EXPONENTIAL FITTED OPERATOR METHOD FOR SOLVING SECOND ORDER SINGULARLY PERTURBED PROBLEM HAVING LARGE DELAY

JIMMA UNIVERSITY COLLEGE OF NATURAL SCIENCES

DEPARTMENT OF MATHEMATICS

A Thesis Submitted to the Department of Mathematics Jimma University in Partial Fulfillment of the Requirements for the Degree of Master of Science in Mathematics.

(Numerical Analysis)

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Approved By The Examined Board

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Declaration

I here by declare that the work which is being presented in this thesis entitled **"exponential fitted operator method for solving second order singularly perturbed problem having large delay"** in partial fulfillment of the requirement for the degree of Masters of Science in Mathematics, submitted to Jimma University, department of Mathematics is my original work and it has not been submitted for the award of any academic degree or the like in any other institution or university, and that all the sources I have used or quoted have been indicated and acknowledged as complete references.

Name: Osman Nurru

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Date: ....

The work has been done under supervision of:

Name: Habtamu Garoma (PhD)

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Date: ....

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Contents

Acronyms

* BVP - Boundary Value Problem.
* SPBVP - Singularly Perturbed Boundary Value Problems.
* SPP - Singularly Perturbed Problem.
* SPDDE -Singularly Perturbed Delay Differential Equation.

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Abstract

In this thesis, exponential fitted operator method has been presented for solving second order singularly perturbed problem having large delay. The stability and parameter uniform convergence of the proposed method are proved. To validate the applicability of the scheme, a model problem is considered for numerical experimentation and solved for different values of the perturbation parameter, ε and number of mesh points,N. Maximum absolute errors and rates of convergence for different values of perturbation parameter and number of mesh points are tabulated for the numerical example considered and it is observed that the present method is more accurate and first order ε- uniformly convergent.

Chapter 1

Introduction

1.1 Background of the study

Numerical analysis is a branch of mathematics concerned with theoretical foundations of numerical algorithms for the solution of problems arising in scientific applications (Wasow , 1942). Science and technology develop many practical problems, such as the mathematical boundary layer theory or approximation of solution of various problems described by differential equations and almost all physical phenomena in nature are modeled using differential equations, and singularly perturbed problems are vital class of these kinds of problems (Cengizci, 2017).

An equation involving derivatives of one or more dependent variables with respect to one or more independent variables is called a differential equation. A differential equation involving ordinary derivatives of one or more dependent variables with respect to a single independent variable is called an ordinary differential equation. A differential equation in which the highest order derivative term is multiplied by a small positive parameter ε where $0<\varepsilon << 1$ is known to be singularly perturbed differential equations and the parameter is known as the perturbation parameter.

The classification of singularly perturbed higher order problems depending on how the order of the original equation is affected if one sets $\varepsilon = 0$, where ε is small positive parameter multiplying the highest derivative occurring in the differential equation. If the order is reduced by one, we say that the problem is convection-diffusion type and of reaction-diffusion type if the order is reduced by two.

Any system involving a feedback control will almost involve time delays. These arise because a finite time is required to sense information and then react to it. If we restrict the class of delay differential equation to a class in which the highest derivative is multiplied by a small positive parameter and involving at least one delay term, then it is said to be singularly perturbed delay differential equation. We call delay differential equations retarded type if the delay argument does not occur in the highest order derivative term, otherwise it is known as neutral delay differential equations. As ε tends to zero, the solution of problems exhibits interesting behaviors (rapid changes) since the order of the equation reduces. The region where these rapid changes occur is called inner region and the region in which the solution changes regularly is called outer region.

In many branches of applied mathematics and engineering, Singularly Perturbed Delay Differential Equations (SPDDEs) are commonly used. In the mathematical modeling of various practical phenomena, certain forms of equations often occur, such as in the modeling of the human pupil - light reflex (Longitin, 1988). The mathematical model for calculating the expected time by random synaptic inputs in the dendrites (Lange, 1994). To generate action potential in nerve cells and variational problems in control theory (Glizer, 2003). Many researchers have been trying to develop numerical methods for solving these problems. For example, Awoke and Reddy (2013) presented parameter fitted scheme to solve singularly perturbed delay differential equations. Chakravarthy et al., (2015) presented fitted numerical scheme to solve singular perturbed delay differential equation. Merga et al.,(2021) presented exponentially fitted numerical scheme for singularly perturbed differential equations involving small delays. Erdogan (2009) presented an exponentially fitted method to solve singular perturbed delay differential equation. Subburayan and Ramanujam (2013) an initial value technique for singularly perturbed convection–diffusion problems with a negative shift.

Recentily, Rai and Sharma (2020) considered singularly perturbed delay differential equations using fitted mesh method. Kumar and Rao (2020) presented a stabilized central difference method for the boundary value problem of singularly perturbed differential equations with a large negative shift. Kumar and Subburayan (2021), presented an improved initial value method for singularly perturbed convection diffusion delay differential equation. But, still there is a room to increase the accuracy. Besides, as far as the researchers' knowledge is concerned the problem under consideration via exponential fitted operator method is not yet considered.

Hence, the aim of this project is to formulate uniformly convergent exponential fitted operator method to solve singularly perturbed problem having large delay.

Therefore, the main objective of this study is to develop more accurate, and ε-uniformly convergent method for solving singularly perturbed convection- diffusion problems having large delay.

Throughout our analysis C is generic positive constant that is independent of the parameter ε and number of mesh points N. We assume that,$\Omega^- = [-1, 0]$, $\overline{\Omega} = [0, 2]$, $\Omega = (0, 2)$, $\Omega_1 = (0, 1)$, $\Omega_2 = (1, 2)$. Further, $\Omega^* = \Omega_1 \cup \Omega_2$, $\overline{\Omega}^{2N}$ is denoted by $\{0, 1, 2, ..., 2N\}$, $\overline{\Omega}_1^{2N}$ is denoted by $\{1, 2, ..., N-1\}$, $\overline{\Omega}_2^{2N}$ is denoted by $\{N+1, N+2, ..., 2N-1\}$.

1.2 Objectives

1.2.1 General objective

The general objective of this study is to develop exponential fitted operator method for second order singularly perturbed problem having large delay.

1.2.2 Specific objectives

The specific objectives of the study are:

* To describe the exponentially fitted operator method for second order singularly perturbed problem having large delay.
* To establish the convergence of the present scheme.
* To investigate the accuracy of the present method.

1.3 Significance of the study

The results obtained in this study may:

* Help the graduate students to acquire research skills and scientific procedures.
* To introduce the application of numerical methods in different field of studies.
* Serve as a reference material for scholars who works on this area.

1.4 Delimitation of the study

This study is delimited to exponential fitted operator method for solving second order singularly perturbed problem having large delay of the form:

$$Ly(x) = -\varepsilon y''(x) + a(x)y'(x) + b(x)y(x) + c(x)y(x-1) = f(x), x \in \Omega, \quad (1.4.1)$$

$$y(x) = \phi(x), x \in [-1, 0], \quad (1.4.2)$$

$$y(2) = l, \quad (1.4.3)$$

where $\phi(x)$ is sufficiently smooth on $[-1, 0]$. For all $x \in \Omega$, it is assumed that the sufficient smooth functions $a(x), b(x)$ and $c(x)$ satisfy at $a(x) \geq \alpha > 0$, $b(x) \geq \beta > 0$, $b(x) + c(x) \geq \gamma > 0$, $c(x) \leq \eta < 0$. The above assumptions ensure that $y \in X = C^0(\bar{\Omega}) \cap C^1(\Omega) \cap C^2(\Omega_1 \cup \Omega_2)$.

The BVP Eq. (1.4.1) exhibits strong boundary layer at $x = 2$ and interior layer at $x = 1$ (Kumar and Subburayan, 2021).

Chapter 2

Review of Related Literature

2.1 Boundary value problem

A boundary value problem is a system of ordinary differential equations with solution and derivative values specified at more than one point. Most commonly, the solution and derivatives are specified at just two points (the boundaries) defining a two-point boundary value problem. A boundary value problem for a given differential equation consists of finding a solution of the given differential equation subject to a given set of boundary conditions.

A boundary condition is a prescription some combinations of values of the unknown solution and its derivatives at more than one point. Finding the numerical solution of a boundary value problem is more difficult than that of corresponding initial value problem. For, there are BVPs for which solutions do not exist; and even if a solution exists there might be many more. Thus, existence and uniqueness generally fail for BVPs. The boundary value problems for such a class of delay differential equations are ubiquitous in the modeling of several physical and biological phenomena like first exit time problem in modeling of activation of neuronal variability (Mackey and Glass, 1994). In the study of bistable device (Derstine et al, 1982). and to describe the human pupil-light reflex (Longtin and Milton, 1988). variation problems in control theory (Glizer, 1988), and in describing the motion of the sunflower (Pena, 1989).

2.2 Singularly Perturbed Delay Differential Equation

Singularly perturbed delay differential equation is an equation in which evolution of system at a certain time depends on the rate at an earlier time. The delay in process arises due to requirement of definite time to sense the instruction and react to it. The delay differential equation in which the highest derivative is multiplied by perturbation parameter is known as perturbed delay differential equation. The delay differential equation can be classified as retarded delay differential equation and neutral differential equation.

If we restrict it to a class in which the highest derivative term is multiplied by a small parameter, then we obtain singularly perturbed delay differential equation of the retarded type. Frequently, delay differential equations have been reduced to differential equations with coefficients that depend on the delay by means Taylor's series expansions of the terms that involve delay. The resulting differential equation have been solved either analytically when the coefficients of these equations are constant or numerically, when they are not. The theory and numerical solution of singularly perturbed delay differential equation are still at the initial stage.

In the past, only every few people had worked in the area of numerical methods on singularly perturbed delay differential equations (SPDDEs). But in the recent years, there has been a growing interest in this area. Gemechis and Reddy (2013) presented computational method for solving singularly delay differential equations with negative shift. When the delay argument is sufficiently small, to tackle the delay term (Kadalbajoo and Sharma,2008) used Taylor's series expansion and presented an asymptotic as well as numerical approach to solve such type boundary value problem. But the existing methods in the literature fail in the case when the delay argument is bigger one because in this case, the use of Taylor's series expansion for the term containing delay may lead to a bad approximation.

The numerical treatment of singularly perturbed problems preserves some major computational difficulties and in recent years a large number of special purpose methods have been proposed to provide accurate numerical solutions. This type of problem has been intensively studied analytically and it is known that its solution generally has boundary layers where the solution varies rapidly. The outer solution corresponds to the reduced problem, i.e., that obtained by setting the small perturbation parameter to zero.

2.3 Recent developments

In fact, Habtamu and Gemechis (2021) proposed an exponentially fitted operator method for singularly perturbed convection-diffusion type problems with nonlocal boundary condition. Senthil et al., (2021) an improved initial value method for singularly perturbed convection diffusion delay differential equation. Pratima et al.,(2019) presented a numerical approximation for a class of singularly perturbed delay differential equations with boundary and interior layer(s). Erdogan (2009) proposed an exponentially fitted operator method for singularly perturbed first order delay differential equation. Lange and Miura (1994) gave an asymptotic approximation to solve singularly perturbed second order delay differential equations. Chakravarthy et al.,(2015) presented an exponentially fitted finite difference scheme to solve singularly perturbed delay differential equation of second order with a large delay. Kumar and Rao (2020) presented a stabilized central difference method for the boundary value problem of singularly perturbed differential equations with a large negative shift.
As introduced in the literature, most researchers have been tried to find approximate solution for singularly perturbed differential equations with a large delay, but mainly focuses on constant coefficients, and some others those who have done for variable coefficients did not get more accurate solutions. Owing this, this study presents a more accurate and convergent numerical method for singularly perturbed differential equations with a large delay, by using exponentially fitted operator method.

Chapter 3

Methodology

3.1 Study Site and Period

This study conducted at Jimma University, College of Natural Science, Department of Mathematics from December 2021 to February 2022 G.C .

3.2 Study Design

This study employed mixed-design (i.e., documentary review design and experimental design) on singularly perturbed delay differential problem.

3.3 Source of Information

The relevant source of information for this study are books, published articles on reputable journals and related study from Internet.

3.4 Mathematical Procedure

In order to achieve the stated objectives, the study was followed the following mathematical procedures.

1. Defining (or describing) the problems.

2 Analyze the properties of continuous solution.

3. Discretizing the solution domain /interval.

4. Describing the method by exponential fitted operator method and obtain the schemes in to system of equation.

5. Establishing the stability and convergence analysis of the formulated schemes.

6. Solve the obtained system of equation using Gaussian elimination method.

7. Writing MATLAB code for validation.

8. Validating the schemes using numerical experimentations and presenting the results using tables and graphs.

9. Discussing and providing conclusions.

Chapter 4

DESCRIPTION OF THE METHOD, RESULTS AND DISCUSSION

4.1 Description of the scheme

Consider the following singularly perturbed problem

$$Ly(x) = -\varepsilon y''(x) + a(x)y'(x) + b(x)y(x) + c(x)y(x-1) = f(x), x \in \Omega, \tag{4.1.1}$$

$$y(x) = \phi(x), x \in [-1, 0], \tag{4.1.2}$$

$$y(2) = l. \tag{4.1.3}$$

As we observed from Eq. (4.1.1) and Eq. (4.1.2), the values of $y(x-1)$ is known for the domain Ω_1 and unknown for the domain Ω_2 due to the large delay at $x = 1$. So, it impossible to treat the problem throughout the domain $(\bar{\Omega})$. Thus, we have to treat the problem at Ω_1 and Ω_2 separately.
So, Eqs. (4.1.1)-(4.1.3) is equivalent to

$$Ly(x) = R(x), \tag{4.1.4}$$

where

$$Ly(x) = \begin{cases} L_1y(x) = -\varepsilon y''(x) + a(x)y'(x) + b(x)y(x), x \in \Omega_1, \\ L_2y(x) = -\varepsilon y''(x) + a(x)y'(x) + b(x)y(x) + c(x)y(x-1), x \in \Omega_2, \end{cases} \tag{4.1.5}$$

$$R(x) = \begin{cases} f(x) - c(x)\phi(x-1), x \in \Omega_1, \\ f(x), x \in \Omega_2, \end{cases} \tag{4.1.6}$$

with boundary conditions

$$\begin{cases} y(x) = \phi(x), x \in [-1, 0], \\ y(1^-) = y(1^+), y'(1^-) = y'(1^+), \\ y(2) = l. \end{cases} \tag{4.1.7}$$

4.2 Properties of continuous solution

Lemma 4.2.1 *(Maximum Principle) Let $\psi(x)$ be any function in X such that $\psi(0) \geq 0, \psi(2) \geq 0, L_1\psi(x) \geq 0, \forall x \in \Omega_1, L_2\psi(x) \geq 0, \forall x \in \Omega_2$ and $[\psi'](1) \leq 0$ then $\psi(x) \geq 0, \forall x \in \bar{\Omega}$.*

Proof: Define the test function ,

$$s(x) = \begin{cases} \dfrac{1}{12} + \dfrac{x}{4}, & x \in [0, 1], \\ \dfrac{2}{12} + \dfrac{x}{6}, & x \in [1, 2]. \end{cases} \tag{4.2.8}$$

Note that $S(x) > 0, \forall_x \in \overline{\Omega}, Ls(x) > 0, \forall_x \in \Omega_1 \cup \Omega_2, s(0) > 0, s(2) > 0, and [s'](1) < 0$. Let $\mu = max\{\dfrac{-\psi(x)}{s(x)}: x \in \overline{\Omega}\}$. Then, there exists $x_0 \in \overline{\Omega}$ such that $\psi(x_0) + \mu s(x_0) = 0$ and $\psi(x) + \mu s(x) \geq 0, \forall_x \in \overline{\Omega}$. Therefore , the function $(\psi + \mu s)$ attains its minimum at $x = x_0$. Suppose the theorem does not hold true,then $\mu > 0$.

Case (i): $x_0 = 0$,

$0 < (\psi) + \mu s)(0) = \psi(0) + \mu s(0) = 0$, it is contradiction.

Case (ii):$x_0 \in \Omega_1$

$0 < L(\psi + \mu s)(x_0) = -\varepsilon(\psi + \mu s)''(x_0) + a(x_0)(\psi + \mu s)'(x_0) + b(x_0)(\psi + \mu s)(x_0) \leq 0$, it is a contradiction.

Case (iii): $x_0 = 1$

$0 \leq [(\psi + \mu s)'](1) = [\psi'](1) + \mu[s'](1) < 0$, it is a contradiction.

Case (iv):$x_0 \in \Omega_2$

$0 < L(\psi+\mu s)(x_0)) = -\varepsilon(\psi+\mu s)''(x_0)+a(x_0)(\psi+\mu s)'(x_0)+b(x_0)(\psi+\mu s)(x_0)+c(x_0)(\psi+\mu s)(x_0 - 1) \leq 0$, it is a contradiction.

Case (v): $x_0 = 2$

$0 \leq [(\psi + \mu s)'](2) = [\psi'](2) + \mu[s'](2) < 0$, it is a contradiction.

All the cases are contradict and giving that $L\psi(x_0) < 0$ which is contradiction to the assumption made above $L\psi(x_0) \geq 0, \forall x \in \Omega$. Therefore, $\psi(x) \geq 0, \forall x \in \bar{\Omega}$.

The uniqueness of the solution is implied by this maximum principle. Its existence follows trivially (as for linear problems, the uniqueness of the solution implies its existence). This principle is applied to prove that the solution of equations(1)-(3) is bounded.

Lemma 4.2.2 *(Stability Result) The solution $y(x)$ of Eqs.* (1.4.1)-(1.4.3)*, satisfies the bound*

$$|y(x)| \leq C \max\{|y(0)|, |y(2)|, \sup_{x \in \Omega*} \left|Ly(x)\right|\}, \quad x \in \overline{\Omega}.$$

Proof: Refer from (Sakar and Tamilsevan, 2018)

Lemma 4.2.3 *Let $y(x)$ be the solution of Eqs.* (1.4.1)-(1.4.3)*. Then, we have the following bounds*

$$|y^{(k)}(x)|_{\Omega^*} \leq C\varepsilon^{-k}, \quad k = 1, 2, 3. \tag{4.2.9}$$

proof: To bound $y'(x)$ on the interval Ω_1, we consider

$$L_1 y(x) = -\varepsilon y''(x) + a(x)y'(x) + b(x)y(x) = f(x). \tag{4.2.10}$$

Integrating the above equation on both sides, we have

$$-\varepsilon\big(y'(x) - y'(0)\big) = -[a(x)y(x) + a(0)y(0)] + \int_0^x a'(t)y(t)dt - \int_0^x b(t)y(t)dt + \int_0^x [f(t) - c(t)\phi(t-1)dt. \tag{4.2.11}$$

Therefore,

$$\varepsilon y'(0) = \varepsilon y'(x) - [a(x)y(x) + a(0)y(0)] + \int_0^x a'(t)y(t)dt - \int_0^x b(t)y(t)dt + \int_0^x [f(t) - c(t)\phi(t-1)dt. \tag{4.2.12}$$

Then by the Mean value theorem, there exits $z \in (0, \varepsilon)$ such that

$$\begin{aligned} &\mid \varepsilon y'(z) \mid \leq C\Big(\mid\mid y(x) \mid\mid, \mid\mid f(x) \mid\mid, \mid\mid \phi(x) \mid\mid_{[-1,0]} \Big). \\ \text{and} \quad &\mid \varepsilon y'(0) \mid \leq C\Big(\mid\mid y(x) \mid\mid + \mid\mid f(x) \mid\mid + \mid\mid \phi(x) \mid\mid \Big). \end{aligned} \tag{4.2.13}$$

Hence,

$$\mid \varepsilon y'(z) \mid \leq C\Big(\mid\mid y(x) \mid\mid, \mid\mid f(x) \mid\mid, \mid\mid \phi(x) \mid\mid \Big). \tag{4.2.14}$$

By a similar argument, we can bound $y'(x)$ on Ω_2, as $\Big|\varepsilon y'(x)\Big| \leq C$. From (4.2.13) and (4.2.14) we have

$$\mid\mid y^{(k)}(x) \mid\mid_{\Omega^*} \leq C\varepsilon^{-k}, \quad for, k = 2, 3.$$

Hence the proof.

Lemma 4.2.4 *The bound for derivative of the solution $y(x)$ of Eqs.* (1.4.1)-(1.4.3) *when $x \in \Omega_1 = (0, 1]$ is given by*

$$|y^{(k)}(x)| \leq C\left(1 + \varepsilon^{-k} \exp\left(\frac{-a(1-x_j)}{\varepsilon}\right)\right), \quad 0 \leq k \leq 4, j = 1, 2, 3, ..., N-1. \tag{4.2.15}$$

Proof: Refer from (Clavero et al., 2005).

4.3 Formulation of the numerical scheme

The linear ordinary differential equation in Eq. (1.4.1) cannot, in general, be solved analytically because of the dependence of $a(x)$, $b(x)$ and $c(x)$ on the spatial coordinate x. We divide the interval $[0, 2]$ into $2N$ equal parts with constant mesh length h. Let $0 = x_0 < x_1 < x_2 < ... < x_N = 1 < x_{N+1} < x_{N+2} < ... < x_{2N} = 2$ be the mesh points. Then, we have $x_i = ih, \quad i = 1, 2, 3, ..., 2N$.

If we consider, the interval $x \in (0, 1]$, the domain [0,1] is discretized into N equal number of subintervals, each of length h. Let $0 = x_0 < x_1 < x_2 < ... < x_N = 1$ be the points such that $x_i = ih, \quad i = 1, 2, 3, ..., N$. We apply an exponentially fitted operator finite difference method (FOFDM).

From Eq. (4.1.5) and Eq. (4.1.6), we have

$$\begin{cases} -\varepsilon y''(x) + a(x)y'(x) + b(x)y(x) = R(x), \quad x \in \Omega_1, \\ y_0 = \phi(0), \quad y(1) = \theta, \end{cases} \tag{4.3.16}$$

where $R(x) = f(x) - c(x)\phi(x - 1)$.

To find the numerical solution of Eq. (4.3.16), we use the theory used in asymptotic method for solving singularly perturbed BVPs. In the considered case, the boundary layer is in the right side of the domain i.e. near $x = 1$. From the theory of singular perturbations given in by (O'Malley, 1991) we get the asymptotic solution up to first order approximation as

$$y(x) = y_0(x) + \frac{a(1)}{a(x)}(\theta - y_0(1))exp\Big(-\int_x^1 \Big(\frac{a(x)}{\varepsilon} - \frac{b(x)}{a(x)}\Big)dx\Big) + O(\varepsilon),$$

by using Taylor series about $x = 1$ for $a(x)$ and $b(x)$ and simplifying we obtain

$$y(x) = y_0(x) + (\theta - y_0(1))exp\Big(-\frac{a^2(1) - \varepsilon b(1)}{\varepsilon a(1)}(1 - x)\Big) + O(\varepsilon), \tag{4.3.17}$$

where $y_0(x)$ is the solution of the reduced problem (obtained by setting $\varepsilon = 0$ of Eq. (4.3.16) which is given by

$$a(x)y'(x) + b(x)y(x) = R(x), \quad y_0 = \phi(0). \tag{4.3.18}$$

Considering h small enough, the discretized form of Eq. (4.3.17) becomes

$$y(ih) = y_0(ih) + (\theta - y_0(1)) \exp\Bigg(- \frac{a^2(1) - \varepsilon b(1)}{a(1)}(1/\varepsilon - i\rho)\Bigg), \tag{4.3.19}$$

where $\rho = \frac{h}{\varepsilon}$, $h = \frac{1}{N}$. Similarly, we write

$$y_{i\pm1} = y_0((i \pm 1)h) + (\theta - y_0(1)) \exp\Bigg(- \frac{a^2(1) - \varepsilon b(1)}{a(1)}(1/\varepsilon - (i \pm 1)\rho)\Bigg).$$

Using Taylors series approximation for $y_0((i+1)h)$ and $y_0((i-1)h)$ up to first order, we obtain

$$\begin{cases} y_{i+1} = y_0(ih) + (\theta - y_0(1)) \exp\Bigg(- \frac{a^2(1)-\varepsilon b(1)}{a(1)}(1/\varepsilon - (i+1)\rho)\Bigg), \\ y_{i-1} = y_0(ih) + (\theta - y_0(1)) \exp\Bigg(- \frac{a^2(1)-\varepsilon b(1)}{a(1)}(1/\varepsilon - (i-1)\rho)\Bigg). \end{cases} \tag{4.3.20}$$

To handle the effect of the perturbation parameter artificial viscosity (exponentially fitting factor $\sigma(\rho)$) is multiplied on the term containing the perturbation parameter as

$$-\varepsilon\sigma(\rho)y''(x) + a(x)y'(x) + b(x)y(x) = R(x), \tag{4.3.21}$$

with boundary conditions $y_0 = \phi(0)$ and $y(1) = \theta$.
Next, we consider the difference approximation of Eq. (4.3.16) on a uniform grid $\overline{\Omega}^N = \{x\}_{i=0}^N$ and denote $h = x_{i+1} - x_i$. When we apply central finite difference formula on Eq.(4.3.22) takes the form

$$-\varepsilon\sigma(\rho)\Big(D^+D^-y(x_i)\Big) + a(x_i)\Big(D^0y(x_i)\Big) + b(x_i)y(x_i) = R(x_i). \tag{4.3.22}$$

Using operator, Eq. (4.3.16) is rewritten as

$$L^N Y_i = R_i, \tag{4.3.23}$$

with boundary conditions $Y_0 = \phi(0)$ and $Y(1) = \theta$, where

$$L^N Y_i = -\varepsilon\sigma(\rho)\left(\frac{Y_{i+1} - 2Y_i + Y_{i-1}}{h^2}\right) + a(x_i)\left(\frac{Y_{i+1} - Y_{i-1}}{2h}\right) + b(x_i)Y_i = R_i. \tag{4.3.24}$$

Multiplying Eq. (4.3.24) by h and considering h small and truncating the term $(R_i - b(x_i)Y_i)h$, results to

$$\frac{-\sigma(\rho)}{\rho}\left(Y_{i+1} - 2Y_i + Y_{i-1}\right) + \frac{a(x_i)}{2}\left(Y_{i+1} - Y_{i-1}\right) = 0. \tag{4.3.25}$$

Substituting the results in Eq.(4.3.19) and Eq.(4.3.20) into Eq. (4.3.25) and simplifying, the exponential fitting factor is obtained as

$$\sigma(\rho) = \frac{\rho a(1)}{2}\coth\left(\frac{\rho a(1)}{2}\right). \tag{4.3.26}$$

Assume that $\overline{\Omega}^{2N}$ denote partition of [0,2] into $2N$ subintervals such that $0 = x_0, x_1, x_2, ..., x_N = 1$ and $x_{N+1}, x_{N+2}, ..., x_{2N} = 2$ with $x_i = ih,\ h = \frac{2}{2N} = \frac{1}{N},\ \ i = 0, 1, 2, ..., 2N$.

Case (1): Consider Eq. (4.1.4) on the domain $\Omega_1 = (0, 1]$ which is given by

$$-\varepsilon y''(x) + a(x)y'(x) + b(x)y(x) = f(x) - c(x)\phi(x - 1), \tag{4.3.27}$$

Hence, the required finite difference scheme becomes

$$\left(\frac{-\varepsilon\sigma(\rho)}{h^2} - \frac{a(x_i)}{2h}\right)Y_{i-1} + \left(\frac{2\varepsilon\sigma(\rho)}{h^2} + b(x_i)\right)Y_i + \left(\frac{-\varepsilon\sigma(\rho)}{h^2} + \frac{a(x_i)}{2h}\right)Y_{i+1} = f_i - c_i\phi(x_{i-N}). \tag{4.3.28}$$

The numerical scheme in Eq. (4.3.28) can be written in three term recurrence relation as

$$E_iY_{i-1} + F_iY_i + G_iY_{i+1} = H_i, \quad i = 1, 2, ..., N, \tag{4.3.29}$$

where $E_i = \frac{-\varepsilon\sigma(\rho)}{h^2} - \frac{a_i}{2h}$, $F_i = \frac{2\varepsilon\sigma(\rho)}{h^2} + b_i$, $G_i = \frac{-\varepsilon\sigma(\rho)}{h^2} + \frac{a_i}{2h}$, $H_i = f_i - c_i\phi(x_{i-N})$.

Case (2): Consider Eq. (4.1.4) on the domain $\Omega_2 = (1, 2)$ using exponentially fitted finite difference method, which is given by

$$-\varepsilon\sigma(\rho)\left(\frac{Y_{i+1} - 2Y_i + Y_{i-1}}{h^2}\right) + a_i\left(\frac{Y_{i+1} - Y_{i-1}}{2h}\right) + b_iY_i + c_iY(x_i - 1) = f_i. \tag{4.3.30}$$

Similarly, this equation can be written as

$$E_iY_{i-1} + F_iY_i + G_iY_{i+1} + C_i = H_i, \quad i = N+1, N+2, ..., 2N-1, \tag{4.3.31}$$

where $E_i = \frac{-\varepsilon\sigma(\rho)}{h^2} - \frac{a_i}{2h}$, $F_i = \frac{2\varepsilon\sigma(\rho)}{h^2} + b_i$, $G_i = \frac{-\varepsilon\sigma(\rho)}{h^2} + \frac{a_i}{2h}$, $C_i = c_iy(x_i - 1)$ and $H_i = f_i$.

Therefore, on the whole domain $\overline{\Omega} = [0, 2]$, the basic schemes to solve Eqs. (1.4.1)-(1.4.3) are the schemes given in Eqs. (4.3.29) and (4.3.31).

4.4 Convergence analysis

The discrete scheme corresponding to Eqs. (1.4.1)-(1.4.3) is as follows

For $i = 1, 2, 3, ..., N$,

$$L_1^NY_i = f_i - c_i\phi_{i-N}. \tag{4.4.32}$$

For $i = N+1, N+2, ..., 2N-1$,

$$L_2^NY_i = f_i, \tag{4.4.33}$$

subject to the boundary conditions

$$Y_i = \phi_i, \quad i = -N, -N+1, ..., 0, \tag{4.4.34}$$

$$Y_{2N} = l, \tag{4.4.35}$$

where

$$\begin{cases} L_1^N y_i = -\varepsilon\delta^2 Y(x_i) + a(x_i)D^0 Y(x_i) + b(x_i)Y(x_i), \\ L_2^N y_i = -\varepsilon\delta^2 Y(x_i) + a(x_i)D^0 Y(x_i) + b(x_i)Y(x_i) + c(x_i)Y(x_{i-N}). \end{cases} \tag{4.4.36}$$

Lemma 4.4.1 *Let $\psi(x)$ be any mesh function then for $0 < i < 2N$,*

$$|\psi(x_i)| \le C \max\{|\psi(x_0)|, |\psi(x_{2N})|, \max_{i \in \Omega_1^{2N} \cup \Omega_2^{2N}} |L^N \psi(x_i)|\}.$$

Proof: Consider the barrier functions

$$\theta^{\pm}(x_i) = CM \pm \phi(x_i), \quad \forall x_i \in \overline{\Omega}^{2N}, \tag{4.4.37}$$

where $M = \max\{|\psi(x_0)|, |\psi(x_{2N})|, \max_{i \in \Omega_1^{2N} \cup \Omega_2^{2N}} |L^N \psi(x_i)|\}$.
From Eq. (4.4.31) it is clear that $\theta^{\pm}(x_i) \ge 0$ and $\theta^{\pm}(x_{2N}) \ge 0$,

$$\begin{aligned} &L_1^N \theta^{\pm}(x_i) \ge 0, \quad \forall x_i \in \Omega_1^{2N}, \\ &L_2^N \theta^{\pm}(x_i) \ge 0, \quad \forall x_i \in \Omega_2^{2N}, \\ &D^+ \theta^{\pm}(x_N) - D^- \theta^{\pm}(x_N) \le 0. \end{aligned} \tag{4.4.38}$$

Using Lemma (4.2.1), $\theta^{\pm}(x_i) \ge 0, \quad \forall x_i \in \overline{\Omega}^{2N}$.
We proved above the discrete operator L^N satisfy the maximum principle and the uniform stability estimate. Next we analyze the uniform convergence analysis. The following theorem shows the parameter uniform convergence of the scheme developed.

Theorem 4.4.2 *Let $y(x_i)$ and Y_i be respectively the exact solution of Eqs.* (1.4.1)-(1.4.3) *and numerical solutions of Eq.* (4.3.23)*. Then, for sufficiently large N, the following parameter uniform error estimate holds*

$$|L^N(y(x_i) - Y_i)| \leq \frac{CN^{-2}}{N^{-1}+\varepsilon}\left(1+\varepsilon^{-3}\exp\left(-\frac{a(1-x_i)}{\varepsilon}\right)\right). \tag{4.4.39}$$

Proof Let us consider the local truncation error defined as

$$\begin{aligned} L^N(y(x_i) - Y_i) &= -\varepsilon\sigma(\rho)(y''(x_i) - D^+D^-y(x_i)) + a(x_i)(y'(x_i) - D^0y(x_i)), \\ &= -\varepsilon\left[\frac{\rho a(1)}{2}\coth\left(\frac{\rho a(1)}{2}\right) - 1\right]D^+D^-y(x_i) \\ &\quad + \varepsilon(y''(x_i) - D^+D^-y(x_i)) + a(x_i)(y'(x_i) - D^0y(x_i)), \end{aligned} \tag{4.4.40}$$

where $\sigma(\rho) = a(1)\frac{\rho}{2}\coth\left(a(1)\frac{\rho}{2}\right)$, and $\rho = \frac{N^{-1}}{\varepsilon}$.
Now, for $z > 0$, C_1 and C_2 are constants, and we have $|z\coth(z) - 1| \leq C_1 z^2, \quad z \leq 1$.
Similarly, for $z \longrightarrow \infty$, since $\lim_{z\longrightarrow\infty}\coth(z) = 1$, $|z\coth(z) - 1| \leq C_1 z$ is given.
In general, for all $z > 0$, as Eq.(4.2.11), we write

$$C_1\frac{z^2}{z+1} \leq z\coth(z) - 1 \leq C_2\frac{z^2}{z+1} \tag{4.4.41}$$

implying that

$$\varepsilon[a(1)\frac{\rho}{2}\coth(a(1)\frac{\rho}{2}) - 1] \leq \varepsilon\left(\frac{(N^{-1}/\varepsilon)^2}{(N^{-1}/\varepsilon)+1}\right) = \frac{N^{-2}}{N^{-1}+\varepsilon}. \tag{4.4.42}$$

Using Taylor series expansion, the bound for $y(x_{i-1})$ and $y(x_{i+1})$ at x_i as

$$\begin{cases} y(x_{i-1}) = y(x_i) - hy'(x_i) + \frac{h^2}{2!}y''(x_i) - \frac{h^3}{3!}y^{(3)}(x_i) + \frac{h^4}{4!}y^{(4)}(x_i) + O(h^5), \\ y(x_{i+1}) = y(x_i) + hy'(x_i) + \frac{h^2}{2!}y''(x_i) + \frac{h^3}{3!}y^{(3)}(x_i) + \frac{h^4}{4!}y^{(4)}(x_i) + O(h^5). \end{cases}$$

We obtain the bound for

$$\begin{cases} |D^+D^-y(x_i)| \leq C|y''(x_i)|, \\ |y''(x_i) - D^+D^-y(x_i)| \leq CN^{-2}|y^{(4)}(x_i)|. \end{cases} \quad (4.4.43)$$

Similarly, for the first derivative term

$$|y'(x_i) - D^0y(x_i)| \leq CN^{-2}|y^{(3)}(x_i)|, \quad (4.4.44)$$

where $|y^{(k)}(x_i)| = \sup_{x_i \in (x_0, x_N)} |y^{(k)}(x_i)|, \quad k = 2, 3, 4.$

Using the bounds in Eq.(4.4.43) and Eq.(4.4.44), we obtain

$$\begin{aligned} |L^N(y(x_i) - Y_i)| &\leq C\frac{N^{-2}}{N^{-1}+\varepsilon}|y''(x_i)| + \varepsilon CN^{-2}|y^{(4)}(x_i)| + CN^{-2}|y^{(3)}(x_i)|, \\ &\leq C\frac{N^{-2}}{N^{-1}+\varepsilon}|y''(x_i)| + CN^{-2}[\varepsilon|y^{(4)}(x_i)| + |y^{(3)}(x_i)|]. \end{aligned}$$

Now, using the bounds for the derivatives of the solution in lemma (4.4.40)and the assumption $\varepsilon \leq N^{-1}$, Eq. (4.2.4), we have

$$\begin{aligned} |L^N(y(x_i) - Y_i)| \leq & \frac{CN^{-2}}{N^{-1}+\varepsilon}\left(1 + \varepsilon^{-2}\exp\left(\frac{-a(1-x_j)}{\varepsilon}\right)\right) \\ & + CN^{-2}\left[\varepsilon\left(1 + \varepsilon^{-4}\exp\left(\frac{-\alpha(1-x_j)}{\varepsilon}\right)\right) + \left(1 + \varepsilon^{-3}\exp\left(\frac{-a(1-x_j)}{\varepsilon}\right)\right)\right] \\ . \leq & \frac{CN^{-2}}{N^{-1}+\varepsilon}\left(1 + \varepsilon^{-2}\exp\left(\frac{-a(1-x_j)}{\varepsilon}\right)\right) \\ & + CN^{-2}\left[\left(\varepsilon + \varepsilon^{-3}\exp\left(\frac{-\alpha(1-x_j)}{\varepsilon}\right)\right) + \left(1 + \varepsilon^{-3}\exp\left(\frac{-a(1-x_j)}{\varepsilon}\right)\right)\right], \end{aligned}$$

which simplifies to

$$|L^N(y(x_i) - Y_i)| \leq \frac{CN^{-2}}{N^{-1}+\varepsilon}\left(1 + \varepsilon^{-3}\exp\left(\frac{-a(1-x_j)}{\varepsilon}\right)\right), \quad \textit{since} \quad \varepsilon^{-3} \geq \varepsilon^{-2}. \quad (4.4.45)$$

Lemma 4.4.3 *For a fixed mesh and for $\varepsilon \to 0$, it holds*

$$\lim_{\varepsilon \to 0} \max_{1 \le j \le N-1} \frac{\exp\left(\frac{-ax_j}{\varepsilon}\right)}{\varepsilon^m} = 0, \quad m = 1, 2, 3, ...$$

$$\lim_{\varepsilon \to 0} \max_{1 \le j \le N-1} \frac{\exp\left(\frac{-a(1-x_j)}{\varepsilon}\right)}{\varepsilon^m} = 0, \quad m = 1, 2, 3, ...$$

Proof: Refer from (Woldaregay and Duressa , 2019)

Theorem 4.4.4 *Let $y(x_i)$ and Y_i be the exact solution of Eqs.* (1.4.1)-(1.4.3) *and numerical solutions of Eq.* (4.3.23) *respectively. Then, the following error bound holds*

$$\sup_{0<\varepsilon<<1} |(y(x_i) - Y_i)| \le \frac{CN^{-2}}{N^{-1} + \varepsilon} \le CN^{-1}. \tag{4.4.46}$$

Proof: By substituting the results in lemma 4.4.3 in to theorem 4.4.2 and applying the discrite maximum principle, we obtain the required bound.

For the case $\varepsilon > N^{-1}$ the scheme secures second order convergence and we expect to lose an order of convergence for $\varepsilon \le N^{-1}$, and in fact it turns out that the scheme is first order uniformly convergent.

Remark: A similar analysis for convergence may be carried out for the finite difference scheme Eq. (4.3.24).

4.5 Numerical Examples and Results

In this section, an example is given to illustrate the numerical method discussed above. The exact solutions of the test problems are not known. Therefore, we use the double mesh principle to estimate the error and compute the experiment rate of convergence to the computed solution. For this we put

$$E_\varepsilon^N = \max_{0 \le i \le 2N} |Y_i^N - Y_{2i}^{2N}|, \tag{4.5.47}$$

where Y_i^N and Y_{2i}^{2N} are the i^{th} components of the numerical solutions on meshes of N and $2N$ respectively. We compute the uniform error and the rate of convergence as

$$E^N = \max_{\varepsilon} E_{\varepsilon}^N, \quad and \quad R^N = \log_2\left(\frac{E^N}{E^{2N}}\right). \tag{4.5.48}$$

The numerical results are presented for the values of the perturbation parameter $\varepsilon \in \{\, 10^{-4}, 10^{-8}, ..., 10^{-20}\}$.

Example 4.5.1 *Consider the model singularly perturbed boundary value problem*

$$-\varepsilon y''(x) + (5+x)y'(x) + 2y(x) - \frac{x}{2}y(x-1) = \exp(x), \quad x \in (0,1) \cup (1,2),$$

subject to the boundary conditions

$$y(x) = 1 + x, \quad x \in [-1, 0], \quad y(2) = 2.$$

Table 4.1: Maximum absolute errors and rate of convergence for Example 4.5.1 at different number of mesh points N

ε	N=16	N=32	N=64	N=128	N=256
10^{-4}	3.6790e-03	2.0411e-03	1.0718e-03	5.4879e-04	2.7764e-04
10^{-8}	3.6790e-03	2.0411e-03	1.0718e-03	5.4879e-04	2.7764e-04
10^{-12}	3.6790e-03	2.0411e-03	1.0718e-03	5.4878e-04	2.7764e-04
10^{-16}	3.6790e-03	2.0411e-03	1.0718e-03	5.4879e-04	2.7764e-04
10^{-20}	3.6790e-03	2.0411e-03	1.0718e-03	5.4879e-04	2.7764e-04
E^N	3.6790e-03	2.0411e-03	1.0718e-03	5.4879e-04	2.7764e-04
R^N	0.8500	0.9293	0.9657	0.9830	

4.6 Discussion

The developed method is based on exponentially fitted operator method for solving singularly perturbed problem having large delay. We investigate the effect of delay and perturbation parameters on the solution of the problem; numerical solutions have been

Table 4.2: Comparison of Maximum absolute errors and rate of convergence for Example 4.5.1 at different number of mesh points N.

$N \rightarrow$	16	32	64	128	256
	present method				
E^N	3.6790e-03	2.0411e-03	1.0718e-03	5.4879e-04	2.7764e-04
R^N	0.8500	0.9293	0.9657	0.9830	
	(Kumar and Subburayan, 2021).				
E^N	2.0055e-2	8.0979e-3	2.7416e-3	9.8601e-4	3.1655e-4
R^N	1.3083	1.5626	1.4753	1.6392	

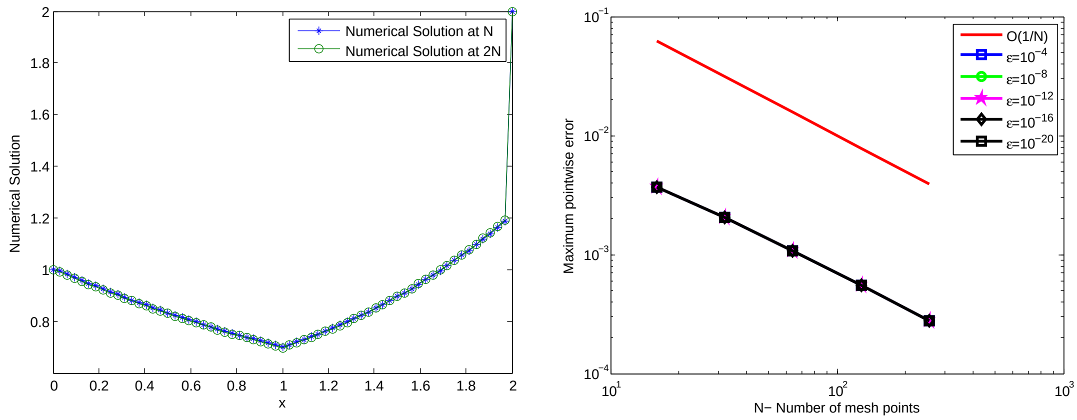

Figure 4.1: The Numerical Solution at $\varepsilon = 10^{-12}$ and $N = 32$ and the maximum point wise errors log-log plot scale of Example 4.5.1 respectively.

presented using Tables and graphs. As shown in Table 4.1,the maximum absolute error of Example 1 is given, as $\varepsilon \rightarrow 0$ it is shown that the maximum absolute error is stable. In Table 4.2 we compare the proposed method with the work of (Senthil et al., 2021), it is clearly shown that the proposed method is more acurate and as $\varepsilon \rightarrow 0$, the maximum absolute error is uniform. The solution of the example given in(4.5.1) has strong boundary layer at the right side of the interval [0,2],(see Figure 4.1). The computed solution of Example 1 for different values of perturbation parameters are also shown in Figure (4.1). The results in the proposed method is better than that obtained in (Senthil et al.,2021).

Chapter 5

CONCLUSSION AND SCOPE OF THE FUTURE WORK

5.1 Conclussion

In this Thesis, we considered exponential fitted operator method for solving singularly perturbed problem having large delay. The behavior of the continuous solution of the problem is studied and shown that it satisfies the continuous stability estimate and the derivatives of the solution are also bounded. The numerical scheme is developed on uniform mesh using exponential fitted operator in the given differential equation. The stability of the developed numerical method is established and its uniform convergence is proved. To validate the applicability of the method, a model problem is considered for numerical experimentation for different values of the perturbation parameter and mesh points. The numerical results are tabulated in terms of maximum absolute errors, numerical rate of convergence and uniform errors.

5.2 Scope for Future Work

In this study, exponential fitted operator method has been presented for solving second order singularly perturbed problem having large delay. Hence, the scheme proposed in this study can also be extended to solve singularly perturbed problems with delay and advance the boundary layer on the left and right side of the domain and internal layer in the middle side of the domain.

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