Generation of Quantum Features of Light Via Non-degenerate Three-Level Atom Coupled to two-mode vacuum reservoir

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By

Demeke Wudu

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DECLARATION

I hereby declare that this MSc thesis is my original work and has not been presented for a degree in any other university, and that all sources of material used for the thesis have been duly acknowledged.

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Abstract

In this thesis, we have studied the squeezing, entanglement and statistical properties of the cavity light beams produced by a coherently driven nondegenerate three-level laser with an open cavity and coupled to a two-mode vacuum reservoir via a single-port mirror.

We have carried out our analysis by putting the noise operators associated with the vacuum reservoir in normal order. Applying the solutions of the equations of evolution for the expectation values of the atomic operators and the quantum Langevin equations for the cavity mode operators, we have calculated the mean and variance of the photon number as well as the quadrature squeezing of the cavity light.

We have found that a large part of the mean and the variance of the photon number are confined in a relatively small frequency interval. The analysis showed that the intracavity quadrature squeezing is enhanced due to the absence of spontaneous emission. It is revealed that the squeezing and entanglement in the two-mode light are directly related. Moreover, We have observed that the degree of cavity-atomic state entanglement is greater than the photon entanglement of light.

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Introduction

Quantum optics deals mainly with the quantum properties of the light generated by various optical systems such as lasers with the effects of light on the dynamics of the atoms. squeezing is one of the nonclassical features of light that have attracted a great deal of interest [1, 2, 3, 4, 5, 6]. In squeezed light, the noise in one quadrature is below the vacuum-state level at the expense of enhanced fluctuations in the other quadrature, with the product of the uncertainties in the two quadratures satisfying the uncertainty relation [5, 6]. Squeezed light has potential applications in low-noise optical communications and weak signal detection [6, 7, 8, 9].

There has been a considerable interest in the analysis of the squeezing and statistical properties of the light generated by three-level lasers [9, 10, 11, 12, 13, 14, 15, 16, 17]. A three-level laser may be defined as a quantum optical system in which three-level atoms in a cascade configuration, initially prepared in a coherent superposition of the top and bottom levels, are injected into a cavity coupled to a vacuum reservoir via a single-port mirror. A three-level laser may have additional or modified features. One interesting additional feature of a three-level laser involves the coupling of the top and bottom levels of the atoms injected into the cavity by a strong coherent light. When a three-level atom in a cascade configuration makes a transition from the top to the bottom level via the intermediate level, two photons are generated. If the two photons have the same frequency, then the three-level atom is called a degenerate three-level atom otherwise it is called nondegenerate.

The squeezing and statistical properties of the light produced by three-level lasers when the atoms are initially prepared in a coherent superposition of the top and bottom levels or when these levels are coupled by a strong coherent light have been studied by several authors [18, 19, 20, 21, 22]. These authors have found that these quantum optical systems can generate squeezed light under certain conditions. The squeezing properties of the cavity modes produced by a nondegenerate three-level laser have been studied [23]. It has been found that the two-mode cavity light exhibits squeezing if the atoms are initially prepared with more atoms in the bottom than in the upper level, and the degree of squeezing increases with the linear gain coefficient [22, 23]. A three-level laser, in which the top and bottom levels of three-level atoms are injected into a cavity and coupled by a strong coherent light can also generate light in a squeezed state [23].

Moreover, Fesseha [6] has studied the squeezing and the statistical properties of the light produced by a three-level laser with the atoms in a closed cavity and pumped by electron bombardment. He has shown that the maximum quadrature squeezing of the light generated by the laser, operating below the threshold, is found to be 50% below the vacuum-state level. In addition, he has also found that the quadrature squeezing of the output light is equal to that of the cavity light. On the other hand, this study shows that the local quadrature squeezing is greater than the global quadrature squeezing. He has also found that a large part of the total mean photon number is confined in a relatively small frequency interval. In addition, Fesseha [5] has studied the squeezing and the statistical properties of the light produced by a degenerate three-level laser with the atoms in a closed cavity and pumped by coherent light. He has shown that the maximum quadrature squeezing is 43% below the vacuum-state level, which is slightly less than the result found with electron bombardment.

Furthermore, detailed analysis of the time evolution of the two-mode squeezing, entanglement, and intensity of the cavity radiation of a two-photon correlated emission laser initially seeded with a thermal light is studied by Tesfa [22]. In this work, he found that the two-mode squeezing, entanglement, and intensity of the radiation are independent of the strength of the initial thermal light. This is due to the effect of the thermal light being sucked from the cavity to an extent that it does not affect the non-classical features of the radiation.

Entanglement is one of the fundamental features of quantum information processing and communication protocols. The generation and manipulation of the entanglement have attracted a great deal of interest with wide applications in quantum teleportation, quantum dense coding, quantum computation, quantum error correction, and quantum cryptography [24, 25, 26, 26]. Recently, much attention is given to the generation of a continuous-variable entanglement to manipulate the discrete counterparts and quantum bits and to perform the quantum information processing. In general, the degree of entanglement decreases, when it interacts with the environment. But the quantum information processing efficiency highly depends on the degree of entanglement. Therefore, it is necessary to generate strongly entangled states which can survive under the external noise.

This thesis aims to analyze the quantum and statistical properties of the light generated by a coherently driven nondegenerate three-level laser with an open cavity coupled to a two-mode vacuum reservoir via a single-port mirror. These properties of light are local and global Photon Statistics of the System, entanglement, and quadrature squeezing for both single-mode and two-mode cavity light produced by the coherently driven non-degenerate three-level laser. To get the results of these properties of the light, we carry out our calculation by putting the noise operators associated with the vacuum reservoir in normal order. We thus first determine the master equation for a coherently driven nondegenerate three-level laser in an open cavity coupled to a two-mode vacuum reservoir and the quantum Langevin equations for the cavity mode operators. In addition, employing the master equation and the large-time approximation scheme, we obtain equations of evolution of the expectation values of atomic operators. Moreover, we determine the solutions of the equations of evolution of the expectation values of the atomic operators and the quantum Langevin equations for cavity mode operators. Then applying the resulting solutions, we calculate the quadrature squeezing and the entanglement of the two-mode cavity light.

The findings of this study can contribute to developing knowledge in the field of quantum optics and information. In addition to this, the study can also provide clear mathematical procedures with their physical explanations for those who wish to carry out related investigations. Moreover, this study can demonstrate the various entanglement quantification criteria which can apply to another optical device.

2

Model and Hamiltonian of the System

In this chapter we consider a nondegenerate three-level laser driven by coherent light and with the cavity modes coupled to a two-mode vacuum reservoir via a single-port mirror as shown in Fig. (2.1). We first obtain the master equation for a coherently driven nondegenerate three-level atom with the cavity modes and the quantum Langevin equations for the cavity mode operators. In addition, employing the master equation

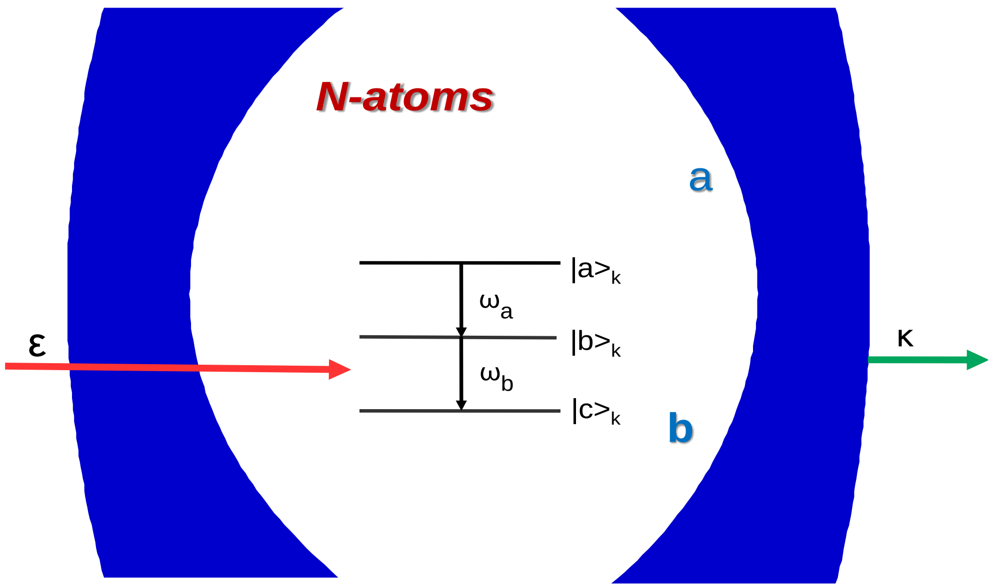

Figure 2.1: Schematic representation of a coherently driven nondegenerate three-level laser coupled to a two-mode vacuum reservoir.

and the large-time approximation scheme, we drive the equations of evolution of the expectation values of the atomic operators. Finally, we determine the steady-state solutions of the resulting equations of evolution. Here we carry out our calculation by putting the noise operators associated with the two-mode vacuum reservoir in normal order.

2.1 Hamiltonian Formulations

We consider here the case in which N nondegenerate three-level atoms in cascade configuration are available in an open cavity. We denote the top, intermediate, and bottom levels of the three-level atom by $|a\rangle_k$, $|b\rangle_k$, and $|c\rangle_k$, respectively. As shown in Fig. (2.1) for nondegenerate cascade configuration, when the atom makes a transition from level $|a\rangle_k$ to $|b\rangle_k$ and from levels $|b\rangle_k$ to $|c\rangle_k$ two photons with different frequencies are emitted. The emission of light when the atoms makes the transition from the top level to the intermediate level is light mode a and the emission of light when the atoms makes the transition from the intermediate level to the bottom level is light mode b. We assume that the cavity mode a is at resonance with transition $|a\rangle_k \rightarrow |b\rangle_k$ and the cavity mode b is at resonance with the transition $|b\rangle_k \rightarrow |c\rangle_k$, with top and bottom levels of the three-level atom coupled by coherent light. The coupling of top and bottom levels of a nondegenerate three-level atom by coherent light can be described by the Hamiltonian[6]

$$\hat{H}' = \frac{i\Omega}{2}\big[\hat{\sigma}_c^{\dagger k} - \hat{\sigma}_c^k\big], \tag{2.1}$$

where

$$\hat{\sigma}_c^k = |c\rangle_{k\,k}\langle a|, \tag{2.2}$$

is lowering atomic operator and

$$\Omega = 2\varepsilon\lambda. \tag{2.3}$$

Here ε, considered to be real and constant, is the amplitude of the driving coherent light and λ is the coupling constant between the driving coherent light and the three-level atom. In addition, the interaction of a three-level atom with the cavity modes can be described by the Hamiltonian [6]

$$\hat{H}'' = ig\big[\hat{\sigma}_a^{\dagger k}\hat{a} - \hat{a}^{\dagger}\hat{\sigma}_a^k + \hat{\sigma}_b^{\dagger k}\hat{b} - \hat{b}^{\dagger}\hat{\sigma}_b^k\big], \tag{2.4}$$

where

$$\hat{\sigma}_a^k = |b\rangle_{k\ k}\langle a|, \tag{2.5}$$

$$\hat{\sigma}_b^k = |c\rangle_{k\ k}\langle b|, \tag{2.6}$$

g is the coupling constant between the atom and cavity mode a or b, and $\hat{a}$ and $\hat{b}$ are the annihilation operators for light modes a and b. Thus upon combining Eqs. (2.1) and (2.4), the interaction of a three-level atom with the cavity modes and the driving coherent light can be described by the Hamiltonian

$$\hat{H}_S = ig\big[\hat{\sigma}_a^{\dagger k}\hat{a} - \hat{a}^{\dagger}\hat{\sigma}_a^k + \hat{\sigma}_b^{\dagger k}\hat{b} - \hat{b}^{\dagger}\hat{\sigma}_b^k\big] + \frac{i\Omega}{2}\big[\hat{\sigma}_c^{\dagger k} - \hat{\sigma}_c^k\big]. \tag{2.7}$$

2.1.1 Dynamics of the System

The quantum analysis of the interaction of a system such as a cavity mode or a three-level atom with the external environment is a relatively complex problem. The external environment, usually referred to as a reservoir, can be thermal light, ordinary or squeezed vacuum. We are interested in the dynamics of a system and this is describable by the master equation, the Fokker-Planck equation, or quantum Langevin equations. Here, we obtain the above set of dynamical equations for a cavity mode coupled to a vacuum reservoir via a single-port mirror. We then focus our study when the cavity mode is couple to a vacuum reservoir. A system coupled with a vacuum reservoir can be described by the Hamiltonian

$$\hat{H} = \hat{H}_S + \hat{H}_{SR}, \tag{2.8}$$

where $\hat{H}_S$ is the Hamiltonian of the system and $\hat{H}_{SR}$ describes the interaction between the system and reservoir. Suppose $\hat{\chi}(t)$ is the density operator for system and reservoir. Then the equation of evolution of this density operator is given by

$$\frac{d}{dt}\hat{\chi}(t) = -i\big[\hat{H}_S(t) + \hat{H}_{SR}, \hat{\chi}(t)\big]. \tag{2.9}$$

We are interested in the quantum dynamics of the system alone. Hence taking into account (2.9), we see that the density operator for the system, also known as the reduced density operator,

$$\hat{\rho}(t) = Tr_R \hat{\chi}(t) \tag{2.10}$$

evolves in time according to

$$\frac{d}{dt}\hat{\rho}(t) = -i\big[\hat{H}_S(t), \hat{\rho}(t)\big] - iTr\big[\hat{H}_{SR}(t), \hat{\chi}(t)\big], \tag{2.11}$$

in which Tr_R indicates the trace over the reservoirs variables only. On the other hand, a formal solution of Eq. (2.9) can be written as

$$\hat{\chi}(t) = \hat{\chi}(0) - i\int_0^t \big[\hat{H}_S(t') + \hat{H}_{SR}(t'), \hat{\chi}(t')\big]dt'. \tag{2.12}$$

In order to obtain mathematically manageable that $\hat{\chi}(t')$ by some approximately valid expression, first we would arrange the reservoir in such a way that its density operator $\hat{R}$ remains constant in time. This can be achieved by letting a beam of light in a vacuum state of constant intensity fall continuously on the system. Moreover, we decouple the system and reservoirs density operators, so that

$$\hat{\chi}(t') = \hat{\rho}(t')\hat{R}. \tag{2.13}$$

Therefore, with the aid of this, one can rewrite Eq. (2.12) as

$$\hat{\chi}(t') = \hat{\rho}(t')\hat{R} - \int_0^t \big[\hat{H}_S(t') + \hat{H}_{SR}(t'), \hat{\rho}(t')\hat{R}\big]dt'. \tag{2.14}$$

Now on substituting (2.14) in to (2.11) there follows

$$\begin{aligned}\frac{d}{dt}\hat{\rho}(t) &= -i\big[\hat{H}_S(t),\hat{\rho}(t)\big] - i\big[\langle\hat{H}_{SR}(t)\rangle_R,\hat{\rho}(0)\big] \\ &\quad - \int_0^t \big[\hat{R}\hat{H}_{SR}(t),\big[\hat{H}_{SR}(t'),\hat{\rho}(t')\big]\big]dt' \\ &\quad - \int_0^t Tr_R\big[\hat{H}_{SR}(t'),\big[\hat{H}_{SR}(t'),\hat{\rho}(t')\hat{R}\big]\big]dt', \qquad (2.15)\end{aligned}$$

where the subscript R indicates that the expectation value is to be calculated using the reservoirs density operator $\hat{R}$. Furthermore, the master equation for a system coupled to a reservoir takes the form

$$\begin{aligned}\frac{d\hat{\rho}(t)}{dt} &= -iTr_A[\hat{H}_S,\hat{\rho}_{AR}(t,t')] - h\langle\hat{H}_{SR}^2\hat{R}\rangle_R\hat{\rho}(t) \\ &\quad +2hTr_R(\hat{H}_{SR}\hat{\rho}(t)\hat{R}\hat{H}_{SR}) - h\hat{\rho}(t)\langle\hat{H}_{SR}^2\hat{R}\rangle_R, \qquad (2.16)\end{aligned}$$

A light mode confined in a cavity, usually formed by two mirrors, is called a cavity mode. A commonly used cavity has a single-port mirror[3]. One side of each cavity is a mirror through which light can enter or leave the cavity. We now proceed to obtain the equation of evolution of the reduced density operator, in short the master equation, for the atoms coupled to a two-mode vacuum reservoir via a single port-mirror. We consider the reservoirs to be composed of large number of submodes. Thus, the interaction Hamilitonian for N nondegenerate three-level atoms coupled to vacuum reservoir is written as

$$\hat{H}_{SR} = i\lambda(\hat{\sigma}_a^{\dagger k}\hat{a}_{in} - \hat{a}_{in}^{\dagger}\hat{\sigma}_a^k + \hat{\sigma}_b^{\dagger k}\hat{b}_{in} - \hat{b}_{in}^{\dagger}\hat{\sigma}_b^k), \qquad (2.17)$$

where λ is the coupling constant, $\hat{a}_{in}$ and $\hat{b}_{in}$ are the annihilation operators of the two-mode vacuum reservoir. By employing Eq. (2.17), we then see that

$$hTr_R(\hat{H}_{SR}^2\hat{R}) = hTr_R\langle(i\lambda(\hat{\sigma}_a^{\dagger k}\hat{a}_{in} - \hat{a}_{in}^{\dagger}\hat{\sigma}_a^k + \hat{\sigma}_b^{\dagger k}\hat{b}_{in} - \hat{b}_{in}^{\dagger}\hat{\sigma}_b^k)^2\rangle. \qquad (2.18)$$

This can be rewritten as

$$\begin{aligned} hTr_R(\hat{H}^2_{SR}\hat{R}) &= -h\lambda^2 Tr_R\big[(\hat{\sigma}^{\dagger k}_a\hat{a}_{in}\hat{\sigma}^{\dagger k}_a\hat{a}_{in})_R - (\hat{\sigma}^{\dagger k}_a\hat{a}_{in}\hat{a}^{\dagger}_{in}\hat{\sigma}^{k}_a)_R + (\hat{\sigma}^{\dagger k}_a\hat{a}_{in}\hat{\sigma}^{\dagger k}_b\hat{b}_{in})_R \\ &-(\hat{\sigma}^{\dagger k}_a\hat{a}_{in}\hat{b}^{\dagger}_{in}\hat{\sigma}^{k}_b)_R - (\hat{a}^{\dagger}_{in}\hat{\sigma}^{k}_a\hat{a}^{\dagger k}\hat{a}_{in})_R + (\hat{a}^{\dagger}_{in}\hat{\sigma}^{k}_a\hat{a}^{\dagger}_{in}\hat{\sigma}^{k}_a)_R - (\hat{a}^{\dagger}_{in}\hat{\sigma}^{k}_a\hat{\sigma}^{\dagger k}_b\hat{b}_{in})_R \\ &+(\hat{a}^{\dagger}_{in}\hat{\sigma}^{k}_a\hat{b}^{\dagger}_{in}\hat{\sigma}^{k}_b)_R + (\hat{\sigma}^{\dagger k}_b\hat{b}_{in}\hat{\sigma}^{\dagger k}_a\hat{a}_{in})_R - (\hat{\sigma}^{\dagger k}_b\hat{b}_{in}\hat{a}^{\dagger}_{in}\hat{\sigma}^{k}_a)_R + (\hat{\sigma}^{\dagger k}_b\hat{b}_{in}\hat{\sigma}^{\dagger k}_b\hat{b}_{in})_R \\ &-(\hat{\sigma}^{\dagger k}_b\hat{b}_{in}\hat{b}^{\dagger}_{in}\hat{\sigma}^{k}_b)_R - (\hat{b}^{\dagger}_{in}\hat{\sigma}^{k}_b\hat{\sigma}^{\dagger k}_a\hat{a}_{in})_R + (\hat{b}^{\dagger}_{in}\hat{\sigma}^{k}_b\hat{a}^{\dagger}_{in}\hat{\sigma}^{k}_a)_R - (\hat{b}^{\dagger}_{in}\hat{\sigma}^{k}_b\hat{\sigma}^{\dagger k}_b\hat{b}_{in})_R \\ &+(\hat{b}^{\dagger}_{in}\hat{\sigma}^{k}_b\hat{b}^{\dagger}_{in}\hat{\sigma}^{k}_b)_R\big]. \end{aligned} \tag{2.19}$$

The atomic and reservoir operators are commute to each other. Then we observe that

$$\begin{aligned} hTr_R(\hat{H}^2_{SR}\hat{R}) &= -h\lambda^2\big[\hat{\sigma}^{\dagger k2}_a\langle\hat{a}^2_{in}\rangle_R - \hat{\sigma}^{\dagger k}_a\hat{\sigma}^{k}_a\langle\hat{a}_{in}\hat{a}^{\dagger}_{in}\rangle_R + \hat{\sigma}^{\dagger k}_a\hat{\sigma}^{\dagger k}_b\langle\hat{a}_{in}\hat{b}_{in}\rangle_R - \hat{\sigma}^{\dagger k}_a\hat{\sigma}^{k}_b\langle\hat{a}_{in}\hat{b}^{\dagger}_{in}\rangle_R \\ &-\hat{\sigma}^{k}_a\hat{\sigma}^{\dagger k}_a\langle\hat{a}^{\dagger}_{in}\hat{a}_{in}\rangle_R + \hat{\sigma}^{\dagger k2}_a\langle\hat{a}^{\dagger 2}_{in}\rangle_R - \hat{\sigma}^{k}_a\hat{\sigma}^{\dagger k}_b\langle\hat{a}^{\dagger}_{in}\hat{b}_{in}\rangle_R + \hat{\sigma}^{k}_a\hat{\sigma}^{k}_a\langle\hat{a}^{\dagger}_{in}\hat{b}^{\dagger}_{in}\rangle_R \\ &+\hat{\sigma}^{\dagger k}_b\hat{\sigma}^{\dagger k}_a\langle\hat{b}_{in}\hat{a}_{in}\rangle_R - \hat{\sigma}^{\dagger k}_b\hat{\sigma}^{k}_a\langle\hat{b}_{in}\hat{a}^{\dagger}_{in}\rangle_R + \hat{\sigma}^{\dagger k2}_b\langle\hat{b}^2_{in}\rangle_R - \hat{\sigma}^{\dagger k}_b\hat{\sigma}^{k}_b\langle\hat{b}_{in}\hat{b}^{\dagger}_{in}\rangle_R \\ &-\hat{\sigma}^{k}_b\hat{\sigma}^{\dagger k}_a\langle\hat{b}^{\dagger}_{in}\hat{a}_{in}\rangle_R + \hat{\sigma}^{k}_b\hat{\sigma}^{k}_a\langle\hat{b}^{\dagger}_{in}\hat{a}^{\dagger}_{in}\rangle_R - \hat{\sigma}^{k}_b\hat{\sigma}^{\dagger k}_b\langle\hat{b}^{\dagger}_{in}\hat{b}_{in}\rangle_R + \hat{\sigma}^{\dagger k2}_b\langle\hat{b}^{\dagger 2}_{in}\rangle_R\big]. \end{aligned} \tag{2.20}$$

Now using the density operator of the vacuum reservoir

$$\hat{R} = |0,0\rangle\langle 0,0|, \tag{2.21}$$

one can easily check that

$$\langle\hat{a}^2_{in}\rangle_R = Tr_R(|0,0\rangle\langle 0,0|\hat{a}^2_{in}). \tag{2.22}$$

It then follows that

$$\langle\hat{a}^2_{in}\rangle_R = \langle n|\hat{a}^2_{in}|n-2\rangle = 0. \tag{2.23}$$

Following the same procedure, we obtain

$$\langle\hat{a}^2_{in}\rangle = \langle\hat{b}^2_{in}\rangle = \langle\hat{a}^{\dagger 2}_{in}\rangle = \langle\hat{b}^{\dagger 2}_{in}\rangle = 0, \tag{2.24}$$

$$\langle\hat{a}^{\dagger}_{in}\hat{b}_{in}\rangle_R = \langle\hat{a}_{in}\hat{b}^{\dagger}_{in}\rangle_R = \langle\hat{b}^{\dagger}_{in}\hat{a}_{in}\rangle_R = \langle\hat{b}_{in}\hat{a}^{\dagger}_{in}\rangle_R = 0, \tag{2.25}$$

$$\langle\hat{a}_{in}\hat{b}_{in}\rangle_R = \langle\hat{b}_{in}\hat{a}_{in}\rangle_R = \langle\hat{b}^{\dagger}_{in}\hat{a}^{\dagger}_{in}\rangle_R = \langle\hat{a}^{\dagger}_{in}\hat{b}^{\dagger}_{in}\rangle_R = 0. \tag{2.26}$$

In addition, applying the commutation relation $[\hat{a}_{in}, \hat{a}^{\dagger}_{in}] = 1$, we then note that

$$\langle \hat{a}_{in}\hat{a}^{\dagger}_{in}\rangle = 1, \tag{2.27}$$

$$\langle \hat{a}^{\dagger}_{in}\hat{a}_{in}\rangle = 0, \tag{2.28}$$

Hence on account of Eqs. (2.24), (2.25), (2.26), (2.27), and (2.28) into Eq. (2.20), there follows

$$hTr_R(\hat{H}^2_{SR}\hat{R})\hat{\rho}(t) = h\lambda^2[\hat{\sigma}^{\dagger k}_a\hat{\sigma}^k_a\hat{\rho} + \hat{\sigma}^{\dagger k}_b\hat{\sigma}^k_b\hat{\rho}]. \tag{2.29}$$

In the same manner, one can readily verify that

$$h\hat{\rho}(t)Tr_R(\hat{H}^2_{SR}\hat{R}) = h\lambda^2[\hat{\rho}\hat{\sigma}^{\dagger k}_a\hat{\sigma}^k_a + \hat{\rho}\hat{\sigma}^{\dagger k}_b\hat{\sigma}^k_b]. \tag{2.30}$$

In addition, one can readily find

$$\begin{aligned} 2hTr_R[\hat{H}_{SR}\hat{\rho}(t)\hat{R}\hat{H}_{SR}] \;=\; & -2h\lambda^2\big[\hat{a}^{\dagger}\hat{\rho}\hat{a}^{\dagger}\langle\hat{a}^2_{in}\rangle_R - \hat{\sigma}^{\dagger k}_a\hat{\rho}\hat{\sigma}^k_a\langle\hat{a}^{\dagger}_{in}\hat{a}_{in}\rangle_R + \hat{\sigma}^{\dagger k}_a\hat{\rho}\hat{\sigma}^{\dagger k}_b\langle\hat{b}_{in}\hat{a}_{in}\rangle_R \\ & -\hat{\sigma}^{\dagger k}_a\hat{\rho}\hat{\sigma}^k_b\langle\hat{b}^{\dagger}_{in}\hat{a}_{in}\rangle_R - \hat{\sigma}^k_a\hat{\rho}\hat{\sigma}^{\dagger k}_a\langle\hat{a}_{in}\hat{a}^{\dagger}_{in}\rangle_R + \hat{\sigma}^k_a\hat{\rho}\hat{\sigma}^k_a\langle\hat{a}^{\dagger 2}_{in}\rangle_R \\ & -\hat{\sigma}^k_a\hat{\rho}\hat{\sigma}^{\dagger k}_b\langle\hat{b}_{in}\hat{a}^{\dagger}_{in}\rangle_R + \hat{\sigma}^k_a\hat{\rho}\hat{\sigma}^k_b\langle\hat{b}^{\dagger}_{in}\hat{a}^{\dagger}_{in}\rangle_R + \hat{\sigma}^{\dagger k}_b\hat{\rho}\hat{\sigma}^{\dagger k}_a\langle\hat{a}_{in}\hat{b}_{in}\rangle_R \\ & -\hat{\sigma}^{\dagger k}_b\hat{\rho}\hat{\sigma}^k_a\langle\hat{a}^{\dagger}_{in}\hat{b}_{in}\rangle_R + \hat{\sigma}^{\dagger k}_b\hat{\rho}\hat{\sigma}^{\dagger k}_b\langle\hat{b}^2_{in}\rangle_R - \hat{\sigma}^{\dagger k}_b\hat{\rho}\hat{\sigma}^k_b\langle\hat{b}^{\dagger}_{in}\hat{b}_{in}\rangle_R \\ & -\hat{\sigma}^k_b\hat{\rho}\hat{\sigma}^{\dagger k}_a\langle\hat{a}_{in}\hat{b}^{\dagger}_{in}\rangle_R + \hat{\sigma}^k_b\hat{\rho}\hat{\sigma}^k_a\langle\hat{a}^{\dagger}_{in}\hat{b}^{\dagger}_{in}\rangle_R - \hat{\sigma}^k_b\hat{\rho}\hat{\sigma}^{\dagger k}_b\langle\hat{b}_{in}\hat{b}^{\dagger}_{in}\rangle_R \\ & +\hat{\sigma}^{\dagger k}_b\hat{\rho}\hat{\sigma}^k_b\langle\hat{b}^{\dagger 2}_{in}\rangle_R\big], \end{aligned} \tag{2.31}$$

so that applying Eqs. (2.24), (2.25), (2.26), (2.27), and (2.28) in Eq. (2.31) leads to

$$2hTr_R[\hat{H}_{SR}\hat{\rho}(t)\hat{R}\hat{H}_{SR}] = 2\lambda^2h[\hat{\sigma}^k_a\hat{\rho}\hat{\sigma}^{\dagger k}_a + \hat{\sigma}^k_b\hat{\rho}\hat{\sigma}^{\dagger k}_b]. \tag{2.32}$$

Taking into account Eq. (2.29), (2.30), and (2.32), the expression in Eq. (2.16) takes the form

$$\begin{aligned} \frac{d}{dt}\hat{\rho}(t) \;=\; & -i\big[\hat{H}_S, \hat{\rho}(t)\big] + \frac{\gamma}{2}\big[2\hat{\sigma}^k_a\hat{\rho}\hat{\sigma}^{\dagger k}_a - \hat{\sigma}^{\dagger k}_a\hat{\sigma}^k_a\hat{\rho} - \hat{\rho}\hat{\sigma}^{\dagger k}_a\hat{\sigma}^k_a\big] \\ & +\frac{\gamma}{2}\big[2\hat{\sigma}^k_b\hat{\rho}\hat{\sigma}^{\dagger k}_b - \hat{\sigma}^{\dagger k}_b\hat{\sigma}^k_b\hat{\rho} - \hat{\rho}\hat{\sigma}^{\dagger k}_b\hat{\sigma}^k_b\big], \end{aligned} \tag{2.33}$$

where $\gamma_a = \gamma_b = \gamma = 2h\lambda^2$, considered to be the same for levels $|a\rangle$ and $|b\rangle$, is the spontaneous emission decay constant. In addition, a nondegenerate three-level atom in an open cavity is coupled to a two-mode vacuum reservoir. Therefore, the master equation for a coherently driven nondegenerate three-level atom in an open cavity and coupled to a two-mode vacuum reservoir, with the aid of (2.7), is found to be

$$\begin{aligned}\frac{d}{dt}\hat{\rho}(t) \;=\;& g\big[\hat{\sigma}_a^{\dagger k}\hat{a}\hat{\rho} - \hat{a}^\dagger\hat{\sigma}_a^k\hat{\rho} + \hat{\sigma}_b^{\dagger k}\hat{b}\hat{\rho} - \hat{b}^\dagger\hat{\sigma}_b^k\hat{\rho} - \hat{\rho}\hat{\sigma}_a^{\dagger k}\hat{a} + \hat{\rho}\hat{a}^\dagger\hat{\sigma}_a^k - \hat{\rho}\hat{\sigma}_b^{\dagger k}\hat{b} + \hat{\rho}\hat{b}^\dagger\hat{\sigma}_b^k\big] \\ &+\frac{\Omega}{2}\big[\hat{\sigma}_c^{\dagger k}\hat{\rho} - \hat{\sigma}_c^k\hat{\rho} - \hat{\rho}\hat{\sigma}_c^{\dagger k} + \hat{\rho}\hat{\sigma}_c^k\big] + \frac{\gamma}{2}\big[2\hat{\sigma}_a^k\hat{\rho}\hat{\sigma}_a^{\dagger k} - \hat{\sigma}_a^{\dagger k}\hat{\sigma}_a^k\hat{\rho} - \hat{\rho}\hat{\sigma}_a^{\dagger k}\hat{\sigma}_a^k\big] \\ &+\frac{\gamma}{2}\big[2\hat{\sigma}_b^k\hat{\rho}\hat{\sigma}_b^{\dagger k} - \hat{\sigma}_b^{\dagger k}\hat{\sigma}_b^k\hat{\rho} - \hat{\rho}\hat{\sigma}_b^{\dagger k}\hat{\sigma}_b^k\big]. \end{aligned} \tag{2.34}$$

This is the master equation for a coherently driven nondegenerate three-level atom in an open cavity and coupled to a two-mode vacuum reservoir.

2.2 Quantum Langevin Equations of the System

We recall that the laser cavity is coupled to a two-mode vacuum reservoir via a single-port mirror. In addition, we carry out our calculation by putting the noise operators associated with the vacuum reservoir in normal order. Thus the noise operators will not have any effect on the dynamics of the cavity mode operators [3]. We can therefore drop the noise operators and write the quantum Langevin equations for the cavity operators $\hat{a}$ and $\hat{b}$ as

$$\frac{d\hat{a}}{dt} = -\frac{\kappa}{2}\hat{a} - i[\hat{a}, \hat{H}_S], \tag{2.35}$$

$$\frac{d\hat{b}}{dt} = -\frac{\kappa}{2}\hat{b} - i[\hat{b}, \hat{H}_S], \tag{2.36}$$

where κ is the cavity damping constant. Then in view of Eq. (2.7), the quantum Langevin equations for cavity mode operators $\hat{a}$ and $\hat{b}$ turns out to be

$$\frac{d\hat{a}}{dt} = -\frac{\kappa}{2}\hat{a} - g\hat{\sigma}_a^k, \tag{2.37}$$

$$\frac{d\hat{b}}{dt} = -\frac{\kappa}{2}\hat{b} - g\hat{\sigma}_b^k. \tag{2.38}$$

2.3 Equations of evolution of the atomic Operators

Here we seek to derive the equations of evolution of the expectation values of the atomic Operators by applying the master equation and the large-time approximation scheme. Moreover, we find the steady-state solutions of the equations of evolution of the atomic Operators. To this end, employing the relation[3]

$$\frac{d}{dt}\langle\hat{A}\rangle = Tr\left(\frac{d\hat{\rho}}{dt}\hat{A}\right) \tag{2.39}$$

along with the master equation (2.34), one can readily establish that

$$\frac{d}{dt}\langle\hat{\sigma}_a^k\rangle = g[\langle\hat{\eta}_b^k\hat{a}\rangle - \langle\hat{\eta}_a^k\hat{a}\rangle + \langle\hat{b}^\dagger\hat{\sigma}_c^k\rangle] + \frac{\Omega}{2}\langle\hat{\sigma}_b^{\dagger k}\rangle - \gamma\langle\hat{\sigma}_a^k\rangle, \tag{2.40}$$

$$\frac{d}{dt}\langle\hat{\sigma}_b^k\rangle = g[\langle\hat{\eta}_c^k\hat{b}\rangle - \langle\hat{a}^\dagger\hat{\sigma}_c^k\rangle - \langle\hat{\eta}_b^k\hat{b}\rangle] - \frac{\Omega}{2}\langle\hat{\sigma}_a^{\dagger k}\rangle - \gamma\langle\hat{\sigma}_b^k\rangle, \tag{2.41}$$

$$\frac{d}{dt}\langle\hat{\sigma}_c^k\rangle = g[\langle\hat{\sigma}_b^k\hat{a}\rangle - \langle\hat{\sigma}_a^k\hat{b}\rangle] + \frac{\Omega}{2}\left[\langle\hat{\eta}_c^k\rangle - \langle\hat{\eta}_a^k\rangle\right] - \gamma\langle\hat{\sigma}_c^k\rangle, \tag{2.42}$$

$$\frac{d}{dt}\langle\hat{\eta}_a^k\rangle = g[\langle\hat{\sigma}_a^{\dagger k}\hat{a}\rangle + \langle\hat{a}^\dagger\hat{\sigma}_a^k\rangle] + \frac{\Omega}{2}\left[\langle\hat{\sigma}_c^k\rangle + \langle\hat{\sigma}_c^{\dagger k}\rangle\right] - \frac{\gamma}{2}\langle\hat{\eta}_a^k\rangle, \tag{2.43}$$

$$\frac{d}{dt}\langle\hat{\eta}_b^k\rangle = g[\langle\hat{\sigma}_b^\dagger\hat{b}\rangle + \langle\hat{b}^\dagger\hat{\sigma}_b^k\rangle - \langle\hat{\sigma}_a^{\dagger k}\hat{a}\rangle - \langle\hat{a}^\dagger\hat{\sigma}_a^k\rangle] - \gamma[\langle\hat{\eta}_a^k\rangle - \langle\hat{\eta}_b^k\rangle], \tag{2.44}$$

where

$$\hat{\eta}_a^k = |a\rangle_{k\,k}\langle a|, \tag{2.45}$$

$$\hat{\eta}_b^k = |b\rangle_{k\,k}\langle b|, \tag{2.46}$$

$$\hat{\eta}_c^k = |c\rangle_{k\,k}\langle c|. \tag{2.47}$$

We see that Eqs. (2.40) - (2.44) are nonlinear differential equations and hence it is not possible to find exact time-dependent solutions of these equations. We intend to overcome this problem by applying the large-time approximation [7,8]. Therefore, employing this approximation scheme, we get from Eqs. (2.37) and (2.38) the approximately valid relations

$$\hat{a} = -\frac{2g}{\kappa}\hat{\sigma}_a^k, \tag{2.48}$$

$$\hat{b} = -\frac{2g}{\kappa}\hat{\sigma}_b^k. \tag{2.49}$$

Evidently, these turn out to be exact relations at steady-state. Now introducing Eqs. (2.48) and (2.49) into Eqs. (2.40)-(2.44), the equations of evolution of the atomic operators take the form

$$\frac{d}{dt}\langle\hat{\sigma}_a^k\rangle = -(\gamma+\gamma_c)\langle\hat{\sigma}_a^k\rangle + \frac{\Omega}{2}\langle\hat{\sigma}_b^{\dagger k}\rangle, \tag{2.50}$$

$$\frac{d}{dt}\langle\hat{\sigma}_b^k\rangle = -\frac{1}{2}(\gamma+\gamma_c)\langle\hat{\sigma}_b^k\rangle - \frac{\Omega}{2}\langle\hat{\sigma}_a^{\dagger k}\rangle, \tag{2.51}$$

$$\frac{d}{dt}\langle\hat{\sigma}_c^k\rangle = -\frac{1}{2}(\gamma+\gamma_c)\langle\hat{\sigma}_c^k\rangle + \frac{\Omega}{2}\big[\langle\hat{\eta}_c^k\rangle - \langle\hat{\eta}_a^k\rangle\big], \tag{2.52}$$

$$\frac{d}{dt}\langle\hat{\eta}_a^k\rangle = -(\gamma+\gamma_c)\langle\hat{\eta}_a^k\rangle + \frac{\Omega}{2}\big[\langle\hat{\sigma}_c^k\rangle + \langle\hat{\sigma}_c^{\dagger k}\rangle\big], \tag{2.53}$$

$$\frac{d}{dt}\langle\hat{\eta}_b^k\rangle = (\gamma+\gamma_c)[\langle\hat{\eta}_b^k\rangle - \langle\hat{\eta}_a^k\rangle], \tag{2.54}$$

where

$$\gamma_c = \frac{4g^2}{\kappa} \tag{2.55}$$

is the stimulated emission decay constant. We next , introducing Eqs.(2.48) and (2.49) into Eqs. (2.50) - (2.54) and summing over the N three-level atoms, so that

$$\frac{d}{dt}\langle\hat{m}_1\rangle = -(\gamma+\gamma_c)\langle\hat{m}_1\rangle + \frac{\Omega}{2}\langle\hat{m}_2^\dagger\rangle, \tag{2.56}$$

$$\frac{d}{dt}\langle\hat{m}_2\rangle = -\frac{1}{2}(\gamma+\gamma_c)\langle\hat{m}_2\rangle - \frac{\Omega}{2}\langle\hat{m}_1^\dagger\rangle, \tag{2.57}$$

$$\frac{d}{dt}\langle\hat{m}_3\rangle = -\frac{1}{2}(\gamma+\gamma_c)\langle\hat{m}_3\rangle + \frac{\Omega}{2}\big[\langle\hat{N}_c\rangle - \langle\hat{N}_a\rangle\big], \tag{2.58}$$

$$\frac{d}{dt}\langle\hat{N}_a\rangle = -(\gamma+\gamma_c)\langle\hat{N}_a\rangle + \frac{\Omega}{2}\big[\langle\hat{m}_3\rangle + \langle\hat{m}_3^\dagger\rangle\big], \tag{2.59}$$

$$\frac{d}{dt}\langle\hat{N}_b\rangle = (\gamma+\gamma_c)[\langle\hat{N}_b\rangle - \langle\hat{N}_a\rangle], \tag{2.60}$$

in which

$$\hat{m}_1 = \sum_{k=1}^{N} \hat{\sigma}_a^k, \tag{2.61}$$

$$\hat{m}_2 = \sum_{k=1}^{N} \hat{\sigma}_b^k, \tag{2.62}$$

$$\hat{m}_3 = \sum_{k=1}^{N} \hat{\sigma}_c^k, \tag{2.63}$$

$$\hat{N}_a = \sum_{k=1}^{N} \hat{\eta}_a^k, \tag{2.64}$$

$$\hat{N}_b = \sum_{k=1}^{N} \hat{\eta}_b^k, \tag{2.65}$$

$$\hat{N}_c = \sum_{k=1}^{N} \hat{\eta}_c^k, \tag{2.66}$$

with the operators $\hat{N}_a$, $\hat{N}_b$, and $\hat{N}_c$ representing the number of atoms in the top, intermediate, and bottom levels, respectively. In addition, employing the completeness relation

$$\hat{\eta}_a^k + \hat{\eta}_b^k + \hat{\eta}_c^k = \hat{I}, \tag{2.67}$$

where $\hat{\eta}_a^k$, $\hat{\eta}_b^k$ and $\hat{\eta}_c^k$ are the probabilities of the atoms on the top, middle, and bottom levels, respectively.
we easily arrive at

$$\langle \hat{N}_a \rangle + \langle \hat{N}_b \rangle + \langle \hat{N}_c \rangle = N. \tag{2.68}$$

Furthermore, using the definition given by Eq. (2.61) and setting for any k

$$\hat{\sigma}_a^k = |b\rangle\langle a|, \tag{2.69}$$

we have

$$\hat{m}_1 = N|b\rangle\langle a|. \tag{2.70}$$

Following the same procedure, one can also easily establish that

$$\hat{m}_2 = N|c\rangle\langle b|, \tag{2.71}$$

$$\hat{m}_3 = N|c\rangle\langle a|, \tag{2.72}$$

$$\hat{N}_a = N|a\rangle\langle a|, \tag{2.73}$$

$$\hat{N}_b = N|b\rangle\langle b|, \tag{2.74}$$

$$\hat{N}_c = N|c\rangle\langle c|. \tag{2.75}$$

Using the definition [6, 28]

$$\hat{m} = \hat{m}_1 + \hat{m}_2 \tag{2.76}$$

and taking into account Eqs. (2.70)-(2.75), it can be readily established that [8]

$$\hat{m}^\dagger \hat{m} = N(\hat{N}_a + \hat{N}_b), \tag{2.77}$$

$$\hat{m}\hat{m}^\dagger = N(\hat{N}_b + \hat{N}_c), \tag{2.78}$$

$$\hat{m}^2 = N\hat{m}_3. \tag{2.79}$$

In the presence of N three-level atoms, we rewrite Eqs. (2.37) and (2.38) as [6]

$$\frac{d\hat{a}}{dt} = -\frac{\kappa}{2}\hat{a} + \theta\hat{m}_1, \tag{2.80}$$

$$\frac{d\hat{b}}{dt} = -\frac{\kappa}{2}\hat{b} + \beta\hat{m}_2, \tag{2.81}$$

in which λ and β are constants whose values remain to be fixed. We note that the steady-state solutions of Eqs. (2.37) and (2.38) are

$$\hat{a} = -\frac{2g}{\kappa}\hat{\sigma}_a^k, \tag{2.82}$$

$$\hat{b} = -\frac{2g}{\kappa}\hat{\sigma}_b^k. \tag{2.83}$$

Now employing Eqs. (2.82) and (2.83), the commutation relations for the cavity mode operators are found to be

$$[\hat{a}, \hat{a}^\dagger]_k = \frac{\gamma_c}{\kappa}\left[\hat{\eta}_b^k - \hat{\eta}_a^k\right], \tag{2.84}$$

$$[\hat{b}, \hat{b}^\dagger]_k = \frac{\gamma_c}{\kappa}\left[\hat{\eta}_c^k - \hat{\eta}_b^k\right], \tag{2.85}$$

and on summing over all atoms, we have

$$[\hat{a}, \hat{a}^{\dagger}] = \frac{\gamma_c}{\kappa}\big[\hat{N}_b - \hat{N}_a\big], \tag{2.86}$$

$$[\hat{b}, \hat{b}^{\dagger}] = \frac{\gamma_c}{\kappa}\big[\hat{N}_c - \hat{N}_b\big], \tag{2.87}$$

where

$$[\hat{a}, \hat{a}^{\dagger}] = \sum_{k=1}^{N}[\hat{a}, \hat{a}^{\dagger}]_k, \tag{2.88}$$

$$[\hat{b}, \hat{b}^{\dagger}] = \sum_{k=1}^{N}[\hat{b}, \hat{b}^{\dagger}]_k. \tag{2.89}$$

We note that Eqs. (2.88) and (2.89) stand for the commutators $\hat{a}$ and $\hat{a}^{\dagger}$, and for $\hat{b}$ and $\hat{b}^{\dagger}$ when the light modes a and b are interacting with all the N three-level atoms. On the other hand, using the steady-state solutions of Eqs. (2.80) and (2.81), one can easily verify that

$$[\hat{a}, \hat{a}^{\dagger}] = N\left(\frac{2\theta}{\kappa}\right)^2\left(\hat{N}_b - \hat{N}_a\right), \tag{2.90}$$

$$[\hat{b}, \hat{b}^{\dagger}] = N\left(\frac{2\beta}{\kappa}\right)^2\left(\hat{N}_c - \hat{N}_b\right). \tag{2.91}$$

Thus on account of Eqs.(2.55), (2.86) and (2.90), we see that

$$\theta = \pm\frac{g}{\sqrt{N}}. \tag{2.92}$$

Similarly, inspection of Eqs. 2.55), (2.87) and (2.91) shows that

$$\beta = \pm\frac{g}{\sqrt{N}}. \tag{2.93}$$

Hence in view of these two results, the equations of evolution of the light modes a and b operators given by Eqs. (2.80) and (2.81) can be written as [6, 28]

$$\frac{d\hat{a}}{dt} = -\frac{\kappa}{2}\hat{a} + \frac{g}{\sqrt{N}}\hat{m}_1, \tag{2.94}$$

$$\frac{d\hat{b}}{dt} = -\frac{\kappa}{2}\hat{b} + \frac{g}{\sqrt{N}}\hat{m}_2. \tag{2.95}$$

Now adding Eqs. (2.86) and (2.87) as well as Eqs. (2.94) and (2.95), we get

$$[\hat{c}, \hat{c}^\dagger] = \frac{\gamma_c}{\kappa}\left[\hat{N}_c - \hat{N}_a\right] \tag{2.96}$$

and

$$\frac{d\hat{c}}{dt} = -\frac{\kappa}{2}\hat{c} + \frac{g}{\sqrt{N}}\hat{m}, \tag{2.97}$$

in which

$$\hat{c} = \hat{a} + \hat{b}. \tag{2.98}$$

We next proceed to obtain the expectation value of the cavity mode operators. One can rewrite Eq. (2.56) and the ad-joint of (2.57) as

$$\frac{d}{dt}\langle\hat{m}_1\rangle = -(\gamma + \gamma_c)\langle\hat{m}_1\rangle + \frac{\Omega}{2}\langle\hat{m}_2^\dagger\rangle, \tag{2.99}$$

$$\frac{d}{dt}\langle\hat{m}_2^\dagger\rangle = -\frac{\Omega}{2}\langle\hat{m}_1\rangle - \frac{1}{2}(\gamma + \gamma_c)\langle\hat{m}_2^\dagger\rangle. \tag{2.100}$$

To solve the coupled differential equations (2.99 and (2.100), we write the single-matrix equation then analyze the eigenvalues and eigenvectors of the matrix. After that, we can get the solution of the differential equation.

$$\begin{pmatrix} \langle\hat{m}_1(t)\rangle \\ \langle\hat{m}_2^\dagger(t)\rangle \end{pmatrix} = \begin{bmatrix} \langle\hat{m}_1(0)\rangle e^{-\beta t} & 0 \\ 0 & \langle\hat{m}_2^\dagger(0)\rangle e^{-\beta t} \end{bmatrix}. \tag{2.101}$$

It then follows that

$$\langle\hat{m}_1(t)\rangle = \langle\hat{m}_1(0)\rangle e^{-\beta t} \tag{2.102}$$

and

$$\langle\hat{m}_2^\dagger(t)\rangle = \langle\hat{m}_2^\dagger(0)\rangle e^{-\beta t}. \tag{2.103}$$

Furthermore, the adjoint of Eq. (2.103) can be written as

$$\langle\hat{m}_2(t)\rangle = \langle\hat{m}_2(0)\rangle e^{-\beta t}. \tag{2.104}$$

With the atoms considered to be initially in the bottom level, Eqs. (2.102) and (2.104) reduce to

$$\langle \hat{m}_1(t) \rangle = 0, \tag{2.105}$$

$$\langle \hat{m}_2(t) \rangle = 0. \tag{2.106}$$

The expectation value of the solution of Eq. (2.94) is expressible as

$$\langle \hat{a}(t) \rangle = \langle \hat{a}(0) \rangle e^{-\kappa t/2} + \frac{g}{\sqrt{N}} \int_0^t e^{\kappa t'/2} \langle \hat{m}_1(t') \rangle dt'. \tag{2.107}$$

With the help of Eq. (2.105) and the assumption that the cavity light is initially in a vacuum state, Eq. (2.107) turns out to be

$$\langle \hat{a}(t) \rangle = 0. \tag{2.108}$$

In view of the linear equation described by Eq. (2.94) and the result given by Eq. (2.108), we claim that $\hat{a}(t)$ is a Gaussian variable with zero mean. Following a similar procedure, one can readily obtain the expectation value of the solution of Eq. (2.95) to be

$$\langle \hat{b}(t) \rangle = 0. \tag{2.109}$$

Then on account of the linear equation described by Eq. (2.95) and the result given by Eq. (2.109), we realize $\hat{b}(t)$ to be a Gaussian variable with zero mean. Now with the aid of Eqs. (2.108) and (2.109) together with (2.98), we have

$$\langle \hat{c}(t) \rangle = 0. \tag{2.110}$$

In view of Eqs. (2.110) and (2.97), we see that $\hat{c}(t)$ is a Gaussian variable with zero mean.

Finally, we seek to determine the steady-state solutions of the expectation values of the atomic operators. We note that the steady-state solutions of Eqs. (2.58), (2.59), and (2.60) are given by

$$\langle \hat{m}_3 \rangle_{ss} = \frac{\Omega}{(\gamma + \gamma_c)} \big[\langle \hat{N}_c \rangle_{ss} - \langle \hat{N}_a \rangle_{ss} \big], \tag{2.111}$$

$$\langle \hat{N}_a \rangle_{ss} = \frac{\Omega}{2(\gamma + \gamma_c)} \big[\langle \hat{m}_3 \rangle_{ss} + \langle \hat{m}_3^\dagger \rangle_{ss} \big], \tag{2.112}$$

$$\langle \hat{N}_b \rangle_{ss} = \langle \hat{N}_a \rangle_{ss}, \tag{2.113}$$

where ss stands for steady-state. Furthermore, with the help of Eq. (2.68) together with (2.113), we see that

$$\langle \hat{N}_c \rangle_{ss} = N - 2\langle \hat{N}_a \rangle_{ss}. \tag{2.114}$$

With the aid of Eq. (2.114), Eq. (2.111) can be written as

$$\langle \hat{m}_3 \rangle_{ss} = \frac{\Omega}{(\gamma + \gamma_c)} \left[N - 3\langle \hat{N}_a \rangle_{ss} \right] \tag{2.115}$$

and in view of Eq. (2.115), we observe that

$$\langle \hat{m}_3 \rangle_{ss} = \langle \hat{m}_3^\dagger \rangle_{ss}. \tag{2.116}$$

Now taking into consideration this result, Eq. (2.116) can be put in the form

$$\langle \hat{N}_a \rangle_{ss} = \frac{\Omega}{(\gamma + \gamma_c)} \langle \hat{m}_3 \rangle_{ss}. \tag{2.117}$$

Using Eqs. (2.115) and (2.117), one readily gets

$$\langle \hat{N}_a \rangle_{ss} = \left[\frac{\Omega^2}{(\gamma_c + \gamma)^2 + 3\Omega^2} \right] N. \tag{2.118}$$

Substitution of Eq. (2.118) into Eqs. (2.113), (2.114), and (2.115) results in

$$\langle \hat{N}_b \rangle_{ss} = \left[\frac{\Omega^2}{(\gamma_c + \gamma)^2 + 3\Omega^2} \right] N, \tag{2.119}$$

$$\langle \hat{N}_c \rangle_{ss} = \left[\frac{(\gamma_c + \gamma)^2 + \Omega^2}{(\gamma_c + \gamma)^2 + 3\Omega^2} \right] N, \tag{2.120}$$

$$\langle \hat{m}_3 \rangle_{ss} = \left[\frac{\Omega(\gamma_c + \gamma)}{(\gamma_c + \gamma)^2 + 3\Omega^2} \right] N. \tag{2.121}$$

These equations represent the steady-state solutions of the equations of evolution of the atomic operators for a coherently driven nondegenerate three-level atom in an open cavity and coupled to a two-mode vacuum reservoir. Furthermore, upon setting $\gamma = 0$, for the case in which spontaneous emission is absent, the steady-state solutions described by Eqs. (2.118)-(2.121) take the form

$$\langle \hat{N}_a \rangle_{ss} = \left[\frac{\Omega^2}{\gamma_c^2 + 3\Omega^2} \right] N, \tag{2.122}$$

$$\langle \hat{N}_b \rangle_{ss} = \left[\frac{\Omega^2}{\gamma_c^2 + 3\Omega^2} \right] N, \tag{2.123}$$

$$\langle \hat{N}_c \rangle_{ss} = \left[\frac{\gamma_c^2 + \Omega^2}{\gamma_c^2 + 3\Omega^2} \right] N, \tag{2.124}$$

$$\langle \hat{m}_3 \rangle_{ss} = \left[\frac{\gamma_c \Omega}{\gamma_c^2 + 3\Omega^2} \right] N. \tag{2.125}$$

These represent the steady-state solutions of the equations of evolution of the expectation values of the atomic operators for a closed cavity. The results described by Eqs. (2.122)-(2.125) are exactly the same as those obtained by Fesseha and Yeshiwas[5, 29]. In addition, we note that for $\Omega \gg \gamma_c$, Eqs. (2.122)-(2.125) reduce to

$$\langle \hat{N}_a \rangle_{ss} = \frac{1}{3} N, \tag{2.126}$$

$$\langle \hat{N}_b \rangle_{ss} = \frac{1}{3} N, \tag{2.127}$$

$$\langle \hat{N}_c \rangle_{ss} = \frac{1}{3} N, \tag{2.128}$$

$$\langle \hat{m}_3 \rangle_{ss} = 0. \tag{2.129}$$

Finally, in the absence of the deriving coherent light, when $\Omega = 0$, Eqs. (2.122)-(2.125) turns out to be

$$\langle \hat{N}_a \rangle = 0, \tag{2.130}$$

$$\langle \hat{N}_b \rangle = 0, \tag{2.131}$$

$$\langle \hat{N}_c \rangle = N, \tag{2.132}$$

$$\langle \hat{m}_3 \rangle = 0. \tag{2.133}$$

These results shows that initially, when the deriving coherent light $(\Omega = 0)$, all the atoms to be in bottom level.

3

Quantum Features of Light

Photon Statistics of the System

In this chapter we seek to study the statistical properties of the light produced by the coherently driven N non degenerate three-level laser with an open cavity and coupled to a two-mode vacuum reservoir via a single-port mirror. Applying the solutions of the equations of evolution of the expectation values of the atomic operators and the quantum Langavin equations for the cavity mode operators, we obtain the global and local photon statistics for light modes a and b. In addition, we determine the global photon statistics of the two-mode cavity light.

3.1 Generation of Single-mode photon statistics

In this section we obtain the global and local mean as well as variance of the photon numbers for light modes a and b.

3.1.1 Global mean photon number light mode a and b

Here we seek to calculate the global mean photon numbers of light mode a, produced by the system under consideration. The mean photon number of light mode a, represented by the operators $\hat{a}$ and $\hat{a}^\dagger$, is defined by

$$\bar{n}_a = \langle \hat{a}^\dagger \hat{a} \rangle. \tag{3.1}$$

We note that the steady-state solution of Eq. (2.94) is

$$\hat{a} = \frac{2g}{\kappa\sqrt{N}}\hat{m}_1, \tag{3.2}$$

so that introducing Eq. (3.2) and its adjoint into (3.1), we see that

$$\bar{n}_a = \frac{\gamma_c}{\kappa N}\langle \hat{m}_1^\dagger \hat{m}_1\rangle. \tag{3.3}$$

With the help of Eq. (2.70), one can write

$$\hat{m}_1^\dagger \hat{m}_1 = N\hat{N}_a. \tag{3.4}$$

On account of Eq. (3.4), Eq. (3.3) can be expressed as

$$\bar{n}_a = \frac{\gamma_c}{\kappa}\langle \hat{N}_a\rangle. \tag{3.5}$$

In view of Eq. (2.118), there follows

$$\bar{n}_a = \frac{\gamma_c}{\kappa}N\left[\frac{\Omega^2}{(\gamma_c+\gamma)^2+3\Omega^2}\right]. \tag{3.6}$$

This is the steady-state mean photon number of light mode a produced by the coherently driven non-degenerate three-level laser with an open cavity and coupled to a two-mode vacuum reservoir.

The plots on Fig. (3.1) describes the global mean photon number of light mode a versus Ω for the values of $\gamma_c = 0.4$, $\kappa = 0.8$, $N = 50$, $\gamma = 0.2$, and different values of γ. In addition, when we observe the plots on the same figure, as the spontaneous emission decay constant, γ, increases the mean photon number decreases. On the other hand, when the parameter of pumping, Ω, increases the mean photon number also increases.

When we consider the case in which spontaneous emission is absent $(\gamma = 0)$, the mean photon number of light mode a for this case has the form

$$\bar{n}_a = \frac{\gamma_c}{\kappa}N\left[\frac{\Omega^2}{\gamma_c^2+3\Omega^2}\right]. \tag{3.7}$$

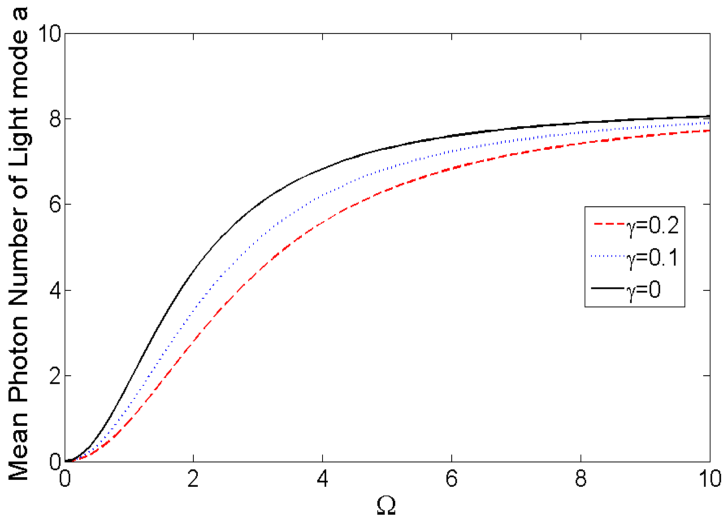

Figure 3.1: Plots of the mean photon number of light mode a [Eq. (3.6)] versus Ω.

This is the mean photon number of light mode a in the absence of spontaneous emission. In addition, we note that for $\Omega \gg \gamma_c$, Eq. (3.7) reduces to

$$\bar{n}_a = \frac{\gamma_c}{3\kappa} N. \tag{3.8}$$

When we observe the plots on Fig. (3.2) that the global mean photon number of light mode a versus the spontaneous emission decay constant (γ) for the values of $\gamma_c = 0.4$, $\kappa = 0.8$, $N = 50$, and different values of Ω. As we seen that when the spontaneous emission decay constant, γ, increases, the global mean photon number decreases. Moreover, as Ω increases the mean photon number increases. This must be due to the atomic coherence induced by the atoms in coherent coupling of the top and bottom levels.

Here we seek to determine the mean photon number of light mode b in the entire frequency interval produced by the system under consideration. The mean photon number of light mode b, represented by the operators $\hat{b}$ and $\hat{b}^\dagger$, is defined by

$$\bar{n}_b = \langle \hat{b}^\dagger \hat{b} \rangle. \tag{3.9}$$

We note that the steady-state solution of Eq. (2.95) is

$$\hat{b} = \frac{2g}{\kappa\sqrt{N}} \hat{m}_2, \tag{3.10}$$

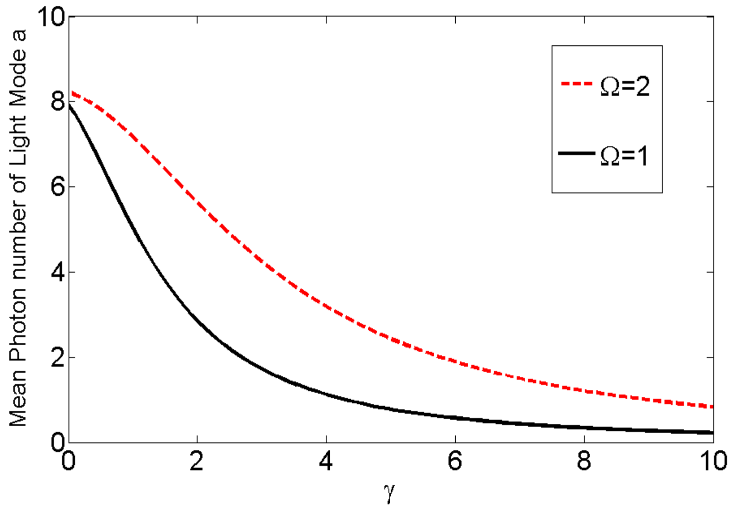

Figure 3.2: Plots of the mean photon number of light mode a [Eq. (3.6)] versus γ.

so that introducing Eq. (3.10) and its adjoint into (3.9), we see that

$$\bar{n}_b = \frac{\gamma_c}{\kappa N}\langle \hat{m}_2^\dagger \hat{m}_2\rangle. \tag{3.11}$$

With the help of Eq. (2.71), one can write

$$\hat{m}_2^\dagger \hat{m}_2 = N\hat{N}_b. \tag{3.12}$$

On account of Eq. (3.11), Eq. (3.12) can be expressed as

$$\bar{n}_b = \frac{\gamma_c}{\kappa}\langle \hat{N}_b\rangle. \tag{3.13}$$

Now on substituting Eq. (2.119) into (3.13), the mean photon number of light mode b takes, at steady-state, the form

$$\bar{n}_b = \frac{\gamma_c}{\kappa}N\left[\frac{\Omega^2}{(\gamma_c+\gamma)^2+3\Omega^2}\right]. \tag{3.14}$$

This is the steady-state mean photon number of light mode b produced by the coherently driven non-degenerate three-level laser with an open cavity and coupled to a two-mode vacuum reservoir.

Furthermore, we consider the case in which spontaneous emission is absent (when $\gamma = 0$). Then the mean photon number of light mode b for this case has the form

$$\bar{n}_b = \frac{\gamma_c}{\kappa} N \left[\frac{\Omega^2}{\gamma_c^2 + 3\Omega^2} \right]. \tag{3.15}$$

This is the mean photon number of light mode b in the absence of spontaneous emission. We would like to point out that this result is exactly the same as that described by Eq. (3.15). In addition, we note that for $\Omega \gg \gamma_c$, Eq. (3.15) reduces to

$$\bar{n}_b = \frac{\gamma_c}{3\kappa} N. \tag{3.16}$$

Finally, we observe that the global mean photon number of light mode a is equal to global mean photon number of light mode b.

3.1.2 Local mean photon number of light mode a and b

Here we seek to determine the local mean photon numbers of light modes a and b, produced by the coherently driven nondegenerate three-level laser with an open cavity and coupled to a two-mode vacuum reservoir. We now proceed to obtain the mean photon number of light mode a in a given frequency interval. To determine the local mean photon number of light mode a, we need to consider the power spectrum of light mode a. The power spectrum of light mode a with central frequency ω_0 is expressible as [6]

$$P_a(\omega) = \frac{1}{\pi} Re \int_0^{\infty} d\tau e^{i(\omega-\omega_0)\tau} \langle \hat{a}^\dagger(t)\hat{a}(t+\tau)\rangle_{ss}. \tag{3.17}$$

Upon integrating both sides of Eq. (3.17) over ω, we readily get

$$\int_{-\infty}^{\infty} P_a(\omega) d\omega = \bar{n}_a, \tag{3.18}$$

in which $\bar{n}_a$ is the steady-state mean photon number of light mode a. From this result, we observe that $P_a(\omega)d\omega$ is the steady-state mean photon number of light mode a in the frequency interval between ω and $\omega + d\omega$ [6].

We now proceed to determine the two-time correlation function that appears in Eq. (3.17). To this end, we realize that the solution of Eq. (2.94) can be written as

$$\hat{a}(t+\tau) = \hat{a}(t)e^{-\kappa\tau/2} + \frac{g}{\sqrt{N}}e^{-\kappa\tau/2}\int_0^\tau e^{\kappa\tau'/2}\hat{m}_1(t+\tau')d\tau'. \quad (3.19)$$

Applying the large time approximation on Eq. (2.100), we have

$$\langle\hat{m}_2^\dagger\rangle = \frac{\Omega}{\gamma+\gamma_c}\langle\hat{m}_1\rangle. \quad (3.20)$$

Employing this result, Eq. Eq. (2.99) takes the form

$$\frac{d}{dt}\langle\hat{m}_1\rangle = -\frac{1}{2}\eta\langle\hat{m}_1\rangle. \quad (3.21)$$

On the basis of Eq. (3.21), we see that

$$\frac{d}{dt}\hat{m}_1(t) = -\frac{1}{2}\eta\hat{m}_1(t) + \hat{F}_a(t), \quad (3.22)$$

in which $\hat{F}_a(t)$ is a noise operator with a vanishing mean and η is given by

$$\eta = \left[\frac{\Omega^2 + 2(\gamma+\gamma_c)^2}{(\gamma+\gamma_c)}\right]. \quad (3.23)$$

The solution of Eq. (3.22) can be put in the form

$$\hat{m}_1(t+\tau) = \hat{m}_1(t)e^{-\eta\tau/2} + e^{-\eta\tau/2}\int_0^\tau e^{\eta\tau'/2}\hat{F}_a(t+\tau')d\tau', \quad (3.24)$$

so that on introducing this into Eq. (3.19), there follows

$$\begin{aligned}\hat{a}(t+\tau) &= \hat{a}(t)e^{-\kappa\tau/2} + \frac{g}{\sqrt{N}}e^{-\kappa\tau/2}\hat{m}_1(t)\int_0^\tau e^{(\kappa-\eta)\tau'/2}d\tau' \\ &+ \frac{g}{\sqrt{N}}e^{-\kappa\tau/2}\int_0^\tau d\tau'\int_0^{\tau'} d\tau'' e^{[(\kappa-\eta)\tau'+\eta\tau'']/2}\hat{F}_a(t+\tau''). \quad (3.25)\end{aligned}$$

Thus on carrying out the first integration, we arrive at

$$\begin{aligned}\hat{a}(t+\tau) &= \hat{a}(t)e^{-\kappa\tau/2} + \frac{2g\hat{m}_1(t)}{\sqrt{N}(\kappa-\eta)}\left[e^{-\eta\tau/2} - e^{-\kappa\tau/2}\right] \\ &+ \frac{g}{\sqrt{N}}e^{-\kappa\tau/2}\int_0^\tau d\tau'\int_0^{\tau'} d\tau'' e^{[(\kappa-\eta)\tau'+\eta\tau'']/2}\hat{F}_a(t+\tau''). \quad (3.26)\end{aligned}$$

Now multiplying on the left by $\hat{a}^\dagger(t)$ and taking the expectation value of the resulting expression, we have

$$\begin{aligned}\langle \hat{a}^\dagger(t)\hat{a}(t+\tau)\rangle &= \langle \hat{a}^\dagger(t)\hat{a}(t)\rangle e^{-\kappa\tau/2} + \frac{2g\langle \hat{a}^\dagger(t)\hat{m}_1(t)\rangle}{\sqrt{N}(\kappa-\eta)}\left[e^{-\eta\tau/2} - e^{-\kappa\tau/2}\right] \\ &+ \frac{g}{\sqrt{N}} e^{-\kappa\tau/2}\int_0^\tau d\tau' \int_0^{\tau'} d\tau'' e^{[(\kappa-\eta)\tau'+\eta\tau'']/2}\langle \hat{a}^\dagger(t)\hat{F}_a(t+\tau'')\rangle. \quad (3.27)\end{aligned}$$

Applying the large-time approximation scheme, one gets from Eq. (2.94)

$$\hat{a}(t) = \frac{2g}{\kappa\sqrt{N}}\hat{m}_1(t), \quad (3.28)$$

so that in view of this result, we get

$$\hat{m}_1(t) = \frac{\kappa\sqrt{N}}{2g}\hat{a}(t). \quad (3.29)$$

Thus substitution of Eq. (3.29) into Eq. (3.27) results in

$$\begin{aligned}\langle \hat{a}^\dagger(t)\hat{a}(t+\tau)\rangle &= \langle \hat{a}^\dagger(t)\hat{a}(t)\rangle \left[\frac{\kappa}{\kappa-\eta}e^{-\eta\tau/2} - \frac{\eta}{\kappa-\eta}e^{-\kappa\tau/2}\right] \\ &+ \frac{g}{\sqrt{N}} e^{-\kappa\tau/2}\int_0^\tau d\tau' \int_0^{\tau'} d\tau'' e^{[(\kappa-\eta)\tau'+\eta\tau'']/2}\langle \hat{a}^\dagger(t)\hat{F}_a(t+\tau'')\rangle. \quad (3.30)\end{aligned}$$

Since a noise operator at a certain time should not affect a light mode operator at an earlier time [6], we note that

$$\langle \hat{a}^\dagger(t)\hat{F}_a(t+\tau'')\rangle = 0. \quad (3.31)$$

It then follows that

$$\langle \hat{a}^\dagger(t)\hat{a}(t+\tau)\rangle = \langle \hat{a}^\dagger(t)\hat{a}(t)\rangle \left[\frac{\kappa}{\kappa-\eta}e^{-\eta\tau/2} - \frac{\eta}{\kappa-\eta}e^{-\kappa\tau/2}\right] \quad (3.32)$$

and at steady-state, we have

$$\langle \hat{a}^\dagger(t)\hat{a}(t+\tau)\rangle_{ss} = \bar{n}_a \left[\frac{\kappa}{\kappa-\eta}e^{-\eta\tau/2} - \frac{\eta}{\kappa-\eta}e^{-\kappa\tau/2}\right]. \quad (3.33)$$

Thus on combining Eq. (3.33) with (3.17), the power spectrum of light mode a with central frequency ω_0 is expressible as

$$P_a(\omega) = \frac{1}{\pi}\left[\frac{\bar{n}_a}{\kappa-\eta}\right] Re\left[\kappa\int_0^\infty d\tau e^{-[\eta/2-i(\omega-\omega_0)]\tau} - \eta\int_0^\infty d\tau e^{-[\kappa/2-i(\omega-\omega_0)]\tau}\right], \quad (3.34)$$

so that on carrying out the integration, we readily arrive at

$$P_a(\omega) = \frac{1}{\pi}\left[\frac{\bar{n}_a}{\kappa - \eta}\right] Re\left[\frac{\kappa}{[\eta/2 - i(\omega - \omega_0)]} - \frac{\eta}{[\kappa/2 - i(\omega - \omega_0)]}\right]. \tag{3.35}$$

This can be rewritten as

$$P_a(\omega) = \frac{\kappa\bar{n}_a}{\kappa - \eta}\left[\frac{\eta/2\pi}{[\eta/2]^2 + (\omega - \omega_0)^2}\right] - \frac{\eta\bar{n}_a}{\kappa - \eta}\left[\frac{\kappa/2\pi}{[\kappa/2]^2 + (\omega - \omega_0)^2}\right]. \tag{3.36}$$

We realize that the mean photon number of light mode a in the interval between $\omega' = -\lambda$ and $\omega' = \lambda$ is expressible as [6]

$$\bar{n}_{a\pm\lambda} = \int_{-\lambda}^{\lambda} P_a(\omega')d\omega', \tag{3.37}$$

in which $\omega' = \omega - \omega_0$. Therefore, upon substituting Eq. (3.36) into (3.37) and carrying out the integration by employing the relation

$$\int_{-\lambda}^{\lambda} \frac{dx}{x^2 + a^2} = \frac{2}{a}\tan^{-1}\left(\frac{\lambda}{a}\right), \tag{3.38}$$

the local mean photon number of light mode a produced by the coherently driven non-degenerate three-level laser with an open cavity and coupled to a two-mode vacuum reservoir is found to be

$$\bar{n}_{a\pm\lambda} = \bar{n}_a z_a(\lambda), \tag{3.39}$$

where $z_a(\lambda)$ is given by

$$z_a(\lambda) = \frac{2\kappa/\pi}{\kappa - \eta}\tan^{-1}\left(\frac{2\lambda}{\eta}\right) - \frac{2\eta/\pi}{\kappa - \eta}\tan^{-1}\left(\frac{2\lambda}{\kappa}\right). \tag{3.40}$$

We see from Eq. (3.39) along with the plot of $z_a(\lambda)$ that $\bar{n}_{a\pm\lambda}$ increases with λ until it reaches the maximum value of the global mean photon number. The plots on Fig. (3.3) describes $z_a(\lambda)$ versus λ for $\gamma_c = 0.4$, $\kappa = 0.8$, $N = 50$, $\Omega = 2$, and for different values of γ. From these plots, we find the values indicated below:

Furthermore, in the absence of thermal light the local mean photon number indicated by Eq. (3.40) plotted on Fig. (3.3). From the plots in Fig. (3.3), we find the values indicated below:

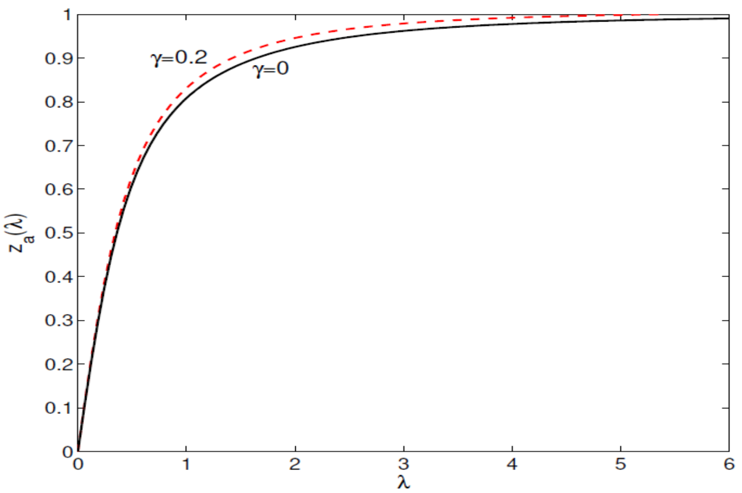

Figure 3.3: Plots of $z_a(\lambda)$[Eq. 3.40] versus λ.

	$z_a(0.5)$	$z_a(1)$	$z_a(2)$
$\gamma = 0.2$	0.63	0.82	0.94
$\gamma = 0$	0.61	0.80	0.92

Table 3.1: Values of $z_a(\lambda)$ for $\gamma_c = 0.4$, $\kappa = 0.8$, and $\Omega = 2$.

We see from these results, $z_a(\lambda)$ in the absence of spontaneous emission $(\gamma = 0)$ is less than in the presence of spontaneous emission $(\gamma = 0.2)$. Moreover, using the above results of $z_a(\lambda)$ and on account of Eq. (3.39) along with Eq. (3.6), we have

	$\bar{n}_{a\pm0.5}$	$\bar{n}_{a\pm1}$	$\bar{n}_{a\pm2}$
$\gamma = 0.2$	5.09	6.63	7.61
$\gamma = 0$	5.01	6.58	7.57

Table 3.2: Values of $\bar{n}_{a\pm\lambda}$ for $N = 50$, $\gamma_c = 0.4$, $\kappa = 0.8$, and $\Omega = 2$.

We see from Table 3.2 These results $\bar{n}_{a\pm\lambda}$ in the absence of spontaneous emission $(\gamma = 0)$ is less than in the presence of spontaneous emission $(\gamma = 0.2)$.

We therefore observe that a large part of the total mean photon number is confined in a relatively small frequency interval.

We now proceed to obtain the mean photon number of a light mode b in a given frequency interval produced by the system under consideration. To determine the local mean photon number of light mode b, we need to consider the power spectrum of light mode b. The power spectrum of light mode b with central frequency ω_0 is expressible as

$$P_b(\omega) = \frac{1}{\pi} Re \int_0^{\infty} d\tau e^{i(\omega-\omega_0)\tau} \langle \hat{b}^\dagger(t)\hat{b}(t+\tau)\rangle_{ss}. \tag{3.41}$$

Upon integrating both sides of Eq. (3.41) over ω, we readily get

$$\int_{-\infty}^{\infty} P_b(\omega) d\omega = \bar{n}_b, \tag{3.42}$$

in which $\bar{n}_b$ is the steady-state mean photon number of light mode b. From this result, we observe that $P_b(\omega)d\omega$ is the steady-state mean photon number of light mode b in the frequency interval between ω and $\omega + d\omega$. We now proceed to calculate the two-time correlation function that appears in Eq. (3.41). To this end, we realize that the solution of Eq. (2.95) can be written as

$$\hat{b}(t+\tau) = \hat{b}(t)e^{-\kappa\tau/2} + \frac{g}{\sqrt{N}} e^{-\kappa\tau/2} \int_0^{\tau} e^{\kappa\tau'/2} \hat{m}_2(t+\tau')d\tau'. \tag{3.43}$$

Applying the large time approximation on Eq. (2.99), we have

$$\langle \hat{m}_1 \rangle = \frac{\Omega}{\gamma + \gamma_c} \langle \hat{m}_2^\dagger \rangle. \tag{3.44}$$

Employing this result, Eq. Eq. (2.100) takes the form

$$\frac{d}{dt}\langle \hat{m}_2^\dagger \rangle = -\frac{1}{2}\mu \langle \hat{m}_2^\dagger \rangle. \tag{3.45}$$

On the basis of Eq. (3.45), we see that

$$\frac{d}{dt}\hat{m}_2(t) = -\frac{1}{2}\mu \hat{m}_2(t) + \hat{F}_b(t), \tag{3.46}$$

in which $\hat{F}_b(t)$ is a noise operator with a vanishing mean and μ is given by

$$\mu = \left[\frac{\Omega^2 + 2(\gamma+\gamma_c)^2}{2(\gamma+\gamma_c)}\right]. \tag{3.47}$$

The solution of equation (3.46) can be put in the form

$$\hat{m}_2(t+\tau') = \hat{m}_2(t)e^{-\mu\tau'/2} + e^{-\mu\tau'/2}\int_0^{\tau'} e^{-\mu\tau''/2}\hat{F}_b(t+\tau'')d\tau'', \tag{3.48}$$

so that on introducing this into Eq. (3.43), we have

$$\begin{aligned}\hat{b}(t+\tau) &= \hat{b}(t)e^{-\kappa\tau/2} + \frac{g}{\sqrt{N}}e^{-\kappa\tau/2}\hat{m}_2(t)\int_0^{\mu} e^{(\kappa-\mu)\tau'/2}d\tau' \\ &+ \frac{g}{\sqrt{N}}e^{-\kappa\tau/2}\int_0^{\tau} d\tau'\int_0^{\tau'} d\tau'' e^{[(\kappa-\mu)\tau'+\mu\tau'']/2}\hat{F}_b(t+\tau''). \end{aligned}\tag{3.49}$$

Thus on carrying out the first integration, we arrive at

$$\begin{aligned}\hat{b}(t+\tau) &= \hat{b}(t)e^{-\kappa\tau/2} + \frac{2g\hat{m}_b(t)}{\sqrt{N}(\kappa-\mu)}\left[e^{-\mu\tau/2} - e^{-\kappa\tau/2}\right] \\ &+ \frac{g}{\sqrt{N}}e^{-\kappa\tau/2}\int_0^{\tau} d\tau'\int_0^{\tau'} d\tau'' e^{[(\kappa-\mu)\tau'+\mu\tau'']/2}\hat{F}_b(t+\tau''). \end{aligned}\tag{3.50}$$

Now multiplying both sides on the left by $\hat{b}^\dagger(t)$ and taking the expectation value of the resulting equation, we have

$$\begin{aligned}\langle\hat{b}^\dagger(t)\hat{b}(t+\tau)\rangle &= \langle\hat{b}^\dagger(t)\hat{b}(t)\rangle e^{-\kappa\tau/2} + \frac{2g\langle\hat{b}^\dagger(t)\hat{m}_2(t)\rangle}{\sqrt{N}(\kappa-\mu)}\left[e^{-\mu\tau/2} - e^{-\kappa\tau/2}\right] \\ &+ \frac{g}{\sqrt{N}}e^{-\kappa\tau/2}\int_0^{\tau} d\tau'\int_0^{\tau'} d\tau'' e^{[(\kappa-\mu)\tau'+\mu\tau'']/2}\langle\hat{b}^\dagger(t)\hat{F}_b(t+\tau'')\rangle. \end{aligned}\tag{3.51}$$

Applying the large-time approximation scheme, one gets from Eq. (2.95)

$$\hat{b}(t) = \frac{2g}{\kappa\sqrt{N}}\hat{m}_2(t). \tag{3.52}$$

In view of Eq. (3.52), we see that

$$\hat{m}_2(t) = \frac{\kappa\sqrt{N}}{2g}\hat{b}(t). \tag{3.53}$$

With this substituted into Eq. (3.51), there follows

$$\begin{aligned}\langle\hat{b}^\dagger(t)\hat{b}(t+\tau)\rangle &= \langle\hat{b}^\dagger(t)\hat{b}(t)\rangle\left[\frac{\kappa}{\kappa-\mu}e^{-\mu\tau/2} - \frac{\mu}{\kappa-\mu}e^{-\kappa\tau/2}\right] \\ &+ \frac{g}{\sqrt{N}}e^{-\kappa\tau/2}\int_0^{\tau} d\tau'\int_0^{\tau'} d\tau'' e^{[(\kappa-\mu)\tau'+\mu\tau'']/2}\langle\hat{b}^\dagger(t)\hat{F}_b(t+\tau'')\rangle \end{aligned}\tag{3.54}$$

and taking into account the fact that

$$\langle \hat{b}^{\dagger}(t)\hat{F}_b(t+\tau'')\rangle = 0, \tag{3.55}$$

we arrive at

$$\langle \hat{b}^{\dagger}(t)\hat{b}(t+\tau)\rangle = \langle \hat{b}^{\dagger}(t)\hat{b}(t)\rangle \left[\frac{\kappa}{\kappa-\mu}e^{-\mu\tau/2} - \frac{\mu}{\kappa-\mu}e^{-\kappa\tau/2}\right]. \tag{3.56}$$

Therefore, at steady-state, Eq. (3.56) takes the form

$$\langle \hat{b}^{\dagger}(t)\hat{b}(t+\tau)\rangle_{ss} = \bar{n}_b \left[\frac{\kappa}{\kappa-\mu}e^{-\mu\tau/2} - \frac{\mu}{\kappa-\mu}e^{-\kappa\tau/2}\right]. \tag{3.57}$$

Thus on combining Eq. (3.57) with (3.41), the power spectrum of light mode b with central frequency ω_0 can be put in the form

$$P_b(\omega) = \frac{1}{\pi}\left[\frac{\bar{n}_b}{\kappa-\mu}\right] Re\left[\kappa\int_0^{\infty} d\tau e^{-[\mu/2-i(\omega-\omega_0)]\tau} - \mu\int_0^{\infty} d\tau e^{-[\kappa/2-i(\omega-\omega_0)]\tau}, \tag{3.58}$$

so that on carrying out the integration, we readily arrive at

$$P_b(\omega) = \frac{\kappa\bar{n}_b}{\kappa-\mu}\left[\frac{\mu/2\pi}{[\mu/2]^2+(\omega-\omega_0)^2}\right] - \frac{\mu\bar{n}_b}{\kappa-\mu}\left[\frac{\kappa/2\pi}{[\kappa/2]^2+(\omega-\omega_0)^2}\right]. \tag{3.59}$$

We realize that the mean photon number of light mode b in the interval between $\omega' = -\lambda$ and $\omega' = \lambda$ is expressible as

$$\bar{n}_{b\pm\lambda} = \int_{-\lambda}^{\lambda} P(\omega')d\omega', \tag{3.60}$$

in which $\omega' = \omega - \omega_0$. Therefore, upon substituting Eq. (3.59) into (3.60), and performing the integration by using the relation given by Eq. (3.38), we readily get

$$\bar{n}_{b\pm\lambda} = \bar{n}_b z_b(\lambda), \tag{3.61}$$

where $z_b(\lambda)$ is given by

$$z_b(\lambda) = \frac{2\kappa/\pi}{\kappa-\mu}\tan^{-1}\left(\frac{2\lambda}{\mu}\right) - \frac{2\mu/\pi}{\kappa-\mu}\tan^{-1}\left(\frac{2\lambda}{\kappa}\right). \tag{3.62}$$

We see from Eq. (3.14) along with the plot of $z_a(\lambda)$ that $\bar{n}_{a\pm\lambda}$ increases with λ until it reaches the maximum value of the global mean photon number.

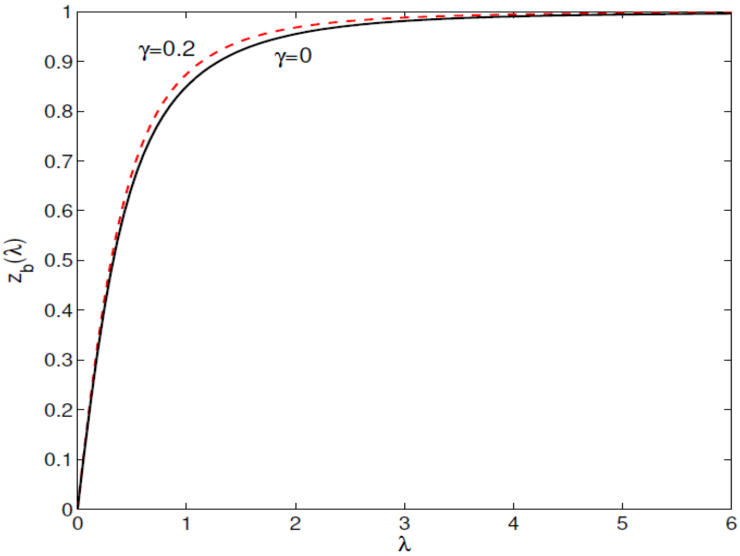

Figure 3.4: Plots of $z_b(\lambda)$[Eq. 3.62] versus λ.

We see from Eq. (3.14) along with the plot of $z_b(\lambda)$ that $\bar{n}_{b\pm\lambda}$ increases with λ until it reaches the maximum value of the global mean photon number.

The plots on Fig. (3.4) describes z_b versus λ for $\gamma_c = 0.4$, $\kappa = 0.8$, $N = 50$, $\Omega = 2$, and for different values of γ. From these plots, we find the values indicated below:

	$z_b(0.5)$	$z_b(1)$	$z_b(2)$
$\gamma = 0.2$	0.67	0.87	0.97
$\gamma = 0$	0.64	0.84	0.95

Table 3.3: Values of $z_b(\lambda)$ for $\gamma_c = 0.4$, $\kappa = 0.8$, and $\Omega = 2$.

We see from these results $z_b(\lambda)$ in the absence of spontaneous emission $(\gamma = 0)$ is less than in the presence of spontaneous emission $(\gamma = 0.2)$. Moreover, using the above results of $z_b(\lambda)$ and on account of Eq. (3.61) along with (3.14), we have

	$\bar{n}_{b\pm0.5}$	$\bar{n}_{b\pm1}$	$\bar{n}_{b\pm2}$
$\gamma = 0.2$	5.57	7.16	7.98
$\gamma = 0$	5.26	6.91	7.81

Table 3.4: Values of $\bar{n}_{b\pm\lambda}$ for $N = 50$, $\bar{n}_{th} = 0$, $\gamma_c = 0.4$, $\kappa = 0.8$, and $\Omega = 2$.

We see from Table 3.4 These results $\bar{n}_{b\pm\lambda}$ in the absence of spontaneous emission $(\gamma = 0)$ is less than in the presence of spontaneous emission $(\gamma = 0.2)$.

From the plots in Fig. (3.4), we therefore observe that a large part of the total mean photon number is confined in a relatively small frequency interval.

3.1.3 Global photon-number variance of light mode a and b

We now proceed to calculate the photon number variance of light mode a in the entire frequency interval. The photon number variance of light mode a is expressible as

$$(\Delta n)_a^2 = \langle \hat{a}^\dagger \hat{a} \hat{a}^\dagger \hat{a} \rangle - \langle \hat{a}^\dagger \hat{a} \rangle^2. \tag{3.63}$$

Applying the fact that $\hat{a}$ is a Gaussian variable with zero mean, we arrive at

$$(\Delta n)_a^2 = \langle \hat{a}^\dagger \hat{a} \rangle \langle \hat{a} \hat{a}^\dagger \rangle + \langle \hat{a}^{\dagger 2} \rangle \langle \hat{a}^2 \rangle. \tag{3.64}$$

In view of Eq. (3.2), we see that

$$\langle \hat{a}^2 \rangle = 0, \tag{3.65}$$

$$\langle \hat{a} \hat{a}^\dagger \rangle = \frac{\gamma_c}{\kappa} \langle \hat{N}_b \rangle. \tag{3.66}$$

Thus on account of Eqs. (3.5), (3.65) and (3.66), the photon number variance (3.64) turns out to be

$$(\Delta n)_a^2 = \left(\frac{\gamma_c}{\kappa} \right)^2 \langle \hat{N}_a \rangle \langle \hat{N}_b \rangle. \tag{3.67}$$

With the aid of Eqs. (2.118) and (2.119), the photon number variance of light mode a takes, at steady-state, the form

$$(\Delta n)_a^2 = \left[\frac{\gamma_c}{\kappa} N \right]^2 \left[\frac{\Omega^2}{(\gamma_c + \gamma)^2 + 3\Omega^2} \right]^2. \tag{3.68}$$

This is the steady-state normally-ordered variance of the photon number for light mode a produced by the coherently driven non-degenerate three-level laser with an open cavity and coupled to a two-mode vacuum reservoir. Hence, in view of Eq. (3.6), we have

$$(\Delta n)_a^2 = \bar{n}_a^2. \tag{3.69}$$

Furthermore, we consider the case in which spontaneous emission is absent $(\gamma = 0)$. Then the photon number variance for this case has the form

$$(\Delta n)_a^2 = \left(\frac{\gamma_c}{\kappa}N\right)^2 \left[\frac{\Omega^2}{\gamma_c^2 + 3\Omega^2}\right]^2. \tag{3.70}$$

This is the photon number variance of light mode a in the absence of spontaneous emission. Therefore, in view of Eq. (3.7), we have

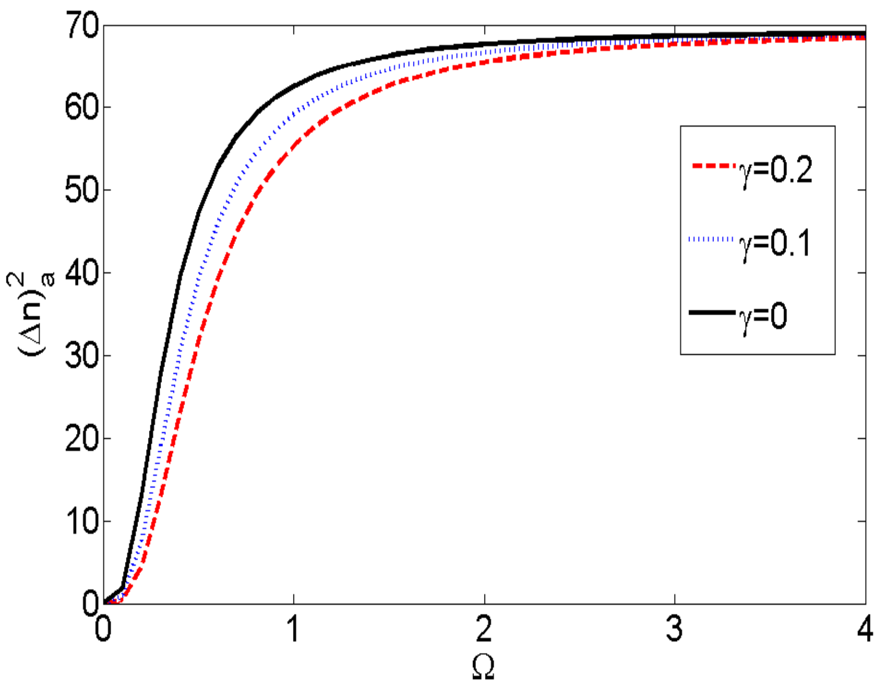

Figure 3.5: Plots of the photon number variance of light mode a [Eq. 3.68] versus Ω.

$$(\Delta n)_a^2 = \bar{n}_a^2. \tag{3.71}$$

We readily observe from the plots in Fig. (3.5) that the photon number variance of light mode a versus Ω for $\gamma_c = 0.4$, $\kappa = 0.8$, $N = 50$, and for different values of γ. From these plots, we observe that $(\Delta n)^2_a$ increases as γ decreases until $\Omega = 4$.

In addition, we note that for $\Omega \gg \gamma_c$, Eq. (3.70) reduces to

$$(\Delta n)^2_a = \left[\frac{\gamma_c}{3\kappa}N\right]^2, \tag{3.72}$$

so that with the aid of Eq. (3.8), we see that

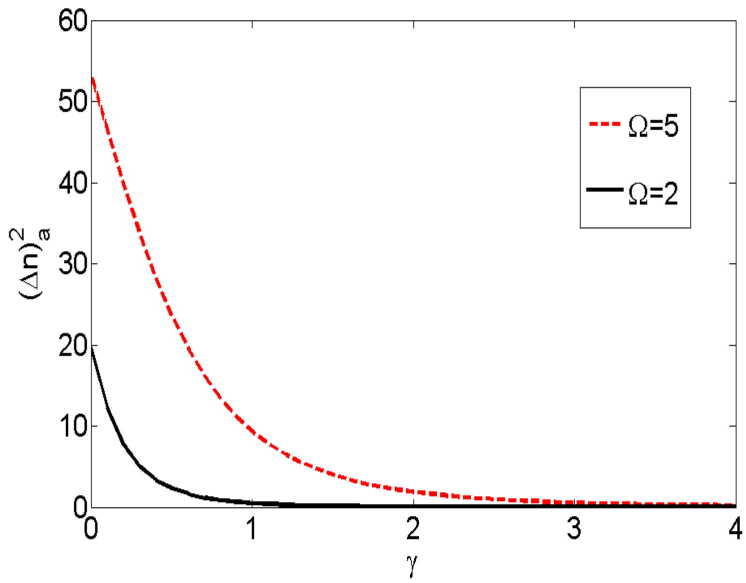

Figure 3.6: Plots of the photon number variance of light mode a [Eq. 3.70] versus γ.

$$(\Delta n)^2_a = \bar{n}^2_a. \tag{3.73}$$

The plots in Fig. (3.6) show that the photon number variance of light mode a versus γ for with values of $\gamma_c = 0.4$, $\kappa = 0.8$, $N = 50$, and for different values of Ω. As we observe from the plots in Fig. (3.6) that the photon number variance of light mode a increases with Ω. On the other hand, when the spontaneous emission decay constant increases the photon number variance decreases.

Here we seek to obtain the photon number variance of light mode b in the entire frequency interval. The photon number variance of light mode b is defined as

$$(\Delta n)_b^2 = \langle \hat{b}^\dagger \hat{b} \hat{b}^\dagger \hat{b} \rangle - \langle \hat{b}^\dagger \hat{b} \rangle^2 \tag{3.74}$$

and using the fact that $\hat{b}$ is a Gaussian variable with zero mean, we readily get

$$(\Delta n)_b^2 = \langle \hat{b}^\dagger \hat{b} \rangle \langle \hat{b} \hat{b}^\dagger \rangle + \langle \hat{b}^{\dagger 2} \rangle \langle \hat{b}^2 \rangle. \tag{3.75}$$

In view of Eq. (3.10), we have

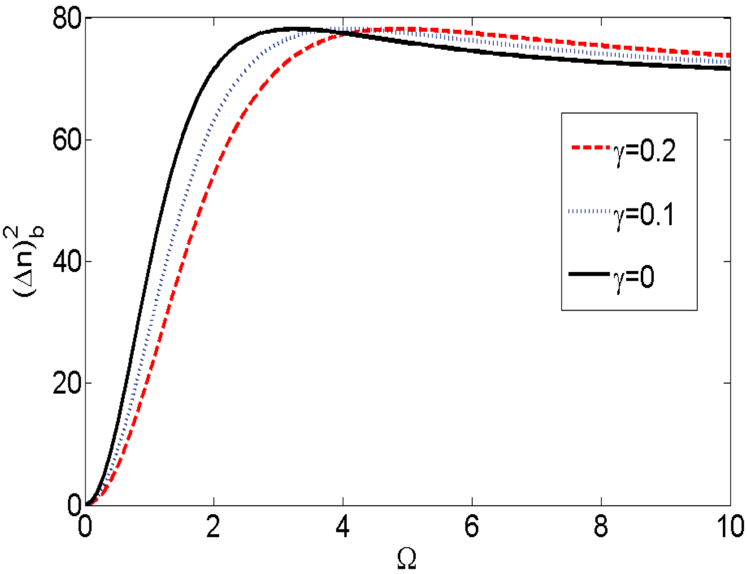

Figure 3.7: Plots of the photon number variance of light mode b [Eq. 3.80] versus Ω.

$$\langle \hat{b}^2 \rangle = 0, \tag{3.76}$$

$$\langle \hat{b} \hat{b}^\dagger \rangle = \frac{\gamma_c}{\kappa} \langle \hat{N}_c \rangle. \tag{3.77}$$

Thus on account of Eqs. (3.13), (3.76) and (3.77), the photon number variance (3.75) turns out to be

$$(\Delta n)_b^2 = \left(\frac{\gamma_c}{\kappa} \right)^2 \langle \hat{N}_c \rangle \langle \hat{N}_b \rangle, \tag{3.78}$$

from which follows

$$(\Delta n)_b^2 = \bar{n}_b \left[\frac{\gamma_c}{\kappa} N - 2\bar{n}_b \right]. \tag{3.79}$$

With the aid of Eq. (3.14), the photon number variance of light mode b takes, at steady-state, the form

$$(\Delta n)_b^2 = \left(\frac{\gamma_c}{\kappa}N\right)^2 \left[\frac{\Omega^2(\gamma_c+\gamma)^2+\Omega^4}{[(\gamma_c+\gamma)^2+3\Omega^2]^2}\right]. \tag{3.80}$$

This is the global photon number variance of light mode b, produced by the coherently driven nondegenerate three-level laser with an open cavity and coupled to a two-mode vacuum reservoir.

The plots on Fig. (3.7) describes the photon number variance of light mode b versus Ω for $\gamma_c = 0.4$, $\kappa = 0.8$, $N = 50$, and for different values of γ. We readily observe from the plots in Fig. (3.7) that the photon number variance of light mode b in the absence of spontaneous emission when $\gamma = 0$ is greater than in the presence of spontaneous emission when $\gamma = 0.2$ until $\Omega = 4$. In addition, the maximum photon number variance $(\Delta n)_b^2 = 78.12$ when $\gamma = 0$, $\gamma = 0.1$, and $\gamma = 0.2$. These results occur when the three-level laser is operating at $\Omega = 0.404$,$\Omega = 0.505$, and $\Omega = 0.606$, respectively.

Furthermore, we consider the case in which spontaneous emission is absent $(\gamma = 0)$. Then the variance of photon number for this case has the form

$$(\Delta n)_b^2 = \left(\frac{\gamma_c}{\kappa}N\right)^2 \left[\frac{\Omega^2(\gamma_c^2+\Omega^2)}{(\gamma_c^2+3\Omega^2)^2}\right]. \tag{3.81}$$

This represents the variance of the photon number of light mode b in the absence of spontaneous emission. In addition, we note that for $\Omega \gg \gamma_c$, Eq. (3.81) reduces to

$$(\Delta n)_b^2 = \left[\frac{\gamma_c}{3\kappa}N\right]^2 \tag{3.82}$$

and in view of Eq. (3.16), there follows

$$(\Delta n)_b^2 = \bar{n}_b^2, \tag{3.83}$$

which represents the normally-ordered variance of the photon number for chaotic light.

The plots on Fig. (3.8) describes the photon number variance of light mode b versus γ for the values of $\gamma_c = 0.4$, $\kappa = 0.8$, $N = 50$, with different values of Ω. We readily

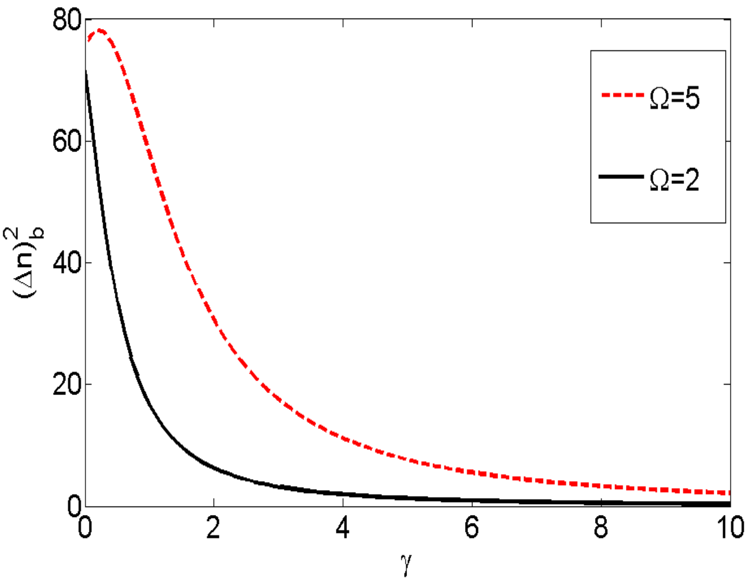

Figure 3.8: Plots of the photon number variance of light mode b [Eq. 3.80] versus γ.

observe from the plots in Fig. (3.8) that the photon number variance of light mode b for $\Omega = 5$ is greater than for $\Omega = 2$. In addition, the maximum photon number variance is found to be $(\Delta n)^2_b = 71.52$ for $\Omega = 2$ and it occurs at $\gamma = 0$. Moreover, the maximum photon number variance is $(\Delta n)^2_b = 78.12$ for $\Omega = 5$ and it occurs at $\gamma = 0$.

3.1.4 Local photon-number variance of light mode a and b

Here we seek to study the local photon number variance of light modes a and b, produced by the coherently driven non degenerate three-level laser with an open cavity and coupled to a two-mode vacuum reservoir. To determine the local photon number variance of light mode a, we need to consider the spectrum of photon number fluctuations of light mode a. The spectrum of photon number fluctuations of light mode a with central frequency ω_0 is expressible as [7]

$$S_a(\omega) = \frac{1}{\pi}\int_0^{\infty} d\tau e^{i(\omega-\omega_0)\tau}\langle\hat{n}_a(t), \hat{n}_a(t+\tau)\rangle_{ss}, \tag{3.84}$$

where

$$\hat{n}_a(t) = \hat{a}^\dagger(t)\hat{a}(t), \tag{3.85}$$

$$\hat{n}_a(t+\tau) = \hat{a}^\dagger(t+\tau)\hat{a}(t+\tau). \tag{3.86}$$

Upon integrating both sides of Eq. (2.89) over ω, we find

$$\int_{-\infty}^{\infty} S_a(\omega)d\omega = (\Delta n)_a^2, \tag{3.87}$$

in which $(\Delta n)_a^2$ is the steady-state photon number variance of light mode a. From this result, we realize that $S_a(\omega)d\omega$ is the photon number variance of light mode a in the frequency interval between ω and $\omega + d\omega$ [7].

We now proceed to evaluate the two-time correlation function that appears in Eq. (3.84). Applying the notation [7]

$$\langle \hat{A}, \hat{B}\rangle = \langle \hat{A}\hat{B}\rangle - \langle \hat{A}\rangle\langle \hat{B}\rangle, \tag{3.88}$$

we see that

$$\langle \hat{n}_a(t), \hat{n}_a(t+\tau)\rangle = \langle \hat{n}_a(t)\hat{n}_a(t+\tau)\rangle - \langle \hat{n}_a(t)\rangle\langle \hat{n}_a(t+\tau)\rangle. \tag{3.89}$$

On account of Eqs. (3.85) and (3.86) and using the fact that a is a Gaussian variable with zero mean given by Eq. (2.108), we have

$$\begin{aligned}\langle \hat{n}_a(t)\hat{n}_a(t+\tau)\rangle &= \langle \hat{a}^\dagger(t)\hat{a}(t)\rangle\langle \hat{a}^\dagger(t+\tau)\hat{a}(t+\tau)\rangle \\ &+ \langle \hat{a}(t)\hat{a}(t+\tau)\rangle\langle \hat{a}^\dagger(t)\hat{a}^\dagger(t+\tau)\rangle \\ &+ \langle \hat{a}^\dagger(t)\hat{a}(t+\tau)\rangle\langle \hat{a}(t)\hat{a}^\dagger(t+\tau)\rangle. \end{aligned}\tag{3.90}$$

Thus substitution of Eq. (3.90) into Eq. (3.89) results in

$$\begin{aligned}\langle \hat{n}_a(t), \hat{n}_a(t+\tau)\rangle &= \langle \hat{a}^\dagger(t)\hat{a}^\dagger(t+\tau)\rangle\langle \hat{a}(t)\hat{a}(t+\tau)\rangle \\ &+ \langle \hat{a}^\dagger(t)\hat{a}(t+\tau)\rangle\langle \hat{a}(t)\hat{a}^\dagger(t+\tau)\rangle. \end{aligned}\tag{3.91}$$

With the help of Eq. (3.25), one can readily establish that

$$\langle\hat{a}^\dagger(t)\hat{a}^\dagger(t+\tau)\rangle = \langle\hat{a}^{\dagger 2}(t)\rangle\left[\frac{\kappa}{\kappa-\eta}e^{-\eta\tau/2} - \frac{\eta}{\kappa-\eta}e^{-\kappa\tau/2}\right], \tag{3.92}$$

$$\langle\hat{a}(t)\hat{a}(t+\tau)\rangle = \langle\hat{a}^2(t)\rangle\left[\frac{\kappa}{\kappa-\eta}e^{-\eta\tau/2} - \frac{\eta}{\kappa-\eta}e^{-\kappa\tau/2}\right], \tag{3.93}$$

$$\langle\hat{a}(t)\hat{a}^\dagger(t+\tau)\rangle = \langle\hat{a}(t)\hat{a}^\dagger(t)\rangle\left[\frac{\kappa}{\kappa-\eta}e^{-\eta\tau/2} - \frac{\eta}{\kappa-\eta}e^{-\kappa\tau/2}\right]. \tag{3.94}$$

Now employing Eqs. (3.33), (3.92), (3.93), and (3.94), we obtain

$$\begin{aligned}\langle\hat{n}_a(t)\hat{n}_a(t+\tau)\rangle &= \left[\langle\hat{a}^\dagger(t)\hat{a}(t)\rangle\langle\hat{a}(t)\hat{a}^\dagger(t)\rangle + \langle\hat{a}^2(t)\rangle\langle\hat{a}^{\dagger 2}(t)\rangle\right]\\ &\times \left[\left(\frac{\eta}{\kappa-\eta}\right)^2 e^{-\kappa\tau} + \left(\frac{\kappa}{\kappa-\eta}\right)^2 e^{-\eta\tau} - \frac{2\kappa\eta}{(\kappa-\eta)^2}e^{-(\kappa+\eta)\tau/2}\right]. \end{aligned}\tag{3.95}$$

This can be rewritten as

$$\langle\hat{n}_a(t)\hat{n}_a(t+\tau)\rangle_{ss} = \frac{(\Delta n)^2_a}{(\kappa-\eta)^2}\left[\eta^2 e^{-\kappa\tau} + \kappa^2 e^{-\eta\tau} - 2\kappa\eta e^{-(\kappa+\eta)\tau/2}\right], \tag{3.96}$$

in which $(\Delta n)^2_a$ is the steady-state photon number variance of light mode a given by Eq. (3.68). Therefore, in view of Eq. (3.96), the spectrum of photon number fluctuations can be put in the form

$$\begin{aligned}S_a(\omega) &= \frac{(\Delta n)^2_a}{\pi(\kappa-\eta)^2}Re\left[\eta^2\int_0^\infty d\tau e^{-[\kappa-i(\omega-\omega_0)]\tau}\right.\\ &+ \left.\kappa^2\int_0^\infty d\tau e^{-[\eta-i(\omega-\omega_0)]\tau} - 2k\eta\int_0^\infty d\tau e^{-[\frac{(\kappa+\eta)}{2}-i(\omega-\omega_0)]\tau}\right]. \end{aligned}\tag{3.97}$$

Thus on carrying out the integration, the spectrum of photon number fluctuations of light mode a turns out to be

$$S_a(\omega) = \frac{(\Delta n)^2_a}{(\kappa-\eta)^2}\left[\frac{\eta^2\kappa/\pi}{\kappa^2+(\omega-\omega_0)^2} + \frac{\kappa^2\eta/\pi}{\eta^2+(\omega-\omega_0)^2} - \frac{2\kappa\eta(\kappa+\eta)/2\pi}{(\frac{\kappa+\eta}{2})^2+(\omega-\omega_0)^2}\right]. \tag{3.98}$$

Now we realize that the photon number variance in the frequency interval between $\omega' = -\lambda$ and $\omega' = \lambda$ is expressible as [3]

$$(\Delta n)^2_{a\pm\lambda} = \int_{-\lambda}^{\lambda} S_a(\omega')d\omega', \tag{3.99}$$

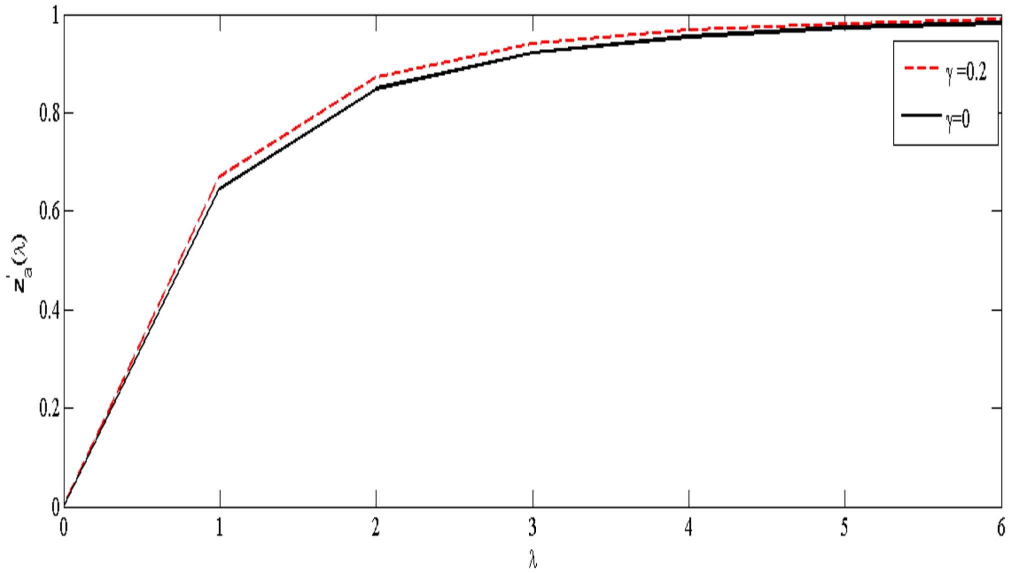

Figure 3.9: Plots of $z_a^{'}(\lambda)$[Eq. 3.102] versus λ.

in which $\omega' = \omega - \omega_0$. Therefore, substituting Eq. (3.98) into Eq. (3.99) leads to

$$(\Delta n)^2_{a\pm\lambda} = \frac{(\Delta n)^2_a}{\pi(\kappa-\eta)^2}\Bigg[\int_{-\lambda}^{\lambda}\frac{\eta^2\kappa d\omega'}{\kappa^2+\omega'^2} - \int_{-\lambda}^{\lambda}\frac{2\kappa\eta(\kappa+\eta)d\omega'}{(\frac{\kappa+\eta}{2})^2+\omega'^2} + \int_{-\lambda}^{\lambda}\frac{\eta\kappa^2 d\omega'}{\eta^2+\omega'^2}\Bigg]. \quad (3.100)$$

Employing the relation given by Eq. (3.38), the local photon number variance of light mode a produced by the coherently driven nondegenerate three-level laser with an open cavity and coupled to a two-mode vacuum reservoir is found to be

$$(\Delta n)^2_{a\pm\lambda} = (\Delta n)^2_a z_a^{'}(\lambda), \quad (3.101)$$

where $z_a^{'}(\lambda)$ is given by

$$z_a^{'}(\lambda) = \frac{2\eta^2/\pi}{(\eta-\kappa)^2}tan^{-1}\left(\frac{\lambda}{\kappa}\right) + \frac{2\kappa^2/\pi}{(\kappa-\eta)^2}tan^{-1}\left(\frac{\lambda}{\eta}\right) - \frac{4\kappa\eta/\pi}{(\kappa-\eta)^2}tan^{-1}\left(\frac{2\lambda}{\kappa+\eta}\right). \quad (3.102)$$

We see from Eq. (3.101) along with the plot $z_a^{'}(\lambda)$ that $(\Delta n)^2_{a\pm\lambda}$ increases with λ until it reaches the maximum value of the global photon number variance.

The plots on Fig. (3.9) describes $z_a^{'}(\lambda)$ versus λ for $\gamma_c = 0.4$, $\kappa = 0.8$, $N = 50$, $\Omega = 2$,and for different values of γ. From these plot, we find the values indicated below:

	$z_a^{'}(0.5)$	$z_a^{'}(1)$	$z_a^{'}(2)$
$\gamma = 0.2$	0.42	0.66	0.86
$\gamma = 0$	0.40	0.64	0.84

Table 3.5: Values of $z_a^{'}(\lambda)$ for $\gamma_c = 0.4$, $\kappa = 0.8$, and $\Omega = 2$.

We see from these results that $z_a^{'}(\lambda)$ in the absence of spontaneous emission $(\gamma = 0)$ is less than in the presence of spontaneous emission $(\gamma \neq 0)$. Moreover, using the above results of $z_a^{'}(\lambda)$ and on account of Eq. (3.102), we have

	$(\Delta n)^2_{a\pm 0.5}$	$(\Delta n)^2_{a\pm 1}$	$(\Delta n)^2_{a\pm 2}$
$\gamma = 0.2$	27.49	43.20	56.29
$\gamma = 0$	26.18	41.89	54.98

Table 3.6: Values of $(\Delta n)^2_{a\pm\lambda}$ for $\gamma_c = 0.4$, $\kappa = 0.8$, $N = 50$, and $\Omega = 2$.

We therefore observe that a large part of the total variance of photon number is confined in a relatively small frequency interval.

We now proceed to obtain the photon number variance of light mode b in a given frequency interval produced by the system under consideration. To determine the local photon number variance of light mode b, we need to consider the spectrum of photon number fluctuations of light mode b. We define the spectrum of photon number fluctuations of light mode b with central frequency ω_0 by

$$S_b(\omega) = \frac{1}{\pi} Re \int_0^\infty d\tau e^{i(\omega-\omega_0)\tau} \langle \hat{n}_b(t), \hat{n}_b(t+\tau) \rangle_{ss}, \tag{3.103}$$

where

$$\hat{n}_b(t) = \hat{b}^\dagger(t)\hat{b}(t), \tag{3.104}$$

$$\hat{n}_b(t+\tau) = \hat{b}^\dagger(t+\tau)\hat{b}(t+\tau). \tag{3.105}$$

Upon integrating both sides of Eq. (3.103) over ω, we easily find

$$\int_{-\infty}^{\infty} S_b(\omega) d\omega = (\Delta n)_b^2, \tag{3.106}$$

in which $(\Delta n)_b^2$ is the steady-state photon number variance of the light mode b. We can then assert that $S_b(\omega)d\omega$ is the steady-state photon number variance of light mode b in the frequency interval between ω and $\omega + d\omega$.

We now proceed to evaluate the two-time correlation function that appears in Eq. (3.106). Applying the relation given by Eq. (3.88), we see that

$$\langle \hat{n}_b(t), \hat{n}_b(t+\tau)\rangle = \langle \hat{n}_b(t)\hat{n}_b(t+\tau)\rangle - \langle \hat{n}_b(t)\rangle\langle \hat{n}_b(t+\tau)\rangle. \tag{3.107}$$

On account of Eqs. (3.104) and (3.105), we have

$$\begin{aligned} \langle \hat{n}_b(t)\hat{n}_b(t+\tau)\rangle &= \langle \hat{b}^\dagger(t)\hat{b}(t)\rangle\langle \hat{b}^\dagger(t+\tau)\hat{b}(t+\tau)\rangle \\ &+ \langle \hat{b}(t)\hat{b}(t+\tau)\rangle\langle \hat{b}^\dagger(t)\hat{b}^\dagger(t+\tau)\rangle \\ &+ \langle \hat{b}^\dagger(t)\hat{b}(t+\tau)\rangle\langle \hat{b}(t)\hat{b}^\dagger(t+\tau)\rangle. \end{aligned} \tag{3.108}$$

Thus substitution of Eq. (3.107) into Eq. (3.108) results in

$$\begin{aligned} \langle \hat{n}_b(t), \hat{n}_b(t+\tau)\rangle &= \langle \hat{b}^\dagger(t)\hat{b}^\dagger(t+\tau)\rangle\langle \hat{b}(t)\hat{b}(t+\tau)\rangle \\ &+ \langle \hat{b}^\dagger(t)\hat{b}(t+\tau)\rangle\langle \hat{b}(t)\hat{b}^\dagger(t+\tau)\rangle. \end{aligned} \tag{3.109}$$

With the help of Eq. (3.50), one can readily obtain the following equations

$$\langle \hat{b}^\dagger(t)\hat{b}^\dagger(t+\tau)\rangle = \langle \hat{b}^{\dagger 2}(t)\rangle\left[\frac{\kappa}{\kappa-\mu}e^{-\mu\tau/2} - \frac{\mu}{\kappa-\mu}e^{-\kappa\tau/2}\right], \tag{3.110}$$

$$\langle \hat{b}(t)\hat{b}(t+\tau)\rangle = \langle \hat{b}^{2}(t)\rangle\left[\frac{\kappa}{\kappa-\mu}e^{-\mu\tau/2} - \frac{\mu}{\kappa-\mu}e^{-\kappa\tau/2}\right], \tag{3.111}$$

$$\langle \hat{b}(t)\hat{b}^\dagger(t+\tau)\rangle = \langle \hat{b}(t)\hat{b}^\dagger(t)\rangle\left[\frac{\kappa}{\kappa-\mu}e^{-\mu\tau/2} - \frac{\mu}{\kappa-\mu}e^{-\kappa\tau/2}\right]. \tag{3.112}$$

Hence on account of Eqs. (3.52), (3.110), (3.111), and (3.112), Eq. (3.109) can be put in the form

$$\begin{aligned} \langle \hat{n}_b(t)\hat{n}_b(t+\tau)\rangle &= \left[\langle \hat{b}^\dagger(t)\hat{b}(t)\rangle\langle \hat{b}(t)\hat{b}^\dagger(t)\rangle + \langle \hat{b}^2(t)\rangle\langle \hat{b}^{\dagger 2}(t)\rangle\right] \\ &\times \left[\left(\frac{\mu}{\kappa-\mu}\right)^2 e^{-\kappa\tau} + \left(\frac{\kappa}{\kappa-\mu}\right)^2 e^{-\mu\tau} - \frac{2\kappa\mu}{(\kappa-\mu)^2}e^{-(\kappa+\mu)\tau/2}\right]. \end{aligned} \tag{3.113}$$

This can be rewritten as

$$\langle\hat{n}_b(t)\hat{n}_b(t+\tau)\rangle_{ss} = \frac{(\Delta n)_b^2}{(\kappa-\mu)^2}\bigg[\mu^2 e^{-\kappa\tau} + \kappa^2 e^{-\mu\tau} - 2\kappa\mu e^{-(\kappa+\mu)\tau/2}\bigg], \tag{3.114}$$

in which $(\Delta n)_b^2$ is the steady-state photon number variance of light mode b given by Eq. (3.80). With the help of Eq. (3.114), the spectrum of photon number fluctuations can be put in the form

$$\begin{aligned} S_b(\omega) &= \frac{(\Delta n)_b^2}{\pi(\kappa-\mu)^2} Re\bigg[\mu^2\int_0^\infty d\tau e^{-[\kappa - i(\omega-\omega_0)]\tau} \\ &+ \kappa^2\int_0^\infty d\tau e^{-[\mu - i(\omega-\omega_0)]\tau} \\ &- 2\kappa\mu\int_0^\infty d\tau e^{-[\frac{(\kappa+\mu)}{2} - i(\omega-\omega_0)]\tau}\bigg] \end{aligned} \tag{3.115}$$

and carrying out the integration, we obtain

$$S_b(\omega) = \frac{(\Delta n)_b^2}{(\kappa-\mu)^2}\bigg[\frac{\mu^2\kappa/\pi}{\kappa^2+(\omega-\omega_0)^2} - \frac{2\kappa\mu(\kappa+\mu)/2\pi}{(\frac{\kappa+\mu}{2})^2+(\omega-\omega_0)^2} + \frac{\kappa^2\mu/\pi}{\mu^2+(\omega-\omega_0)^2}\bigg]. \tag{3.116}$$

Now we realize that the photon number variance in the frequency interval between $\omega' = -\lambda$ and $\omega' = \lambda$ is expressible as

$$(\Delta n)_{b\pm\lambda}^2 = \int_{-\lambda}^{\lambda} S_b(\omega')d\omega', \tag{3.117}$$

in which $\omega' = \omega - \omega_0$. Therefore, substitution of Eq. (3.116) into Eq. (3.117) leads to

$$(\Delta n)_{b\pm\lambda}^2 = \frac{(\Delta n)_b^2}{\pi(\kappa-\mu)^2}\bigg[\int_{-\lambda}^{\lambda}\frac{\mu^2\kappa d\omega'}{\kappa^2+\omega'^2} - \int_{-\lambda}^{\lambda}\frac{2\kappa\mu(\kappa+\mu/2)d\omega'}{(\frac{\kappa+\mu}{2})^2+\omega'^2} + \int_{-\lambda}^{\lambda}\frac{\kappa^2\mu d\omega'}{\mu^2+\omega'^2}\bigg]. \tag{3.118}$$

Employing the relation given by Eq. (3.38), the local photon number variance of light mode b produced by the coherently driven nondegenerate three-level laser with an open cavity and coupled to a two-mode vacuum reservoir is found to be

$$(\Delta n)_{b\pm\lambda}^2 = (\Delta n)_b^2 z_b'(\lambda), \tag{3.119}$$

where $z_b'(\lambda)$ is given by

$$z_b'(\lambda) = \frac{2\mu^2/\pi}{(\mu-\kappa)^2}tan^{-1}\bigg(\frac{\lambda}{\kappa}\bigg) + \frac{2\kappa^2/\pi}{(\kappa-\mu)^2}tan^{-1}\bigg(\frac{\lambda}{\mu}\bigg) - \frac{4\kappa\mu/\pi}{(\kappa-\mu)^2}tan^{-1}\bigg(\frac{2\lambda}{\kappa+\mu}\bigg). \tag{3.120}$$

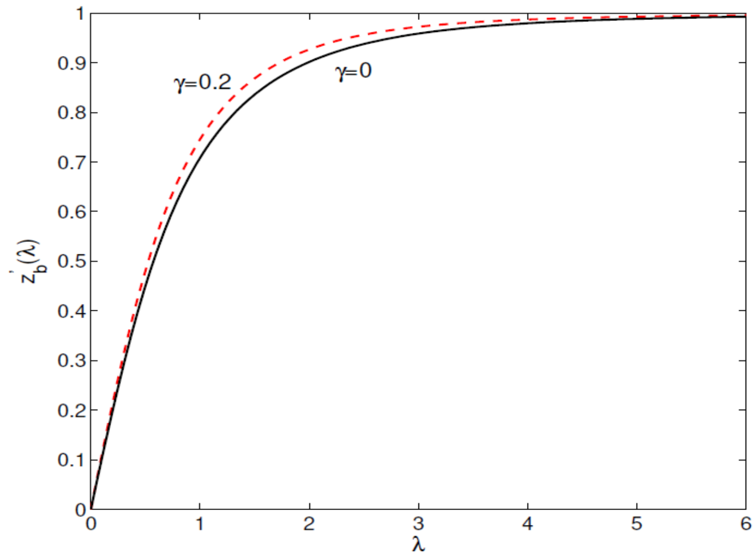

Figure 3.10: Plots of $z_b^{'}(\lambda)$[Eq. 3.120] versus λ.

We see from Eq. (3.119) along with the plot $z_b^{'}(\lambda)$ that $(\Delta n)^2_{b\pm\lambda}$ increases with λ until it reaches the maximum value of the global photon number variance.

We see from these results, $z_b^{'}(\lambda)$ in the absence of spontaneous emission $(\gamma = 0)$ is less than in the presence of spontaneous emission $(\gamma \neq 0)$. Moreover, the plots in Fig. (3.10) shows that $z_b^{'}(\lambda)$ versus λ and for the values of $\gamma_c = 0.4$, $\kappa = 0.8$, $N = 50$, $\Omega = 2$, and for different values of γ. From the plots in Fig. (3.10), we find the indicated below:

	$z_b^{'}(0.5)$	$z_b^{'}(1)$	$z_b^{'}(2)$
$\gamma = 0.2$	0.47	0.73	0.92
$\gamma = 0$	0.44	0.70	0.89

Table 3.7: Values of $z_b^{'}(\lambda)$ for $\gamma_c = 0.4$, $\kappa = 0.8$, and $\Omega = 2$.

We see from these results, $z_b^{'}(\lambda)$ in the absence of spontaneous emission $(\gamma = 0)$ is less than in the presence of spontaneous emission $(\gamma \neq 0)$. Moreover, using the above results of $z_b^{'}(\lambda)$ and on account of Eq. (3.119), we have

	$(\Delta n)^2_{b\pm 0.5}$	$(\Delta n)^2_{b\pm 1}$	$(\Delta n)^2_{b\pm 2}$
$\gamma = 0.2$	31.78	49.37	62.22
$\gamma = 0$	29.76	47.34	60.19

Table 3.8: Values of $(\Delta n)^2_{b\pm\lambda}$ for $\gamma_c = 0.4$, $\kappa = 0.8$, $N = 50$, and $\Omega = 2$.

We therefore observe that a large part of the total variance of photon number is confined in a relatively small frequency interval.

3.2 Generation of Two-mode photon statistics

In this section, applying the steady-state solutions of the equations of evolution of the expectation values of the atomic operators and the quantum Langavin equations for the cavity mode operators, we seek to obtain the mean and variance of the photon numbers for the two-mode light beam.

3.2.1 Two-mode mean photon number

Here we seek to calculate the steady-state mean photon number of the two-mode cavity light beam. The mean photon number of the two-mode light beam, represented by the operators $\hat{c}$ and $\hat{c}^\dagger$, is defined by

$$\bar{n} = \langle \hat{c}^\dagger \hat{c} \rangle. \tag{3.121}$$

The steady-state solution of Eq. (2.97) is found to be

$$\hat{c} = \frac{2g}{\kappa\sqrt{N}} \hat{m}. \tag{3.122}$$

Hence at steady state the mean photon number goes over into

$$\bar{n} = \frac{\gamma_c}{\kappa} \left[\langle \hat{N}_a \rangle + \langle \hat{N}_b \rangle \right]. \tag{3.123}$$

We see from Eq. (3.123) that the mean photon number of the two-mode light beam is the sum of the mean photon numbers of the separate single-mode light beams given

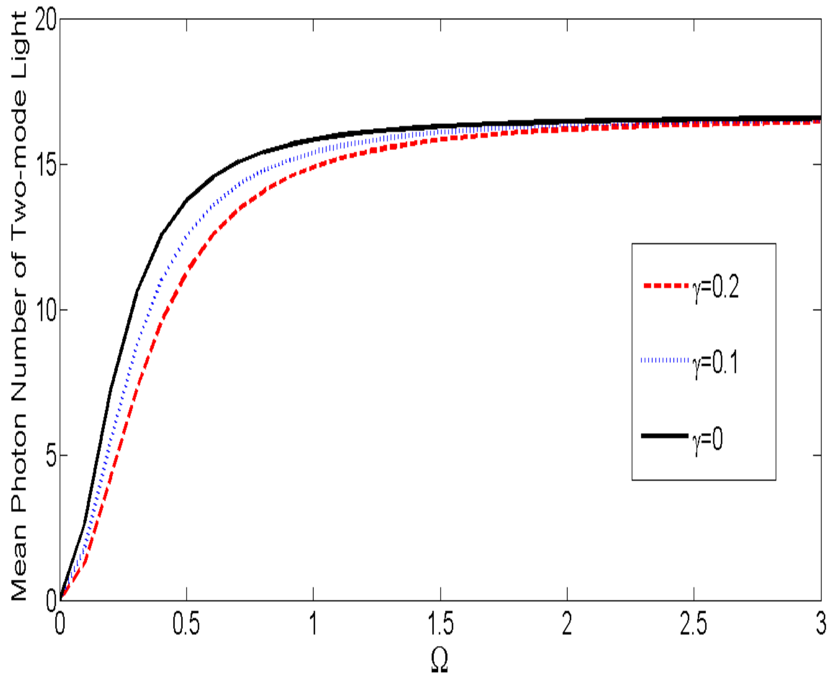

Figure 3.11: Plots of mean photon number $\bar{n}$ [Eq. 3.124] versus Ω.

by Eqs. (3.5) and (3.13). Therefore, on account of Eqs. (2.118) and (2.119), Eq. (3.123) turns out to be

$$\bar{n} = \left(\frac{\gamma_c}{\kappa}N\right)\left[\frac{2\Omega^2}{(\gamma_c+\gamma)^2+3\Omega^2}\right]. \tag{3.124}$$

This is the steady-state mean photon number of the two-mode light beam, produced by the coherently driven nondegenerate three-level laser with an open cavity and coupled to a two-mode vacuum reservoir.

The plots on Fig. (3.11) describes the two-mode mean photon number $\bar{n}$ Ω for $\gamma_c = 0.4$, $\kappa = 0.8$, $N = 50$, and for different values of γ. When we see the plots on Fig. (3.11) that as the spontaneous emission decay constant γ increases the global mean photon number of two-mode cavity light decreases. In addition, the plots of Fig. (3.11) indicates that as Ω increases, the mean photon number also increases.

We next proceed to consider the case in which spontaneous emission is absent ($\gamma =$

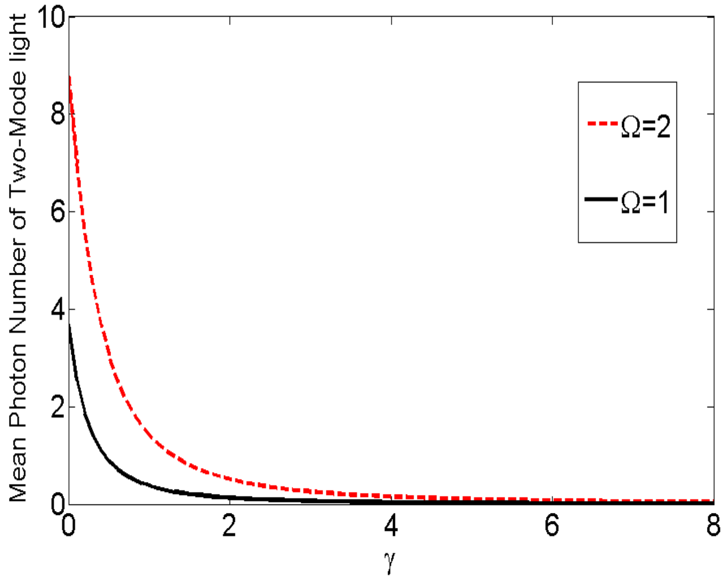

Figure 3.12: Plots of mean photon number $\bar{n}$ [Eq. 3.124] versus γ.

0). Then the mean photon number for this case takes the form

$$\bar{n} = \frac{\gamma_c}{\kappa} N \left[\frac{2\Omega^2}{\gamma_c^2 + 3\Omega^2} \right]. \tag{3.125}$$

This represents the steady-state mean photon number of the two-mode light beam in the absence of spontaneous emission. The result described by Eq. (3.125) is exactly the same as the one obtained by Fesseha [1].

Furthermore, we note that for $\Omega \gg \gamma_c$, Eq. (3.125) reduces to

$$\bar{n} = \frac{2\gamma_c}{3\kappa} N. \tag{3.126}$$

The plots on Fig. (3.12) describes the two-mode mean photon number $\bar{n}$ versus γ for $\gamma_c = 0.4$, $\kappa = 0.8$, $N = 50$, and for different values of Ω. We observe from the plots in Fig. (3.12) that the mean photon number of the two-mode light beam has greater value for the case $\Omega = 2$ than when $\Omega = 1$. This indicates that the greater in mean photon number is depending on the amplitude of pumping. This means more pumping gives more bright light. On the other hand, Fig. (3.12) shows that as the spontaneous emission decay constant, γ, increase the mean photon number decrease.

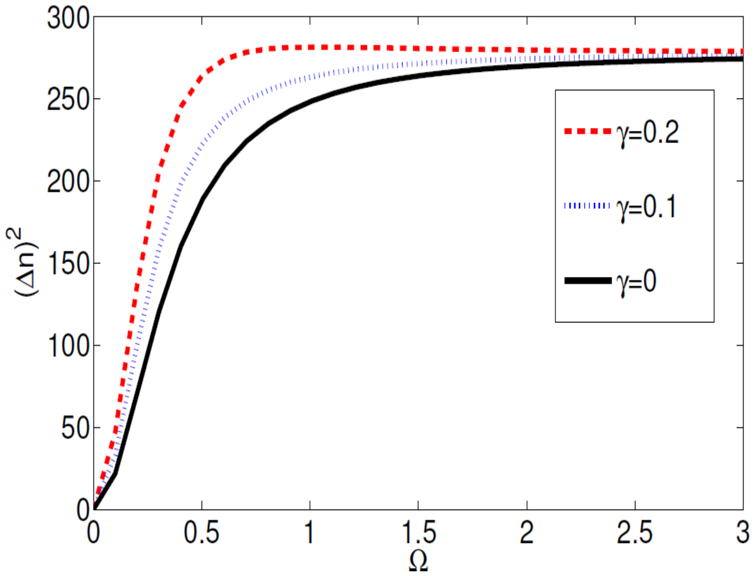

Figure 3.13: Plots of the photon number variance $(\Delta n)^2$ of [Eq. 3.132] versus Ω

3.2.2 Two-mode photon-number variance

Here we proceed to study the steady-state photon number variance of the two-mode light beam, produced by the coherently driven nondegenerate three-level laser with an open cavity and coupled to a two-mode vacuum reservoir.

The photon number variance for the two-mode cavity light is expressible as

$$(\Delta n)^2 = \langle \hat{c}^\dagger \hat{c} \hat{c}^\dagger \hat{c} \rangle - \langle \hat{c}^\dagger \hat{c} \rangle^2. \tag{3.127}$$

Since $\hat{c}$ is Gaussian variable with zero mean, the variance of the photon number can be written as

$$(\Delta n)^2 = \langle \hat{c}^\dagger \hat{c} \rangle \langle \hat{c} \hat{c}^\dagger \rangle + \langle \hat{c}^{\dagger 2} \rangle \langle \hat{c}^2 \rangle. \tag{3.128}$$

With the aid of Eq. (3.122), one can easily establish that

$$\langle \hat{c} \hat{c}^\dagger \rangle = \frac{\gamma_c}{\kappa} \big[\langle \hat{N}_b \rangle + \langle \hat{N}_c \rangle \big] \tag{3.129}$$

$$\langle \hat{c}^2 \rangle = \frac{\gamma_c}{\kappa} \langle \hat{m}_3 \rangle. \tag{3.130}$$

Since $\langle \hat{m}_3 \rangle$ is real, then $\langle \hat{c}^2 \rangle = \langle \hat{c}^{\dagger 2} \rangle$. Therefore, with the aid of Eqs. (3.123), (3.129) and (3.130), the variance of the photon number for the two-mode cavity light turns out to be

$$(\Delta n)^2 = \left(\frac{\gamma_c}{\kappa}\right)^2 \left[\left(\langle \hat{N}_a \rangle + \langle \hat{N}_b \rangle\right)\left(\langle \hat{N}_b \rangle + \langle \hat{N}_c \rangle\right) + \langle \hat{m}_3 \rangle^2\right]. \tag{3.131}$$

We observe from Eq.(3.131) that the photon number variance of the two-mode light beam does not happen to be the sum of the photon number variance of the separate single-mode light beams given by Eqs. (3.67) and (3.78). Furthermore, upon substituting of Eqs. (2.118)-(2.119) into Eq. (3.131), the steady-state variance of the photon number goes over into

$$(\Delta n)^2 = \left(\frac{\gamma_c}{\kappa} N\right)^2 \left[\frac{4\Omega^4 + 3\Omega^2(\gamma_c + \gamma)^2}{\left[(\gamma_c + \gamma)^2 + 3\Omega^2\right]^2}\right]. \tag{3.132}$$

This is the steady-state photon number variance of the two-mode light beam, produced by the coherently driven nondegenerate three-level laser with an open cavity and coupled to a two-mode vacuum reservoir. Finally, we obtained Fig. (3.13), the plots of global photon number variance of the two-mode cavity light beams versus Ω with $\gamma_c = 0.4$, $\kappa = 0.8$, $N = 50$, and for different values of γ. From the plots we see that as spontaneous emission decay constant γ increases the global photon variance increases of two mode light beam increases. In addition, the plot of Fig. (3.13)indicates that Ω increase, the two mode photon variance increases.

We next proceed to consider the case in which spontaneous emission is absent ($\gamma = 0$). Then the variance of the photon number for this case takes the form

$$(\Delta n)^2 = \left(\frac{\gamma_c}{\kappa} N\right)^2 \left[\frac{4\Omega^4 + 3\Omega^2\gamma_c^2}{(\gamma_c^2 + 3\Omega^2)^2}\right]. \tag{3.133}$$

This represent the steady state variance of photon number in the absence of spontaneous emission. Furthermore, we note that for $\Omega \gg \gamma_c$, Eq. (3.133) reduces to

$$(\Delta n)^2 = \left[\frac{2\gamma_c}{3\kappa} N\right]^2 \tag{3.134}$$

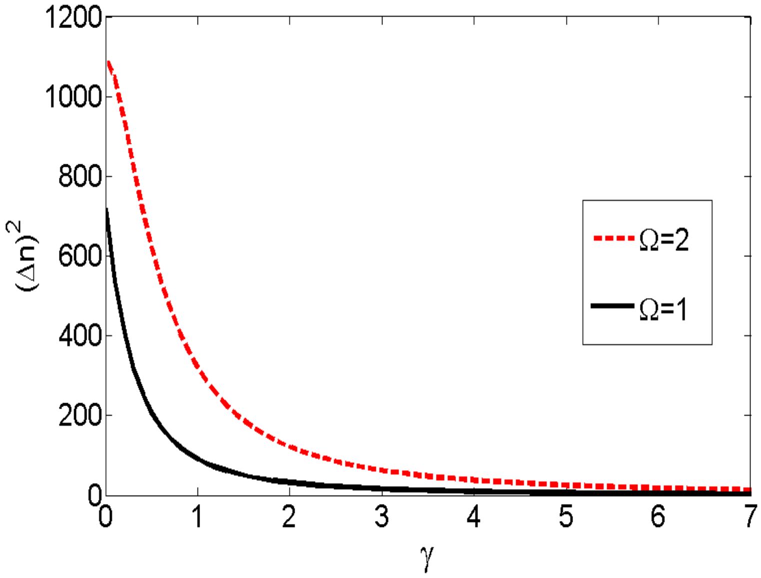

Figure 3.14: Plots of the photon number variance$(\Delta n)^2$ of [Eq. 3.132] versus γ

and in view of Eq. (3.126), we have

$$(\Delta n)^2 = \bar{n}^2, \tag{3.135}$$

which represents the normally-ordered variance of the photon number for chaotic light.

Finally, we obtained Fig. (3.14), the plots of global photon number variance of the two-mode cavity light beams versus γ with $\gamma_c = 0.4$, $\kappa = 0.8$, $N = 50$, and for different values of Ω. From these plots we see that the global photon variance decreases when the spontaneous emission decay constant, γ, increases. On the other hand, when the pumping increases the photon number variance of the two-mode cavity light beams increases. This is due to the coupling of the top and bottom levels of the atom by coherent light or the atomic coherence of the two-mode cavity light.

4

Generation of the System of Quadrature Squeezing

In this chapter we seek to study the quadrature variance and the quadrature squeezing of the light produced by the coherently driven nondegenerate three-level laser with an open cavity and coupled to a two-mode vacuum reservoir via a single-port mirror. Applying the steady-state solutions of the equations of evolution of the expectation values of the atomic operators and the quantum Langevin equations for the cavity mode operators, we obtain the global quadrature variances for light modes a and b. In addition, we determine the global quadrature squeezing of the two-mode cavity light.

4.1 Single-mode quadrature variance

In this section we obtain the global quadrature variances of light modes a and b, produced by the system under consideration.

4.1.1 Global quadrature variance of light mode a

We now proceed to calculate the quadrature variance of light mode a in the entire frequency interval. The squeezing properties of light mode a are described by two quadrature operators

$$\hat{a}_+ = \hat{a}^\dagger + \hat{a}, \tag{4.1}$$

$$\hat{a}_- = i(\hat{a}^\dagger - \hat{a}), \tag{4.2}$$

where $\hat{a}_+$ and $\hat{a}_-$ are Hermitian operators representing physical quantities called plus and minus quadratures, respectively, while $\hat{a}^\dagger$ and $\hat{a}$ are the creation and annihilation oprators for light mode a. With the help of Eqs. (4.1) and (4.2), we can show that the two quadrature operators satisfy the commutation relation

$$[\hat{a}_-, \hat{a}_+] = 2i\frac{\gamma_c}{\kappa}\bigg[\hat{N}_a - \hat{N}_b\bigg]. \tag{4.3}$$

In view of this result, the uncertainity relation for the plus and minus quadrature operators of mode a is expressible as

$$\begin{aligned} \Delta a_+ \Delta a_- &\geq \frac{1}{2}\bigg|\langle[\hat{a}_+, \hat{a}_-]\rangle\bigg| \\ &\geq \bigg|\langle\hat{a}\hat{a}^\dagger\rangle - \langle\hat{a}^\dagger\hat{a}\rangle\bigg|, \end{aligned} \tag{4.4}$$

so that using Eqs. (3.5) and (3.66), there follows

$$\Delta a_+ \Delta a_- \geq \frac{\gamma_c}{\kappa}\bigg|\langle\hat{N}_a\rangle - \langle\hat{N}_b\rangle\bigg|. \tag{4.5}$$

On account of Eq. (2.113), the uncertainity relation for the quadrature operators can be put in the form

$$\Delta a_+ \Delta a_- \geq 0. \tag{4.6}$$

Next we proceed to calculate the quadrature variance of light mode a. The variance of the plus and minus quadrature operators are defined by

$$(\Delta\hat{a}_+)^2 = \langle\hat{a}_+^2\rangle - \langle\hat{a}_+\rangle^2, \tag{4.7}$$

$$(\Delta\hat{a}_-)^2 = \langle\hat{a}_-^2\rangle - \langle\hat{a}_-\rangle^2. \tag{4.8}$$

With the aid of Eq. (4.1), Eq. (4.7) can be expressed in terms of the creation and annihilation operators as

$$(\Delta\hat{a}_+)^2 = \langle\hat{a}\hat{a}^\dagger\rangle + \langle\hat{a}^\dagger\hat{a}\rangle + \langle\hat{a}^2\rangle + \langle\hat{a}^{\dagger 2}\rangle - \langle\hat{a}\rangle^2 - \langle\hat{a}^\dagger\rangle^2 - 2\langle\hat{a}\rangle\langle\hat{a}^\dagger\rangle. \tag{4.9}$$

In addition, on account of Eqs. (4.2) and (4.8), we get

$$(\Delta\hat{a}_-)^2 = \langle\hat{a}\hat{a}^\dagger\rangle + \langle\hat{a}^\dagger\hat{a}\rangle - \langle\hat{a}^2\rangle - \langle\hat{a}^{\dagger 2}\rangle + \langle\hat{a}\rangle^2 + \langle\hat{a}^\dagger\rangle^2 - 2\langle\hat{a}\rangle\langle\hat{a}^\dagger\rangle, \tag{4.10}$$

so that inspection of Eqs. (4.9) and (4.10) shows that

$$(\Delta \hat{a}_{\pm})^2 = \langle \hat{a}\hat{a}^{\dagger} \rangle + \langle \hat{a}^{\dagger}\hat{a} \rangle \pm \langle \hat{a}^2 \rangle \pm \langle \hat{a}^{\dagger 2} \rangle \mp \langle \hat{a} \rangle^2 \mp \langle \hat{a}^{\dagger} \rangle^2 - 2\langle \hat{a} \rangle \langle \hat{a}^{\dagger} \rangle. \tag{4.11}$$

Moreover, with the help of Eqs. (2.108) and (3.65), we have

$$(\Delta \hat{a}_{\pm})^2 = \langle \hat{a}\hat{a}^{\dagger} \rangle + \langle \hat{a}^{\dagger}\hat{a} \rangle \tag{4.12}$$

and in view of Eqs. (3.5) and (3.66), there follows

$$(\Delta \hat{a}_{\pm})^2 = \frac{\gamma_c}{\kappa} \left[\langle \hat{N}_a \rangle + \langle \hat{N}_b \rangle \right]. \tag{4.13}$$

On account of Eq. (2.113), we see that

$$(\Delta \hat{a}_{\pm})^2 = \frac{2\gamma_c}{\kappa} \langle \hat{N}_a \rangle. \tag{4.14}$$

Now substitution of Eq. (2.118) into Eq. (4.13) results in

$$(\Delta \hat{a}_{\pm})^2 = \frac{\gamma_c}{\kappa} N \left[\frac{2\Omega^2}{(\gamma_c + \gamma)^2 + 3\Omega^2} \right]. \tag{4.15}$$

This is the steady-state quadrature variance of light mode a produced by the coherently driven non-degenerate three-level laser with an open cavity and coupled to a two-mode vacuum reservoir.

Furthermore, we consider the case in which spontaneous emission is absent ($\gamma = 0$). Then the quadrature variance for this case takes the form

$$(\Delta \hat{a}_{\pm})^2 = \frac{\gamma_c}{\kappa} N \left[\frac{2\Omega^2}{\gamma_c^2 + 3\Omega^2} \right]. \tag{4.16}$$

In addition, we note that for $\Omega \gg \gamma_c$, Eq. (4.16) reduces to

$$(\Delta \hat{a}_{\pm})^2 = \frac{2\gamma_c}{3\kappa} N. \tag{4.17}$$

On account of Eq. (3.8), Eq. (4.17) can be put in the form

$$(\Delta \hat{a}_{\pm})^2 = 2\bar{n}_a, \tag{4.18}$$

which is the normally ordered quadrature variance for chaotic light.

4.1.2 Global quadrature variance of light mode b

Here we wish to calculate the quadrature variance of light mode b in the entire frequency interval, produced by the system under consideration. The squeezing properties of light mode b are described by two quadrature operators

$$\hat{b}_+ = \hat{b}^\dagger + \hat{b}, \tag{4.19}$$

$$\hat{b}_- = i(\hat{b}^\dagger - \hat{b}), \tag{4.20}$$

where $\hat{b}_+$ and $\hat{b}_-$ are Hermitian operators representing physical quantities called plus and minus quadratures, respectively, while $\hat{b}^\dagger$ and $\hat{b}$ are the creation and annihilation oprators for light mode b. With the help of Eqs. (4.19) and (4.20), we can show that the two quadrature operators satisfy the commutation relation

$$[\hat{b}_-, \hat{b}_+] = 2i\frac{\gamma_c}{\kappa}\Big[\hat{N}_b - \hat{N}_c\Big]. \tag{4.21}$$

In view of this result, the uncertainity relation for the plus and minus quadrature operators of mode b is expressible as

$$\begin{aligned} \Delta b_+ \Delta b_- &\geq \frac{1}{2}\Big|\langle[\hat{b}_+, \hat{b}_-]\rangle\Big| \\ &\geq \Big|\langle\hat{b}\hat{b}^\dagger\rangle - \langle\hat{b}^\dagger\hat{b}\rangle\Big|, \end{aligned} \tag{4.22}$$

so that using Eqs. (3.13) and (3.77), there follows

$$\Delta b_+ \Delta b_- \geq \frac{\gamma_c}{\kappa}\Big|\langle\hat{N}_b\rangle - \langle\hat{N}_c\rangle\Big|, \tag{4.23}$$

On account of Eqs. (2.119) and (2.120), the uncertainity relation of the quadrature operators can be put the form

$$\Delta b_+ \Delta b_- \geq \frac{\gamma_c}{\kappa}N\Big|\frac{(\gamma_c + \gamma)^2}{(\gamma_c + \gamma)^2 + 3\Omega^2}\Big|. \tag{4.24}$$

Now setting $\gamma = 0$, one finds

$$\Delta b_+ \Delta b_- \geq \frac{\gamma_c}{\kappa}N\Big|\frac{\gamma_c^2}{\gamma_c^2 + 3\Omega^2}\Big|. \tag{4.25}$$

Moreover, we consider the case in which the deriving coherent light is absent. Thus upon setting $\Omega = 0$ in Eq. (4.25), we readily get

$$\Delta b_+ \Delta b_- \geq \frac{\gamma_c}{\kappa} N. \tag{4.26}$$

We therefore notice that the product of the uncertainties in the two quadrature satisfies the minimum uncertainty relation.

Next we proceed to calculate the quadrature variance of light mode b. The variance of the plus and minus quadrature operators for light mode b are defined by

$$(\Delta \hat{b}_+)^2 = \langle \hat{b}_+^2 \rangle - \langle \hat{b}_+ \rangle^2, \tag{4.27}$$

$$(\Delta \hat{b}_-)^2 = \langle \hat{b}_-^2 \rangle - \langle \hat{b}_- \rangle^2. \tag{4.28}$$

On account of Eq. (4.19), Eq. (4.27) can be expressed in terms of the creation and annihilation operators as

$$(\Delta \hat{b}_+)^2 = \langle \hat{b}\hat{b}^\dagger \rangle + \langle \hat{b}^\dagger \hat{b} \rangle + \langle \hat{b}^2 \rangle + \langle \hat{b}^{\dagger 2} \rangle - \langle \hat{b} \rangle^2 - \langle \hat{b}^\dagger \rangle^2 - 2\langle \hat{b} \rangle \langle \hat{b}^\dagger \rangle. \tag{4.29}$$

In addition, with the help of Eqs. (4.20) and (4.28), we see that

$$(\Delta \hat{b}_-)^2 = \langle \hat{b}\hat{b}^\dagger \rangle + \langle \hat{b}^\dagger \hat{b} \rangle - \langle \hat{b}^2 \rangle - \langle \hat{b}^{\dagger 2} \rangle + \langle \hat{b} \rangle^2 + \langle \hat{b}^\dagger \rangle^2 - 2\langle \hat{b} \rangle \langle \hat{b}^\dagger \rangle, \tag{4.30}$$

so that inspection of Eqs. (4.29) and (4.30) shows that

$$(\Delta \hat{b}_\pm)^2 = \langle \hat{b}\hat{b}^\dagger \rangle + \langle \hat{b}^\dagger \hat{b} \rangle \pm \langle \hat{b}^2 \rangle \pm \langle \hat{b}^{\dagger 2} \rangle \mp \langle \hat{b} \rangle^2 \mp \langle \hat{b}^\dagger \rangle^2 - 2\langle \hat{b} \rangle \langle \hat{b}^\dagger \rangle. \tag{4.31}$$

Moreover, with the aid of Eqs. (2.109) and (3.76), we get

$$(\Delta \hat{b}_\pm)^2 = \langle \hat{b}\hat{b}^\dagger \rangle + \langle \hat{b}^\dagger \hat{b} \rangle \tag{4.32}$$

and in view of Eqs. (3.13) and (3.77), there follows

$$(\Delta \hat{b}_\pm)^2 = \frac{\gamma_c}{\kappa} \left[\langle \hat{N}_b \rangle + \langle \hat{N}_c \rangle \right]. \tag{4.33}$$

Now on account of Eqs. (2.119) and (2.120), the quadrature variance of light mode b takes, at steady-state, the form

$$(\Delta\hat{b}_{\pm})^2 = \left(\frac{\gamma_c}{\kappa}N\right)\left[\frac{(\gamma_c+\gamma)^2+2\Omega^2}{(\gamma_c+\gamma)^2+3\Omega^2}\right]. \tag{4.34}$$

This represents the quadrature variance of light mode b, produced by the coherently driven nondegenerate three-level laser with an open cavity and coupled to a two-mode vacuum reservoir.

Furthermore, we consider the case in which spontaneous emission is absent $(\gamma = 0)$. Then the quadrature variance for this case has the form

$$(\Delta\hat{b}_{\pm})^2 = \frac{\gamma_c}{\kappa}N\left[\frac{\gamma_c^2+2\Omega^2}{\gamma_c^2+3\Omega^2}\right]. \tag{4.35}$$

In addition, we note that for $\Omega \gg \gamma_c$, Eq. (4.35) reduces to

$$(\Delta\hat{b}_{\pm})^2 = \frac{2\gamma_c}{3\kappa}N. \tag{4.36}$$

In view of Eq. (3.16), this can be expressed as

$$(\Delta\hat{b}_{\pm})^2 = 2\bar{n}_b, \tag{4.37}$$

which is the normally ordered quadrature variance for chaotic light.

4.2 Two-mode quadrature squeezing

Now we seek to determine the quadrature variances of the two-mode light beam. The squeezing properties of the two-mode cavity light are described by two quadrature operators

$$\hat{c}_+ = \hat{c}^\dagger + \hat{c}, \tag{4.38}$$

$$\hat{c}_- = i(\hat{c}^\dagger - \hat{c}), \tag{4.39}$$

where $\hat{c}_+$ and $\hat{c}_-$ are Hermitian operators representing the physical quantities called plus and minus quadrature, respectively while $\hat{c}^\dagger$ and $\hat{c}$ are the creation and annihilation operators of the two-mode cavity light. With the aid of Eqs. (4.38) and (4.39), we show that the two quadrature operators satisfy the commutation relation

$$[\hat{c}_-, \hat{c}_+] = 2i\frac{\gamma_c}{\kappa}\left[\hat{N}_a - \hat{N}_c\right]. \tag{4.40}$$

In view of this result, the uncertainty relation for the plus and minus quadrature operators of the two-mode cavity light is expressible as

$$\begin{aligned} \Delta c_+ \Delta c_- &\geq \frac{1}{2}\left|\langle[\hat{c}_+, \hat{c}_-]\rangle\right| \\ &\geq \left|\langle\hat{c}\hat{c}^\dagger\rangle - \langle\hat{c}^\dagger\hat{c}\rangle\right|, \end{aligned} \tag{4.41}$$

so that using Eqs. (3.123) and (3.129), there follows

$$\Delta c_+ \Delta c_- \geq \frac{\gamma_c}{\kappa}\left|\langle\hat{N}_a\rangle - \langle\hat{N}_c\rangle\right|. \tag{4.42}$$

On account of Eqs. (2.118) and (2.120), the uncertainty relation for the plus and minus quadrature operators is found to be

$$\Delta c_+ \Delta c_- \geq \frac{\gamma_c}{\kappa}N\left|\frac{(\gamma_c + \gamma)^2}{(\gamma_c + \gamma)^2 + 3\Omega^2}\right|. \tag{4.43}$$

In addition, we consider the case in which spontaneous emission is absent ($\gamma = 0$). Then the uncertainity relation for this case takes the form

$$\Delta c_+ \Delta c_- \geq \frac{\gamma_c}{\kappa}N\left|\frac{\gamma_c^2}{\gamma_c^2 + 3\Omega^2}\right|. \tag{4.44}$$

Moreover, we consider the case in which the deriving coherent light is absent. Thus upon setting $\Omega = 0$ in Eq. (4.43), we readily get

$$\Delta c_+ \Delta c_- \geq \frac{\gamma_c}{\kappa}N, \tag{4.45}$$

which is the minimum uncertainty relation for the two-mode cavity vacuum state. We therefore notice that the uncertainties for the two quadratures are equal and their product satisfies the minimum uncertainty relation.

Next we proceed to calculate the quadrature variance of the two-mode cavity light. The variance of the plus and minus quadrature operators of the two-mode cavity light are defined by

$$(\Delta\hat{c}_+)^2 = \langle\hat{c}_+^2\rangle - \langle\hat{c}_+\rangle^2. \tag{4.46}$$

$$(\Delta\hat{c}_-)^2 = \langle\hat{c}_-^2\rangle - \langle\hat{c}_-\rangle^2. \tag{4.47}$$

On account of Eqs. (4.38) and (4.47), the plus quadrature variance can be expressed in terms of the creation and annihilation operators as

$$(\Delta\hat{c}_+)^2 = \langle\hat{c}\hat{c}^\dagger\rangle + \langle\hat{c}^\dagger\hat{c}\rangle + \langle\hat{c}^2\rangle + \langle\hat{c}^{\dagger 2}\rangle - \langle\hat{c}\rangle^2 - \langle\hat{c}^\dagger\rangle^2 - 2\langle\hat{c}\rangle\langle\hat{c}^\dagger\rangle \tag{4.48}$$

and with the help of Eqs. (4.39) and (4.47), we get

$$(\Delta\hat{c}_-)^2 = \langle\hat{c}\hat{c}^\dagger\rangle + \langle\hat{c}^\dagger\hat{c}\rangle - \langle\hat{c}^2\rangle - \langle\hat{c}^{\dagger 2}\rangle + \langle\hat{c}\rangle^2 + \langle\hat{c}^\dagger\rangle^2 - 2\langle\hat{c}\rangle\langle\hat{c}^\dagger\rangle, \tag{4.49}$$

so that inspection of Eqs. (4.48) and (4.49) shows that

$$(\Delta\hat{c}_\pm)^2 = \langle\hat{c}\hat{c}^\dagger\rangle + \langle\hat{c}^\dagger\hat{c}\rangle \pm \langle\hat{c}^2\rangle \pm \langle\hat{c}^{\dagger 2}\rangle \mp \langle\hat{c}\rangle^2 \mp \langle\hat{c}^\dagger\rangle^2 - 2\langle\hat{c}\rangle\langle\hat{c}^\dagger\rangle. \tag{4.50}$$

In view of Eqs. (2.110), we see that

$$(\Delta\hat{c}_\pm)^2 = \langle\hat{c}\hat{c}^\dagger\rangle + \langle\hat{c}^\dagger\hat{c}\rangle \pm \langle\hat{c}^2\rangle \pm \langle\hat{c}^{\dagger 2}\rangle. \tag{4.51}$$

In addition, with the aid of Eqs. (3.123), (3.129) and (3.130), expression (4.51) goes over into

$$(\Delta\hat{c}_\pm)^2 = \frac{\gamma_c}{\kappa}\left[\langle\hat{N}_a\rangle + 2\langle\hat{N}_b\rangle + \langle\hat{N}_c\rangle \pm \langle\hat{m}_3\rangle \pm \langle\hat{m}_3^\dagger\rangle\right]. \tag{4.52}$$

Now using Eqs. (2.114) and (2.116), the quadrature variance of the two-mode cavity light is found to be

$$(\Delta\hat{c}_\pm)^2 = \frac{\gamma_c}{\kappa}\left[N + \langle\hat{N}_b\rangle \pm 2\langle\hat{m}_3\rangle\right]. \tag{4.53}$$

Finally, on account of Eqs. (2.119) and (2.121), the quadrature variance of the two-mode cavity light takes, at steady-state, the form

$$(\Delta\hat{c}_{\pm})^2 = \frac{\gamma_c}{\kappa}N\left[\frac{(\gamma_c+\gamma)^2 + 4\Omega^2 \pm 2\Omega(\gamma_c+\gamma)}{(\gamma_c+\gamma)^2+3\Omega^2}\right]. \tag{4.54}$$

This represents the quadrature variance of the two-mode cavity light produced by the coherently driven nondegenerate three-level laser with an open cavity and coupled to a two-mode vacuum reservoir.

Furthermore, we consider the case in which spontaneous emission is absent $(\gamma = 0)$. Thus the quadrature variance for this case has the form

$$(\Delta\hat{c}_{\pm})^2 = \frac{\gamma_c}{\kappa}N\left[\frac{4\Omega^2 + \gamma_c^2 \pm 2\Omega\gamma_c}{\gamma_c^2+3\Omega^2}\right]. \tag{4.55}$$

This result is exactly the same as the one obtained by Fesseha [6]. In addition, we note that for $\Omega \gg \gamma_c$, Eq. (4.55) reduces to

$$(\Delta\hat{c}_{\pm})^2 = \frac{4\gamma_c}{3\kappa}N. \tag{4.56}$$

This can be rewritten as

$$(\Delta\hat{c}_{\pm})^2 = 2\bar{n}, \tag{4.57}$$

where $\bar{n}$ is given by Eq. (3.126). We see that Eq. (4.57) represents the normally ordered quadrature variance for chaotic light. Moreover, we consider the case in which the deriving coherent light is absent. Thus upon setting $\Omega = 0$ in Eq. (4.55), we get

$$(\Delta\hat{c}_{+})^2_v = (\Delta\hat{c}_{-})^2_v = \frac{\gamma_c}{\kappa}N, \tag{4.58}$$

which is the normally ordered quadrature variance of the two-mode cavity vacuum state. We note that for $\Omega = 0$ the uncertainty in the plus and minus quadratures are equal and satisfy the minimum uncertainty relation.

When we see the plots on Fig. (4.1) that the minus quadrature variance of the two-mode cavity light beams versus Ω for $\gamma_c = 0.4$, $N = 50$, $\kappa = 0.8$, and for different values

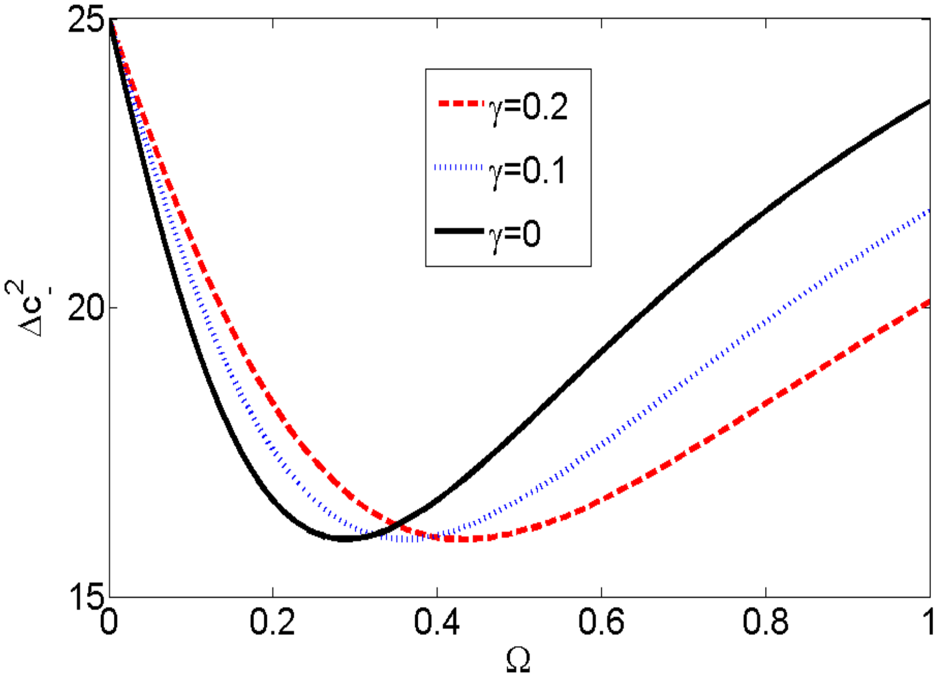

Figure 4.1: Plots of quadrature variance $(\Delta c_-)^2$ [Eq. (4.54)] versus Ω

of γ. The plots in Fig. (4.1) show that the minus quadrature variance when $\gamma = 0$ is less than when $\gamma = 0.1$ in the interval $0 \leq \Omega \leq 0.36$ and minus quadrature variance when $\gamma = 0$ is greater than when $\gamma = 0.1$ for $\Omega > 0.361$. And the minus quadrature variance when $\gamma = 0$ is less than when $\gamma = 0.2$ in the interval $0 \leq \Omega \leq 0.38$ and the minus quadrature variance when $\gamma = 0$ is greater than when $\gamma = 0.2$ for $\Omega > 0.38$. Moreover, plots in the same figure show that the minus quadrature variance when $\gamma = 0.1$ is less than when $\gamma = 0.2$ in the interval $0 \leq \Omega \leq 0.4$ and the minus quadrature variance when $\gamma = 0.1$ is greater than when $\gamma = 0.2$ for $\Omega > 0.4$. Furthermore, from the same plots the quadrature variance increases as spontaneous emission γ increases for Ω is small.

Next we proceed to calculate the quadrature squeezing of the two-mode cavity light in the entire frequency interval relative to the quadrature variance of the two-mode vacuum state. We then define the quadrature squeezing of the two-mode cavity light by [6]

$$S = \frac{(\Delta \hat{c}_-)^2_v - (\Delta \hat{c}_-)^2}{(\Delta \hat{c}_-)^2_v}. \tag{4.59}$$

It then follows that

$$S = 1 - \frac{(\Delta \hat{c}_-)^2}{(\Delta \hat{c}_-)^2_v}. \tag{4.60}$$

In view of Eqs. (4.54) and (4.58), the quadrature squeezing of the two-mode cavity light takes, at steady-state, the form

$$S = \left[\frac{2\Omega(\gamma_c + \gamma) - \Omega^2}{(\gamma_c + \gamma)^2 + 3\Omega^2}\right]. \tag{4.61}$$

This represents the quadrature squeezing of the two-mode cavity light produced by the coherently driven nondegenerate three-level laser with an open cavity and coupled to a two-mode vacuum reservoir.

We observe from this equation that unlike the mean photon number and the quadrature variance, the quadrature squeezing does not depend on the number of atoms. This implies that the quadrature squeezing of the two-mode cavity light is independent of the number of atoms.

Applying Eqs. (3.2) and (3.10), we find

$$\langle \hat{b}\hat{a} \rangle = \frac{\gamma_c}{\kappa}\langle \hat{m}_3 \rangle. \tag{4.62}$$

Since $\langle \hat{b} \rangle = \langle \hat{a} \rangle = 0$, we see that light modes a and b are correlated. The squeezing of the two-mode cavity light is due to this correlation. The two-mode light can be used in experiments involving entangled light modes.

In addition, we consider the case in which spontaneous emission is absent $(\gamma = 0)$. Then the quadrature squeezing for this case takes the form

$$S = \left[\frac{2\Omega\gamma_c - \Omega^2}{\gamma_c^2 + 3\Omega^2}\right]. \tag{4.63}$$

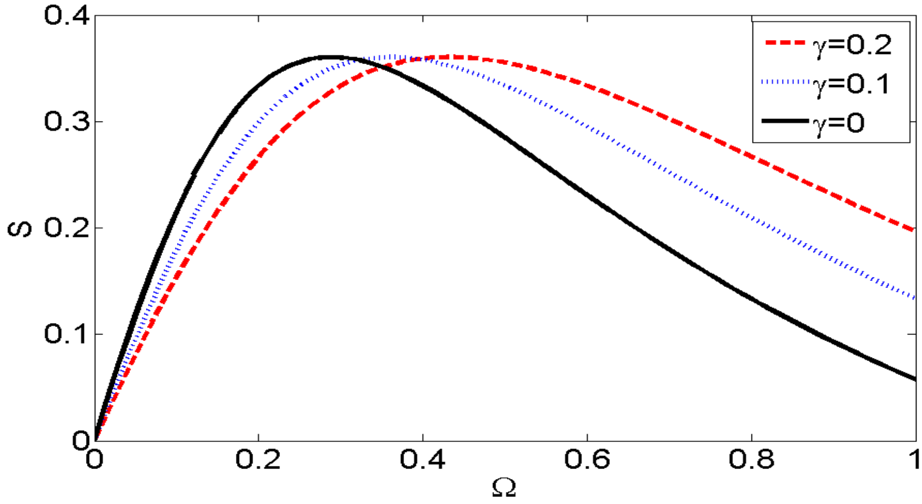

Figure 4.2: Plot of the quadrature squeezing [Eq. (4.61)] versus Ω.

The result described by Eq. (4.63) is exactly the same as the one obtained by Tamirat [4]. The plots on Fig. (4.2) indicates that describes the quadrature squeezing versus Ω for $\gamma_c = 0.4$, $\kappa = 0.8$, $N = 50$, and for different values of γ. From these plots, we find that the maximum quadrature squeezing to be the same in the presence as well as in the absence of spontaneous emission. The plots in Fig. (4.2) show that the quadrature squeezing when $\gamma = 0$ is greater than when $\gamma = 0.1$ in the interval $0 \leq \Omega \leq 0.36$ and the quadrature squeezing when $\gamma = 0$ is less than when $\gamma = 0.1$ for $\Omega > 0.361$. And the quadrature squeezing when $\gamma = 0$ is greater than when $\gamma = 0.2$ in the interval $0 \leq \Omega \leq 0.38$ and the quadrature squeezing when $\gamma = 0$ is less than when $\gamma = 0.2$ for $\Omega > 0.38$. Moreover, plots in the same figure show that the quadrature squeezing when $\gamma = 0.1$ is greater than when $\gamma = 0.2$ in the interval $0 \leq \Omega \leq 0.4$ and the quadrature squeezing when $\gamma = 0.1$ is less than when $\gamma = 0.2$ for $\Omega > 0.4$. Furthermore, from the same plots the maximum squeezing is found to be 36% for $\gamma = 0.2$ (dashed curve), for $\gamma = 0.1$ (dotted curve), and for $\gamma = 0$ (solid curve) below the vacuum-state level. These occur when the three-level laser is operating at $\Omega = 0.4343$, $\Omega = 0.3636$, and $\Omega = 0.2929$, respectively.

5

Generation of Entanglement quantification of the two-mode light

In this chapter we seek to study the photon entanglement as well as atom entanglement of a two-mode laser light beams produced by the coherently driven nondegenerate three-level lasers with open cavities and coupled to the two-mode vacuum reservoirs via single-port mirrors. Applying the solutions of the equations of evolution of the expectation values of the atomic operators and the quantum Langevin equations for the cavity mode operators, we obtain the entanglement of the two-mode light beams.

5.1 Photon Entanglement

Here, we prefer to analyze the entanglement of photon-states in the laser cavity. Quantum entanglement is a physical phenomenon that occurs when pairs or groups of particles cannot be described independently instead, a quantum state may be given for the system as a whole. Measurements of physical properties such as position, momentum, spin, polarization, etc. performed on entangled particles are found to be appropriately correlated. A pair of particles is taken to be entangled in quantum theory, if its states cannot be expressed as a product of the states of its individual constituents. The preparation and manipulation of these entangled states that have nonclassical and nonlocal properties lead to a better understanding of the basic quantum principles. It is in this spirit that this section is devoted to the analysis of the entanglement of the two-mode photon states. In other words, it is a well-known fact that a quantum system is said to be entangled, if it is not separable. That is, if the density operator for the combined state cannot be described as a combination of the product density operators of the constituents,

$$\hat{\rho} \neq \sum_k p_k \hat{\rho}_k^{(1)} \otimes \hat{\rho}_k^{(2)}, \tag{5.1}$$

in which $p_k \gg 0$ and $\sum_k p_k = 1$ to verify the normalization of the combined density states. On the other hand, a maximally entangled CV state can be expressed as a co-eigenstate of a pair of EPR-type operators [31] such as $\hat{x}_a - \hat{x}_b$ and $\hat{P}_a - \hat{P}_b$. The total variance of these two operators reduces to zero for maximally entangled CV states. According to the inseparable criteria given by Duan et al[30],the cavity photon-states of a system are entangled, whereas the sum of the variance of a pair of EPR-like operators is stated as follows,

$$\hat{s} = \hat{x}_a - \hat{x}_b, \tag{5.2}$$

$$\hat{t} = \hat{p}_a + \hat{p}_b, \tag{5.3}$$

where

$$\hat{x}_a = \frac{1}{\sqrt{2}}\left(\hat{a} + \hat{a}^\dagger\right), \tag{5.4}$$

$$\hat{x}_b = \frac{1}{\sqrt{2}}\left(\hat{b} + \hat{b}^\dagger\right), \tag{5.5}$$

$$\hat{p}_a = \frac{i}{\sqrt{2}}\left(\hat{a}^\dagger - \hat{a}\right), \tag{5.6}$$

$$\hat{p}_b = \frac{i}{\sqrt{2}}\left(\hat{b}^\dagger - \hat{b}\right), \tag{5.7}$$

are quadrature operators for modes a and b, satisfy

$$\Delta s^2 + \Delta t^2 < 2N \tag{5.8}$$

and recalling the cavity mode operators $\hat{a}$ and $\hat{b}$ are Gaussian variables with zero mean, we readily get

$$\Delta s^2 + \Delta t^2 = \Big[\langle \hat{a}^\dagger \hat{a}\rangle + \langle \hat{a}\hat{a}^\dagger\rangle + \langle \hat{b}^\dagger \hat{b}\rangle + \langle \hat{b}\hat{b}^\dagger\rangle\Big] - \Big[\langle \hat{a}\hat{b}\rangle + \langle \hat{a}^\dagger \hat{b}^\dagger\rangle + \langle \hat{b}\hat{a}\rangle + \langle \hat{b}^\dagger \hat{a}^\dagger\rangle\Big]. \tag{5.9}$$

Thus with the aid of Eqs. (3.2) and (3.10), we see that

$$\Delta s^2 + \Delta t^2 = \frac{2\gamma_c}{\kappa}\Big[N + \langle \hat{N}_b\rangle - 2\langle \hat{m}_3\rangle\Big]. \tag{5.10}$$

It then follows that

$$\Delta s^2 + \Delta t^2 = 2\Delta c_-^2. \tag{5.11}$$

where Δc_-^2 is given by (4.54). One can readily see from Eq.(5.11), the degree of entanglement is directly proportional to the the minus quadrature variance of the two-mode light.

One can immediately notice that, this particular entanglement measure is directly related the two-mode squeezing. This direct relationship shows that whenever there is a two-mode squeezing in the system there will be entanglement in the system as well. It is noted that the entanglement disappears when the squeezing vanishes. This is due to the fact that the entanglement is directly related to the squeezing as given by (4.53). It also follows that like the mean photon number and quadrature variance the degree of entanglement depends on the number of atom. With the help of the criterion (5.8) that a significant entanglement between the states of the light generated in the cavity. This is due to the strong correlation between the radiation emitted when the atoms decay from the upper energy level to the lower via the intermediate level.

On account of Eqs. (2.119) and (2.121), the photon entanglement of the two-mode cavity light takes, at steady-state, the form

$$\Delta s^2 + \Delta t^2 = \frac{2\gamma_c}{\kappa}N\left[\frac{(\gamma_c+\gamma)^2 + 4\Omega^2 - 2\Omega(\gamma_c+\gamma)}{(\gamma_c+\gamma)^2 + 3\Omega^2}\right]. \tag{5.12}$$

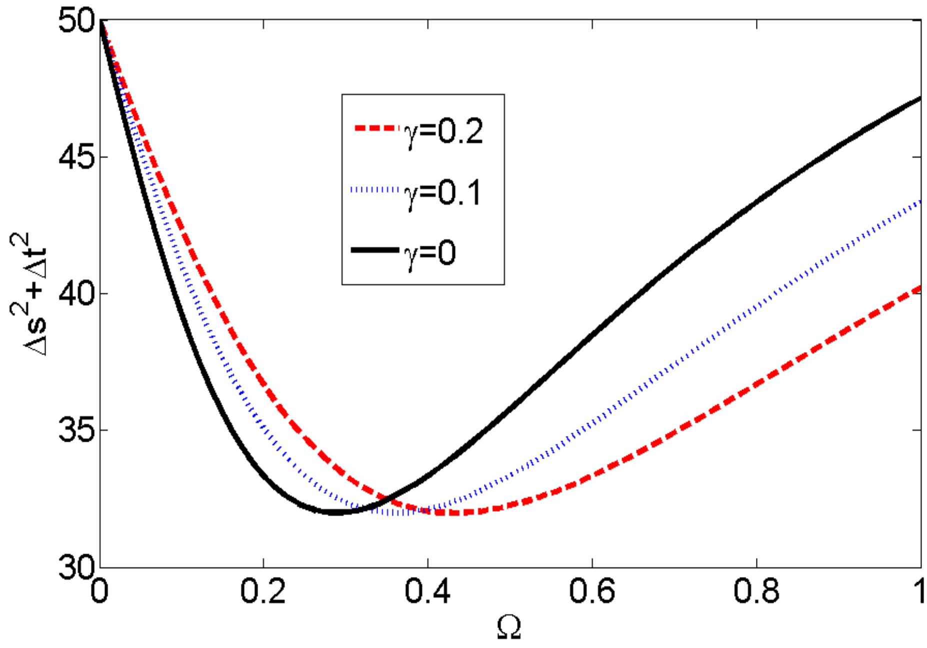

Figure 5.1: Plot of the photon entanglement of the two-mode cavity light [Eq. (5.12)] versus Ω

This is the steady-state the photon entanglement of a two-mode cavity light produced by the coherently driven non-degenerate three-level laser with an open cavity and coupled to a two-mode vacuum reservoir.

We next proceed to consider the case in which spontaneous emission is absent ($\gamma = 0$). Then the photon entanglement for this case takes the form

$$\Delta s^2 + \Delta t^2 = \frac{2\gamma_c}{\kappa} N \left[\frac{\gamma_c^2 + 4\Omega^2 - 2\Omega\gamma_c}{\gamma_c^2 + 3\Omega^2} \right]. \tag{5.13}$$

In addition, we note that for $\Omega \gg \gamma_c$, Eq. (5.13) reduces to

$$\Delta s^2 + \Delta t^2 = \frac{8\gamma_c}{3\kappa} N. \tag{5.14}$$

This can be rewritten as

$$\Delta s^2 + \Delta t^2 = 4\bar{n}, \tag{5.15}$$

where $\bar{n}$ is given by Eq. (3.126).

The plots on Fig. (5.1) describes the photon entanglement of the two-mode cavity light Ω for $\gamma_c = 0.4$, $\kappa = 0.8$, $N = 50$, and for different values of γ. From these plots along with Eq. (5.12), we have

γ	$\Delta s^2 + \Delta t^2$	Ω
$\gamma = 0.2$	32	0.4343
$\gamma = 0.1$	32	0.3636
$\gamma = 0$	32	0.2929

Table 5.1: Values of $\Delta s^2 + \Delta t^2$ for $\gamma_c = 0.4$, $\kappa = 0.8$, and $N = 50$.

When we see the plots on Fig. (5.1) that as the spontaneous emission decay constant increases the photon entanglement also decreases. Similarly, as we observe from the data on Table 5.1, the photon entanglement is decreased with increasing of the spontaneous emission decay constant. From these plots and values of $\kappa = 0.8$, $\gamma_c = 0.4$, and $N = 50$, we determined the maximum photon entanglement is 34% and it occurs at $\Omega = 0.2929$, $\Omega = 0.3636$, and $\Omega = 0.4343$ for $\gamma = 0$, $\gamma = 0.1$, and $\gamma = 0.2$ respectively.

5.2 Cavity Atomic-States Entanglement

The quantum entanglement between the two cavity modes a and b proposed by Duan-Giedke-Cirac-Zoller (DGCZ) [30], which is a sufficient condition for entangled quantum states. According to DGCZ, a quantum state of a system is said to be entangled if the sum of the variances of the EPR-like quadrature operators, $\hat{u}$ and $\hat{v}$, satisfy the inequality

$$\Delta u^2 + \Delta v^2 < 2N^2. \tag{5.16}$$

On the other hand, cavity atomic-states of a system are entangled, if the sum of the variance of a pair of EPR-like operators,

$$\hat{u} = \hat{x}'_a - \hat{x}'_b, \tag{5.17}$$

$$\hat{v} = \hat{p}'_a + \hat{p}'_b, \tag{5.18}$$

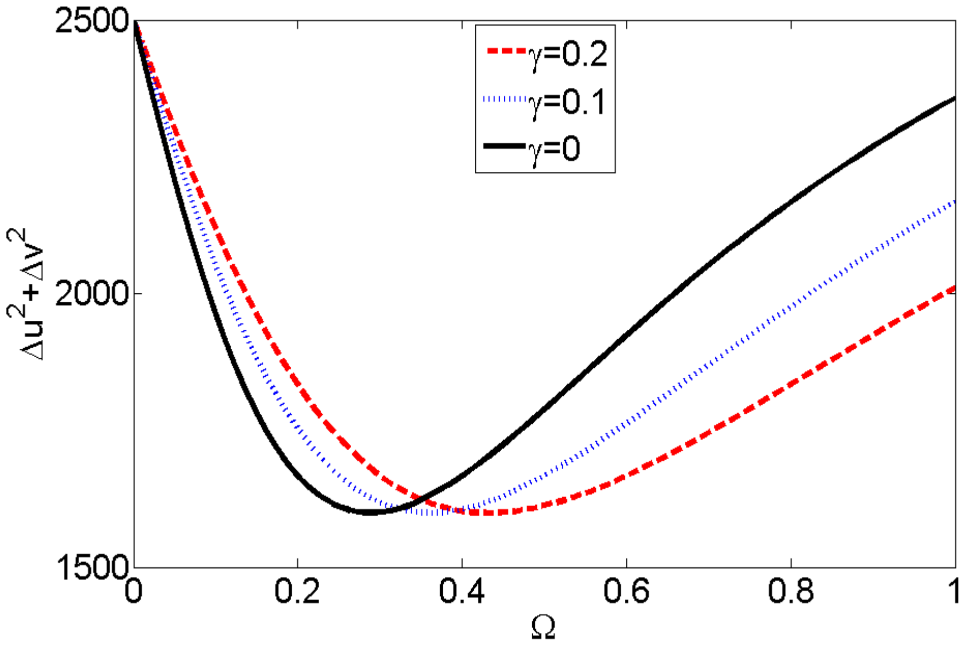

Figure 5.2: Plot of the atom entanglement of the two-mode cavity light [Eq. (5.25)] versus Ω.

where

$$\hat{x}'_a = \frac{1}{\sqrt{2}}\left(\hat{m}_1 + \hat{m}_1^\dagger\right), \tag{5.19}$$

$$\hat{x}'_b = \frac{1}{\sqrt{2}}\left(\hat{m}_2 + \hat{m}_2^\dagger\right), \tag{5.20}$$

$$\hat{p}'_a = \frac{i}{\sqrt{2}}\left(\hat{m}_1^\dagger - \hat{m}_1\right), \tag{5.21}$$

$$\hat{p}'_b = \frac{i}{\sqrt{2}}\left(\hat{m}_2^\dagger - \hat{m}_2\right), \tag{5.22}$$

Since $\hat{m}_1$ and $\hat{m}_2$ are Gaussian variables with zero means, so one can easily verify that

$$\Delta u^2 + \Delta v^2 = \Big[\langle\hat{m}_1^\dagger\hat{m}_1\rangle + \langle\hat{m}_1\hat{m}_1^\dagger\rangle + \langle\hat{m}_2^\dagger\hat{m}_2\rangle + \langle\hat{m}_2\hat{m}_2^\dagger\rangle - \langle\hat{m}_2^\dagger\hat{m}_1^\dagger\rangle - \langle\hat{m}_1\hat{m}_2\rangle\Big]. \tag{5.23}$$

Now with the aid of (2.70) and (2.71), Eq. (5.23) takes the form

$$\Delta u^2 + \Delta v^2 = N\big[N + \langle\hat{N}_a\rangle - 2\langle\hat{m}_3\rangle\big]. \tag{5.24}$$

On account of Eqs. (2.119) and (2.121), the cavity atomic-states entanglement of the two-mode cavity light takes, at steady-state, the form

$$\Delta u^2 + \Delta v^2 = N^2\left[\frac{(\gamma_c+\gamma)^2 + 4\Omega^2 - 2\Omega(\gamma_c+\gamma)}{(\gamma_c+\gamma)^2 + 3\Omega^2}\right]. \tag{5.25}$$

This is the steady-state the cavity atomic-states entanglement of a two-mode cavity light produced by the coherently driven non-degenerate three-level laser with an open cavity and coupled to a two-mode vacuum reservoir. When we observe Eq. (5.25), the cavity atomic-states entanglement of the two-mode cavity light highly depends on the number of atoms.

We next proceed to consider the case in which spontaneous emission is absent ($\gamma = 0$). Then the cavity atomic-states entanglement for this case takes the form

$$\Delta u^2 + \Delta v^2 = N^2\left[\frac{\gamma_c^2 + 4\Omega^2 - 2\Omega\gamma_c}{\gamma_c^2 + 3\Omega^2}\right]. \tag{5.26}$$

In addition, we note that for $\Omega \gg \gamma_c$, Eq. (5.26) reduces to

$$\Delta u^2 + \Delta v^2 = \frac{4}{3}N^2. \tag{5.27}$$

Furthermore, when $\Omega = 0$, Eq. (5.26) turns out to be

$$\Delta u^2 + \Delta v^2 = N^2. \tag{5.28}$$

The plots on Fig. (5.2) describes atom entanglement of the two-mode cavity light versus Ω for $\gamma_c = 0.4$, and for different values of γ. From the plots on Fig. (2.93) along with Eq. (5.25), we have

γ	$\Delta u^2 + \Delta v^2$	Ω
$\gamma = 0.2$	1599	0.4343
$\gamma = 0.1$	1599	0.3636
$\gamma = 0$	1599	0.2929

Table 5.2: Values of $\Delta u^2 + \Delta v^2$ for $\gamma_c = 0.4$, $\kappa = 0.8$, and $N = 50$.

When we see the plots on Fig. (5.2) that as the spontaneous emission decay constant increases the atom entanglement also decreases when it occurs at the same value of Ω. Similarly, as we observe from the data on Table 5.2, the atom entanglement is decreased with increasing of the spontaneous emission decay constant, γ. From these plots and values of $\kappa = 0.8$, $\gamma_c = 0.4$, and $N = 50$, we determined the maximum atom entanglement is 17% and it occurs at $\Omega = 0.2929$, $\Omega = 0.3636$, and $\Omega = 0.4343$ for $\gamma = 0$, $\gamma = 0.1$, and $\gamma = 0.2$, respectively.

On the basis of the criteria (5.8) and (5.16), we clearly see that the two states of the generated light are strongly entangled at steady-state. Moreover, the absence of thermal light leads to an increase in the degree of entanglement. Furthermore, when we observe from Figs. (5.1) and (5.2) that the degree of cavity-atomic state entanglement is greater than the photon entanglement of light.

6

Conclusion

In conclusion, the squeezing and entanglement properties of a non-degenerate three-level laser driven by coherent light and coupled to a two-mode vacuum reservoir via a single-port mirror whose open cavity contains N non-degenerate three-level atoms, are thoroughly analyzed. We carried out the analysis by putting the noise operators associated with the vacuum reservoir in normal order and by considering the interaction of a three-level atoms with the vacuum reservoir outside the cavity. The master equation and the quantum Langevin equations for the cavity light is obtained. Applying these equations, the equations of evolution of the cavity mode and the atomic operators are solved. Making use of the steady-state solutions of atomic and cavity mode operators, the quadrature variance, the quadrature squeezing, and the entanglement for the two-mode cavity light, at steady state, are determined. .

The analysis showed that the interactivity quadrature squeezing is enhanced due to in the absence spontaneous emission. It is found that the squeezing and entanglement in the two-mode light is directly related. As a result, an increase in the degree of squeezing directly implies an increase in the degree of entanglement and vise versa. This shows that whenever there is squeezing in the two-mode light, there exists entanglement in the system. It is found to be the maximum quadrature squeezing is 36% for $\gamma = 0.2$. Moreover, the more the mean photon number is the more bright light. Hence the absence of spontaneous emission gives bright and squeezed light beams.

Bibliography

[1] K. Fesseha, Phys. Rev. A **63**, 33811 (2001).

[2] N. A. Ansari, Phys. Rev. A **48**, 4686 (1993)

[3] Fesseha Kassahun, *Fundamentals of Quantum Optics* (Lulu, North Carolina, 2008).

[4] N. Lu and S. Y. Zhu, Phys. Rev. A **41**, 2865 (1990).

[5] Fesseha Kassahun, e-print, arXiv:1105.1438v

[6] Fesseha Kassahun, *Refined Quantum Analysis of Light* (Create Space Independent Publishing Platform, 2014).

[7] D. F. Walls and J. G. Milburn, *Quantum Optics* (Springer-Verlag, Berlin, 1994).

[8] M. O. Scully and M. S. Zubairy, *Quantum Optics* (Cambridge University Press, Cambridge,1997).

[9] C. C. Gerry and P. L. Knight, *Introductory Quantum Optics* (Cambridge University Press, Cambridge, 2004).

[10] Eyob Alebachew and K. Fesseha, Opt. Commun. **265**, 314 (2006).

[11] J. Anwar and M.S. Zubairy, Phys. Rev. A **49**, 481 (1994).

[12] E. Alebachew, Phys. Rev. A **76**, 023808 (2007).

[13] B. Daniel and K. Fesseha, Opt. Commun. **151**, 384 (1998).

[14] C. Benkert and O.M. Scully, Phys. Rev. A **41**, 2756 (1990).

[15] L. Davidovich, Rev. Mod. Phys. **68**, 127 (1997).

[16] R. Vyas , Phys. Rev. A **46**, 395 (1992).

[17] M. Xiao, L. A. Wu, H. J. Kimble, Phys. Rev. Lett. **59**, 278 (1992).

[18] J. Anwar and M. S. Zubairy, Phys. Rev. A **45**, 1804 (1992).

[19] N. A. Ansari, Phys. Rev. A **46**, 1560 (1992).

[20] S. M. Barnett and P. M. Radmore, *Methods in Theoretical Quantum Optics*, (Oxford University Press, New York, 1997).

[21] Eyob Alebachew, Opt. Commun. **280**, 133 (2007).

[22] Sintayehu Tesfa, Phys. Scr. **84**, 045403 (2011).

[23] S. Tesfa, *Phys. Rev.* A *74*, 043816 (2006).

[24] J.M. Liu, B.S. Shi, X.F. Fan, J. Li, G.C. Guo. *Wigner function description of continuous variable entanglement swapping.* J. Opt.B: Quant. Semiclass. Opt. *3*, 189 (2001)

[25] S.L. Braunstein, H.J. Kimble. *Dense coding for continuous variables. Phys. Rev.* A *61*, 42302 (2000)

[26] S.L. Braunstein. *Quantum error correction for communication with linear optics.* Nature 394, 47 (1998).

[27] T.C. Ralph. *Continuous variable quantum cryptography. Phys. Rev.* A *61*, 010302 (2000).

[28] F. Kassahun, *Stimulated emission by two-level atoms pumped to the upper level, Optics Communication*, vol. 284, no. 5, pp. 13571363, (2011).

[29] Yeshiwas, A. *Dynamics of Coherently Driven Nondegrnerate Three-Level Atom In Open Cavity* (AAU, 2014).

[30] L. M. Duan, G. Giedke, J. J. Cirac, and P. Zoller, *Phys. Rev.Lett.* A **84**, 2722 (2000)

[31] A. Einstein, B. Podolsky, and R. Rosen, Phys. Rev. **47**, 777 (1935)