BOUNDEDNESS AND COMPACTNESS OF GENERALIZED INTEGRATION OPERATORS ON FOCK SPACES

A THESIS SUBMITTED TO THE DEPARTMENT OF MATHEMATICS IN PARTIAL FULFILLMENT FOR THE REQUIREMENTS OF THE DEGREE OF MASTERS OF SCIENCE IN MATHEMATICS

By: Hafiz A/Bor
Main-Advisor: Mafuz Humer (PhD)
Co-Advisor: Nigussa Lamessa (MSc)

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Declaration

I, Hafiz A/Bor, with student ID number RM/0649/13, declare that this thesis entitled ”Boundedness and compactness of generalized integration operators on Fock spaces” is my own original work and it has not been submitted to any institution or University elsewhere for the award of any academic degree, and sources of information that I have been used or quoted are indicated and acknowledged.

Signature _______________
Date _______________
Hafiz A/Bor.

The work has been done under the supervision and approval of:

Name: Mafuz Humer Worku (PhD)
Signature _______________
Date _______________

Name: Nigissa Lamessa (MSc)
Signature _______________
Date _______________

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Abstract

Different properties of Volterra-type integral operator have been studied in the past two decades on several functional spaces. In particular, on Fock spaces boundedness and compactness of the operator was studied by (Constantin, 2012) and (Mengestie, 2013). Boundedness and compactness of generalized integration operator $V_g^{(n,m)}$ have been studied also on spaces of analytic functions defined over a unit disc by (Du et al., 2021) and (Qian and Zhu, 2021). However, it was not studied on Fock spaces. So, the purpose of this thesis is to fill this gap and study bounded and compact properties of the operator on Fock spaces. The result of this thesis generalizes the works of (Constantin, 2012) and (Mengestie, 2013).

Contents

Chapter 1

Introduction

1.1 Background of the study

Let X and Y be Banach spaces and $T : X \to Y$ is a linear operator. If there is a constant $c > 0$ such that $\|Tx\| \leq c\|x\|$, $x \in X$, then we say that T is a bounded linear operator. Moreover, if $\|Tx_n\| \to 0$ whenever $x_n \to 0$ weakly in X, then we say that T is compact.
The study of boundedness and compactness of different linear operators on spaces of analytic functions defined over a domain $U \subseteq \mathbb{C}$ is a rich history, where many authors are participated and many papers and books are written on. In particular, integral operators including the Volterra-type are among widely studied linear operators. This is due to their applicability in solving real world problems.
For a given a space $\mathcal{H}(U)$ of analytic functions on U, the Volterra-type integral operator on $\mathcal{H}(U)$ induced by a analytic symbol function g,

$$V_g f(z) = \int_0^z f(w)g'(w)dw,$$

is among the linear operators studied a lot acting between different spaces. The operator is first introduced by (Pommerenke, 1977) and studied by other authors with the aim to explore the connection between their operator theoretic behaviors with the function-theoretic properties of the symbols g. (Pommerenke, 1977) studied continuity of the operator on the Hilbert space of Hardy space H^2 and this result is extended to $H^p, 0 < p < \infty$, in general by (Aleman and Siskakis, 1995) and furthermore they studied compactness property also. Later (Aleman and Siskakis, 1997), gave the analogous characterization on the Bergman space. But, those studies are considered on spaces of analytic functions defined over a disk. (Constantin, 2012) and

(Mengestie, 2013) considered the problem over a space defined over the whole complex plane $\mathbb{C}$, namely Fock spaces $\mathcal{F}_\alpha^p$.
(Li and Stevic, 2008) raised an idea to extend the Volterra-type integral operator V_g by considering its product with composition operator $C_\psi f = f(\psi)$ and they studied their operator theoretic properties in terms of the inducing pair of symbols on some spaces of analytic functions on the unit disk. They eventually considered the following operator induced by analytic functions g and ψ

$$V_{(g,\psi)}f(z) = \int_0^z f(\psi(w))g'(w)dw.$$

Since a particular choice of $\psi(z) = z$ reduce $V_{(g,\psi)}$ to the Volterra-type integral operator V_g, the operator $V_{(g,\psi)}$ is called the generalized Volterra-type integral operator. Boundedness and compactness of this operator have been studied on different spaces and the characterization of these properties on Fock space have been given by (Mengestie, 2014) and later by (Mengestie and Worku, 2018).
(Chalmoukis, 2020) introduced a new generalization of Volterra-type integral operators, which is defined as follows. For a nonnegative integers m and n with $0 \leq m < n$, and an entire function g, the generalized integration operator, $V_g^{n,m}$, is defined by

$$V_g^{n,m}f = I^n(f^{(m)}g^{(n-m)}),$$

Where I^n is the n-th iterate of the integration operator $I(f)(z) = \int_0^z f(w)dw$. (Chalmoukis, 2020) studied the operator on the Hardy spaces and very recently other researchers considered the operator on other spaces. See (Du et al., 2021) and (Qian and Zhu, 2021). In particular, for the values of $n = 1$ and $m = 0$, it gives the Volterra-type integral operator V_g.
The aim of this thesis is to investigate boundedness and compactness of the generalized integraton operator $V_g^{n,m}$ acting between Fock spaces.

1.1.1 Fock Spaces

Let $\alpha > 0$. An entire function $f(z)$ of complex variable is said to belong to the Fock space $\mathcal{F}_\alpha^2$ if $\int_{\mathbb{C}} |f(z)|^2 e^{-\alpha|z|^2} dm(z) < \infty$ where dm is Lebesgue area measure. The space $\mathcal{F}_\alpha^2$ is equipped with inner product defined by

$$\langle f, g\rangle_\alpha = \frac{\alpha}{\pi}\int_{\mathbb{C}} f(z)\overline{g(z)}e^{-\alpha|z|^2}dm(z) \qquad f, g \in F_\alpha^2$$

So the norm of a function f in $\mathcal{F}^2_\alpha$ is

$$\|f\|_{(2,\alpha)} = (\frac{\alpha}{\pi}\int_{\mathbb{C}} |f(z)|^2 e^{-\alpha|z|^2} dm(z))^{\frac{1}{2}}$$

The constant $\frac{\alpha}{\pi}$ ensures that the function $f(z) = 1$ has norm 1. The Gaussian weight $e^{-\alpha|z|^2}$ in the Fock space $\mathcal{F}^2_\alpha$ arises from adjoint condition on the inner product $\langle , \rangle_\alpha$ imposed by Fock. For two polynomials $p(z)$ and $q(z)$, we have

$$\langle \alpha z p(z), q(z)\rangle_\alpha = \langle p(z), q'(z)\rangle_\alpha \tag{1.1.1}$$

Suppose that $w(z)$ is the weight in the inner product

$$\langle p, q\rangle_\alpha = \int_{\mathbb{C}} p(z)\overline{q(z)} w(z) dm(z)$$

for which the adjoint condition holds. Apply ((1.1.1)) to $p(z) = z^{n-1}$ and $q(z) = z^m, n, m \in \mathbb{N}$ to obtain

$$\alpha \int_{\mathbb{C}} z^n \bar{z}^m w(z) dm(z) = m \int_{\mathbb{C}} z^{n-1}\bar{z}^{m-1} w(z) dm(z)$$

Let $z = re^{i\theta}$. If we suppose that w is a radial weight,i.e $w(z) = w(r)$, then

$$2\pi\alpha \int_0^\infty r^{2n+1} w(r) dr = 2\pi n \int_0^\infty r^{2n-1} w(r) dr$$

let $\gamma_n = 2\pi \int_0^\infty r^{2n+1} w(r) dr, n = 0, 1, 2, ...$ then we have $\gamma_n = \frac{n}{\alpha}\gamma_{n-1}, n = 1, 2, 3, ...$ The convention that $f(z) = 1$ has norm 1 means that $1 = \int_{\mathbb{C}} w(z) dm(z) = 2\pi \int_0^\infty r w(r) dr = \gamma_0$. So we get $\gamma_n = \frac{n!}{\alpha^n}$, but a calculation using the gamma function shows that

$$2\alpha \int_0^\infty r^{2n} e^{-\alpha r^2} r dr = \frac{n!}{\alpha^n} = 2\pi \int_0^\infty r^{2n} w(r) r dr$$

If we further suppose that w is continuous, then we have $w(z) = \frac{\alpha}{\pi} e^{-\alpha|z|^2}$ to generalize $\mathcal{F}^2_\alpha$ to Banach spaces, for $0 < p \leq \infty$, define L^p_α to be the spaces of measurable function f on $\mathbb{C}$ such that $f(z)e^{-\alpha|z|^2} \in L^p(\mathbb{C}, dm)$, where dm is lebesgue measure. The subspace of L^p_α consisting of entire functions is denoted by $\mathcal{F}^p_\alpha$ and is called a Fock space. Thus, Fock space is defined as:

Definition 1.1.1. For $0 < p \leq \infty$ and $\alpha > 0$ the Fock space $\mathcal{F}^p_\alpha$ consists of entire functions for which

$$\|f\|_{(p,\alpha)}^p = \frac{p\alpha}{2\pi}\int_{\mathbb{C}} |f(z)|^p e^{-\frac{p\alpha}{2}|z|^2}\, dm(z) < \infty$$

for $0 < p < \infty$ where dm is the usual lebesgue measure. And

$$\|f\|_{(\infty,\alpha)} = \sup_{z\in\mathbb{C}} |f(z)| e^{-\frac{\alpha}{2}|z|^2} < \infty \quad \text{for} \quad p = \infty$$

Example 1.1.2. The function $f(z) = z^n$ is an entire function in $\mathcal{F}_\alpha^p$, since

$$\begin{aligned}\|z^n\|_{(p,\alpha)}^p &= \frac{p\alpha}{2\pi}\int_{\mathbb{C}} |z^n|^p e^{-\frac{p\alpha}{2}|z|^2} dm(z) \\ &= \alpha p \int_0^\infty r^{np} e^{-p\alpha\frac{r^2}{2}} r dr = (\frac{1}{\alpha p})^{\frac{np}{2}} \Gamma(\frac{np}{2}+1) < \infty,\end{aligned}$$

where $\Gamma(z)$ is gamma function.

The space $\mathcal{F}_\alpha^p$ for $1 \le p \le \infty$ is Banach space. For each $f \in \mathcal{F}_\alpha^p$, we have a pointwise estimate given by, $|f(z)| \le e^{\frac{\alpha}{2}|z|^2}\|f\|_{(p,\alpha)}$. In particular for $p = 2$, it ensures that for any fixed $w \in \mathbb{C}$, the mapping $f \mapsto f(w)$ is bounded linear functional on $\mathcal{F}_\alpha^2$. By Riesz representation theorem in functional analysis, there exists a unique function K_w in $\mathcal{F}_\alpha^2$ such that $f(w) = \langle f, K_w\rangle_\alpha$ for all $f \in \mathcal{F}_\alpha^2$. This function K_w is called the reproducing kernel of $\mathcal{F}_\alpha^2$ and $\mathcal{F}_\alpha^2$ is a reproducing kernel Hilbert space. To find an explicit expression for kernel function, for any orthonormal bases $\{e_n\}$ and $f \in \mathcal{F}_\alpha^2$, we have $f(z) = \sum_{k=0}^\infty \langle f, e_n\rangle_\alpha e_n(z)$ which implies

$$\begin{aligned}K_{w,\alpha}(z) = \sum_{k=0}^\infty \langle K_{w,\alpha}, e_n\rangle_\alpha e_n(z) &= \sum_{k=0}^\infty \overline{\langle e_n, K_w\rangle_\alpha} e_n(z) \\ &= \sum_{k=0}^\infty \overline{e_n(w)} e_n(z).\end{aligned}$$

Since $\{e_n(z) = \sqrt{\frac{\alpha^n}{n!}} z^n\}$ is an orthonormal base for $\mathcal{F}_\alpha^2$, the reproducing kernel for $\mathcal{F}_\alpha^2$ is $K_{w,\alpha}(z) = e^{\alpha z\overline{w}}$ and the normalized kernel function is $k_{w,\alpha}(z) = e^{\alpha z\overline{w} - \frac{\alpha}{2}|w|^2}$. The following local estimate finds lots of application in the spaces.

Lemma 1.1.3. *For any $r > 0$ and $p > 0$ there exists a constant C such that*

$$|f(z)|^p e^{-\frac{\alpha p}{2}|z|^2} \le C \int_{D(z,r)} |f(w)|^p e^{-\frac{\alpha p}{2}|w|^2}\, dm(w) \quad \forall z \in \mathbb{C}$$

where $D(z,r)$ is a disc of center z and radius r.

The next theorem gives the inclusion property of the space.

Theorem 1.1.4. *Let $0 < p \leq q \leq \infty$. Then $\mathcal{F}^p_\alpha \subseteq \mathcal{F}^q_\alpha$.*

Proof. If $q = \infty$, then applying the above lemma, we have

$$|f(z)|e^{-\frac{\alpha}{2}|z|^2} \leq (C\int_{D(z,1)} |f(w)|^p e^{-\frac{p\alpha}{2}|w|^2} dm(w))^{\frac{1}{p}}$$

$$= (C\frac{2\pi}{p\alpha})^{\frac{1}{p}}(\int_{D(z,1)} |f(w)|^p e^{-\frac{p\alpha}{2}|w|^2} dm(w))^{\frac{1}{p}}$$

$$\leq (C\frac{2\pi}{p\alpha})^{\frac{1}{p}}(\int_{\mathbb{C}} |f(w)|^p e^{-\frac{p\alpha}{2}|w|^2} dm(w))^{\frac{1}{p}}$$

$$= (C\frac{2\pi}{p\alpha})^{\frac{1}{p}}\|f\|_{(p,\alpha)} < \infty,$$

which implies,

$$\|f\|_{(\infty,\alpha)} = \sup_{z\in\mathbb{C}} |f(z)|e^{-\frac{\alpha}{2}|z|^2} \leq (C\frac{2\pi}{p\alpha})^{\frac{1}{p}}\|f\|_{(p,\alpha)} < \infty.$$

For $q < \infty$ assume $f \in \mathcal{F}^p_\alpha$. Then,

$$\|f\|^q_{(q,\alpha)} = \frac{q\alpha}{2\pi}\int_{\mathbb{C}} |f(z)|^q e^{-\frac{q\alpha}{2}|z|^2} dm(z)$$

$$= \frac{q\alpha}{2\pi}\int_{\mathbb{C}} |f(z)|^p |f(z)|^{q-p} e^{-\frac{q\alpha}{2}|z|^2} dm(z)$$

Applying the pointwise estimate, we have the right hand side is

$$\leq \|f\|^{q-p}_{(p,\alpha)} \frac{q}{p}\frac{p\alpha}{2\pi}\int_{\mathbb{C}} |f(z)|^p e^{\frac{\alpha(q-p)}{2}|z|^2} e^{-\frac{q\alpha}{2}|z|^2} dm(z)$$

$$= \frac{q}{p}\|f\|_{(p,\alpha)}$$

Thus, $\|f\|_{(q,\alpha)} \leq (\frac{q}{p})^{\frac{1}{q}}\|f\|_{(p,\alpha)}$. Therefore, $\mathcal{F}^p_\alpha \subseteq \mathcal{F}^q_\alpha$. ☐

For more detail and the above note we refer to the book by (Zhu, 2012).

1.2 Statement of the problem

As noted in the background of the study, boundedness and compactness of Volterra-type integral operator on the Fock space was studied by (Constantin, 2012) and (Mengestie, 2013), expressing boundedness in terms of the function g to be complex polynomial of degree atmost two and compactness was expressed interms of in terms of of the function g to be complex polynomial of degree atmost one. For the generalized Volterra-type integral operators it was studied by (Mengestie, 2013) interms of Berezin type integral transforms. Later, (Mengestie and Worku, 2018) simplified the Berezin type characterization to a new simpler function. But, the characterization of boundedness and compactness of $V_g^{n,m}$ acting between Fock spaces is not studied yet, except for the case $n = 1$ and $m = 0$, which is studied in (Constantin, 2012) and (Mengestie, 2013). Therefore, this thesis studies boundedness and compactness of $V_g^{n,m}$ on Fock spaces $\mathcal{F}_\phi^p$.

1.3 Objectives of the study

1.3.1 General objectives

The general objective of this thesis is to study the boundedness and compactness properties of generalized integration operators on Fock spaces.

1.3.2 Specific objectives

The specific objectives of this thesis are;

- Describing boundedness of the generalized integration operators on Fock spaces by giving sufficient and necessary condition for it.
- Describing compactness of generalized integration operators on Fock spaces by giving sufficient and necessary condition for it.
- Finding a condition on which boundedness and compactness are equivalent.

1.4 Significance of the study

The result of this study have the following importance:

- It generalizes study of Volterra-type integral operators into more general operators.

- It can be used as a base for any researcher who is interested to study other properties of generalized integration operators on Fock spaces.
- Help the graduate students to acquire research skills and scientific procedures.

1.5 Delimitation of the study

This study focused only on establishing bounded and compact generalized integration operators acting between Fock spaces.

Chapter 2

Review of Related Literature

Ever since introduced by (Pommerenke, 1977) and after the works of (Aleman and Siskakis, 1995), a number of researchers are motivated to study different properties of the Volterra-type integral operator V_g on different spaces. (Constantin, 2012) studied boundededness, compactness and other properties of V_g on the Fock spaces $\mathcal{F}_\alpha^p$. Then the study was continued by (Mengestie, 2013) on the growth type Fock space $\mathcal{F}_\alpha^\infty$. We will state the two results by the following theorem.

Theorem 2.0.1 (Constantin, 2012 and Mengestie, 2013).
Let $0 < p \leq q \leq \infty$. Then $V_g : \mathcal{F}_\alpha^p \to \mathcal{F}_\alpha^q$ is

(i) bounded if and only if $g(z) = az^2 + bz + c,\ a, b, c \in \mathbb{C}$.

(ii) compact if and only if $g(z) = az + b,\ a, b \in \mathbb{C}$.

For the case when the operator maps from larger space to the smaller, there is a stronger condition in which boundedness and compactness are equivalent.

Theorem 2.0.2 (Constantin, 2012 and Mengestie, 2013).
Let $0 < q < p \leq \infty$. Then the following are equivalent.

a) $V_g : \mathcal{F}_\alpha^p \to \mathcal{F}_\alpha^q$ is bounded,

b) $V_g : \mathcal{F}_\alpha^p \to \mathcal{F}_\alpha^q$ is compact,

c) $q > \begin{cases} \frac{2p}{p+2}, & p < \infty \\ 2, & p = \infty \end{cases}$ *and $g(z) = az + b$ for some $a, b \in \mathbb{C}$.*

(Mengestie, 2014) studied the extended operator, namely the generalized Volterra type integral operators, on Fock spaces $\mathcal{F}_\alpha^p$. Recently, (Mengestie and Worku, 2018) studied also bounded and compact generalized Volterra type integral operator $V_{(g,\psi)}$ with simpler characterization on Fock spaces $\mathcal{F}_\alpha^p$. They obtained the following results.

Theorem 2.0.3 (Mengestie and Worku, 2018)**.**
Let $0 < p \leq q \leq \infty$ and (g, ψ) be pairs of nonconstant entire functions. Then

i) $V_{(g,\psi)} : \mathcal{F}_\alpha^p \to \mathcal{F}_\alpha^q$ *is bounded if and only if* $\frac{|g'(z)|}{1+|z|} e^{\frac{1}{2}(|\psi(z)|^2 - |z|^2)} \in L^\infty(\mathbb{C}, dm)$.

ii) $V_{(g,\psi)} : \mathcal{F}_\alpha^p \to \mathcal{F}_\alpha^q$ *is compact if and only if* $\lim_{|z|\to\infty} \frac{|g'(z)|}{1+|z|} e^{\frac{1}{2}(|\psi(z)|^2 - |z|^2)} = 0$.

Their result is different for the cases $p \leq q$ and $q < p$. For the latter case, we have a stronger condition under which the boundedness implies compactness as stated below.

Theorem 2.0.4 (Mengestie and Worku, 2018)**.**
Let $0 < q < p \leq \infty$ and (g, ψ) be pairs of nonconstant entire functions. Then the following statements are equivalent.

i) $V_{(g,\psi)} : \mathcal{F}_\alpha^p \to \mathcal{F}_\alpha^q$ *is bounded;*

ii) $V_{(g,\psi)} : \mathcal{F}_\alpha^p \to \mathcal{F}_\alpha^q$ *is compact;*

iii) $\frac{|g'(z)|}{1+|z|} e^{\frac{1}{2}(|\psi(z)|^2 - |z|^2)} \in \begin{cases} L^{\frac{pq}{p-q}}(\mathbb{C}, dm), & p < \infty \\ L^q(\mathbb{C}, dm), & p = \infty. \end{cases}$

For the generalized integration operator $V_g^{n,m}$ boundedness and compactness properties have not studied on Fock spaces. So, this thesis is devoted for such a study.

Chapter 3

Methodology of the study

3.1 Study area and Period

The study was conducted in Jimma University department of mathematics under the functional analysis stream from September, 2020 G.C. to June, 2021 G.C. Conceptually, the study focused on generalized Volterra-type integral operators acting between generalized Fock spaces.

3.2 Study design

In this research work we employed analytical method of design.

3.3 Source of information

The relevant sources of information for this study were journals, books, published articles and related studies from Internet.

3.4 Mathematical Procedure of the study

The mathematical procedure that the researcher follows for this research work is the following:

- Providing a sufficient and necessary condition for boundedness and compactness of the generalized integration operators.
- Characterizing boundedness and compactness of Volterra type integral operator.
- Giving conclusion based on the main findings.

Chapter 4

Main Result and Discussion

We begin the section with the following lemma.

Lemma 4.0.1. *Let $0 < p \leq \infty$ and m be a nonnegative integer. Then for each $f \in \mathcal{F}_\alpha^p$, it holds that*

$$|f^{(m)}(z)| \lesssim (1+|z|)^m e^{\frac{\alpha}{2}|z|^2} \|f\|_{(p,\alpha)}.$$

Proof. The pointwise estimate (that is when $m = 0$) was proved in Corollary 2.8 of (Zhu, 2012) and the proof for the case when $m = 1$ was given in Lemma 2.1 of (Tien and Khoi, 2019). For $m > 1$, we have the following. If $|z| \leq 1$, then by Cauchy formula and the pointwise estimate,

$$\begin{aligned}|f^{(m)}(z)| \leq \frac{m!}{2\pi} \int_{|w-z|=1} \frac{|f(w)|}{|w-z|^{m+1}} |dw| &\leq m! \max_{|w-z|=1} |f(w)| \\ &\leq m! \|f\|_{(p,\alpha)} \max_{|w-z|=1} e^{\frac{\alpha}{2}|w|^2} \leq m! e^{2\alpha} \|f\|_{(p,\alpha)}.\end{aligned}$$

If $|z| > 1$, then arguing as above,

$$\begin{aligned}|f^{(m)}(z)| \leq \frac{m!}{2\pi} \int_{|w-z|=\frac{1}{|z|}} \frac{|f(w)|}{|w-z|^{m+1}} |dw| &\leq m! |z|^m \max_{|w-z|=\frac{1}{|z|}} |f(w)| \\ \leq m! |z|^m \|f\|_{(p,\alpha)} \max_{|w-z|=\frac{1}{|z|}} e^{\frac{\alpha}{2}|w|^2} &\leq m! |z|^m e^{\frac{\alpha}{2}(|z|+\frac{1}{|z|})^2} \|f\|_{(p,\alpha)} \\ &\leq m! |z|^m e^{\frac{\alpha}{2}|z|^2} \|f\|_{(p,\alpha)} e^{2\alpha}.\end{aligned}$$

Where the last inequality follows from the fact that

$$e^{\frac{\alpha(|z|+|z|^{-1})^2}{2}} = e^{\frac{\alpha(|z|^2+|z|^{-2}+2)}{2}} \le e^{\frac{\alpha(|z|^2+3)}{2}} \le e^{2\alpha}e^{\frac{\alpha|z|^2}{2}}.$$

Combining the two estimates give the conclusion. $\square$

In the study of integral operators, description of the working space in terms derivative has a crucial role. Such a description (in terms of first derivative) for Fock space was given by (Constantin, 2012) and (Mengestie, 2013), and later extended to m^{th} ($m \ge 1$) order derivative characterization,

$$\|f\|_{p,\alpha} \asymp \begin{cases} \left(|f^{(m-1)}(0)|^p + \int_{\mathbb{C}} |f^{(m)}(z)|^p(1+|z|)^{-mp}e^{-\frac{p\alpha}{2}|z|^2}dm(z)\right)^{\frac{1}{p}}, & 0 < p < \infty \\ |f^{(m-1)}(0)| + \sup_{z\in\mathbb{C}} |f^{(m)}(z)|(1+|z|)^{-m}e^{-\frac{\alpha}{2}|z|^2}, & p = \infty, \end{cases} \tag{4.0.1}$$

by (Hu, 2013) and (Ueki, 2016).

Proposition 4.0.2. *Let $0 < p, q \le \infty$ and $g \in \mathcal{H}(\mathbb{C})$ with $g \not\equiv 0$.*

(I) If $V_g^{n,m} : \mathcal{F}_\alpha^p \to \mathcal{F}_\lambda^q$ is bounded, then the function $\frac{|g^{(n-m)}(z)||z|^m}{(1+|z|)^n}e^{\frac{\alpha}{2}|z|^2-\frac{\lambda}{2}|z|^2}$ is bounded.

(II) If $V_g^{n,m} : \mathcal{F}_\alpha^p \to \mathcal{F}_\lambda^q$ is compact, then $\frac{|g^{(n-m)}(z)||z|^m}{(1+|z|)^n}e^{\frac{\alpha}{2}|z|^2-\frac{\lambda}{2}|z|^2} \to 0$ as $|z| \to \infty$.

Proof. (I) Using the inclusion $\mathcal{F}_\lambda^q \subseteq \mathcal{F}_\lambda^\infty$ for $q \le \infty$, Littlewood-Paley estimate (4.0.1) and then setting in particular $w = z$ gives

$$\begin{aligned} \|k_{w,\alpha}\|_{(p,\alpha)}\|V_g^{n,m}\| \ge \|V_g^{n,m}k_{w,\alpha}\|_{(q,\lambda)} &\gtrsim \sup_{z\in\mathbb{C}} |V_g^{n,m}k_{w,\alpha}(z)|e^{-\frac{\lambda}{2}|z|^2} \\ &\gtrsim \frac{|w\alpha|^m|g^{(n-m)}(z)|}{(1+|z|)^n}|e^{\alpha z\bar{w}-\frac{\alpha|w|^2}{2}}|e^{-\frac{\lambda}{2}|z|^2} \\ &= \frac{|g^{(n-m)}(z)||z|^m}{(1+|z|)^n}e^{\frac{\alpha}{2}|z|^2-\frac{\lambda}{2}|z|^2}. \end{aligned} \tag{4.0.2}$$

From this we have that, $\frac{|g^{(n-m)}(z)||z|^m}{(1+|z|)^n}e^{\frac{\alpha}{2}|z|^2-\frac{\lambda}{2}|z|^2}$ is bounded whenever the operator is bounded.

(II) This part follows from (4.0.2) and the fact that $k_{w,\alpha} \to 0$ uniformly on a compact subsets of $\mathbb{C}$.

$\square$

If $V_g^{n,m} : \mathcal{F}_\alpha^p \to \mathcal{F}_\lambda^q$, $0 < p, q \le \infty$, is bounded, then an application of $V_g^{n,m}$ to the function $f(z) = z^m \in \mathcal{F}_\alpha^p$ shows that $g^{(n-m)} \in \mathcal{F}_{q,\lambda}$.
Clearly, if $g^{(n-m)}$ is identically zero, then the operator is compact. Thus, in our results below we assume $g^{(n-m)}$ to be not identically zero function in $\mathcal{F}_\lambda^q$. We now state our main result.

Theorem 4.0.3. *Let $0 < p, q \le \infty$, m, n nonnegative integers with $0 \le m < n$ and $V_g^{n,m}$ maps from $\mathcal{F}_\alpha^p$ into $\mathcal{F}_\lambda^q$.*

(I) If $p \le q$, then $V_g^{n,m}$ is bounded (respectively, compact) if and only if the function $\frac{|g^{(n-m)}(z)||z|^m}{(1+|z|)^n} e^{\frac{(\alpha-\lambda)}{2}|z|^2}$ is bounded (respectively, $\lim_{|z|\to\infty} \frac{|g^{(n-m)}(z)||z|^m}{(1+|z|)^n} e^{\frac{(\alpha-\lambda)}{2}|z|^2} = 0$).

(II) If $q < p$, then $V_g^{n,m}$ is bounded or compact if and only if

$$\begin{cases} \int_{\mathbb{C}} \left(\frac{|g^{(n-m)}(z)||z|^m}{(1+|z|)^n} e^{\frac{(\alpha-\lambda)}{2}|z|^2}\right)^{\frac{pq}{p-q}} dm(z) < \infty, & \text{for } p < \infty \\ \int_{\mathbb{C}} \left(\frac{|g^{(n-m)}(z)||z|^m}{(1+|z|)^n} e^{\frac{(\alpha-\lambda)}{2}|z|^2}\right)^{q} dm(z) < \infty, & \text{for } p = \infty. \end{cases}$$

Proof. (I) The forward implication have been shown in Proposition 4.0.2.
Suppose $\frac{|g^{(n-m)}(z)||z|^m}{(1+|z|)^n}|z|^m e^{\frac{\alpha}{2}|z|^2 - \frac{\lambda}{2}|z|^2}$ is bounded. Then for $q < \infty$, the use of the estimate in (4.0.1) and inclusion property (Theorem 2.10 of (Zhu, 2012)) of the space gives,

$$\begin{aligned} \|V_g^{n,m} f\|_{(q,\lambda)}^q &\asymp \int_{\mathbb{C}} (\frac{|g^{(n-m)}(z)|}{(1+|z|)^n})^q |f^{(m)}(z)|^q e^{-\frac{q\lambda}{2}|z|^2} dm(z) \\ &\le \sup_{z\in\mathbb{C}}((\frac{|g^{(n-m)}(z)|}{(1+|z|)^n})^q (1+|z|)^{mq} e^{(\frac{q\alpha}{2}|z|^2 - \frac{q\lambda}{2}|z|^2)}) \int_{\mathbb{C}} \frac{|f^{(m)}(z)|^q}{(1+|z|)^{mq}} e^{-\frac{q\alpha}{2}|z|^2} dm(z) \\ &\lesssim \sup_{z\in\mathbb{C}}((\frac{|g^{(n-m)}(z)||z|^m}{(1+|z|)^n})^q e^{(\frac{q\alpha}{2}|z|^2 - \frac{q\lambda}{2}|z|^2)}) \int_{\mathbb{C}} \frac{|f^{(m)}(z)|^q}{(1+|z|)^{mq}} e^{-\frac{q\alpha}{2}|z|^2} dm(z) \\ &\asymp \|f\|_{(q,\alpha)}^q \Big(\sup_{z\in\mathbb{C}} (\frac{|g^{(n-m)}(z)||z|^m}{(1+|z|)^n})^q e^{\frac{q\alpha}{2}|z|^2 - \frac{q\lambda}{2}|z|^2}\Big) \\ &\lesssim \|f\|_{(p,\alpha)}^q \Big(\sup_{z\in\mathbb{C}} (\frac{|g^{(n-m)}(z)||z|^m}{(1+|z|)^n})^q e^{\frac{q\alpha}{2}|z|^2 - \frac{q\lambda}{2}|z|^2}\Big). \end{aligned} \tag{4.0.3}$$

Similarly, for $q = \infty$,

$$\begin{aligned}
\|V_g^{n,m} f\|_{\infty,\lambda} &\asymp \sup_{z\in\mathbb{C}} \Big(\frac{|g^{(n-m)}(z)|}{(1+|z|)^n}\Big)|f^{(m)}(z)|e^{-\frac{\lambda|z|^2}{2}} \\
&\leq \Big(\sup_{z\in\mathbb{C}} \Big(\frac{|g^{(n-m)}(z)|}{(1+|z|)^n}\Big)(1+|z|)^m e^{\frac{\alpha}{2}|z|^2-\frac{\lambda|z|^2}{2}}\Big)\Big(\sup_{z\in\mathbb{C}} \frac{|f^{(m)}(z)|}{(1+|z|)^m} e^{-\frac{\alpha|z|^2}{2}}\Big) \\
&\lesssim \Big(\sup_{z\in\mathbb{C}} \Big(\frac{|g^{(n-m)}(z)||z|^m}{(1+|z|)^n}\Big)e^{\frac{\alpha}{2}|z|^2-\frac{\lambda|z|^2}{2}}\Big)\Big(\sup_{z\in\mathbb{C}} \frac{|f^{m}(z)|}{(1+|z|)^m} e^{-\frac{\alpha|z|^2}{2}}\Big) \\
&\asymp \|f\|_{(\infty,\alpha)}\Big(\sup_{z\in\mathbb{C}} \Big(\frac{|g^{(n-m)}(z)||z|^m}{(1+|z|)^n}\Big)e^{\frac{\alpha}{2}|z|^2-\frac{\lambda|z|^2}{2}}\Big) \\
&\lesssim \|f\|_{(p,\alpha)}\Big(\sup_{z\in\mathbb{C}} \Big(\frac{|g^{(n-m)}(z)||z|^m}{(1+|z|)^n}\Big)e^{\frac{\alpha}{2}|z|^2-\frac{\lambda|z|^2}{2}}\Big).
\end{aligned} \tag{4.0.4}$$

Hence, from (4.0.3) and (4.0.4), $V_g^{n,m}$ is bounded.

(II) Necessity of $\lim_{|z|\to\infty} \big(\frac{|g^{(n-m)}(z)||z|^m}{(1+|z|)^n}\big)e^{\frac{\alpha}{2}|z|^2-\frac{\lambda}{2}|z|^2} = 0$ for the compactness of the operator was shown in Proposition 4.0.2. For the sufficiency, we let f_l to be arbitrary bounded sequence in $\mathcal{F}_\alpha^p$, that converges to 0 uniformly on a compact subsets of $\mathbb{C}$ as $l \to \infty$. Then for $\mathcal{R} > 0$ and $q < \infty$, using (4.0.1), Proposition 4.0.2 and Theorem 2.10 of (Zhu, 2012),

$$\begin{aligned}
\|V_g^{n,m} f_l\|_{(q,\lambda)}^q &\asymp \int_{\mathbb{C}} \Big(\frac{|g^{(n-m)}(z)|}{(1+|z|)^n}\Big)^q |f_l^{(m)}(z)|^q e^{-\frac{q\lambda}{2}|z|^2} dm(z) \\
&= \Big(\int_{|z|\leq\mathcal{R}} + \int_{|z|>\mathcal{R}}\Big)\Big(\frac{|g^{(n-m)}(z)|}{(1+|z|)^n}\Big)^q |f_l^{(m)}(z)|^q e^{-\frac{q\lambda}{2}|z|^2} dm(z) \\
&\lesssim \max_{|z|\leq\mathcal{R}} |f_l^{(m)}(z)|^q \int_{|z|\leq\mathcal{R}} \Big(\frac{|g^{(n-m)}(z)|}{(1+|z|)^n}\Big)^q e^{-\frac{q\lambda}{2}|z|^2} dm(z) \\
&\quad + \sup_{|z|>R} \Big(\frac{|g^{(n-m)}(z)|}{(1+|z|)^n}\Big)^q |z|^{mq} e^{\frac{q\alpha}{2}|z|^2-\frac{q\lambda}{2}|z|^2} \int_{|z|>\mathcal{R}} \frac{|f_l^{(m)}(z)|^q}{(1+|z|)^{mq}} e^{-\frac{q\alpha}{2}|z|^2} dA(z) \\
&\lesssim \|g^{(n-m)}\|_{(q,\lambda)}^q \max_{|z|\leq\mathcal{R}} |f_l^{(m)}(z)|^q \\
&\quad + \|f_l\|_{q,\alpha}^q \sup_{|z|>\mathcal{R}} \Big(\frac{|g^{(n-m)}(z)||z|^m}{(1+|z|)^n}\Big)^q e^{\frac{q\alpha}{2}|z|^2-\frac{q\lambda}{2}|z|^2} \\
&\lesssim \max_{|z|\leq\mathcal{R}} |f_l^{(m)}(z)|^q + \sup_{|z|>\mathcal{R}} \Big(\frac{|g^{(n-m)}(z)||z|^m}{(1+|z|)^n}\Big)^q e^{\frac{q\alpha}{2}|z|^2-\frac{q\lambda}{2}|z|^2}.
\end{aligned} \tag{4.0.5}$$

Similarly, for $\mathcal{R} > 0$ and $q = \infty$, using Lemma 4.0.1,

$$\begin{aligned}
\|V_g^{n,m} f_l\|_{\infty,\lambda} &\asymp \sup_{z\in\mathbb{C}} \big(\frac{|g^{(n-m)}(z)|}{(1+|z|)^n}\big)|f_l^{(m)}(z)|e^{-\frac{\lambda}{2}|z|^2} \\
&\le \Big(\sup_{|z|\le\mathcal{R}} + \sup_{|z|>\mathcal{R}}\Big)\big(\frac{|g^{(n-m)}(z)|}{(1+|z|)^n}\big)|f_l^{(m)}(z)|e^{-\frac{\lambda}{2}|z|^2} \\
&\lesssim \|g^{(n-m)}\|_{(\infty,\lambda)} \sup_{|z|\le\mathcal{R}} |f_l^{(m)}(z)| + \|f_l\|_{(p,\alpha)} \sup_{|z|>\mathcal{R}} \big(\frac{|g^{(n-m)}(z)||z|^m}{(1+|z|)^n}\big)e^{\frac{\alpha}{2}|z|^2-\frac{\lambda}{2}|z|^2} \\
&\lesssim \sup_{|z|\le\mathcal{R}} |f_l^{(m)}(z)| + \sup_{|z|>\mathcal{R}} \big(\frac{|g^{(n-m)}(z)||z|^m}{(1+|z|)^n}\big)e^{\frac{\alpha}{2}|z|^2-\frac{\lambda}{2}|z|^2}.
\end{aligned} \tag{4.0.6}$$

Letting $l \to \infty$ and then $\mathcal{R} \to \infty$ in (4.0.5) and (4.0.6) gives $\|V_g^{n,m} f_l\|_{(q,\lambda)} \to 0$.and hence $V_g^{n,m}$ is compact.

(III) We will first show that the integral conditions imply compactness of the operator and then boundedness of the operator imply the integral conditions. Let (f_l) be a uniformly bounded sequence in $\mathcal{F}_{(p,\alpha)}$ and $f_l \to 0$ uniformly on compact subsets of $\mathbb{C}$ as $l \to \infty$. Then for $\mathcal{R} > 0$

$$\begin{aligned}
\|V_g^{n,m} f_l\|_{(q,\lambda)}^q &\asymp \int_{\mathbb{C}} \big(\frac{|g^{(n-m)}(z)|}{(1+|z|)^n}\big)^q |f_l^{(m)}(z)|^q e^{-\frac{q\lambda}{2}|z|^2} dm(z) \\
&= \Big(\int_{|z|\le\mathcal{R}} + \int_{|z|>\mathcal{R}}\Big)\big(\frac{|g^{(n-m)}(z)|}{(1+|z|)^n}\big)^q |f_l^{(m)}(z)|^q e^{-\frac{q\lambda}{2}|z|^2} dm(z) \\
&\lesssim \max_{|z|\le\mathcal{R}} |f_l^{(m)}(z)|^q \int_{|z|\le\mathcal{R}} \big(\frac{|g^{(n-m)}(z)|}{(1+|z|)^n}\big)^q e^{-\frac{q\lambda}{2}|z|^2} dm(z) \\
&\quad + \int_{|z|>\mathcal{R}} \frac{|f_l^{(m)}(z)|^q}{(1+|z|)^{mq}} e^{-\frac{q\alpha}{2}|z|^2} \Big(\big(\frac{|g^{(n-m)}(z)||z|^m}{(1+|z|)^n}\big)^q e^{\frac{q\alpha}{2}|z|^2-\frac{q\lambda}{2}|z|^2}\Big) dm(z) \\
&\lesssim \|g^{(n-m)}\|_{(q,\lambda)}^q \max_{|z|\le\mathcal{R}} |f_l^{(m)}(z)|^q \\
&\quad + \int_{|z|>\mathcal{R}} \frac{|f_l^{(m)}(z)|^q}{(1+|z|)^{mq}} e^{-\frac{q\alpha}{2}|z|^2} \Big(\big(\frac{|g^{(n-m)}(z)||z|^m}{(1+|z|)^n}\big)^q e^{\frac{q\alpha}{2}|z|^2-\frac{q\lambda}{2}|z|^2}\Big) dm(z).
\end{aligned} \tag{4.0.7}$$

For $p=\infty$, applying (4.0.1) with Lemma 4.0.1 to the above estimate gives,

$$\begin{aligned}
\|V_g^{n,m}f_l\|_{q,\lambda}^q &\lesssim \|g^{(n-m)}\|_{(q,\lambda)}^q \max_{|z|\le\mathcal{R}} |f_l^{(m)}(z)|^q \\
&\quad + \sup_{|z|>\mathcal{R}} \frac{|f_l^{(m)}(z)|^q}{(1+|z|)^{mq}} e^{-\frac{q\alpha}{2}|z|^2} \int_{|z|>\mathcal{R}} \Big(\frac{|g^{(n-m)}(z)||z|^m}{(1+|z|)^n}\Big)^q e^{\frac{q\alpha}{2}|z|^2-\frac{q\lambda}{2}|z|^2} dm(z) \\
&\lesssim \|g^{(n-m)}\|_{(q,\lambda)}^q \max_{|z|\le\mathcal{R}} |f_l^{(m)}(z)|^q + \|f\|_{(\infty,\alpha)}^q \int_{|z|>\mathcal{R}} \Big(\frac{|g^{(n-m)}(z)||z|^m}{(1+|z|)^n}\Big)^q e^{\frac{q\alpha}{2}|z|^2-\frac{q\lambda}{2}|z|^2} dm(z) \\
&\lesssim \max_{|z|\le\mathcal{R}} |f_l^{(m)}(z)|^q + \int_{|z|>\mathcal{R}} \Big(\frac{|g^{(n-m)}(z)||z|^m}{(1+|z|)^n}\Big)^q e^{\frac{q\alpha}{2}|z|^2-\frac{q\lambda}{2}|z|^2} dm(z).
\end{aligned} \tag{4.0.8}$$

For $p<\infty$, applying Hölder's inequality and then using (4.0.1), the integral in (4.0.7) is estimated as,

$$\begin{aligned}
&\int_{|z|>\mathcal{R}} \frac{|f_l^{(m)}(z)|^q}{(1+|z|)^{mq}} e^{-\frac{q\alpha}{2}|z|^2} \bigg(\Big(\frac{|g^{(n-m)}(z)||z|^m}{(1+|z|)^n}\Big)^q e^{\frac{q\alpha}{2}|z|^2-\frac{q\lambda}{2}|z|^2}\bigg) dm(z) \\
&\quad\le \Big(\int_{|z|>\mathcal{R}} \frac{|f_l^{(m)}(z)|^p}{(1+|z|)^{mp}} e^{-\frac{p\alpha}{2}|z|^2} dm(z)\Big)^{\frac{q}{p}} \Big(\int_{|z|>\mathcal{R}} \Big(\Big(\frac{|g^{(n-m)}(z)||z|^m}{(1+|z|)^n}\Big) e^{\frac{\alpha}{2}|z|^2-\frac{\lambda}{2}|z|^2}\Big)^{\frac{pq}{p-q}} dm(z)\Big)^{\frac{p-q}{p}} \\
&\quad\lesssim \|f_l\|_{(p,\alpha)}^q \Big(\int_{|z|>\mathcal{R}} \Big(\Big(\frac{|g^{(n-m)}(z)||z|^m}{(1+|z|)^n}\Big) e^{\frac{\alpha}{2}|z|^2-\frac{\lambda}{2}|z|^2}\Big)^{\frac{pq}{p-q}} dm(z)\Big)^{\frac{p-q}{p}} \\
&\quad\quad\lesssim \Big(\int_{|z|>\mathcal{R}} \Big(\Big(\frac{|g^{(n-m)}(z)||z|^m}{(1+|z|)^n}\Big) e^{\frac{\alpha}{2}|z|^2-\frac{\lambda}{2}|z|^2}\Big)^{\frac{pq}{p-q}} dm(z)\Big)^{\frac{p-q}{p}}
\end{aligned}$$

and hence

$$\|V_g^{n,m}f_l\|_{(q,\lambda)}^q \lesssim \max_{|z|\le\mathcal{R}} |f_l^{(m)}(z)|^q + \Big(\int_{|z|>\mathcal{R}} \Big(\Big(\frac{|g^{(n-m)}(z)||z|^m}{(1+|z|)^n}\Big) e^{\frac{\alpha}{2}|z|^2-\frac{\lambda}{2}|z|^2}\Big)^{\frac{pq}{p-q}} dm(z)\Big)^{\frac{p-q}{p}}. \tag{4.0.9}$$

Letting $l\to\infty$ and then $\mathcal{R}\to\infty$ in (4.0.8) and (4.0.9), $\|V_g^{n,m}f_l\|_{(q,\lambda)}\to 0$ as $l\to\infty$, which shows that $V_g^{n,m}$ is compact.

Now, we assume $V_g^{n,m}$ is bounded and proceed to show the integral conditions hold. For this, we let a sequence (z_j) to be an $\frac{r}{2}$-lattice for $\mathbb{C}$ (see (Hu and Lv, 2011) and (Mengestie, 2016)) and use a technique initiated by (Luecking, 1993). From the atomic decomposition of functions in $\mathcal{F}_\alpha^p$ (Theorem 2.34 of (Zhu, 2012)) each function $f\in\mathcal{F}_\alpha^p$, $0<p\le\infty$, is generated

by an ℓ^p sequence as

$$f = \sum_{j=1}^{\infty} c_j k_{z_j,\alpha} \quad \text{and} \quad \|f\|_{(p,\alpha)} \asymp \|(c_j)\|_{\ell^p}.$$

If $(r_j(t))$ is the Rademacher sequence of functions in $[0,1]$ (as in (Luecking, 1993)), then the sequence $(c_j r_j(t))$ is in ℓ^p with $\|(c_j r_j(t))\|_{\ell^p} = \|(c_j)\|_{\ell^p}$ for all t and $\sum_{j=1}^{\infty} c_j r_j(t) k_{z_j,\alpha} \in \mathcal{F}_\alpha^p$ with

$$\|\sum_{j=1}^{\infty} c_j r_j(t) k_{z_j,\alpha}\|_{(p,\alpha)} \asymp \|(c_j)\|_{\ell^p}. \tag{4.0.10}$$

From Khinchine's inequality (Luecking, 1993) we have

$$\Big(\sum_{j=1}^{\infty} |c_j z_j^m|^2 |k_{z_j,\alpha}(z)|^2\Big)^{\frac{q}{2}} \lesssim \int_0^1 |\sum_{j=1}^{\infty} c_j z_j^m r_j(t) k_{z_j,\alpha}(z)|^q dt. \tag{4.0.11}$$

Setting $d\xi_g(z) = \big(\frac{|g^{(n-m)}(z)|}{(1+|z|)n}\big)^q e^{-\frac{\lambda q}{2}|z|^2} dm(z) \circ \Phi^{-1}(z)$ and using (4.0.11) with Fubini's theorem,

$$\begin{aligned}
\int_{\mathbb{C}} \Big(\sum_{j=1}^{\infty} |c_j z_j^m|^2 |k_{z_j,\alpha}(z)|^2\Big)^{\frac{q}{2}} d\xi_g(z) &\lesssim \int_{\mathbb{C}} \int_0^1 |\sum_{j=1}^{\infty} c_j z_j^m r_j(t) k_{z_j,\alpha}(z)|^q dt d\xi_g(z) \\
&= \int_0^1 \int_{\mathbb{C}} |\sum_{j=1}^{\infty} c_j z_j^m r_j(t) k_{z_j,\alpha}(z)|^q d\xi_g(z) dt \\
&\asymp \int_0^1 \|V_g^{n,m} \sum_{j=1}^{\infty} c_j r_j(t) k_{z_j,\alpha}\|_{(q,\lambda)}^q dt \\
&\lesssim \|V_g^{n,m}\|^q \|(c_j)\|_{\ell^p}^q.
\end{aligned} \tag{4.0.12}$$

The last estimate above is from boundedness of $V_g^{n,m}$ and equation (4.0.10). Then, using the estimate $|z_j| \asymp |z|$ for each $z \in D(z_j, 2r)$ and

$$|k_{z_j,\alpha}(z)|^q = e^{\frac{\alpha q}{2}(|z|^2 - |z-z_j|^2)} \geq e^{-\frac{\alpha q r}{2} + \frac{\alpha q}{2}|z|^2},$$

$$\sum_{j=1}^{\infty}|c_j|^q\int_{D(z_j,2r)}|z|^{mq}e^{\frac{\alpha q}{2}|z|^2}d\xi_g(z)\lesssim\sum_{j=1}^{\infty}|c_j|^q|z_j|^{mq}\int_{D(z_j,2r)}|k_{z_j,\alpha}(z)|^q d\xi_g(z)$$
$$=\int_{\mathbb{C}}\sum_{j=1}^{\infty}|c_j|^q|z_j|^{mq}|k_{z_j,\alpha}(z)|^q\chi_{D(z_j,2r)}(z)d\xi_g(z)$$
$$\lesssim\max\{1,N^{1-\frac{q}{2}}\}\int_{\mathbb{C}}\Big(\sum_{j=1}^{\infty}|c_j|^2|z_j|^{2m}|k_{z_j,\alpha}(z)|^2\Big)^{\frac{q}{2}}d\xi_g(z).$$

From the estimate in (4.0.12) and the above estimate, we get

$$\sum_{j=1}^{\infty}|c_j|^q\int_{D(z_j,2r)}|z|^{mq}e^{\frac{\alpha q}{2}|z|^2}d\xi_g(z)\lesssim\|V_g^{n,m}\|^q\|(c_j)\|_{l^p}^q. \tag{4.0.13}$$

If $p=\infty$, then putting $c_j=1$ for all j in (4.0.13)and substituting back $d\xi_g$,

$$\infty>\sum_{j=1}^{\infty}\int_{D(z_j,2r)}|z|^{mq}e^{\frac{\alpha q}{2}|z|^2}d\xi_g(z)\gtrsim\sum_{j=1}^{\infty}\int_{D(z_j,3\frac{r}{2})}|z|^{mq}e^{\frac{\alpha q}{2}|z|^2}d\xi_g(z)$$
$$\geq\int_{\mathbb{C}}\Big(\frac{|g^{(n-m)}(z)|}{(1+|z|)^n}|z|^m e^{\frac{\alpha}{2}|z|^2-\frac{\lambda}{2}|z|^2}\Big)^q dm(z).$$

From which and (4.0.13) the conclusion follows. Moreover, we have the following norm estimate, which gives the lower estimate of the norm for $p=\infty$,

$$\|V_g^{n,m}\|\gtrsim\Big(\int_{\mathbb{C}}\Big(\frac{|g^{(n-m)}(z)|}{(1+|z|)^n}|z|^m e^{\frac{\alpha}{2}|z|^2-\frac{\lambda}{2}|z|^2}\Big)^q dm(z)\Big)^{\frac{1}{q}}.$$

If $p<\infty$, then since $(|c_j|^q)\in l^{\frac{p}{q}}$ from (4.0.13) and duality argument between $l^{\frac{p}{q}}$ and $l^{\frac{p}{p-q}}$,

$$\Big(\sum_{j=1}^{\infty}\Big(\int_{D(z_j,2r)}|z|^{mq}e^{\frac{\alpha q}{2}|z|^2}d\xi_g(z)\Big)^{\frac{p}{p-q}}\Big)^{\frac{p-q}{p}}<\infty.$$

Thus, substituting $d\xi_g$, it follows that

$$\begin{aligned}\infty &> \Big(\sum_{j=1}^{\infty}\Big(\int_{D(z_j,2r)}|z|^{mq}e^{\frac{\alpha q}{2}|z|^2}d\xi_g(z)\Big)^{\frac{p}{p-q}}\Big)^{\frac{p-q}{p}}\\ &\gtrsim \Big(\sum_{j=1}^{\infty}\int_{D(z_j,3\frac{r}{2})}\big(\frac{|g^{(n-m)}(z)|}{(1+|z|)^n}|z|^m e^{\frac{\alpha}{2}|z|^2-\frac{\lambda}{2}|z|^2}\big)^{\frac{pq}{p-q}}dm(z)\Big)^{\frac{p-q}{p}}\\ &\geq \Big(\int_{\mathbb{C}}\big(\frac{|g^{(n-m)}(z)|}{(1+|z|)^n}|z|^m e^{\frac{\alpha}{2}|z|^2-\frac{\lambda}{2}|z|^2}\big)^{\frac{pq}{p-q}}dm(z)\Big)^{\frac{p-q}{p}}.\end{aligned}$$

From which and (4.0.13), we get the other estimate

$$\|V_g^{n,m}\| \gtrsim \Big(\int_{\mathbb{C}}\big(\frac{|g^{(n-m)}(z)|}{(1+|z|)^n}|z|^m e^{\frac{\alpha}{2}|z|^2-\frac{\lambda}{2}|z|^2}\big)^{\frac{pq}{p-q}}dm(z)\Big)^{\frac{p-q}{pq}}.$$

Next, we prove the other direction of norm estimate. First, for $p=\infty$,

$$\begin{aligned}\|V_g^{n,m}f\|_{(q,\lambda)}^q &\asymp \int_{\mathbb{C}}\big(\frac{|g^{(n-m)}(z)|}{(1+|z|)^n}\big)^q|f^{(m)}(z)|^q e^{-\frac{q\lambda}{2}|z|^2}dm(z)\\ &\lesssim \int_{\mathbb{C}}\frac{|f^{(m)}(z)|^q}{(1+|z|)^{mq}}e^{-\frac{q\alpha}{2}|z|^2}\big(\frac{|g^{(n-m)}(z)|}{(1+|z|)^n}|z|^m e^{\frac{\alpha}{2}|z|^2-\frac{\lambda}{2}|z|^2}\big)^q dm(z)\\ &\leq \|f\|_{\infty,\alpha}^q\int_{\mathbb{C}}\big(\frac{|g^{(n-m)}(z)|}{(1+|z|)^n}|z|^m e^{\frac{\alpha}{2}|z|^2-\frac{\lambda}{2}|z|^2}\big)^q dm(z),\end{aligned}$$

which shows that,

$$\|V_g^{n,m}\| \lesssim \Big(\int_{\mathbb{C}}\big(\frac{|g^{(n-m)}(z)|}{(1+|z|)^n}|z|^m e^{\frac{\alpha}{2}|z|^2-\frac{\lambda}{2}|z|^2}\big)^q dm(z)\Big)^{\frac{1}{q}}.$$

Similarly, for the case $p < \infty$, using Hölder's inequality and the estimate in (4.0.1),

$$\begin{aligned}
\|V_g^{n,m} f\|_{(q,\lambda)}^q &\asymp \int_{\mathbb{C}} \Big(\frac{|g^{(n-m)}(z)|}{(1+|z|)^n}\Big)^q |f^{(m)}(z)|^q e^{-\frac{q\lambda}{2}|z|^2} dm(z) \\
&\lesssim \int_{\mathbb{C}} \frac{|f^{(m)}(z)|^q}{(1+|z|)^{mq}} e^{-\frac{q\alpha}{2}|z|^2} \Big(\frac{|g^{(n-m)}(z)|}{(1+|z|)^n} |z|^m e^{\frac{\alpha}{2}|z|^2 - \frac{\lambda}{2}|z|^2}\Big)^q dm(z) \\
&\leq \Big(\int_{\mathbb{C}} \frac{|f^{(m)}(z)|^p}{(1+|z|)^{mp}} e^{-\frac{p\alpha}{2}|z|^2} dm(z)\Big)^{\frac{q}{p}} \\
&\qquad \times \Big(\int_{\mathbb{C}} \Big(\frac{|g^{(n-m)}(z)|}{(1+|z|)^n} |z|^m e^{\frac{\alpha}{2}|z|^2 - \frac{\lambda}{2}|z|^2}\Big)^{\frac{pq}{p-q}} dm(z)\Big)^{\frac{p-q}{p}} \\
&\asymp \|f\|_{(p,\alpha)}^q \Big(\int_{\mathbb{C}} \Big(\frac{|g^{(n-m)}(z)|}{(1+|z|)^n} |z|^m e^{\frac{\alpha}{2}|z|^2 - \frac{\lambda}{2}|z|^2}\Big)^{\frac{pq}{p-q}} dm(z)\Big)^{\frac{p-q}{p}}.
\end{aligned}$$

Thus,

$$\|V_g^{n,m}\| \lesssim \Big(\int_{\mathbb{C}} \Big(\frac{|g^{(n-m)}(z)|}{(1+|z|)^n} |z|^m e^{\frac{\alpha}{2}|z|^2 - \frac{\lambda}{2}|z|^2}\Big)^{\frac{pq}{p-q}} dm(z)\Big)^{\frac{p-q}{pq}}.$$

□

Chapter 5

Conclusion

This thesis includes a number of results, which characterize generalized integration operators acting between Fock spaces. Our results in chapter 4, which is about boundedness and compactness are new and may be applied to study other properties defined whenever the operator is bounded. In addition, our results generalizes some of the results that have been obtained for the Volterra-type integral operators. In particular, it generalizes the results of (Constantin, 2012) and (Mengestie, 2013) from Volterra-type integral to the generalized integration operators, which is stated in Theorem 2.0.1 and 2.0.2.

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