}{2} y_i'' - \frac{h_i^3}{6} y_i''' + O(h_i^4) \tag{4.8}$$

Re-arranging Eq.(4.8) and solving for y_i', we get

$$\delta^- y_i = y_i' = \frac{y_i - y_{i-1}}{h_i} + \frac{h_i}{2} y_i'' + TE_2, \tag{4.9}$$

where, $TE_2 = \frac{-h_i^2}{6} y_i'''$.
From Eq(4.6) and Eq (4.8) taking the difference of the two we get the central finite difference, and solving for y_i', we get

$$\delta^o y_i = y_i' = \frac{y_{i+1} - y_{i-1}}{h_{i+1} + h_i} + TE_3, \tag{4.10}$$

where, $TE_3 = -\frac{h_{i+1} - h_i}{2} y_i''$.
Subtracting Eq.(4.9) from Eq.(4.7) and re-aranging gives:

$$\frac{y_{i+1} - y_i}{h_{i+1}} - \frac{y_i - y_{i-1}}{h_i} = \frac{(h_{i+1} + h_i)}{2} y_i'' + TE_2 - TE_1, \tag{4.11}$$

Multiplying both sides by $\frac{2}{h_{i+1} + h_i}$ and solving for y_i'', we get

$$\delta^2 y_i = y_i'' = \frac{2}{h_{i+1} + h_i} \left(\frac{y_{i+1} - y_i}{h_{i+1}} - \frac{y_i - y_{i-1}}{h_i}\right) + TE_4, \tag{4.12}$$

where, $TE_4 = \frac{2}{h_{i+1} + h_i}(TE_2 - TE_1)$.
Denoting this discretization of the solution domain by Ω^N , and y_i is the approximation

of $y(x_i)$, so that the discretization form of Eq. (4.1) on Ω^N , for $i = 1, 2, ...N - 1$ is given by:

$$- \varepsilon y_i'' + p_i y_i' + q_i y_i = f_i, \tag{4.13}$$

$$\frac{-2\varepsilon}{h_{i+1} + h_i}(\delta^+ y_i - \delta^- y_i) + p_i \delta^o y_i + q_i y_i = f_i \tag{4.14}$$

where, $\delta^+ y_i = \dfrac{y_{i+1} - y_i}{h_{i+1}}$, $\delta^- y_i = \dfrac{y_i - y_{i-1}}{h_i}$ and $\delta^o y_i = \dfrac{y_{i+1} - y_{i-1}}{h_{i+1} + h_i}$.

This can be Re-written in 3-term recurrence relation as:

$$[\frac{-2\varepsilon}{h_i(h_{i+1} + h_i)} - \frac{p_i}{h_{i+1} + h_i}]y_{i-1} + [\frac{2\varepsilon}{h_i h_{i+1}} + q_i]y_i + [\frac{-2\varepsilon}{h_{i+1}(h_{i+1} + h_i)} + \frac{p_i}{h_{i+1} + h_i}]y_{i+1} = f_i, \tag{4.15}$$

$$- E_i y_{i-1} + F_i y_i - G_i y_{i+1} = H_i, \tag{4.16}$$

where,

$$E_i = \frac{2\varepsilon}{h_i(h_{i+1} + h_i)} + \frac{p_i}{h_{i+1} + h_i}, \qquad F_i = \frac{2\varepsilon}{h_i h_{i+1}} + q_i,$$
$$G_i = \frac{2\varepsilon}{h_{i+1}(h_{i+1} + h_i)} - \frac{p_i}{h_{i+1} + h_i}, \qquad H_i = f_i.$$

4.2 Thomas Algorithm

In this section, the stability of solving the tri-diagonal system is provided. A brief description for solving the tri-diagonal system using the discrete invariant embedding algorithm, also called the Thomas Algorithm, is presented as follows. Consider the scheme above in Eq. (4.16), for $i = 1, 2, ...N - 1$ and subject to the boundary conditions in Eq. (4.2) that can be re-written as: $y(0) = y_0 = \alpha$ and $y(1) = y_N = \beta$

Assume that the solution of Eq.(4.16), is given by:

$$y_i = W_i y_{i+1} + T_i, i = N, N-1,\ N-2, ..., 2, 1, \tag{4.17}$$

where W_i and T_i are to be determined.

Considering Eq.(4.17) at the nodal point x_{i-1}, we have

$$y_{i-1} = W_{i-1} y_i + T_{i-1}, \tag{4.18}$$

Substituting Eq.(4.18) in to Eq.(4.16) gives:

$-E_i(W_{i-1}y_i + T_{i-1}) + F_i y_i - G_i y_{i+1} = H_i,$

which leads to obtaining the equation

$$y_i = \frac{G_i}{F_i - E_i W_{i-1}} y_{i+1} + \frac{H_i + E_i T_{i-1}}{F_i - E_i W_{i-1}}, \tag{4.19}$$

Comparing Eq.(4.19) with Eq.(4.17), the two values determined as:

$$W_i = \frac{G_i}{F_i - E_i W_{i-1}},\ T_i = \frac{H_i + E_i T_{i-1}}{F_i - E_i W_{i-1}}, \tag{4.20}$$

To solve these recurrence relations $i = 1, 2, ...N - 1$, we need the initial conditions for $W_0 = 0$ and we take $T_0 = y_0 = y(0) = \alpha$. With these starting points of initial values, we compute W_i and T_i for $i = 1, 2, ...N - 1$ from Eq. (4.19) in the forward process, and then obtain y_i in the backward process from Eq. (4.16) and from the boundary condition

$y(1) = y_N = \beta$. Further, the conditions for the discrete invariant embedding algorithm to be stable, (See Fasika et. al., (2017)) if and only if:

$$|E_i| = |\frac{2\varepsilon}{h_i(h_{i+1}+h_i)} + \frac{p_i}{h_{i+1}+h_i}| > 0,\ |G_i| = |\frac{2\varepsilon}{h_{i+1}(h_{i+1}+h_i)} - \frac{p_i}{h_{i+1}+h_i}| > 0,$$
$$|F_i| = |\frac{2\varepsilon}{h_i h_{i+1}} + q_i| \geq 0,\ and,\ |F_i| \geq |E_i| + |G_i| \tag{4.21}$$

Hence, the Thomas Algorithm is stable for the proposed method.

4.3 Truncation error

In this section, the truncation error for the described method will be investigated. The local truncation error is given by:

$$T(h_i) = -\varepsilon y''(x_i) + p(x_i)y'(x_i) + q(x_i)y(x_i) - [\frac{-2\varepsilon}{h_{i+1}+h_i}(\delta^+ y_i - \delta^- y_i) + p_i\delta^o y_i + q_i y_i]$$

$$\begin{aligned} T(h_i) = -\varepsilon y''(x_i) + p(x_i)y'(x_i) + q(x_i)y(x_i) + \frac{2\varepsilon}{h_{i+1}+h_i} \\ (\frac{y_{i+1}-y_i}{h_{i+1}} - \frac{y_i - y_{i-1}}{h_i}) - p_i(\frac{y_{i+1}-y_{i-1}}{h_{i+1}+h_i}) - q_i y_i, \end{aligned} \tag{4.22}$$

Using Taylor's series expansion to y_i around x_i, we have the approximation for $y_{i\pm1}$ as:

$$y_{i+1} = y_i + h_{i+1}y_i' + \frac{h_{i+1}^2}{2}y_i'' + \frac{h_{i+1}^3}{6}y_i''' + \frac{h_{i+1}^4}{24}y_i^4 + O(h_{i+1}^5) \tag{4.23}$$

$$y_{i-1} = y_i - h_i y_i' + \frac{h_i^2}{2}y_i'' - \frac{h_i^3}{6}y_i''' + \frac{h_i^4}{24}y_i^4 + O(h_i^5) \tag{4.24}$$

From these two basic equations, we obtain:

$$\begin{cases} \frac{y_{i+1}-y_i}{h_{i+1}} = y_i' + \frac{h_{i+1}}{2}y_i'' + \frac{h_{i+1}^2}{6}y_i''' + \frac{h_{i+1}^3}{24}y_i^4 + O(h_{i+1}^4) \\ \frac{y_i - y_{i-1}}{h_i} = y_i' - \frac{h_i}{2}y_i'' + \frac{h_i^2}{6}y_i''' - \frac{h_i^3}{24}y_i^4 + O(h_i^4) \end{cases} \tag{4.25}$$

Substituting Eq.(4.25) in to Eq. (4.22), we get:

$$T(h_i) = \frac{\varepsilon(h_{i+1} - h_i)}{3} y'''(x_i) + \frac{\varepsilon(h_{i+1}^2 + h_i^2 - h_i h_{i+1})}{12} y^4(x_i) + ... \tag{4.26}$$

Since at the nodal point x_i, we have:

$y''(x_i) = y_i^{''}$, $y'(x_i) = y_i'$ and $q(x_i)y(x_i) = q_i y_i$.

Then, Eq.(4.26) can be simplified as:

$$T(h_i) = \frac{\varepsilon(h_{i+1} - h_i)}{3} y_i''' + \frac{\varepsilon(h_{i+1}^2 + h_i^2 - h_i h_{i+1})}{12} y_i^4 + ... \tag{4.27}$$

From the considered piecewise discretization of the solution domain in Eq.(4.5), and from the values of $\tau = 2\sqrt{\varepsilon}\ln(N)$, we have:

Case 1: In the layer region, we have:

$h_i = h_{i+1} = h_s$ and $h_s = \dfrac{4\tau}{N} = \dfrac{8\sqrt{\varepsilon}\ln(N)}{N}$, that gives the truncation error from Eq.(4.27)as:

$$T(h_i) = \frac{\varepsilon(h_{i+1} - h_i)}{3} y_i''' + \frac{\varepsilon(h_{i+1}^2 + h_i^2 - h_i h_{i+1})}{12} y_i^4 + ... \tag{4.28}$$

$$T(h_i) = \frac{\varepsilon(h_{i+1}^2 + h_i^2 - h_i h_{i+1})}{12} y_i^4 + ... \tag{4.29}$$

$$T(h_i) = \frac{\varepsilon h_s^2}{12} y_i^4 + ... = \frac{16\varepsilon^2 \ln(N^2)}{3N^2} y_i^4 + ... \tag{4.30}$$

Case 2: In the outer layer region, we have:

$h_i = h_{i+1} = h_b$ and $h_b = \dfrac{2}{N} - \dfrac{4\tau}{N} = \dfrac{2}{N} - \dfrac{8\sqrt{\varepsilon}\ln(N)}{N}$, that gives the truncation error from Eq.(4.27) as:

$$T(h_i) = \frac{\varepsilon(h_{i+1} - h_i)}{3} y_i''' + \frac{\varepsilon(h_{i+1}^2 + h_i^2 - h_i h_{i+1})}{12} y_i^4 + ... \tag{4.31}$$

$$T(h_i) = \frac{\varepsilon(h_{i+1}^2 + h_i^2 - h_i h_{i+1})}{12} y_i^4 + ... \tag{4.32}$$

$$T(h_i) = \varepsilon \frac{h_b^2}{12} y_i^4 + ... = \frac{\varepsilon(2 - 8\sqrt{\varepsilon}\ln(N^2))}{12N^2} y_i^4 + ... \tag{4.33}$$

Case 3: In the neighborhoods between the inner and outer layer region, we have: $h_m = h_{i+1} - h_i$, or $h_m = h_i - h_{i+1}$. Hence,for the first point,we have:

$$\begin{cases} h_m = h_{i+1} - h_i = \dfrac{2(1-2\tau)}{N} - \dfrac{4\tau}{N} = 2\dfrac{((1 - 8\sqrt{\varepsilon}\ln(N)))}{N}, \ \textit{Similarly}, \\ h_m = h_i - h_{i+1} = \dfrac{4\tau}{N} - \dfrac{2(1-2\tau)}{N} = 2\dfrac{(8\sqrt{\varepsilon}\ln(N) - 1)}{N}. \end{cases}$$

Thus the truncation error in Eq.(4.27) become:

$$\begin{aligned} T(h_i) &= \frac{\varepsilon(h_{i+1} - h_i)}{3} y_i''' + ... \\ T(h_i) &= \frac{2\varepsilon(1 - 8\sqrt{\varepsilon})\ln(N)}{3N} y_i''' + ... \end{aligned} \tag{4.34}$$

Hence, one can observe from the three cases that the first truncation error for the described method is provided in Eq. (4.34). Moreover, due to the considered problem exhibits two boundary layers, the described scheme on piecewise discretization. Thus, we have to consider the values of the perturbation parameter $\varepsilon \leq \frac{4}{N}$.
Substituting these values into Eq.(4.34) gives:

$$\begin{aligned} T(h_i) &= \frac{2\varepsilon(1 - 8\sqrt{\varepsilon})\ln(N)}{3N} y_i''' + ... \\ T(h_i) &\leq \frac{2(\frac{4}{N})}{3N}(1 - 8\sqrt{\frac{4}{N}}\ln N) y_i''' + ... \leq \frac{8}{3N^2} y_i''' + ... \end{aligned} \tag{4.35}$$

Thus, the norm of this truncation error is given by:

$$||T(h_i)|| \leq CN^{-2}, \tag{4.36}$$

where, $C = \frac{8}{3}||y_i'''||_\infty$ is arbitrary constant.
Furthermore, within each sub-interval $[0, \tau]$, $[1-\tau, 1]$, and $[\tau, 1-\tau]$, we have the uniform

mesh length $h \leq \frac{1}{N}$. Thus, $h^2 \leq N^{-2}$.
Therefore, the described method is second-order convergent. Truncation errors refer to the differences between the original differential equation and its finite difference approximations at grid points. This error measure how well a finite difference discretization approximates the differential equation,(Siraj et al., 2019).Thus, the developed scheme is second-order accurate.
As a book by Zhilin et al., (2008), a finite difference scheme is called consistent if the limit of truncation error is equal to zero as the mesh size goes to zero. Hence, this definition of consistency on the described method with the local truncation error in Eq. (4.36) is satisfied as:

$$\lim_{h_i \to 0} T(h_i) = \lim_{h_i \to 0} \frac{8}{3N^2} y_i''' = \lim_{h_i \to 0} C h_i^2 = 0$$

Therefore, the proposed method is consistent.

4.4 Richardson extrapolation

The basic idea behind extrapolation is that whenever the leading term in the error for an approximation formula is known, we can combine two approximations obtained from that formula using different values of the mesh sizes h_i, $\frac{h_i}{2}$, $\frac{h_i}{4}$, $\frac{h_i}{8}$,... to obtain a higher-order approximation and the technique is known as Richardson extrapolation. This procedure is a convergence acceleration technique which consists of considering a linear combination of two computed approximations of a solution (on two nested meshes). The linear combination turns out to be a better approximation.

From Eq.(4.36), we have

$$|y(x_i) - y_N| \leq C(h^2), \tag{4.37}$$

where, $y(x_i)$ and y_N are exact and approximate solutions respectively, C is constant independent of mesh sizes h and perturbation parameter.

Let Ω^{2N} be the mesh obtained by bisecting each mesh interval in Ω^N and denote the approximation of the solution on Ω^{2N} by y_{2N}.

Consider Eq.(4.37) works for any $h \neq 0$, which implies:

$$y(x_i) - y_N \leq C(h^2) + R^N, x_i \in \Omega^N. \tag{4.38}$$

So that, it works for any $\frac{h}{2} \neq 0$ yields:

$$y(x_i) - y_{2N} \leq C(\frac{h}{2})^2 + R^{2N}, x_i \in \Omega^{2N}, \tag{4.39}$$

where, the remainders R^N and R^{2N} are $O(h^4)$.

A combination of inequalities in Eq.(4.38) and (4.39) leads to $3y(x_i) - (4y^{2N} - y_N) = O(h^4)$, which suggests that:

$$(y_N)^{ext} = \frac{1}{3}(4y_{2N} - y_N), \tag{4.40}$$

is also an approximation of $y(x_i)$. Using this approximation to evaluate the truncation error, we obtain:

$$|y(x_i) - (y_N)^{ext}| \leq Ch^4 \tag{4.41}$$

Now,using these two different solutions which are obtained by the same scheme given by Eq.(4.16), we get another third solution in terms of the two by Eq.(4.41). This is Richardson extrapolation method for the second-order finite difference scheme only to accelerate the rate of convergence to fourth-order. Similar to these procedures eliminating the constant from the system of equations:

$$\begin{cases} y(x_i) - y_i^N \approx C_1 h^2 + C_2 h^4 + C_3 h^6 + ... \\ y(x_i) - y_i^{2N} \approx C_1 (\frac{h}{2})^2 + C_2 (\frac{h}{2})^4 + C_3 (\frac{h}{2})^6 + ... \\ y(x_i) - y_i^{4N} \approx C_1 (\frac{h}{4})^2 + C_2 (\frac{h}{4})^4 + C_3 (\frac{h}{4})^6 + ... \\ y(x_i) - y_i^{8N} \approx C_1 (\frac{h}{8})^2 + C_2 (\frac{h}{8})^4 + C_3 (\frac{h}{8})^6 + ... \end{cases} \tag{4.42}$$

Eliminating C_1 from Eq.(4.42), we obtain

$$\begin{cases} y(x_i) - y_i^N \approx C_1 h^2 + C_2 h^4 + C_3 h^6 + ... \\ y(x_i) - y_i^{2N} \approx C_1 (\frac{h}{2})^2 + C_2 (\frac{h}{2})^4 + C_3 (\frac{h}{2})^6 + ... \end{cases}$$

$$y(x) - \frac{1}{3}[4y_i^{2N} - y_i^N] = -\frac{1}{4} C_2 h^4 - \frac{5}{16} C_3 h^6 + ... \tag{4.43}$$

Similarly,

$$\begin{cases} y(x_i) - y_i^{2N} \approx C_1 (\frac{h}{2})^2 + C_2 (\frac{h}{2})^4 + C_3 (\frac{h}{2})^6 + ... \\ y(x_i) - y_i^{4N} \approx C_1 (\frac{h}{4})^2 + C_2 (\frac{h}{4})^4 + C_3 (\frac{h}{4})^6 + ... \end{cases}$$

$$y(x_i) - \frac{1}{3}[4y_i^{4N} - y_i^{2N}] = -\frac{1}{64} C_2 h^4 - \frac{5}{1024} C_3 h^6 + ... \tag{4.44}$$

And also,

$$\begin{cases} y(x_i) - y_i^{4N} \approx C_1 (\frac{h}{4})^2 + C_2 (\frac{h}{4})^4 + C_3 (\frac{h}{4})^6 + ... \\ y(x_i) - y_i^{8N} \approx C_1 (\frac{h}{8})^2 + C_2 (\frac{h}{8})^4 + C_3 (\frac{h}{8})^6 + ... \end{cases}$$

$$y(x_i) - \frac{1}{3}[4y_i^{8N} - y_i^{4N}] = -\frac{1}{1024} C_2 h^4 - \frac{5}{65536} C_3 h^6 - \frac{21}{4194304} C_4 h^8 ... \tag{4.45}$$

Thus, the approximations

$$\begin{cases} (y_i^N)^{4ext} = \frac{1}{3}[4y_i^{2N} - y_i^N] \\ (y_i^{2N})^{4ext} = \frac{1}{3}[4y_i^{4N} - y_i^{2N}] \\ (y_i^{4N})^{4ext} = \frac{1}{3}[4y_i^{8N} - y_i^{4N}] \end{cases}$$

in Eq.(4.43) Eq.(4.44) and Eq.(4.45) are $O(h^4)$ approximations to $y(x_i)$

Eliminating C_2 from Eq. (4.43) and Eq. (4.44), gives:

$$\begin{cases} y(x) - \frac{1}{3}[4y_i^{2N} - y_i^N] = -\frac{1}{4}C_2h^4 - \frac{5}{16}C_3h^6 + ... \\ y(x_i) - \frac{1}{3}[4y_i^{4N} - y_i^{2N}] = -\frac{1}{64}C_2h^4 - \frac{5}{1024}C_3h^6 - \frac{21}{16384}C_4h^8 ... \end{cases}$$

$$y(x_i) - \frac{1}{45}[16(4y_i^{4N} - y_i^{2N}) - (4y_i^{2N} - y_i^N)] = -\frac{1}{64}C_3h^6 + \frac{21}{1024}C_4h^8 + ... \tag{4.46}$$

Eliminating C_2 from Eq. (4.44) and Eq. (4.45), gives:

$$\begin{cases} y(x_i) - \frac{1}{3}[4y_i^{4N} - y_i^{2N}] = -\frac{1}{64}C_2h^4 - \frac{5}{1024}C_3h^6 - \frac{21}{16384}C_4h^8 ... \\ y(x_i) - \frac{1}{3}[4y_i^{8N} - y_i 4N] = -\frac{1}{1024}C_2h^4 - \frac{5}{65536}C_3h^6 - \frac{21}{4194304}C_4h^8 ... \end{cases}$$

$$y(x_i) - \frac{1}{45}[16(4y_i^{8N} - y_i^{4N}) - (4y_i^{4N} - y_i^{2N})] = -\frac{1}{4096}C_3h^6 + \frac{21}{262144}C_4h^8 + ... \tag{4.47}$$

Thus, the approximations

$$\begin{cases} (y_i^N)^{6ext} = \frac{1}{45}[16(4y_i^{4N} - y_i^{2N}) - (4y_i^{2N} - y_i^N)] \\ (y_i^{2N})^{6ext} = \frac{1}{45}[16(4y_i^{8N} - y_i^{4N}) - (4y_i^{4N} - y_i^{2N})] \end{cases}$$

in Eq.(4.46) and Eq.(4.47) are $O(h^6)$ approximation to $y(x_i)$.

This is Richardson extrapolation method for the fourth order FDS only to accelerate the rate of convergence to sixth order.

4.5 Numerical Examples and Results

To validate the applicability of the method, two model examples of second order self-adjoint singularly perturbed boundary value problems have been considered.

Example 1: Consider the singularly perturbed problem (Terefe et al., 2016, Seraj et al., 2019)

$$\begin{cases} -\varepsilon((1+x^2)y'(x))' + (1+x-x^2)y(x) = f(x), \quad 0 < x < 1, \\ y(0) = 0 = y(1), \end{cases}$$

where $f(x)$ is chosen such that the exact solution is given by:

$$y(x) = 1 + (x-1)^{\frac{-x}{\sqrt{\varepsilon}}} - x^{\frac{1-x}{\sqrt{\varepsilon}}}$$

. **Example 2**: Consider the singularly perturbed problem (Yitbarek et al., 2017),

$$\begin{cases} -\varepsilon((1+x^2)y'(x))' + (\frac{cosx}{3-x^3})y(x) = 4(3x^2-3x+1)((x-0.5)^2)+2), \quad 0 < x < 1, \\ y(0) = -1, \\ y(1) = 0. \end{cases}$$

The exact solution for this problem is not available. The numerical results are obtained by using the double mesh principle (Yitbarek et al., 2017) and tabulated in terms of maximum absolute errors in Tables and Figures as follow.

Table 4.1: The comparison of maximum absolute errors of Example 1

$\varepsilon \downarrow N \rightarrow$	16	32	64	128	256
Present Method					
2^{-8}	3.0246e-06	5.4027e-08	9.1254e-10	1.4391e-11	2.2626e-13
2^{-12}	1.9527e-04	3.5828e-06	5.7493e-08	2.4432e-09	8.7385e-11
Seraj et al., (2019)					
2^{-8}	1.6978e-06	2.8399e- 08	4.5168e-10	7.0878e-12	1.0836e-13
2^{-12}	4.1790e-04	2.1462e-05	9.9671e-07	1.8913e-08	3.0955e-10
Terefe et al., (2016)					
2^{-8}	1.65e-03	2.33e-04	6.09e-05	1.73e-05	4.37e-06
2^{-12}	3.76e-02	7.40e-03	6.17e-04	3.54e-05	4.74e-06

Table 4.2: The comparison of maximum absolute errors of Example 2

$\varepsilon \downarrow N \rightarrow$	32	64	128	256
Present Method				
2^{-6}	6.2489e-11	3.0926e-12	1.0786e-11	2.8320e-11
2^{-8}	2.8815e-10	6.8070e-12	1.9501e-11	7.0386e-11
2^{-10}	9.9511e-08	2.1603e-10	9.8623e-11	3.9485e-11
2^{-12}	5.1278e-05	4.9372e-07	4.1857e-09	1.1885e-10
2^{-14}	5.0581e-03	7.3639e-05	8.5161e-07	9.1729e-09
Yitbarik et al., 2017				
2^{-6}	6.1739e-04	1.5402e-04	3.8510e-05	9.6262e-06
2^{-8}	1.2821e-03	3.2272e-04	8.0755e-05	2.0193e-05
2^{-10}	1.4094e-03	3.9358e-04	1.0097e-04	2.5404e-05
2^{-12}	3.1577e-03	3.2669e-04	9.0865e-05	2.3923e-05
2^{-14}	8.1928e-02	5.0797e-03	2.6048e-04	2.6985e-05

Table 4.3: Computed maximum absolute errors and Rate of convergence for Example 1

$\varepsilon \downarrow N \rightarrow$	32	64	128	256
Sixth order				
2^{-8}	5.4027e-08	9.1254e-10	1.4391e-11	2.2626e-13
	5.8876	5.9866	5.9910	
2^{-10}	1.2446e-06	4.8857e-08	8.2758e-10	1.3062e-11
	4.6710	5.8835	5.9855	
2^{-12}	3.5828e-06	5.7493e-08	2.4432e-09	8.7385e-11
	5.9616	4.5565	4.8052	
Fourth order				
2^{-8}	3.3700e-05	2.1492e-06	1.3501e-07	8.4491e-09
	3.9709	3.9927	3.9981	
2^{-10}	2.4113e-04	2.8615e-05	1.8253e-06	1.1467e-07
	3.0750	3.9706	3.9926	
2^{-12}	2.2043e-04	3.0584e-05	3.5790e-06	3.8759e-07
	2.8495	3.0951	3.2070	
Second order				
2^{-8}	6.3033e-03	1.6011e-03	4.0354e-04	1.0099e-04
	1.9770	1.9883	1.9985	
2^{-10}	1.6052e-02	5.7063e-03	1.4527e-03	3.6623e-04
	1.4921	1.9738	1.9879	
2^{-12}	1.5208e-02	5.8687e-03	2.0339e-03	6.6725e-04
	1.3737	1.5288	1.6079	

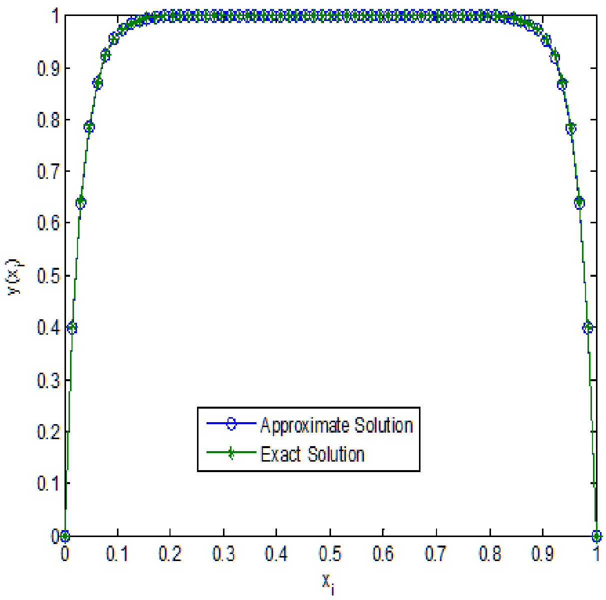

Figure 4.1: Numerical solution for Example 1, when $\varepsilon = 2^{-10}$ and $N = 64$.

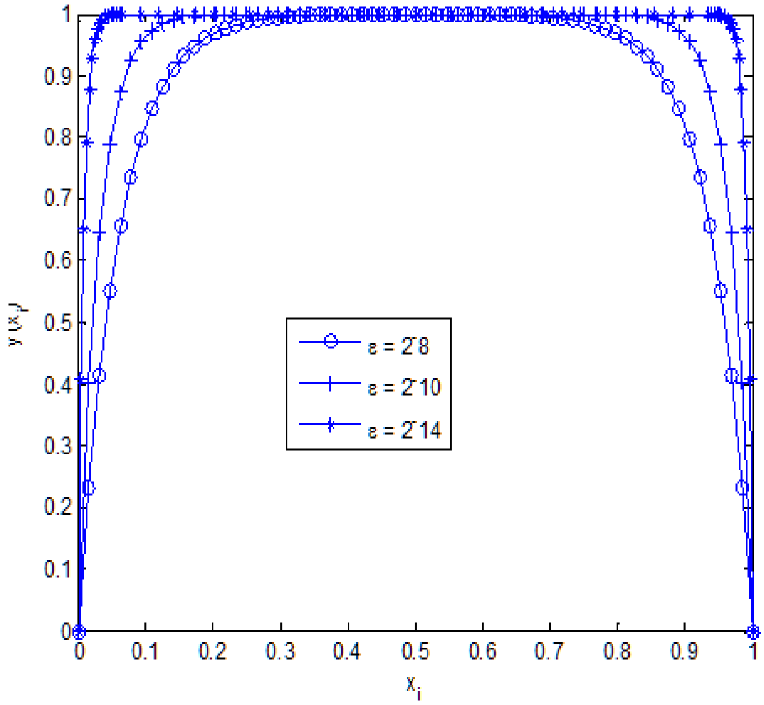

Figure 4.2: Numerical solution for Example 1, when $\varepsilon = 2^{-8}, 2^{-10}, 2^{-14}$ and $N = 64$.

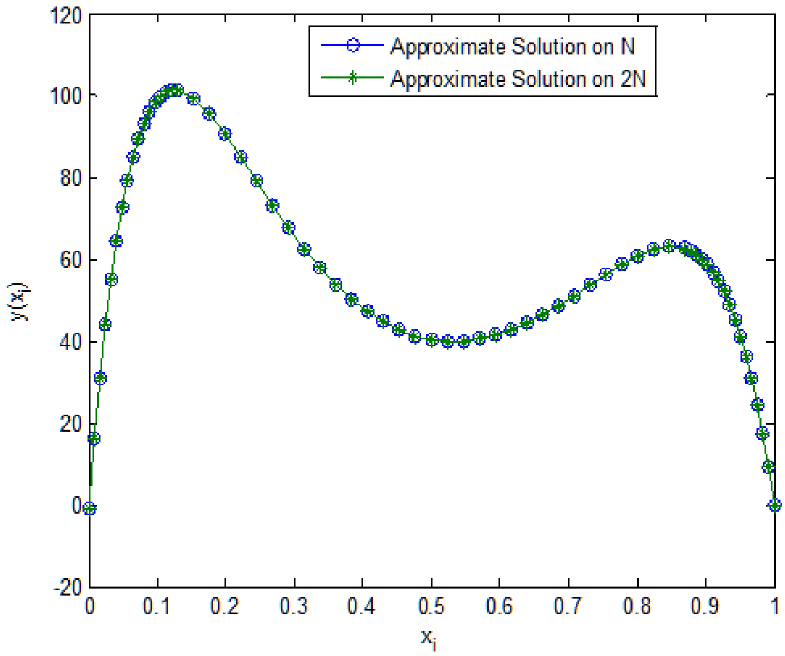

Figure 4.3: Numerical solution for Example 2, when $\varepsilon = 2^{-12}$ and $N = 64$

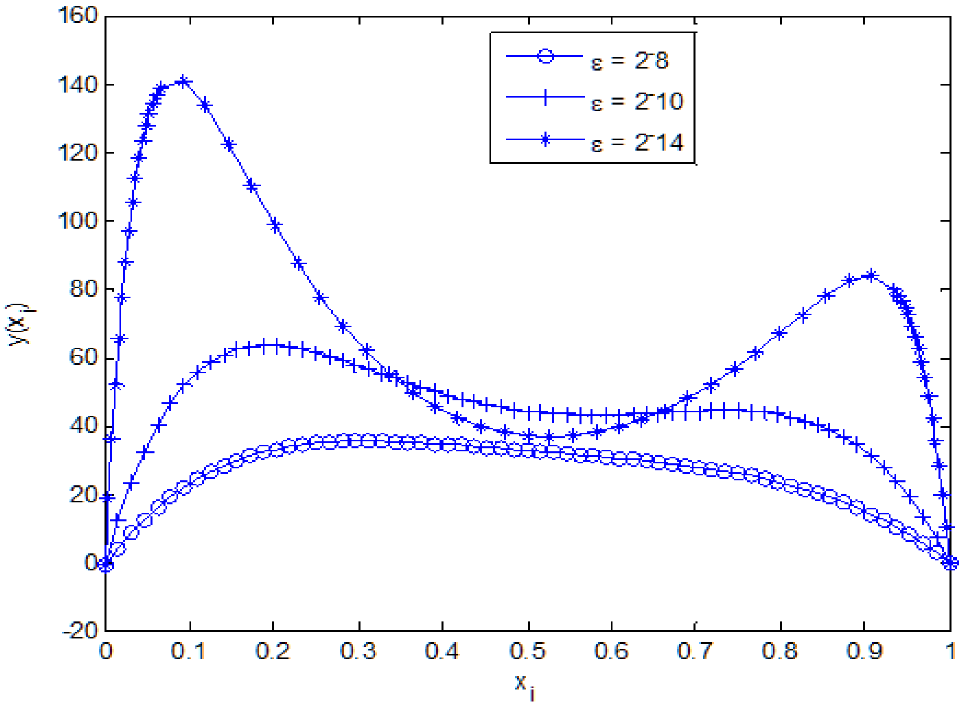

Figure 4.4: Numerical solution for Example 2, when $\varepsilon = 2^{-8}, 2^{-10}, 2^{-14}$ and $N = 64$.

Chapter 5

Conclusion,Discussion and Recommendation

5.1 Discussion and Conclusion

In this paper, we described an accelerated fitted mesh finite difference method for singularly perturbed self-adjoint boundary value problems. To demonstrate the competence of the method, we applied it on two model examples by taking different values for the perturbation parameter, ε and mesh size h.

The results obtained can be obtained from the tables that the present method gives better results than the findings of the existing methods' in the literature. Moreover, the maximum absolute errors decrease rapidly as the number of mesh points N increases. Further, as shown in Figs. 4.1 and 4.3, the proposed method approximates the exact solution very well for $h \geq \varepsilon$, for which most of the current methods fail to give good results. To further verify the applicability of the planned method, graphs are plotted aimed at Examples 1 and 2 for exact solutions versus the numerical solutions obtained. As Figs. 4.1 and 4.3 indicate good agreement of the results, presenting exact as well as numerical solutions, which proves the reliability of the method. Also, Figs. 4.2 and 4.4 specify the effects of perturbation parameter and mesh sizes of the solution domain.

Further, the numerical results presented in this paper validate the improvement of the proposed method over some of the existing methods described in the literature. Both the theoretical and numerical error bounds have been established for the fourth and sixth-order methods.

Hence, the Richardson extrapolation method accelerates fourth order into sixth order convergent as given in Table 4.3. The results in Table 4.3 further confirmed that the computed maximum absolute errors, rate of convergence and theoretical estimates are in agreement(Table 4.3).

Generally, the present method is consistent, stable, and gives more accurate numerical solution for solving second-order singularly perturbed self- adjoint boundary value problems.

5.2 Recommendation

In this thesis, an accelerated fitted mesh finite difference method for singularly perturbed self-adjoint boundary value problems. The scheme proposed in this study can also be extended to fourth order to six or more orders accelerated fitted mesh finite difference method for singularly perturbed self-adjoint boundary value problems.

References

- Aruna, K., and Kanth, A. R. (2012). A spline based computational simulations for solving self-adjoint singularly perturbed two-point boundary value problems. J Inform Comput Sci, 7, 303-314

- Asrat, T., File, G., and Aga, T. (2016). Fourth-order stable central difference method for self-adjoint singular perturbation problems. Ethiopian Journal of Science and Technology, 9(1), 53-68.

- Friedrichs, K. O., and Wasow, W. R. (1946). Singular perturbations of non-linear oscillations. Duke Mathematical Journal, 13(3), 367-381.

- Galu, F. W., Duressa, G. F., and Bullo, T. A. (2016). Tenth order compact finite difference method for solving singularly perturbed 1D reaction-diffusion equations. International Journal of Engineering and Applied Sciences, 8(3), 15-24.

- Gelu, F. W., Duressa, G. F., and Bullo, T. A. (2017). Sixth-order compact finite difference method for singularly perturbed 1D reaction diffusion problems. Journal of Taibah University for Science, 11(2), 302-308.

- Kadalbajoo, M. K., and Aggarwal, V. K. (2005). Fitted mesh B-spline collocation method for solving self-adjoint singularly perturbed boundary value problems. Applied Mathematics and Computation, 161(3), 973-987.

- Kadalbajoo, M. K., and Gupta, V. (2010). A brief survey on numerical methods for solving singularly perturbed problems. Applied mathematics and computation, 217(8), 3641-3716.

- Kadalbajoo, M. K., and Kumar, D. (2008). Parameter-uniform fitted operator B-spline collocation method for self-adjoint singularly perturbed two-point boundary value problems. Electronic Transactions on Numerical Analysis, 30, 346-358.

- Kadalbajoo, M. K., and Kumar, D. (2010). Variable mesh finite difference method for self-adjoint singularly perturbed two-point boundary value problems. Journal of Computational Mathematics, 711-724.

- Kadalbajoo, M. K., and Patidar, K. C. (2003). Spline approximation method for solving self-adjoint singular perturbation problems on non-uniform grids. Journal of Computational Analysis and Applications, 5(4), 425-451.

- Khuri, S. A., and Sayfy, A. (2014). Self-adjoint singularly perturbed second-order two-point boundary value problems: A patching approach. Applied Mathematical Modelling, 38(11-12), 2901-2914.

- Miller, J. J., O'riordan, E., and Shishkin, G. I. (1996). Fitted numerical methods for singular perturbation problems: error estimates in the maximum norm for linear problems in one and two dimensions. World scientific.

- Murad. H., Gemechis.F. and Habtamu G. (2020) have presented NM solution of Sp self-adjoint BVP using Galarkin method.International Journal of Engineering, Science and Technology, 12(3),26-37.

- Patidar, K. C. (2005). High order fitted operator numerical method for self-adjoint singular perturbation problems. Applied mathematics and computation, 171(1), 547-566.

- Rashidinia, J., Mohammad, R.,and Moatamedoshariati, S. H. (2010). Quintic spline methods for the solution of singularly perturbed boundary-value problems. International Journal for Computational Methods in Engineering Science and Mechanics, 11(5), 247-257.

- Roos, H. G., Stynes, M., and Tobiska, L. (2008). Robust numerical methods for singularly perturbed differential equations: convection-diffusion-reaction and flow problems (Vol. 24). Springer Science and Business Media.

- Siraj, M. K., Duressa, G. F., and Bullo, T. A. (2019). Fourth-order stable central difference with Richardson extrapolation method for second-order self-adjoint singularly perturbed boundary value problems. Journal of the Egyptian Mathematical Society, 27(1), 1-14.

- Yüzbaşı, Ş., and Şahin, N. (2013). Numerical solutions of singularly perturbed one-dimensional parabolic convection–diffusion problems by the Bessel collocation method. Applied Mathematics and Computation, 220, 305-315.

- Zeslassie, Y., File, G.,and Aga, T. (2017). Sixth order stable central difference method for self-adjoint singularly perturbed two-point boundary value problems. Ethiopian Journal of Education and Sciences, 13(1), 23-41.