Almost Second Order Finite Difference Method for Singularly Perturbed Delay Differential Equation

A Thesis Submitted to the Department of Mathematics, Jimma University in Partial Fulfillment for the Requirements of the Degree of Masters of Sciences in Mathematics

(Numerical Analysis)

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DECLARATION

I, undesired, declare that “Almost second order finite difference method for singularly perturbed delay differential equations” is Original and it has not been submitted to any institutional elsewhere for the award of any academic degree or like and that all the sources I have used or quoted have been indicated and acknowledged as complete references.

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ABSTRACT

In this thesis, we consider singularly perturbed differential equation containing delay parameter on the convection and reaction terms. The considered problem exhibits left or right boundary layer, depending on the sign of the coefficient of convection term. The terms with delay treated using Taylor's series approximation. The resulting asymptotically equivalent singularly perturbed boundary value problem is solved using the technique of fitted mesh finite difference method. The stability and consistency of the scheme is investigated to guarantee the convergence of the scheme. Further, the theoretical finding is validated using numerical examples that confirm the betterment of the present method than some existing method in the literature.

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CHAPTER ONE

INTRODUCTION

1.1 Background of the Study

In real life, we often encounter many problems which are described by parameter dependent differential equations. The behaviors of the solutions of these types of differential equation depend on the magnitude of the parameters. Any differential equation in which the highest order derivative is multiplied by a small positive parameter $\varepsilon(0< \varepsilon<<1)$ is called singular perturbation problem and the parameter is known as the perturbation parameter. Singularly perturbed second order two – point boundary value problem occur very frequently in fluid motion, chemical reactor theory, elasticity, diffusion in polymer, reaction – diffusion equation, control of chaotic system and so on (Kadalbajoo and Kumar, 2008).

In numerical analysis, finite-difference methods (FDM) are a class of numerical techniques for solving differential equations by approximating derivatives with finite differences. Finite difference methods convert ordinary differential equations (ODE) or partial differential equations (PDE), which may be nonlinear, into a system of linear equations that can be solved by appropriate iterative techniques. Modern computers can perform these linear algebra computations efficiently which, along with their relative ease of implementation, has led to the widespread use of FDM in modern numerical analysis. Today, FDM are one of the most common approaches to the numerical solution of ODE, along with finite element methods. Singularly perturbed delay differential equation (SPDDE) is an ordinary differential equation in which the highest derivative is multiplied by a small parameter and containing delay term.

If the order of singularly perturbed differential equation is reduced by one then the problem called as convection-diffusion type and if the order is reduced by two it is called reaction-diffusion type. Hence, Second order singularly perturbed self-adjoint ordinary differential equations are types of reaction-diffusion problem. Due to the importance of these problems in real life situations, the need to develop numerical methods for approximating its solution is advantageous. But, numerically solving the singularly perturbed differential equations depends up on the small positive parameters; so that, the solution varies rapidly in some parts of the domain and varies slowly in some other parts of the domain because of the existence of boundary layer.

Recently Kadalbajoo and Gupta (2010) one can find a number of papers dealing with the numerical solutions of singularly perturbed BVPs, singularly perturbed problems having delay on the convection or reaction term only. Singularly perturbed differential equations having delay on both the convection and reaction terms are not studied well. To review the numerical schemes developed for solving such problems so far; Kumar Kadalbajoo (2012) considered a singularly perturbed problem having delays on the convection and reaction terms. The authors used Taylor's series approximation for the delay terms and converted the problem into equivalent BVPs. The authors computed the numerical solutions using B-spline collocation method on shishkin mesh. In (2016) the authors used Tailor's series approximation for the delay terms and apply fifth and sixth order finite difference approximation for the derivative terms and develop finite difference scheme

Erdogan and Amiraliyev (2012) presented fitted finite difference method for singularly perturbed delay differential equations. An exponentially fitted difference scheme is constructed in an equidistant mesh, which gives first order uniform convergence in the discrete maximum norm. The difference scheme is shown to be uniformly convergent to the continuous solution with respect to the perturbation parameter.

Debela and Duressa (2020) presented finite difference scheme for singularly perturbed reaction diffusion problem of delay differential equation with nonlocal boundary condition. A small parameter is multiplied in the higher order derivative, which gives boundary layers, and due to the delay term, one more layer occurs on the domain. A simple but novel numerical method is developed to approximate the numerical solution of this types. The method gives accurate solutions for mesh size discritized is greater than or equal to singular perturbation parameter in the inner region of the boundary layer where other classical numerical methods fail to give smooth solution.

The above authors presented convergent methods have not been sufficiently developed for a wide class of singularly perturbed delay differential equations. Thus, this study is aimed at presenting a numerical method that is more satisfactory accurate solution and uniformly convergent method for solving singularly perturbed delay differential equations with negative shifts in convection and reaction terms.

1.2. Statement of the problem

The numerical analysis of singular perturbation problems has always been far from trivial because of the boundary layer behavior of the solution. Such problems undergo rapid changes within very thin layers near the boundary or inside the domain of the problem. The field of DDE attracted mathematicians and engineers due to the following reasons. Firstly, we have to find an appropriate approximation of the solution at the delayed arguments. Secondly, the algorithm has to take care of the jump in the discontinuity due to the delay parameter and thirdly, its solution behavior is very interesting with layers.

However the competition of its solution has been a great challenge and has been of great importance due to the versatility of such equations. In mathematical modeling of process in various application fields, where they provide the best simulation of observed phenomena and hence the numerical approximation of such equations has growing more and more, (Amiraliyev & Erdogan, 2007)

Ongoing this, the present study attempt to answer the following questions:

1. How does investigate the convergence of proposed method?
2. How do the present methods be described for singularly perturbed DDE?
3. How to illustrate the applicability and advantageous of the proposed method?

1.3 Objectives of the study

1.3.1 General Objective

The general objective of this study is to present almost second order finite difference method for singularly perturbed delay differential equation.

1.3.2 Specific objective

- To develop finite difference method for singularly perturbed DDE.
- To establish the convergence of the proposed method
- To investigate the accuracy of the proposed method.

1.3.3 Significance of the study

The outcomes of this study may have the following importance:

- Provide some background information for other researchers who work on this area.
- To introduce the application of numerical methods in different field of studies
- Help graduate students to acquire research skills and scientific procedures.

1.3.4 Delimitation of the study

The singularly perturbed DDEs perhaps arise in variety of applied mathematics that contributes for the advancement of science and technology. Though singularly perturbed DDEs are vast topics and have many applications in the real world, this study is delimited to almost second order finite difference method for solving singularly perturbed DDEs of the form:

$$-\varepsilon u''(x)+\alpha(x)u'(x-\delta)+\beta(x)u(x)+\omega(x)u(x-\delta)=f(x), \quad x\in\Omega=(0,1),$$

with interval and boundary conditions:

$$u(x)=\varphi(x),\ -\delta\le x\le 0, \qquad u(1)=\gamma,$$

where ε , $0<\varepsilon<<1$ is perturbation parameter and δ is delay parameter satisfying $0<\delta<\varepsilon$.

The functions $\alpha(x),\ \beta(x),\ \omega(x)$ and $f(x)$ are assumed to be smooth and bounded to guarantee the existence and unique solution.

CHAPTER TWO

LITERATURE REVIEW

2.1. Finite difference method

Finite difference methods were made during the period of, and immediately following, the Second World War, when large-scale practical applications become possible with the aid of computers. A major role was played by the work of von Neumann, partly reported in O'Brien, Hyman and Kaplan (1951).

Finite difference methods are always a convenient choice for solving boundary value problems because of their simplicity. Finite difference methods are one of the most widely used numerical schemes to solve differential equations and their application in science and technology. In finite difference methods, derivatives appearing in the differential equations are replaced by finite difference approximations at the grid points. This gives a large algebraic system of linear equations to be solved by Thomas Algorithm or other methods in place of the differential equation to give the solution value at the grid points and hence the solution is obtained at grid points. Some of the finite difference methods include forward difference methods, backward difference method, central difference method, etc.

Present-day scientific research concerns on the methods of numerical solutions to mathematical problems which are simpler to use and solve difficult problems. Accordingly, obtaining stable, accurate, uniformly convergent and fast numerical solutions for singularly perturbed delay differential equations has a great importance due to its wide applications in science and engineering research, since they are difficult or impossible to solve analytically. Owing to this, this study presents parametric uniform numerical methods for solving finite element method for singularly perturbed delay differential equations by the methods of second order.

2.2. Singularly perturbed Delay Differential Equation

The theory and numerical solution of singularly perturbed delay differential equations are still at the initial stage. In the past, only very few people had worked in the area of numerical methods on singularly perturbed delay differential equations (SPDDEs). But in the resent years there have been a growing interest in this area. Kadalbajoo and Sharma (2008) and Mohapatra and Netesan (2010) proposed some numerical methods for SPDDEs with small delay. It may be noted that Lange and Miura (1982) gave an asymptotic approximation to solve singularly perturbed second order delay differential equations. In the present work a numerical method named as Initial Value Technique (IVT) is suggested to solve the boundary value problems for second order ordinary differential equations of reaction diffusion type with negative shift in the differentiated term. The initial value method was introduced by the authors Gasparo and Macconi (1990). In fact they applied this method to solve singularly perturbed boundary value problems for differential equations without negative shift/delay.

A delay differential equation (DDE) is an equation where the evaluation of the system at a certain time, depends on the state of the system at an earlier time. This is distinct from ordinary differential equation (ODEs) where the derivatives depend on the current value of the independent variable. A DDE is said to be of retarded delay differential equation (RDDE) if the delay argument does not occur in the highest order derivative term, otherwise it is known as neutral delay differential equation (NDDE). If we restrict it to a class in which the highest derivative term is multiplied by small parameter, then we obtain singularly perturbed delay differential equations of the retarded type. Frequently, delay differential equations have been reduced to differential equations with coefficients that depend on the delay by means of first order accurate Taylor's series expansions of the terms that involve delay the resulting differential equations have been solved either analytically when the coefficients of these equations are constant or numerically, when they are not. When the delay argument is sufficiently small, to tackle the delay term Kadalbajoo and Sharma (2004) used Taylor's series expansion and presented an asymptotic as well as numerical approach to solve such type boundary value problem. But the existing methods in the literature fail in the case when the delay argument is bigger one because in this case, the use of Taylor's series expansion for the term containing delay may lead to a bad approximation.

2.3. Recent development

Amiraliyev and Erdogan (2007) presented a uniformly almost second order convergent numerical method for singularly perturbed delay differential equations. The problem is solved by using a hybrid difference scheme on a Shishkin-type mesh. The method is shown to be uniformly convergent with respect to the perturbation parameter. Numerical experiments illustrate in practice the result of convergence proved theoretically.

Kadalbajoo and Sharma (2008) presented A numerical method based on finite difference for boundary value problems for singularly perturbed delay differential equations When the delay argument is sufficiently small, to tackle the delay term, the researchers Kadalbajoo, Sharma, Numerical analysis of singularly perturbed delay differential equations with layer behavior.

Erdogan and Amiraliyev (2012) presented fitted finite difference method for singularly perturbed delay differential equations. An exponentially fitted difference scheme is constructed in an equidistant mesh, which gives first order uniform convergence in the discrete maximum norm. The difference scheme is shown to be uniformly convergent to the continuous solution with respect to the perturbation parameter.

Erdogan and Cen (2018) presented a uniformly almost second order convergent numerical method for singularly perturbed delay differential equations. The problem is solved by using a hybrid difference scheme on a Shishkin-type mesh. The method is shown to be uniformly convergent with respect to the perturbation parameter. Numerical experiments illustrate in practice the result of convergence proved theoretically.

Gemechis File (2021) presented singularly perturbed boundary value problems with negative shift parameter are special types of differential-difference equations whose solution exhibits boundary layer behavior. A simple but novel numerical method is developed to approximate the numerical solution of the problems of these types. The method gives accurate solutions for $h \geq \varepsilon$ in the inner region of the boundary layer where other classical numerical methods fail to give smooth solution.

Furthermore, Woldaregay and Duressa (2021) developed a robust numerical method for solving singularly perturbed differential difference equations with small negative shifts both in convection and reaction terms. The authors applied nonstandard finite difference method and investigated for the stability and convergence. However, it is proved that the method is almost first order convergent.

Selvakumar (2022) two new optimal and uniform third-order schemes for Singular Perturbation Problems with Initial Layers. This article presents two numerical methods of the order of three for singular perturbation problems, with a small positive parameter using finite differences. It is a problem with an initial layer in the neighborhood of the initial nodal point whose width is of the order of the small parameter.

The above authors presented convergent methods have developed for a wide class of singularly perturbed delay differential equations.

Thus, this study is presented accurate and uniformly convergent method. But we are going to find more accurate and uniformly convergent solution by using almost second order finite difference method for singularly perturbed delay differential equation.

CHAPTER THREE

METHODOLOGY

3.1 Study Area and period

The study is conducted at Jimma University under the department of Mathematics from September 2021 to June 2022.

3.2. Study Design

This study employ mixed-design (documentary review design and experimental design) on finite difference method for singularly perturbed delay differential equations.

3.3. Source of Information

The relevant sources of information for this study are books, published articles & related studies from internet.

3.4. Mathematical Procedure

In order to achieve the stated objectives, the study follows the following procedures:

1. Defining the problem,
2. Develop asymptotically equivalent to the defined problem.
3. Discretizing the solution domain
4. Constructing finite difference method that give the systems of algebraic equations,
5. Establishing the stability and convergence of the proposed scheme,
6. Writing MATLAB code for the proposed scheme
7. Validate using numerical examples.

CHAPTER 4

DESCRIPTIONS OF THE METHODS AND RESULTS

4.1. Description of the method

Consider the singularly perturbed DDEs having delay in the convection and reaction terms of the problem with interval-boundary conditions have the form:

$$\begin{cases} -\varepsilon u''(x)+\alpha(x)\,\mathrm{u}'(x-\delta)+\beta(x)\,\mathrm{u}(x)+\omega(x)\,\mathrm{u}(x-\delta)= f(x), \quad x\in\Omega=(0,1), \\ \mathrm{u}(x)=\phi(x), \ -\delta\le x\le 0, \\ \mathrm{u}(1)=\gamma, \end{cases} \tag{4.1}$$

where $\varepsilon,(0<\varepsilon<<1)$ is singular perturbation parameter and δ is delay parameter satisfying $0<\delta<\varepsilon$. The functions $\alpha(x),\ \ \beta(x),\ \omega(x)$ and $f(x)$ are assumed to be smooth, bounded and not a function of ε for guaranteeing the existence of unique solution. Further, we assume that $\beta(x)+\omega(x)\ge\zeta>0,\ \forall x\in[0,1]$ to ensure the problem in Eq. (4.1) exhibits boundary layer of thickness $O(\varepsilon)$. Then position of the boundary layer depends on the convection terms: For $\alpha(x)-\delta\omega(x)<0$ left boundary layer exist and for $\alpha(\mathrm{x})-\delta\omega(\mathrm{x})>0$ right boundary layer exist, Mesfin and Gemechis (2021).

To review the numerical schemes developed for solving the problem in Eq. (4.1) different authors used Taylor's series approximation for the delay terms and convert the considered problem into equivalent BVPs. For instance, Mesfin and Gemechis (2021) solve this BVP using Robust Numerical Scheme For Solving Singularly Perturbed Differential Equations Involving Small Delays.

However, due to the applicability of the problem in real-life phenomena, accuracy of the solution takes attention. Thus in this thesis, we developed almost second order Finite Difference scheme using piecewise uniform shishkin mesh for solving singularly perturbed DDEs. In singular perturbation problems, when the delay parameter is smaller than the perturbation parameter, treating the delay terms using Taylor's series approximation is acceptable Tian(2002), Mesfin and Gemechis (2021).

So we approximate both the delay function from Eq. (4.1) as

$$\begin{cases} u'(x-\delta) \approx u'(x) - \delta u''(x) + O(\delta^2) \\ u(x-\delta) \approx u(x) - \delta u'(x) + \dfrac{\delta^2}{2} u''(x) + O(\delta^3) \end{cases} \tag{4.2}$$

Since $0 < \delta < \varepsilon,$ for sufficiently small $\varepsilon,$ substituting the approximation in Eq. (4.2) into Eq. (4.1) yields the boundary value problem of the form:

$$\begin{cases} Lu(x) \equiv -c(x)u''(x) + p(x)u'(x) + q(x)u(x) = f(x), \;\; x \in \Omega = (0,1), \\ \qquad u(0) = \phi(0), \\ \qquad u(1) = \gamma, \end{cases} \tag{4.3}$$

where L is differential operator and coefficient functions are:

$$c(x) = \varepsilon + \delta\alpha(x) - \frac{\delta^2}{2}\omega(x), \qquad 0 < c(x) \le \varepsilon$$
$$p(x) = \alpha(x) - \delta\omega(x)$$
$$q(x) = \beta(x) + \omega(x)$$

Now, for small ε the problem in Eq. (4.3) is asymptotically equivalent to Eq. (4.1). Assume that $p(x) \ne 0,$ and consider the case $p(x) \ge p_0 > 0,$ (p_0 is constant), which implies the existence of right boundary layer. If $p(x) \le p_1 < 0,$ (p_1 is constant) then Eq. (4.3) exhibits left boundary layer.

4.1.1. Properties of continuous solution

Lemma 1: (Maximum principle.) Let z be a sufficiently smooth function defined on Ω which satisfies $z(x) \ge 0, x \in \{0,1\}$. Then $Lz(x) > 0, \forall x \in \Omega$ implies that $z(x) \ge 0, \forall x \in \bar{\Omega}.$ by Woldaregay and Duressa, (2020).

Proof. Let x^*be such that $z(x^*) = \min_{(x)\in\bar{\Omega}} z(x)$ and suppose that $z(x^*) < 0$. It is clear that $x^* \notin \{0,1\}$. Since $z(x^*) = \min_{(x)\in\bar{\Omega}} z(x)$ from extreme values in calculus we have $z'(x^*) = 0$ and $z''(x^*) \ge 0$ and implies that $Lz(x^*) < 0$which is contradiction to the assumption that made above $Lz(x^*) > 0, \forall x \in \Omega.$ Therefore $z(x) \ge 0, \forall x \in \bar{\Omega}.$.

Lemma 2 (Stability): Let $u(x)$ be the solution of the problem in Eq. (4.3). then we obtain the bound

$$|u(x)| \leq \frac{\|f\|}{\zeta} + \max\{|\phi|, |\psi|\},$$

For $q(x) \geq \zeta > 0,$ and ζ is lower bound of $q(x)$.

Proof. Defining barrier functions $v_{\pm}(x,t)$ as $v_{\pm}(x,t) = \frac{\|f\|}{\zeta} + \max\{|\phi|, |\psi|\} \pm u(x)$ and applying the maximum principle, we obtain the required bound. At the boundary points,

$$v_{\pm}(0) = \frac{\|f\|}{\zeta} + \max\{|\phi|, |\psi|\} \pm u(0) \geq 0,$$
$$v_{\pm}(0) = \frac{\|f\|}{\zeta} + \max\{|\phi|, |\psi|\} \pm u(0) \geq 0.$$

On differential operator

$$\begin{aligned} Lv_{\pm}(x) &= -cv''_{\pm}(x) + p(x)v'_{\pm}(x) + q(x)v_{\pm}(x) \\ &= -c\left(0 \pm u''(x)\right) + p(x)(0 \pm u'(x)) + q(x)\left(\frac{\|Lu\|}{\zeta} + \max\left(|\phi|, |\psi|\right) \pm u\left(x\right)\right) \\ &= q(x)\left(\frac{\|Lu\|}{\zeta} + \max\{|\phi|, |\psi|\} \pm f\left(x\right)\right) \\ &\geq 0, \quad \text{since} \quad q(x) \geq \zeta > 0, \end{aligned}$$

which implies $\overline{L}v_{\pm}(x) \geq 0.$ Hence, by maximum principle we obtain, $v_{\pm}(x) \geq 0, \forall x \in \overline{\Omega}.$

Lemma 3 (Boundedness condition): The bounded on the derivative of the solution $u(x)$ of the problem in (4.3) is given by

$$\left|u^{(i)}(x)\right| \leq C\left(1 + c(\mathrm{x})^{-i} \exp(-\frac{p^* x}{c(\mathrm{x})})\right), x \in \overline{\Omega},\ 0 < i \leq 4, \text{ for left boundary layer.}$$

$$\left|u^{(i)}(x)\right| \leq C\left(1 + c(\mathrm{x})^{-i} \exp(-\frac{p^*(1-x)}{c(x)})\right), x \in \overline{\Omega},\ 0 < i \leq 4, \text{for right boundary layer.}$$

Proof. See Woldaregay and Duressa (2020).

4.1.2. Formulation of Numerical Scheme

To define a piecewise-uniform mesh, we consider a positive integer $N \geq 4$. Since the solution of the problem in Eq. (4.3) exhibits left or right boundary layer depending on the function of $p(x) > 0$ or $p(x) < 0$ respectively, we choose the transition parameter τ defined by

$$\tau = \min\left\{\frac{1}{2},\ 2\varepsilon\ln(N)\right\} \tag{4.4}$$

Now divide the solation domain $\overline{\Omega} = [0,1]$ in to two subintervals $(0,\tau)$ and $(\tau,1)$ or $(0,1-\tau)$ and $(1-\tau,1)$ with $\frac{N}{2}$ points. Thus, the mesh spacing in the subintervals is given by:

$$h_i = \begin{cases} \dfrac{2\tau}{N}, & i = 1,2,\ldots,\dfrac{N}{2} \\ \dfrac{2(1-\tau)}{N}, & i = \dfrac{N}{2}+1,\ldots N, \end{cases} \tag{4.5}$$

for left boundary layer cases and

$$h_i = \begin{cases} \dfrac{2(1-\tau)}{N} & i = 1,2,\ldots,\dfrac{N}{2} \\ \dfrac{2\tau}{N}, & i = \dfrac{N}{2}+1,\ldots N, \end{cases} \quad 4.6)$$

for right boundary layer respectively. The mesh nodal points are given by:

$$x_i = \begin{cases} 0, & i = 0, \\ ih_i, & i = 1,2,\ldots N-1, \\ 1, & i = N. \end{cases}$$

Denoting this discretization of the solution domain by Ω^N, and u_i is the approximation of $u(x_i)$, so that the discrete form of (4.3) on Ω^N, for $i = 1,2,\ldots N-1$ is given by

$$-c_i\sigma^2 u_i + p_i\sigma^0 u_i + q_i u_i = f_i, \tag{4.7}$$

where $\sigma^2 u_i = \dfrac{2}{h_i + h_{i+1}}(\sigma^+ u_i - \sigma^- u_i)$, $\sigma^0 u_i = \dfrac{u_{i+1} - u_{i-1}}{h_i + h_{i+1}}$, $\sigma^+ u_i = \dfrac{u_{i+1} - u_i}{h_{i+1}}$, $\sigma^- u_i = \dfrac{u_i - u_{i-1}}{h_i}$.

for σ is an operator

Further, the scheme in (4.7) can be re-written as in the form of three-term recurrence relation:

$$E_i u_{i-1} + F_i u_i + G_i u_{i+1} = H_i, \tag{4.8}$$

where

$$E_i = -\frac{2c_i}{h_i(h_i + h_{i+1})} - \frac{p_i}{h_i + h_{i+1}}, \quad F_i = \frac{2c_i}{h_i h_{i+1}} + q_i, \quad G_i = \frac{-2c_i}{h_{i+1}(h_i + h_{i+1})} + \frac{p_i}{h_i + h_{i+1}}, \text{ and } H_i = f_i$$

4.1.3. Stability Analysis

The matrix form of obtained scheme can be written as

$$MU = B, \tag{4.9}$$

where $M = \begin{pmatrix} F_1 & G_1 & 0 & 0 & 0 & \cdots & 0 \\ E_2 & F_2 & G_2 & 0 & 0 & \cdots & 0 \\ 0 & E_3 & F_3 & G_3 & 0 & \cdots & 0 \\ 0 & 0 & E_4 & F_4 & G_4 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \vdots & 0 & E_{N-2} & F_{N-2} & G_{N-2} \\ 0 & \cdots & \cdots & \cdots & \cdots & E_{N-1} & F_{N-1} \end{pmatrix}$, $u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-1} \end{pmatrix}$, $B = \begin{pmatrix} H_1 - E_1 U_0 \\ H_2 \\ H_3 \\ \vdots \\ H_{N-1} - G_{N-1} U_N \end{pmatrix}$.

Definition: (M-Matrices) (from the book by Martin and David, 2018): A given square matrix $M = (\mathrm{m}_{ij})$ is said to be M-matrix if $m_{ij} \le 0, \ \forall i \ne j$, $m_{ii} > 0, \forall i$ and then the inverse, M^{-1} exist with each entries are greater or equal to zero. Then, the difference schemes whose coefficient matrices satisfy M-matrices are generally stable. Further, a square matrix $M = (\mathrm{m}_{ij})$ is said to be strictly diagonally dominant if

$$m_{ii} > \sum_{i \ne j} |m_{ij}|, \ \forall i.$$

In our case, from system of Eq. (4.9), we have:

$$\begin{aligned} m_{ii} &= F_i, \\ m_{ij} &= E_i, \quad i+1 = j, \\ m_{ij} &= G_i, \quad i = j+1, \\ m_{ij} &= 0, \quad \text{otherwise} \end{aligned}$$

Thus, the diagonal dominance defined above can be verified as:

$$m_{ii} > \left|m_{i+1j}\right| + \left|m_{ij+1}\right| \ge \left|m_{i+1j} + m_{ij+1}\right|$$
$$F_i > \left|E_i + G_i\right|$$
$$\frac{2c_i}{h_i h_{i+1}} + q_i > \left|\frac{-2c_i}{h_i(\mathrm{h}_i + \mathrm{h}_{i+1})} - \frac{p_i}{h_i + h_{i+1}} + \frac{-2c_i}{h_{i+1}(\mathrm{h}_i + \mathrm{h}_{i+1})} + \frac{p_i}{h_i + h_{i+1}}\right|$$

which can be satisfied by: $q_i \ge \zeta > 0$.

Hence, the difference schemes in Eq. (4.9) that employ $M-$matrix is stable. Thus, the formulated scheme in Eq. (4.8) satisfies the definition of M-matrix that consequences stability of scheme.

4.1.4. Truncation Error

The truncation error for the described method will be investigated. To achieve this investigation, the local truncation error $T(h_i)$ between the exact solution $U(x_i)$, and the approximate solution Y_i is given by

$$T\left(h_i\right) = -c(x_i)\,\mathrm{u}''(x_i) + p(x_i)\,\mathrm{u}'(x_i) + q(x_i)\,\mathrm{u}(x_i) - \left\{-\frac{2c_i}{h_i + h_{i+1}}(\sigma^+ u_i - \sigma^- u_i) + p_i \sigma^0 u_i + q_i u_i\right\}$$

Using Taylor's series expansion to u_i around x_i, we have the approximation for $u_{i\pm1}$ as

$$u_{i+1} = u_i + h_{i+1}u_i' + \tfrac{h_{i+1}^2}{2}u_i'' + \tfrac{h_{i+1}^3}{6}u_i''' + \frac{h_{i+1}^4}{24}u_i^{(4)} + O(h_{i+1}^5) \quad (4.12)$$

$$u_{i-1} = u_i - h_i u_i' + \tfrac{h_i^2}{2}u_i'' - \tfrac{h_i^3}{6}u_i''' + \tfrac{h_i^4}{24}u_i^{(4)} + O(h_i^5)\ .$$

From these two basic equations, we obtain the following:

$$\sigma^+ u_i = \tfrac{u_{i+1} - u_i}{h_{i+1}} = u_i' + \tfrac{h_{i+1}}{2}u_i'' + \tfrac{h_{i+1}^3}{6}u_i''' + \tfrac{h_{i+1}^4}{24}u_i^{(4)} + O(h_{i+1}^4) \quad (4.13)$$

$$\sigma^- u_i = \tfrac{u_i - u_{i-1}}{h_i} = u_i' - \tfrac{h_i}{2}u_i'' + \tfrac{h_i^2}{6}u_i''' - \tfrac{h_i^3}{24}u_i^{(4)} + O(h_i^4) \quad (4.14)$$

$$\sigma^0 u_i = \frac{u_{i+1} - u_{i-1}}{h_{i+1} + h_i} = \frac{h_{i+1} - h_i}{2}u_i'' - \frac{h_{i+1}^3 + h_i^3}{6(\mathrm{h}_{i+1} + \mathrm{h}_i)}u_i''' + \ldots, \quad (4.15)$$

Substituting Eq.(4.13)-(4.14) in to (4.11) and recall at the nodal point x_i:

$u''(x_i) = \mathrm{u}_i''$, $\quad a(x_i)\,\mathrm{u}'(x_i) = a_i u_i'$, and $b(x_i)\,\mathrm{u}(x_i) = b_i u_i$, we get

$$\mathrm{T}(h_i) = -\tfrac{h_{i+1}-h_i}{2} u_i'' - \left\{ \tfrac{\varepsilon}{3}\left(h_{i+1} - h_i\right) + \tfrac{h_{i+1}^3 + h_i^3}{6(h_{i+1}+h_i)} \right\} u_i'' +, ..., \tag{4.16}$$

From the considered piecewise discretization of the solution domain, assume that the value of chosen transition parameter is $\tau = 2\varepsilon \ln N$. Thus, we have

$$h_{i+1} - h_i = \tfrac{2\tau}{N} - \tfrac{2(1-\tau)}{N} = \tfrac{2-4\tau}{N} = \tfrac{2-4(2\varepsilon \ln(N))}{N} = \frac{2 - 8\varepsilon \ln(\mathrm{N})}{N} \tag{4.17}$$

Since the considered problem exhibits a layer region the described scheme works on piecewise discretization. Thus, we have to consider the values of the perturbation parameter, $\varepsilon \le (\tfrac{2}{N})$. Substituting this inequality in to Eq.(4.17) gives

$$h_{i+1} - h_i \le \tfrac{2N - 16\ln(N)}{N^2} \le N^{-2} \ln(\mathrm{N}) \tag{4.18}$$

Thus, from Eq.(4.17) and Eq. (4.18) the norm of truncation error for the formulated scheme is

$$\|T\| \le CN^{-2} \ln(\mathrm{N}) \tag{4.19}$$

where C is arbitrary constant.

Therefore, the described method is almost second order convergent. Hence, stable and consistent scheme is convergent by Lax's equivalence theorem, (Siraj et al., 2019).

4.2. Numerical Examples and Results

In this section, we consider numerical examples to confirm the theoretical analysis made in the previous sections. Since the exact solution for these examples are not known, the maximum absolute errors are not estimated by using the double mesh principle. Mesfin and Gemechis (2021) defined by:-

$$E_\varepsilon^N = \max\left|u_i^N - u_i^{2N}\right|,$$

where u_i^N stands for numerical solution of the problem on N number of mesh points and U_i^{2N} stands for the numerical solution of the problem on $2N$ Number of mesh points.

The rate of convergence of the scheme is obtained as:

$$r_\varepsilon^N = \frac{\log\left(E_\varepsilon^N\right) - \log\left(E_\varepsilon^{2N}\right)}{\log(2)}$$

Example1. Consider the singularly perturbed problem

$$-\varepsilon u''(x) - (1+x)u'(x-\delta) + e^{-x}u(x) - \sin(2x)u(x-\delta) = -\sin(2x) - 3e^{-x},\ 0 < x < 1,$$

with interval and boundary conditions $u(x) = -1, \quad -\delta \le x \le 0$ and $u(1) = 1,$

Example 2: consider the problem

$$-\varepsilon u''(x) + (1+x)u'(x-\delta) + e^{-x}u(x) - e^{-2x}u(x-\delta) = e^{x-1},\quad 0 < x < 1,$$

with interval and boundary conditions $u(x) = 1, \quad -\delta \le x \le 0$ and $u(1) = -1,$

Table 4.1: Comparison of maximum absolute error for Example 1 when $\delta = 0.3\varepsilon$.

$\varepsilon \downarrow N \rightarrow$	32	64	128	256	512	1024
Present Method						
10^{-4}	2.6387e-03	6.9347e-04	1.7241e-04	4.0783e-05	1.2784e-05	4.0639e-06
10^{-5}	2.6630e-03	7.0593e-04	1.8133e-04	4.5596e-05	1.3008e-05	4.0620e-06
10^{-6}	2.6654e-03	7.0699e-04	1.8189e-04	4.6108e-05	1.3269e-05	4.1653e-06
10^{-7}	2.6657e-03	7.0710e-04	1.8194e-04	4.6133e-05	1.3278e-05	4.1887e-06
10^{-8}	2.6657e-03	7.0711e-04	1.8194e-04	4.6135e-05	1.3278e-05	4.1891e-06
10^{-9}	2.6657e-03	7.0711e-04	1.8194e-04	4.6135e-05	1.3278e-05	4.1891e-06
10^{-10}	2.6657e-03	7.0711e-04	1.8194e-04	4.6135e-05	1.3278e-05	4.1891e-06
Results for Mesfin (2020)						
10^{-4}	1.2360e-02	6.2211e-03	3.1206e-03	1.5628e-03	7.8204e-04	3.9117e-04
10^{-5}	1.2360e-02	6.2211e-03	3.1206e-03	1.5628e-03	7.8204e-04	3.9117e-04
10^{-6}	1.2360e-02	6.2211e-03	3.1206e-03	1.5628e-03	7.8204e-04	3.9117e-04
10^{-7}	1.2360e-02	6.2211e-03	3.1206e-03	1.5628e-03	7.8204e-04	3.9117e-04
10^{-8}	1.2360e-02	6.2211e-03	3.1206e-03	1.5628e-03	7.8204e-04	3.9117e-04
10^{-9}	1.2360e-02	6.2211e-03	3.1206e-03	1.5628e-03	7.8204e-04	3.9117e-04
10^{-10}	1.2360e-02	6.2211e-03	3.1206e-03	1.5628e-03	7.8204e-04	3.9117e-04

Table 4.2: Rate of convergence for example 1.

$\varepsilon \downarrow N \rightarrow$	32	64	128	256	512
10^{-4}	1.9279	2.0080	2.0798	1.6736	1.6534
10^{-5}	1.9155	1.9609	1.9916	1.8095	1.6791
10^{-6}	1.9146	1.9586	1.9800	1.7970	1.6716
10^{-7}	1.9145	1.9585	1.9796	1.7968	1.6645
10^{-8}	1.9145	1.9585	1.9795	1.7968	1.6643
10^{-9}	1.9145	1.9585	1.9795	1.7968	1.6643
10^{-10}	1.9145	1.9585	1.9795	1.7968	1.6643

Table 4.3: Maximum absolute error of example 2 for different values of perturbation parameter for delay $\delta = 0.5\varepsilon$

$\varepsilon \downarrow N \rightarrow$	32	64	128	256	512	1024
10^{-4}	1.5461e-02	5.4971e-03	1.8642e-03	6.0856e-04	1.9256e-04	5.9468e-05
10^{-5}	1.5456e-02	5.4971e-03	1.8598e-03	6.0779e-04	1.9254e-04	5.9472e-05
10^{-6}	1.5456e-02	5.4937e-03	1.8579e-03	6.0689e-04	1.9222e-04	5.9428e-05
10^{-7}	1.5455e-02	5.4934e-03	1.8576e-03	6.0672e-04	1.9211e-04	5.9364e-05
10^{-8}	1.5455e-02	5.4933e-03	1.8576e-03	6.0670e-04	1.9210e-04	5.9355e-05
10^{-9}	1.5455e-02	5.4933e-03	1.8576e-03	6.0670e-04	1.9210e-04	5.9354e-05
10^{-10}	1.5455e-02	5.4933e-03	1.8576e-03	6.0670e-04	1.9210e-04	5.9354e-05

Table 4.4: Rate of Convergence for example 2.

$\varepsilon \downarrow N \rightarrow$	32	64	128	256	512
10^{-4}	1.4914	1.5635	1.6151	1.6601	1.6951
10^{-5}	1.4914	1.5635	1.6135	1.6584	1.6949
10^{-6}	1.4923	1.5641	1.6142	1.6587	1.6935
10^{-7}	1.4923	1.5643	1.6143	1.6591	1.6943
10^{-8}	1.4923	1.5642	1.6144	1.6591	1.6944
10^{-9}	1.4923	1.5642	1.6144	1.6591	1.6944
10^{-10}	1.4923	1.5642	1.6144	1.6591	1.6944

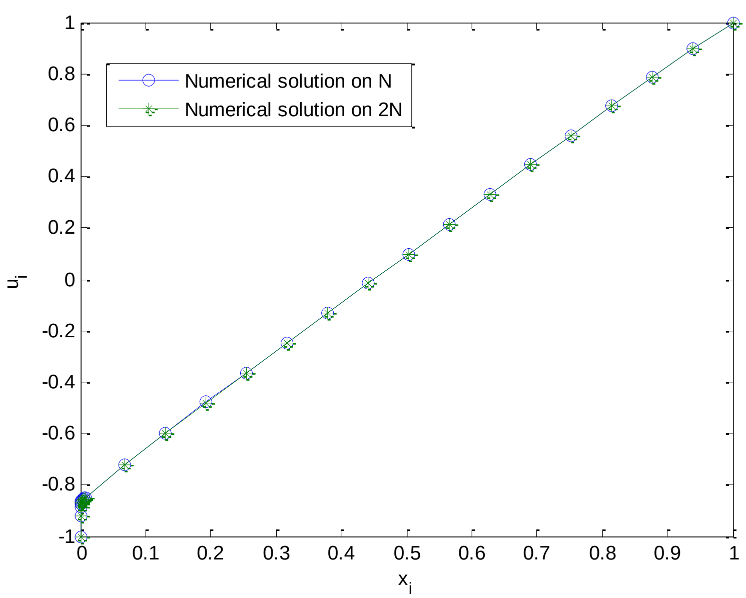

Figure 1: Solution profile for Example 1, when $\varepsilon = 10^{-3}$, $\delta = 0.5 * \varepsilon$ and $N = 32$

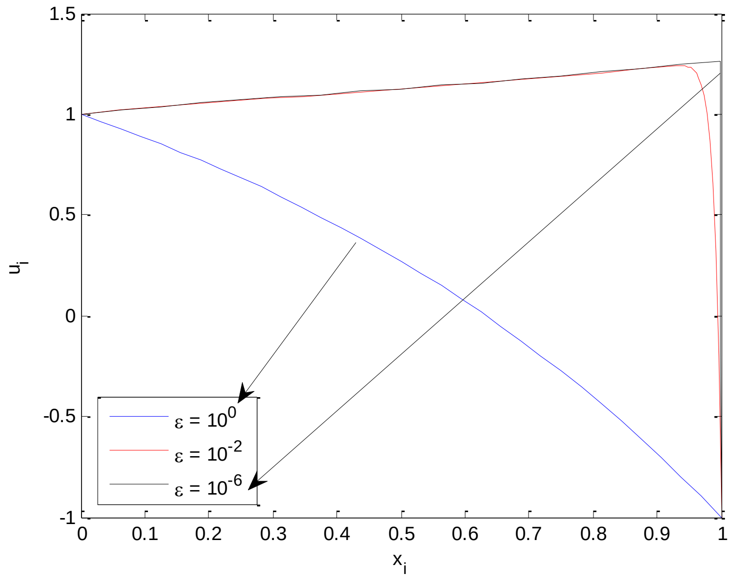

Figure 2: Effect of delay parameter on the solution of example 2 for $\delta = 0.5\varepsilon$

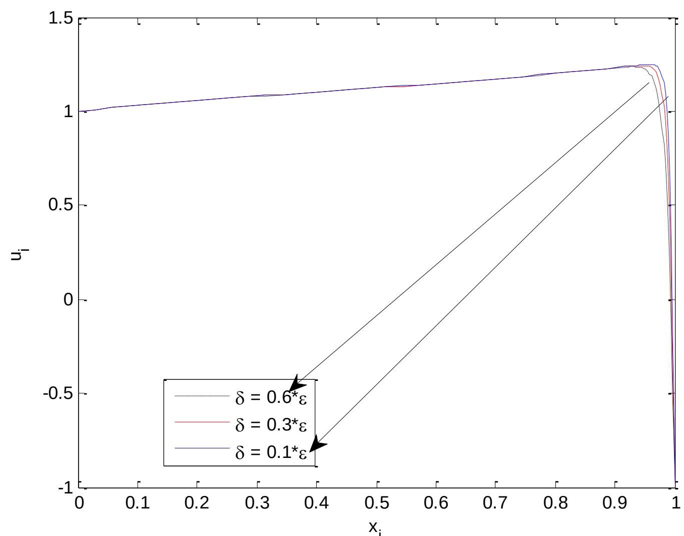

Figure 3: Effect of delay parameter on the solution of example 2 for $N = 32$ and $\varepsilon = 10^{-2}$

In tables 4.1 and 4.3, the maximum absolute errors are presented to show the applicability and effect of the formulated scheme. Besides table 4.2 and table 4.4 demonstrates the rate of convergence to confirm the method is almost second order finite difference method. As one observes the result in the provided tables confirm the theoretical investigations with betterment of accurate solution than some existing method. Further, figures 1-3 shows that the solution profile via layer regions, effects of perturbation and delay parameters.

CHAPTER FIVE

CONCLUSION AND SCOPE FOR FUTURE WORK

5.1 Conclusion

In this thesis, almost second- order finite difference method is presented for solving singularly perturbed delay differential equations. To achieve this method, first, singularly perturbed delay differential equations are written in its asymptotically equivalent of BVPs form. Then the FD scheme is formulated for the BVPs that can be solved using Thomas algorithm. The stability and consistency of the method are investigated to guarantee the convergence of the method. To validate and authorize the effectiveness of the method, two model examples considered. Further, the method gives more accurate solution than some existing methods in the literature. Thus, the formulated method produces more accurate and convergent numerical solution for singularly perturbed delay differential equations.

5.2. Scope of the Future Work

In this thesis, the numerical method based on almost second order finite difference method for singularly perturbed delay differential equations (SPDDEs). Hence the scheme proposed in this thesis can also be extended to fourth order and higher order finite difference methods for solving singularly perturbed differential equations.

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