COMMON FIXED POINT RESULTS FOR RATIONAL TYPE GERAGHTY CONTRACTION MAPPINGS IN b-METRIC SPACES

A THESIS SUBMITTED TO THE DEPARTMENT OF MATHEMATICS IN PARTIAL FULFILLMENT FOR THE REQUIREMENTS OF THE DEGREE OF MASTERS OF SCIENCE IN MATHEMATICS

By: Desalegn Oljira

Advisor: Kidane Koyas (PhD)

Co-advisor: Bontu Nasir (M.Sc.)

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Jimma, Ethiopia

Declaration

I, the undersigned declare that, the thesis entitled "Common Fixed Point Results for Rational type Geraghty Contraction Mappings in b-Metric Spaces**"** is original and it has not been submitted to any institution elsewhere for the award of any academic degree or like, where other sources of information that have been used, were acknowledged .

Name: Desalegn Oljira

Signature: ________________

Date: ____________________

Advisor**:-**Kidane Koyas (PhD)

Signature: _________________

Date: _____________________

Co-advisor:- Bontu Nasir (M.Sc.)

Signature: __________________

Date: ______________________

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Abstract

In this thesis work, we established common fixed point results for Suzuki-rational Geraghty contractive mappings in b-metric spaces and proved the existence and uniqueness of commons fixed point for a pair of self-mappings satisfying the established results. Our results extend fixed point to common fixed point for a pair of mappings satisfying Suzuki-rational Geraghty contraction condition in the setting of complete b- metric spaces. In this study we have followed analytical method of design and used secondary sources of data such as published articles and related books. Finally, we provided examples in support of our main findings.

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CHAPTR ONE

INTRODUCTION

1.1 Background of the study

Fixed point theory is one of the corner stone in development of mathematics and it plays a basic role in application of many branches of mathematics and also it is useful in different fields of studies. The famous Banach contraction principle is one of the powerful tools in metric fixed point theory. Banach contraction principle appeared in explicit form on Banach’s thesis in 1922.

Another category of contraction which is separate from Banach's contraction and does not imply continuity was proposed by (Kannan, 1968). He proved that such type of mappings have necessarily unique fixed points in complete metric spaces. Mappings belonging to this category are known as Kannan type Mappings.

In 1972, a new concept which is different from that of (Banach, 1922) and (Kannan, 1968) for contraction type mapping was introduced by (Chatterjea, 1972) which gives a new direction to the study of fixed point theory. There are classes of contractive mappings which are different from Banach's contraction and have unique fixed point in complete metric spaces. The family of contractive mappings in metric spaces is a great interest and has already been studied in the literature since long time. Banach contraction principle has been extended and generalized by many researchers using different forms of contractive conditions in various spaces.

For more details we refer (Czerwik, 1993, Branciari, 2000, Aydi, 2012, , 2014, Lin *et al*, 2014, George *et al*, 2015, Alharbi et *al*, 2018).

The family of contraction mappings in metric spaces is a great interest and has already been studied in the literature since long time. In a very recent time (Rahrovi and Piri, 2019) investigated the existence of fixed point result for a map satisfying Suzuki-rational Geraghty contraction in the setup of complete b- metric space.

Inspired and motivated by the work of (Rahrovi and Piri, 2019) the main purpose of this study is to establish common fixed point results and prove the existence and uniqueness of common fixed points for a pair of maps satisfying Suzuki- rational Geraghty Contraction in the setting of complete b- metric spaces.

Notation: Throughout this research work we denote:

$\mathbb{R}^+ = [0, \infty)$**:** The set of non-negative real numbers.

$\mathbb{N}$**:** The set of natural numbers.

1.2 Statement of the problem

In 1982 Sessa, introduced a generalization of commutativity of a pair of maps and proved some common fixed point theorems for weakly commuting pair of maps. Then after, (Jungck, 1986) initially proved a common fixed point result for a pair commuting mappings, which generalizes the well-known Banach fixed point theorem. Next, (Abbas and Jungck, 2008) introduced the notion of weakly compatibility of pair of maps and proved common fixed point results. Recently, (Rahrovi and Piri, 2019) investigated the existence of fixed point results for self-maps satisfying Suzuki-rational Geraghty contraction condition in the setup of complete b-metric spaces. However, common fixed point results for a pair of maps were established and generalized in b-metric spaces using different contraction condition, yet now; common fixed point result for a pair of maps satisfying Suzuki- rational Geraghty contraction condition in b -metric space is not conducted. Thus, in this research work we investigated the existence and uniqueness of common fixed point results for a pair of maps satisfying Suzuki- rational Geraghty contraction condition in the context of complete b-metric spaces.

1.3 Objectives of the study

1.3.1 General objective

The general objective of this research work was to study common fixed point results for a pair of maps satisfying Suzuki-rational Geraghty contraction condition in the setting of complete *b*-metric spaces.

1.3.2 Specific objective

This study has the following specific objectives.

- To establish common fixed point theorems for a pair of maps satisfying Suzuki-rational Geraghty contraction condition in the setting of complete b-metric spaces.
- To prove the existence of common fixed point for a pair of maps satisfying Suzuki-rational Geraghty contraction condition in the setup of complete b-metric spaces.
- To verify the uniqueness of common fixed point for a pair of maps satisfying Suzuki-rational Geraghty contraction condition in complete b-metric spaces.
- To provide examples in support of the results obtained.

1.4 Significance of the study

The study may have the following significance.

- It may give basic research skill for the researcher.
- It may be used as a reference for any researcher who has an interest to conduct a research in this line of research.
- May be applied in solving the existence of solutions for some integral and differential equations.

1.5 Delimitation of the Study

This study focused on establishing common fixed point results for a pair of self-maps satisfying Suzuki-rational Geraghty contraction condition in the setting of complete b- metric spaces.

CHAPTER TWO

LITERATURE REVIEW

Fixed point theory is an important tool in the study of non-linear analysis as it is considered to be the key connection between pure and applied mathematics with wide applications in all branches of mathematics, Economics, Biology, Chemistry, Physics and almost all engineering fields. The famous Banach contraction principle which appeared in explicit form on Banach's thesis in 1922 is one of the powerful tools in fixed point theory. Banach contraction principle has been extended and generalized by many researchers using different types of contractive conditions in various spaces.

Let (X, d) be a metric space, a self-map T: $X \to X$ is said to be contraction mapping if there exists a constant $k \in [0, 1)$ such that $d(Tx, Ty) \leq kd(x, y)$ for all $x, y \in X$. Banach Contraction Principle stated that 'Any contraction mapping on complete Metric space has a unique fixed point'. The purpose of Banach contraction is not only to proof that the mapping satisfying the inequality described above has a unique fixed point, but also to show that the Picard iteration converges to the fixed point. There are many results which extend Banach's contraction principle.

(Frechet, 1906) introduced the notions of metric space, which is one of the corner stone in Mathematics and also in several quantitative sciences. Later on, (Bahktin, 1989) and (Czerwik, 1993) introduced the concept of b-metric spaces which is one of the generalization of Banach's fixed point result. (Geraghty, 1973) introduced a contraction in which the Banach's constant of contraction was replaced by a function having some specific properties. Some of the generalizations of Banach contraction principles were obtained by contraction conditions containing rational expressions. Since then, several papers dealt with fixed point theory for rational Geraghty contractive mappings. (Jungck, 1986) proved a common fixed point theorem for commuting mappings, which generalizes the well-known Banach's fixed point result. In 1982 Sessa, introduced a generalization of commutativity for a pair of maps and proved some common fixed point theorems for weakly commuting pair of mappings.

Definition 2.1: (Parasad *et al*, 2020*).* Let (X, d) be a b-metric space with parameter $s \geq 1$ and $f, g: X \to X$ be two self-maps on X. We say that (f, g) is pair of an almost Geraghty-Suzuki type (I) maps, if there exist $L \geq 0$ and $\mathfrak{P} \in \mathfrak{F}$ such that

$$\frac{1}{2s}\mathrm{Min}\{d(x, fx), d(x, y)\} \leq d(x, y) \Longrightarrow sd(fx, gy) \leq \beta\big(M_1(x, y)\big)M_1(x, y) + LN_1(x, y),$$

where

$$\mathfrak{F} = \{\, \beta: \mathbb{R}^+ \to \ [0, \tfrac{1}{s}) \,/\, \lim_{n\to\infty} \beta(t_n) = \tfrac{1}{s} \Rightarrow \lim_{n\to\infty} t_n = 0\},$$

$$M_1(x, y) = max\left\{d(x, y), d(x, fx), d(y, gy), \frac{d(x,gy)+d(fx,y)}{2s}\right\},$$

$$M_2(x, y) = max\{d(x, y), d(x, fx), d(y, gy)\}$$

and

$$N_1(x, y) = min\{d(y, gy), d(y, fx)\}.$$

Theorem 2.1: (Faraji *et al*, 2019**).** Let (X, d) be a complete b-metric space with coefficient

$s \geq 1$. Let $T, S: X \to X$ be self-maps on X which satisfies: there exists $\beta \in \mathfrak{F}$, where β and $\mathfrak{F}$ are that given in Definition 2.1 such that:

$$d(Tx, sy) \leq \beta\big(M(x, y)\big)M(x, y) \text{ for all } x, y \in X,$$

where

$$M(x, y) = max\{d(x, y), d(x, Tx), d(y, Sy)\}.$$

If T or S is b-continuous, then T and S have a unique common fixed point.

Theorem 2.2: **(**Babu and Rantna, 2019). Let (X, d) be a b-metric space with parameter $s \geq 1$ and let $f, g : X \longrightarrow X$ are self-maps on X satisfying the following condition, there exist $\beta \in \mathfrak{F}$ and $L \geq 0$ such that

$$sd(fx, gy) \leq \beta\big(\mathrm{M(x, y)}\big)\mathrm{M(x, y)} + \mathrm{LN(x, y)} \text{ for all } x, y \in X,$$

where

$M(x, y) = max\{d(x, y), d(x, fx), d(y, gy)\}$

and

$N(x, y) = min\{d(x, fx), d(x, gy), d(y, fx)\}$.

If either f or g is b-continuous, then f and g have a unique common fixed point in X.

Recently, (Rahrovi and Piri, 2019) established fixed point theorem for a mappings satisfying Suzuki-rational Geraghty contraction and also proved the existence and uniqueness of fixed point for the result they have established.

CHAPTER THREE

METHODOLOGY

3.1 Study period and site

The study was conducted from September 2020 to February, 2022 in Jimma University under Mathematics department.

3.2 Study Design

To achieve the objective of the study, in this research work, Analytical method of design was used.

3.3 Source of Information

Relevant sources of information for this study were books and published articles related to the area of the study.

3 .4 Mathematical Procedure of the Study

The mathematical procedures that the researcher followed for this research work were the following:

- Establishing common fixed point theorems.
- Constructing sequences.
- Showing the sequences constructed are b-Cauchy.
- Showing the b- convergences of the sequences.
- Proving the existence of common fixed point.
- Verifying the uniqueness of the common fixed point.
- Giving examples in support of the main findings of the research work.

CHAPTER FOUR

DISCUSSION AND RESULT

4.1 Preliminaries

Definition 4.1.1: (Frechet, 1906). Let X be a nonempty set and $d : X \times X \rightarrow \mathbb{R}^+$ be a function satisfies the following conditions.

(i) $d(x, y) = 0$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$;

(iii) $d(x, z) \leq d(x, y) + d(y, z)$, for all $x, y, z \in X$.

Then, d is called a metric on X and the pair (X, d) is called a metric space.

Definition 4.1.2: (Czerwik, 1993). Let X be a non-empty set and $s \geq 1$ be given real number, a function $d : X \times X \rightarrow \mathbb{R}^+$ is b - metric on X if, for all $x, y, z \in X$ the following conditions holds:

(i) $d(x, y) = 0$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$;

(iii) $d(x, z) \leq s[d(x, y) + d(y, z)]$. In this case, the pair (X, d) is called a b - metric space.

It should be noted that, the class of b- metric spaces is effectively larger than that of metric spaces. Every metric space is a b-metric space with $s = 1$ but the converse is not true.

Example: Let $X = \mathbb{R}$ (the set of real numbers) and $d : X \times X \rightarrow \mathbb{R}^+$ be given by $d(x, y) = |x - y|^2$ for all $x,\ y \in X$, then d is a b-metric on X with $s = 2$ but it is not a metric on X for $x = 3,\ y = 5$ and $z = 8$. Since $d(3, 8) = 25 \nleq 13 = d(3,5) + d(5,8)$. Hence the triangle inequality for a metric does not hold.

Definition 4.1.3: (Boriceanu, 2010). Let (X, d) be a b-metric space and $\{x_n\}$ be sequence in X.

(i) $\{x_n\}$ is called b-convergent if there exists $x \in X$ such that $d(x_n, x) \to 0$ as $n \to \infty$, in this case we write $\lim_{n\to\infty} (x_n, x) = 0$.

(ii) $\{x_n\}$ is b – Cauchy if $d(x_{n\,,}\, x_m) \to 0$ as $n, m \to \infty$. Where $m, n \in \mathbb{N}$.

(iii) A b-metric space (X, d) is said to be b-complete if every b-Cauchy sequence in X is b-convergent.

Definition 4.1.4: **Geraghty contraction** (Geraghty, 1973). Let (X, d) be a metric space. A self-map $T : X \to X$ is said to be Geraghty contraction if there exists $\beta \in S$ such that $d(Tx, Ty) \leq \beta\big(d(x, y)\big)\, d(x, y)$ for all $x, y \in X$.

Where, $S = \{\, \beta : \mathbb{R}^+ \to [0,1) / \beta(t_n) \to 1 \Rightarrow t_n \to 0\}$.

Definition 4.1.5: Let X be a non-empty set and $g, T : X \to X$ are self-maps, a point $x \in X$ is said to be a common fixed point of the maps g and T if $gx = Tx = x$.

Definition 4.1.6: Let X be a non-empty set and f, g are self- maps on X, then f and g are said to be commuting maps if $fgx = gfx$ for all $x \in X$.

(Jungck, 1986) proved a common fixed point theorem for a pair of commuting mappings which generalizes the well-known Banach's fixed point result.

Definition 4.1.7: (Abbas and Jungck, 2008). Let X be a non-empty set and $g\,, T : X \to X$ be self-maps. If $gx = Tx = y$ for some $x \in X$, then y is called a point of coincidence of g and T while x is called a coincidence point of g and T.

Definition 4.1.8: (Boriceanu, *et al*, *2014*). Let (X, d_1) and (Y, d_2) be two b-metric spaces. A function $f : X \to Y$ is said to be b-continuous at a point $x \in X$ if it is b-sequentially continuous at x. That is, whenever $\{x_n\}$ is b- convergent to x, the sequence $\{fx_n\}$ is b-convergent to fx.

Definition 4.1.9: (Rahrovi and Piri, 2019). Let (X,d) be a b-metric space with a parameter $s \geq 1$, a self-map $T : X \to X$ is said to be Suzuki-rational Geraghty contraction, if there exist $\delta \in (s,\infty)$, $k \in (0,\infty)$ and the function $\varphi : \mathbb{R}^+ \to [0,\frac{1}{\delta})$ such that for all $x,y \in X$ with $x \neq y$,

$$\frac{1}{2s} d(x,Tx) < d(x,y) \Longrightarrow d(Tx,Ty) \leq \varphi\big(M(x,y)\big)M(x,y),$$

where

$$M(x,y) = Max\left\{d(x,y), \frac{d(x\,,Tx\,)\,d(y,Ty)}{\max\{k,d(x,y)\}}, \frac{d(x,Tx)\,d(y,Ty)}{\max\{\,k,d\,(Tx,Ty)\}}\right\}.$$

Theorem 4.1.10 : (Rahrovi and Piri, 2019). Let (X,d) be a complete b-metric space with constant $s \geq 1$ and $T : X \to X$ be a Suzuki-rational Geraghty contraction. Then T has a unique fixed point $x^\star \in X$ and for every $x \in X$ the sequence $\{T^n x\}_{n=1}^{\infty}$ converges to $x^\star$.

We use the following Lemma in proving the main result of our work.

Lemma 4.1.1: (Roshan *et al*, 2014). Let (X,d) be a b-metric space with coefficient $s \geq 1$ and $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty} d(x_n\,,x_{n+1}) = 0$. If $\{x_n\}$ is not a b- Cauchy sequence then there exist $\varepsilon > 0$ and sequences of positive integers $\{m_k\}$ and $\{n_k\}$ with $n_k > m_k > k$ such that $d(x_{m_k}, x_{n_k}) \geq \varepsilon$ for each $k > 0$, corresponding to m_k we can choose n_k to be the smallest positive integer such that $d(x_{m_k}, x_{n_k-1}) < \varepsilon$ and for the following four sequences.

$d(x_{m_k}, x_{nk})$, $d(x_{m_k}, x_{nk+1})$, $d(x_{m_{k+1}}, x_{nk})$, $d(x_{m_k+1}, x_{n_k+1})$ it holds:

(i) $\varepsilon \leq \lim_{k\to\infty} \inf d(x_{m_k}, x_{n_k}) \leq \lim_{k\to\infty} \sup d(x_{m_k}, x_{n_k}) \leq s\varepsilon$;

(ii) $\frac{\varepsilon}{s} \leq \lim_{k\to\infty} \inf d(x_{m_k+1}, x_{n_k}) \leq \lim_{k\to\infty} \sup d(x_{mk+1}, x_{n_k}) \leq s^2\varepsilon$;

(iii) $\frac{\varepsilon}{s} \leq \lim_{k\to\infty} \inf d(x_{m_k}, x_{n_k+1}) \leq \lim_{k\to\infty} \sup d(x_{m_k}, x_{n_k+1}) \leq s^2\varepsilon$;

(iv) $\frac{\varepsilon}{s^2} \leq \lim_{k\to\infty} \inf d(x_{m_k+1}, x_{n_k+1}) \leq \lim_{k\to\infty} \sup d(x_{m_k+1}, x_{n_k+1}) \leq s^3\varepsilon$.

4. 2 Main Result

Definition 4.2.1: Let (X, d) be a b-metric space with coefficient $s \geq 1$. Self-maps $g, T : X \to X$ are said to be Suzuki-rational Geraghty contraction maps if there exist $\delta \in (s, \infty)$, $k \in (0, \infty)$ and the function $\varphi: \mathbb{R}^+ \to [0, \frac{1}{\delta})$ such that, for all $x, y \in X$ with $x \neq y$

$$\frac{1}{2s} d(x, Tx) < d(x, y) \Rightarrow d(Tx, gy) \leq \varphi\big(M(x, y)\big) M(x, y)$$

$$\text{where,} \qquad M(x, y) = \left\{ \begin{matrix} d(x, y), & d(x, Tx), & d(y, gy) \\ \frac{d(x,Tx)\, d(y,gy)}{Max\,\{ k, d(x,y)\}}, & \frac{d(x,Tx)\; d(y,gy)}{Max\,\{k, d(Tx,gy)\}} & \end{matrix} \right\}. \tag{1}$$

Theorem 4.2.1: Let (X, d) be a b-metric space with constant $s \geq 1$. Suppose $g, T: X \to X$ are Suzuki-rational Geraghty contraction maps. Then g and T have a unique common fixed point $u \in X$ and a sequence $\{x_n\}$ in X converges to u provided that either T or g is b-continuous.

Proof : Let $x_0 \in X$ be arbitrary, we construct a sequence $\{x_n\}$ in X inductively by

$$Tx_{2n} = x_{2n+1} \qquad \text{and} \qquad gx_{2n+1} = x_{2n+2} \quad \text{for all} \quad n \geq 0.$$

Which gives $Tx_0 = x_1$, $gx_1 = x_2$, $Tx_2 = x_3$, $gx_3 = x_4$, ...etc.

If there exists $n \in \mathbb{N}$ such that $d(x_{2n+1}, gx_{2n+1}) = d(x_{2n+1}, x_{2n+2}) = 0$ this completes the proof.

Assume that $d(x_{2n+1}, gx_{2n+1}) > 0$ for every $n \in \mathbb{N}$.

$$\frac{1}{2s} d(x_{2n+1}, gx_{2n+1}) < d(x_{2n+1}, gx_{2n+1}) \quad \text{for all } n \in \mathbb{N}. \text{ Since } s \geq 1$$

By (1) we get,

$$\begin{aligned} d(x_{2n+1}, x_{2n+2}) &= d(Tx_{2n}, gx_{2n+1}) \\ &\leq \varphi\big(M(x_{2n}, x_{2n+1})\big) M(x_{2n}, x_{2n+1}) \\ &\leq \frac{1}{\delta} M(x_{2n}, x_{2n+1}), \end{aligned} \tag{2}$$

where

$$M(x_{2n}, x_{2n+1}) = \text{Max}\left\{\begin{matrix} d(x_{2n}, x_{2n+1}), d(x_{2n}, Tx_{2n}), d(x_{2n+1}, gx_{2n+1}) \\ \dfrac{d(x_{2n}, Tx_{2n})\, d(x_{2n+1}, gx_{2n+1})}{Max\{k, d(x_{2n}, x_{2n+1})\}} \\ \dfrac{d(x_{2n}, Tx_{2n})\, d(x_{2n+1}, gx_{2n+1})}{Max\{k, d(Tx_{2n}, gx_{2n+1})\}} \end{matrix}\right\}$$

$$= Max\left\{\begin{matrix} d(x_{2n}, x_{2n+1}), d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}) \\ \dfrac{d(x_{2n}, x_{2n+1})\, d(x_{2n+1}, x_{2n+2})}{Max\{k, d(x_{2n}, x_{2n+1})\}} \\ \dfrac{d(x_{2n}, x_{2n+1})\, d(x_{2n+1}, x_{2n+2})}{Max\{k, d(x_{2n+1}, x_{2n+2})\}} \end{matrix}\right\}$$

$$\leq Max\left\{\begin{matrix} d(x_{2n}, x_{2n+1}), d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}) \\ \dfrac{d(x_{2n}, x_{2n+1})\, d(x_{2n+1}, x_{2n+2})}{d(x_{2n+1}, x_{2n+2})} \\ \dfrac{d(x_{2n}, x_{2n+1})\, d(x_{2n+1}, x_{2n+2})}{d(x_{2n+1}, x_{2n+2})} \end{matrix}\right\}$$

$$= Max\,\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\}. \tag{3}$$

If there exists $n \in \mathbb{N}$, such that

$$\text{Max}\,\{d(x_{2n}, x_{2n+1}),\ d(x_{2n+1}, x_{2n+2})\} = d(x_{2n+1}, x_{2n+2}),$$

from (2) and (3) we get

$d(x_{2n+1}, x_{2n+2}) \leq \frac{1}{\delta} d(x_{2n+1}, x_{2n+2}) < d(x_{2n+1}, x_{2n+2})$.

Which is a contradiction, since $\delta > s \geq 1$. Thus,

$Max\,\{d(x_{2n}, x_{2n+1}),\ d(x_{2n+1}, x_{2n+2})\} = d(x_{2n}, x_{2n+1})$ for all $n \in \mathbb{N} \cup \{0\}$.

Again from (2) and (3) we have

$$d(x_{2n+1}, x_{2n+2}) \leq \frac{1}{\delta} d(x_{2n}, x_{2n+1}) < d(x_{2n}, x_{2n+1}). \tag{4}$$

Therefore, $\{d(x_{2n+1}, x_{2n+2})\}$ is a non-negative decreasing sequence of positive real numbers.

Hence, there exists $h \geq 0$ such that $\lim_{n\to\infty} d(x_{2n+1}, x_{2n+2}) = h$.

Now, we show that $h = 0$.

If possible, assume that $h > 0$. Letting $n \to \infty$ in (4) we get $h \leq \frac{h}{\delta} < h$. Which is a contradiction. Therefore,

$$\lim_{n\to\infty} d(x_{2n+1}, x_{2n+2}) = 0. \tag{5}$$

Now, we show that the sequence $\{x_n\}$ in X is b-Cauchy.

It is sufficient to show that the sub-sequence $\{x_{2n}\}$ in X is b-Cauchy.

Suppose that $\{x_{2n}\}$ in X is not a b-Cauchy sequence. Then, by Lemma 4.1.1 there exists $\varepsilon > 0$ and sequences of positive integers $\{2m_k\}$ and $\{2n_k\}$ in X such that $2n_k$ is the smallest index for which $2n_k > 2m_k > k$ for all $k \in \mathbb{N}$ satisfying

$$d(x_{2m_k}, x_{2n_k}) \geq \varepsilon, \quad d(2m_k, x_{2n_k-1}) < \varepsilon. \tag{6}$$

By (5) and using the triangle inequality for all $k \in \mathbb{N}$.

$$\varepsilon \leq d(x_{2m_k}, x_{2n_k}) \leq s[d(x_{2m_k}, x_{2n_k+1}) + d(x_{2n_k+1}, x_{2n_k})]$$

$$\leq s^2[d(x_{2m_k}, x_{2m_k}) + d(x_{2m_k}, x_{2n_k+1})] + sd(x_{2n_k+1}, x_{2n_k}).$$

Taking the upper and the lower limits, using (5) and Lemma (4.1.1), we have

$$\frac{\varepsilon}{s} \leq \lim_{k\to\infty} \inf d(x_{2m_k}, x_{2n_k+1}) \leq \lim_{k\to\infty} \sup d(x_{2m_k}, x_{2n_k+1}) \leq \varepsilon s.$$

Again in a similar way from (6) we can get

$$\varepsilon \leq d(x_{2m_k}, x_{2n_k}) \leq s[d(x_{2m_k}, x_{2m_k-1}) + d(x_{2m_k-1}, x_{2n_k})],$$

$$\leq sd(x_{2m_k}, x_{2m_k-1}) + s^2[d(x_{2m_k-1}, x_{2n_k}) + d(x_{2n_k}, x_{2n_k})].$$

Taking the upper and the lower limits and using Lemma (4.1.1) we have

$$\frac{\varepsilon}{s} \leq \lim_{k\to\infty} \inf d(x_{2m_k-1}, x_{2n_k}) \leq \lim_{k\to\infty} \sup d(x_{2m_k-1}, x_{2n_k}) \leq \varepsilon s.$$

Therefore,

$$\frac{\varepsilon}{s} \leq \lim_{k\to\infty} \inf d\left(x_{2m_k}, x_{2n_k+1}\right) \leq \lim_{k\to\infty} \sup d\left(x_{2m_k}, x_{2n_k+1}\right) \leq \varepsilon s$$

and

$$\frac{\varepsilon}{s} \leq \lim_{k\to\infty} \inf d\left(x_{2m_k-1}, x_{2n_k}\right) \leq \lim_{k\to\infty} \sup d\left(x_{2m_k-1}, x_{2n_k}\right) \leq \varepsilon s. \tag{7}$$

From (7) we observe that

$$\frac{\varepsilon}{s} \leq \lim_{k\to\infty} \inf d\left(x_{2m_k}, x_{2n_k+1}\right) \quad \text{and} \quad \frac{\varepsilon}{s} \leq \lim_{k\to\infty} \inf d\left(x_{2m_k-1}, x_{2n_k}\right).$$

From (5) and (7), there exists $k_1 \in \mathbb{N}$ such that

$$\frac{1}{2s} d\left(x_{2m_k}, x_{2m_k+1}\right) < \frac{\varepsilon}{s} \leq d\left(x_{2m_k}, x_{2n_k+1}\right) \text{ for all } k \geq k_1.$$

Again from (7) there exists $k_2 \in \mathbb{N}$ such that

$$\frac{\varepsilon}{s} \leq d(x_{2m_k-1}, x_{2n_k}) \quad \text{for every} \quad k \geq k_2.$$

By (2) for all $k \geq k_3 = Max\{k_1, k_2\}$ we have,

$$\begin{aligned} d\left(x_{2m_k}, x_{2n_k+1}\right) &= d(Tx_{2m_k-1}, gx_{2n_k}) \\ &\leq \varphi(M(x_{2m_k-1}, x_{2n_k}))M(x_{2m_k-1}, x_{2n_k}) \\ &\leq \frac{1}{\delta} M\left(x_{2m_k-1}, x_{2n_k}\right), \end{aligned} \tag{8}$$

where

$$M\left(x_{2m_k-1}, x_{2n_k}\right) = Max \left\{ \begin{matrix} d\left(x_{2m_k-1}, x_{2n_k}\right), d\left(x_{2m_k-1}, Tx_{2m_k-1}\right), d\left(x_{2n_k}, gx_{2n_k}\right) \\ \dfrac{d\left(x_{2m_k-1}, Tx_{2m_k-1}\right) d\left(x_{2n_k}, gx_{2n_k}\right)}{Max\left\{k, d\left(x_{2m_k-1}, x_{2n_k}\right)\right\}} \\ \dfrac{d\left(x_{2m_k-1}, Tx_{2m_k-1}\right) d\left(x_{2n_k}, gx_{2n_k}\right)}{Max\left\{k, d\left(Tx_{2m_k-1}, gx_{2n_k}\right)\right\}} \end{matrix} \right\}$$

$$= Max\left\{\begin{matrix} d(x_{2m_k-1}, x_{2n_k}), d(x_{2m_k-1}, x_{2m_k}), d(x_{2n_k}, x_{2n_k+1}) \\ \dfrac{d(x_{2m_k-1}, x_{2m_k})\ d(x_{2n_k}, x_{2n_k+1})}{Max\{k\ , d(x_{2m_k-1}, x_{2n_k})\}} \\ \dfrac{d(x_{2m_k-1}, x_{2m_k})\ d(x_{2n_k}, x_{2n_k+1})}{Max\{k, d(x_{2m_k}, x_{2n_k+1})\}} \end{matrix}\right\}$$

$$\le Max\left\{\begin{matrix} d(x_{2m_k-1}, x_{2n_k}), d(x_{2m_k-1}, x_{2m_k}), d(x_{2n_k}, x_{2n_k+1}) \\ \dfrac{d(x_{2m_k-1}, x_{2m_k})\ d(x_{2n_k}, x_{2n_k+1})}{d(x_{2m_k-1}, x_{2n_k})} \\ \dfrac{d(x_{2m_k-1}, x_{2m_k})\ d(x_{2n_k}, x_{2n_k+1})}{d(x_{2m_k}, x_{2n_k+1})} \end{matrix}\right\}.$$

By the triangle inequality, we have

$$d(x_{2m_k-1}, x_{2m_k}) \le s[d(x_{2m_k-1}, x_{2n_k}) + d(x_{2n_k}, x_{2m_k})].$$

Since $d(x_{2n_k}, x_{2m_k-1}) < \varepsilon$, using (7) we get

$$M(x_{2m_k-1}, x_{2n_k}) \le Max\left\{\begin{matrix} \varepsilon,\ d(x_{2m_k-1}, x_{2m_k}),\ d(x_{2n_k}, x_{2n_k+1}) \\ \frac{\varepsilon}{s} d(x_{2m_k-1}, x_{2m_k}) d(x_{2n_k}, x_{2n_k+1}) \\ \frac{\varepsilon}{s} d(x_{2m_k-1}, x_{2m_k}) d(x_{2n_k}, x_{2n_k+1}) \end{matrix}\right\}. \qquad .$$

Letting $k \to \infty$ and using (5), the above inequality becomes

$$\lim_{k\to\infty} Max \left\{\begin{matrix} \varepsilon,\ \frac{\varepsilon}{s} d(x_{2m_k-1}, x_{2m_k})\ d(x_{2n_k}, x_{2n_k+1}) \\ \frac{\varepsilon}{s} d(x_{2m_k-1}, x_{2m_k})\ d(x_{2n_k}, x_{2n_k+1}) \end{matrix}\right\} = \varepsilon. \tag{9}$$

From (8) and (9) for all $k \ge k_3$ we get,

$$d(x_{2m_k}, x_{2n_k+1}) \le \frac{1}{\delta} \varepsilon.$$

Using (5) and taking upper limit in the above inequality we have

$$\lim_{k\to\infty} \sup d(x_{2m_k}, x_{2n_k+1}) \le \frac{\varepsilon}{\delta}. \tag{10}$$

From (7) we have $\frac{\varepsilon}{s} \le \lim_{k\to\infty} \sup d(x_{2m_k}, x_{2n_k+1})$. (11)

From (10) and (11) we have $\frac{\varepsilon}{s} \leq \lim_{k\to\infty} d\left(x_{2n_k}, x_{2n_k+1}\right) \leq \frac{\varepsilon}{\delta}$.

which is a contradiction since $\delta > \mathrm{s}$.

Therefore, $\{x_{2n}\}$ is a b-Cauchy sequence in X.

Since X is b-complete, there exists $u \in X$ such that $\lim_{n\to\infty} x_n = u$.

Now, we claim that u is a common fixed point of g and T.

To prove this we consider two cases.

Case I: Suppose that T is b-continuous. Then, $Tu = u$.

Now, we show that u is also a fixed point of g.

To the contrary, assume that u is not a fixed point of g. That is, $gu \neq u$.

Consider the inequality

$$d(Tu, gu) \leq \varphi\left(M(u,u)\right)M(u,u) \leq \frac{1}{\delta} M(u,u),$$

where

$$M(u,u) = Max\left\{\begin{matrix} d(u,u), d(u,Tu), d(u,gu) \\ \frac{d(u,Tu)\, d(u,gu)}{Max\{k,\ d(u,u)\}}, \quad \frac{d(u,Tu)\, d(u,gu)}{Max\{k,\ d(Tu,gu)\}} \end{matrix}\right\} = d(u,gu).$$

Thus, we have

$$d(u,gu) \leq \frac{1}{\delta} d(u,gu) < d(u,gu).$$ Which is a contradiction.

Thus, u is also a fixed point of g.

Hence we have,

$$gu = Tu = u.$$

Similarly if we assume that g is b-continuous, then we can show that u is also a fixed point of T.

Hence u is a common fixed point of g and T.

ow, we show that u is a unique common fixed point of g and T.

Suppose that u and v are two common fixed points of T and g with $u \neq v$. That is,

$$d(u,v) > 0.$$

We have,

$$\frac{1}{2s} d(u,Tu) = 0 < d(u,v).$$

By (1) we get,

$$d(u,v) = d(Tu,gv) \leq \varphi\big(M(u\,,v)\big)M(u,v)$$

$$\leq \tfrac{1}{\delta} M(u,v),$$

where

$$M(u,v) = Max\left\{\begin{matrix} d(u,v), d(u,Tu), d(v,gv) \\ \dfrac{d(u\,,\ Tu)\, d(v,\ gv)}{Max\{k,\ d(u,\ v)\}},\ \dfrac{d(u,\ Tu)\, d(u\,,\ gu)}{Max\{k,\ d(Tu,\ gu)\}} \end{matrix}\right\} = d(u,v)$$

So, we have,

$$d(u,v) \leq \tfrac{1}{\delta} d(u,v) < d(u,v),$$

Which is a contradiction. Hence $u = v$.

Therefore, u is a unique common fixed point of g and T.

Definition 4.2.2: (Rahrovi and Piri, 2019). Let (X,d) be a b-metric space with parameter $s \geq 1$. Self-Map $T: X \to X$ is said to be generalized rational Geraghty contraction of type A, if there exist $\delta \epsilon (s,\infty)$, $k \in (0,\infty)$ and the function $\varphi: \mathbb{R}^+ \to [0, \frac{1}{\delta})$ such that

$$d(Tx,Ty) \leq \begin{cases} \varphi\big(M_A(x,y)\big)M_{A,}(x,y) & \text{if} \quad Max\{d(x,Ty),d(Tx,y)\} \neq 0 \\ 0 & \text{if} \quad Max\{d(x,Ty),d(Tx,y)\} = 0 \end{cases},$$

where,

$$M_A(x,y) = Max\left\{\begin{array}{c} d(x,y) \\ \dfrac{d(x,Tx)\,d(x,Ty)+d(y,Ty)\,d(y,Tx)}{Max\{k,\ S[d(x,Tx)\ +\ d(y,Ty)]\}} \\ \dfrac{d(x,Tx)\,d(x,Ty)+\ d\,(y,Ty)\,d(y,Tx)}{Max\{d(x,Ty),\ d(Tx,y)\}} \end{array}\right\}.$$

Definition 4.2.3: Let (X,d) be a b-metric space with parameter $s \geq 1$. Self-maps $g,T: X \to X$ are said to be generalized rational Geraghty contraction of type A if there exist $\delta \epsilon (s,\infty)$, $k \in (0,\infty)$ and the function $\varphi: \mathbb{R}^+ \to [0,\frac{1}{\delta})$ such that

$$d(Tx,gy) \leq \begin{cases} \varphi\big(M_A(x,y)\big)M_A(x,y) & \text{if} \quad Max\{d(x,gy),d(Tx,y)\} \neq 0 \\ 0 & \text{if} \quad Max\{d(x,gy),d(Tx,y)\} = 0 \end{cases} \tag{12}$$

where, $$M_A(x,y) = Max\left\{\begin{array}{c} d(x,y),d(x,Tx),d(y,gy) \\ \dfrac{d(x,Tx)\,d(x,gy)\ +\ d(y,gy)\,d(y,Tx)}{Max\{k,\ S[d(x,\ Tx)\ +\ d(y,gy)]\}} \\ \dfrac{d(x,Tx)\,d(x,gy)\ +\ d\,(y,gy)\,d(y,Tx)}{Max\{d(x,gy),\ d(Tx,y)\}} \end{array}\right\}.$$

Theorem 4.2.2: Let (X,d) be a complete b-metric space with constant $s \geq 1$ and self-maps $g,T: X \to X$ be generalized rational Geraghty contraction mappings of type A. Then T and g have a unique common fixed point $u \in X$ provided that either T or g is b-continuous.

Proof: As in the proof of theorem 4.2.1 we construct a sequence (x_n) in X as follows:

Let $x_0 \in X$ be arbitrary, put $Tx_{2n} = x_{2n+1}$ and $gx_{2n+1} = x_{2n+2}$ for all $n \geq 0$.

If there exists $n \in \mathbb{N}$ such that $x_{2n} = x_{2n-1}$ then x_{2n-1} is a common fixed point of T and g. This completes the proof.

Suppose that $x_{2n} \neq x_{2n-1}$ for every $n \in \mathbb{N}$. We have two cases to consider.

Case I: Assume that

$$Max\,\{d(x_{2m}, gx_{2n}), d(Tx_{2m}, x_{2n})\} \neq 0 \text{ for all } m, n \in \mathbb{N} \cup \{0\} \tag{13}$$

Observe that by (12)

$$\begin{aligned} d(x_{2n+1}, x_{2n+2}) &= d(Tx_{2n}, gx_{2n+1}) \\ &\leq \varphi(M_A(x_{2n}, x_{2n+1}))M_A(x_{2n}, x_{2n+1}) \\ &\leq \frac{1}{\delta} M_A(x_{2n}, x_{2n+1}), \end{aligned}$$

where,

$$M_A(x_{2n}, x_{2n+1}) = Max \left\{ \begin{array}{c} d(x_{2n}, x_{2n+1}), d(x_{2n}, Tx_{2n}), d(x_{2n+1}, gx_{2n+1}) \\ \dfrac{d(x_{2n}, Tx_{2n})d(x_{2n}, gx_{2n+1}) + d(x_{2n+1}, gx_{2n+1})d(x_{2n+1}, Tx_{2n})}{Max\{k, s[d(x_{2n}, Tx_{2n}) + d(x_{2n+1}, gx_{2n+1})]\}} \\ \dfrac{d(x_{2n}, Tx_{2n})d(x_{2n}, gx_{2n+1}) + d(x_{2n+1}, gx_{2n+1})d(x_{2n+1}, Tx_{2n})}{Max\{d(x_{2n}, gx_{2n+1}),\ d(x_{2n+1}, Tx_{2n})\}} \end{array} \right\}$$

$$\leq \frac{1}{\delta} Max \left\{ \begin{array}{c} d(x_{2n}, x_{2n+1}), d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}) \\ \dfrac{d(x_{2n}, x_{2n+1})\, d(x_{2n}, x_{2n+2}) + d(x_{2n+1}, x_{2n+2})\, d(x_{2n+1}, x_{2n+1})}{Max\{k, s[d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})]\}} \\ \dfrac{d(x_{2n}, x_{2n+1})\, d(x_{2n}, x_{2n+2}) + d(x_{2n+1}, x_{2n+2})\, d(x_{2n+1}, x_{2n+1})}{Max\{d(x_{2n}, x_{2n+2}),\ d(x_{2n+1}, x_{2n+1})\}} \end{array} \right\}$$

$$= \frac{1}{\delta}\, Max \left\{ \begin{array}{c} d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}) \\ \dfrac{d(x_{2n}, x_{2n+1})\, d(x_{2n}, x_{2n+2})}{Max\{k,\ s[d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})]\}} \end{array} \right\}$$

$$= \frac{1}{\delta} Max\,\{d(x_{2n}, x_{2n+1}), (x_{2n+1}, x_{2n+2})\}$$

$$= \frac{1}{\delta}\, d(x_{2n}, x_{2n+1}). \tag{14}$$

Thus we get,

$$d(x_{2n+1}, x_{2n+2}) \leq \frac{1}{\delta}\, d(x_{2n}, x_{2n+1}) \quad \text{for all } \ n \geq 0. \tag{15}$$

As in the proof of Theorem 4.2.1 we can prove that

$$\lim_{n\to\infty} d(x_{2n+1}, gx_{2n+1}) = \lim_{n\to\infty} d(x_{2n+1}, x_{2n+2}) = 0. \tag{16}$$

Now, we show that the sequence $\{x_n\}$ in X is b-Cauchy.

We show a sub-sequence $\{x_{2n}\}$ in X is b- Cauchy sequence in X. Suppose that $\{x_{2n}\}$ in X is not a b-Cauchy sequence. Then by Lemma 4.1.1 there exist $\varepsilon > 0$ and sequences of positive integers $\{2n_k\}$ and $\{2m_k\}$ such that $2n_k > 2m_k > k$ with $2n_k$ is the smallest index and

$$d\left(x_{2m_k}, x_{2n_k}\right) \geq \varepsilon\,, \quad d\left(x_{2m_k}, x_{2n_k-1}\right) < \varepsilon \ \text{ for all } \ k \in \mathbb{N}. \tag{17}$$

From the triangle inequality, for every $k \in \mathbb{N}$, we have

$$\varepsilon \leq \lim_{k\to\infty} \inf d\left(x_{2m_k}, x_{2n_k}\right) \leq s\left[d\left(x_{2m_k}, x_{2n_k+1}\right) + d\left(x_{2n_k+1}, x_{2n_k}\right)\right].$$

It follows from (15) that

$$\frac{\varepsilon}{s} \leq \lim\inf_{k\to\infty} d\left(x_{2m_k}, x_{2n_k+1}\right). \tag{18}$$

So there exists $k_1 \in \mathbb{N}$ such that,

$$\text{Max}\left\{d\left(x_{2m_k}, gx_{2n_k}\right),\ d\left(Tx_{2m_k}, x_{2n_k}\right)\right\} \geq d\left(x_{2m_k}, gx_{2n_k}\right) \geq \frac{\varepsilon}{s} \quad \text{for all } \ k \geq k_1.$$

By (13) for every $k \geq k_1$ we have

$$\begin{aligned} d\left(x_{2m_k}, x_{2n_k+1}\right) &= d\left(Tx_{2m_k-1}, gx_{2n_k}\right) \\ &\leq \varphi(M_A(x_{2n_k-1}, x_{2n_k}))M_A(x_{2m_k-1}, x_{2n_k}) \\ &\leq \frac{1}{\delta} M_A(x_{2m_k-1}, x_{2n_k}), \end{aligned} \tag{19}$$

where

$$\mathrm{M_A}(x_{2m_k-1}, x_{2n_k})$$

$$= Max\left\{\begin{array}{c} d(x_{2m_k-1}, x_{2n_k}), d(x_{2m_k-1}, Tx_{2m_k-1}), d(x_{2n_k}, gx_{2n_k}), \\ \dfrac{d(x_{2m_k-1}, Tx_{2m_k-1})d(x_{2m_k-1}, gx_{2n_k}) + d(x_{2n_k}, gx_{2n_k})d(x_{2n_k}, Tx_{2m_k-1})}{Max\{k, \ s[d(x_{2m_k-1}, Tx_{2m_k-1}) + d(x_{2n_k}, gx_{2n_k})]\}}, \\ \dfrac{d(x_{2m_k-1}, Tx_{2m_k-1})d(x_{2m_k-1}, gx_{2n_k}) + d(x_{2n_k}, gx_{2n_k})d(x_{2n_k}, Tx_{2m_k-1})}{Max\{d(x_{2m_k-1}, gx_{2n_k}), \ d(Tx_{2m_k-1}, x_{2n_k})\}} \end{array}\right\}$$

$$= Max\left\{\begin{array}{c} d(x_{2m_k-1}, x_{2n_k}), d(x_{2m_k-1}, x_{2m_k}), \ d(x_{2n_k}, x_{2n_k+1}), \\ \dfrac{d(x_{2m_k-1}, x_{2m_k})\, d(x_{2m_k-1}, x_{2n_k+1}) + d(x_{2n_k}, x_{2n_k+1})\, d(x_{2n_k}, x_{2m_k})}{Max\,\{k, \ s[\, d(x_{2m_k-1}, x_{2m_k}) + d(x_{2n_k}, x_{2n_k+1})]\}}, \\ \dfrac{d(x_{2m_k-1}, x_{2m_k})\ d(x_{2m_k-1}, x_{2n_k+1}) + d(x_{2n_k}, x_{2n_k+1})\, d(x_{2n_k}, x_{2m_k})}{Max\,\{d(x_{2m_k-1}, x_{2n_k+1}), \ d(x_{2m_k}, x_{2n_k})\}} \end{array}\right\}$$

$$\leq Max\left\{\begin{array}{c} d(x_{2m_k-1}, x_{2n_k}), d(x_{2m_k-1}, x_{2m_k}), d(x_{2n_k}, x_{2n_k+1}), \\ \dfrac{d(x_{2m_k-1}, x_{2m_k})\, d(x_{2m_k-1}, x_{2n_k+1}) + d(x_{2n_k}, x_{2n_k+1})\, d(x_{2n_k}, x_{2m_k})}{k}, \\ \dfrac{d(x_{2m_k-1}, x_{2m_k})\, d(x_{2m_k-1}, x_{2n_k+1})}{d(x_{2m_k-1}, x_{2n_k+1})} + \dfrac{d(x_{2n_k}, x_{2n_k+1})\, d(x_{2n_k}, x_{2m_k})}{d(x_{2m_k}, x_{2n_k})} \end{array}\right\}$$

$$= Max\left\{\begin{array}{c} d(x_{2m_k-1}, x_{2n_k}), d(x_{2m_k-1}, x_{2m_k}), d(x_{2n_k}, \ x_{2n_k+1}), \\ \dfrac{d(x_{2m_k-1}, x_{2m_k})\, d(x_{2m_k-1}, x_{2n_k+1}) + d(x_{2n_k}, x_{2n_k+1})\, d(x_{2n_k}, x_{2m_k})}{k}, \\ d(x_{2m_k-1}, x_{2m_k}) + d(x_{2n_k}, x_{2n_k+1}) \end{array}\right\}$$

$$\leq Max\left\{\begin{array}{c} d(x_{2m_k-1}, x_{2n_k}), \ d(x_{2m_k-1}, x_{2m_k}), \ d(x_{2n_k}, x_{2n_k+1}), \\ \dfrac{d(x_{2m_k-1}, x_{2m_k})\, s[d(x_{2m_k-1}, x_{2m_k}) + \ d(x_{2n_k}, x_{2n_k+1})]}{k} + \\ \dfrac{(x_{2n_k}, x_{2n_k+1})\, s[d(x_{2n_k}, x_{2m_k-1}) + \ d(x_{2m_k-1}, x_{2n_k})]}{k}, \\ (x_{2m_k-1}, x_{2m_k}) + d(x_{2n_k}, x_{2n_k+1}) \end{array}\right\}$$

$$\leq \text{Max}\left\{\begin{array}{c} \varepsilon\,,\ \ d(x_{2mk-1}, x_{2mk}),\qquad d(x_{2nk}, x_{2nk+1}), \\ \frac{d(x_{2mk-1}, x_{2mk})\ \ s[\varepsilon + d(x_{2nk}, x_{2nk+1})]}{k} + \frac{(x_{2nk},\ x_{2nk+1})\ \ s[\ \varepsilon + d(x_{2mk-1}, x_{2nk})]}{k}, \\ d(x_{2mk-1}, x_{2mk})\ +\ d(x_{2nk}, x_{2nk+1}) \end{array}\right\}.$$

Letting $k \to \infty$ in the above inequality and using (15) we get

$$\lim_{k\to\infty} \sup d(x_{2mk},\ \ x_{2nk+1}) \leq \frac{\varepsilon}{s}. \tag{20}$$

Thus, from (18) and (20), we get

$$\frac{\varepsilon}{s} \ \leq\ \frac{\varepsilon}{\delta}.$$

This is a contradiction.

Therefore, $\{x_{2n}\}$ in X is a b-Cauchy sequence in X.

Now we show g and T have a common fixed point.

Since X is b- complete, there exists $u \in X$ such that

$x_n \to u$.

That is,

$$\lim_{n\to\infty} d(x_n, u) = u.$$

Also, we have $\lim_{n\to\infty} Tx_{2n} = \lim_{n\to\infty} x_{2n+1} = u$ and $u = \lim_{n\to\infty} x_{2n+2} = \lim_{n\to\infty} \mathrm{g}x_{2n+1.}$

We assume that T is b-continuous.

Since, $x_{2n} \to u$ as $n \to \infty$ we have $Tx_{2n} \to Tu$ as $n \to \infty,$ thus we have $Tu = u$.

That is, u is a fixed point of T.

Now, we show u is a fixed point of g.

Assume to the contrary that u is not a fixed point of g. That is, $gu \neq u$.

In (12), taking $x = u$ and $y = u$ we get that

$$d(u, gu) = d(Tu, gu) \leq \varphi\big(M_A(u, u)\big)M_A(u, u),$$

where

$$M_A(u,u) = Max\left\{\begin{matrix} d(u,u), d(u,Tu), d(u,gu) \\ \dfrac{d(u,\ Tu)\ d(u,\ gu) +\ d(u,gu)\ d(u,\ Tu)}{Max\{k,\ \ S[d(u,\ Tu)\ +\ d(u,\ gu)]\}} \\ \dfrac{d(u,\ \ Tu)\ d(u,\ \ gu) + d\ (u,\ \ gu)\ d(u,\ Tu)}{Max\{d(u,\ \ gu),\ \ d(Tu,\ u)\}} \end{matrix}\right\} = d(u, gu).$$

Thus, we have

$$d(u, gu) \leq \varphi\big(d(u, gu)\big)d(u, gu) \leq \tfrac{1}{\delta}\ d(u, gu) < d(u, gu),$$

which is a contradiction. So we have,

$$gu = u.$$

Hence, we have

$$gu = Tu = u.$$

Now, we show that the common fixed point is unique.

Suppose u and v are two common fixed points of T and g with $u \neq v$.

That is, $d(u, v) > 0$.

Using (12), we have

$$d(u, v) = (Tu, gv) \leq \varphi(M_A(u, v))M_A(u, v),$$

where

$$M_A(u,v) = Max\left\{\begin{matrix} d(u,v), d(u,Tu), d(v,gv) \\ \dfrac{d(u,Tu)\ d(u,gv) +\ d(v,gv)\ d(v,Tu)}{Max\{k, S[d(u,Tu)\ +\ d(v,gv)]\}} \\ \dfrac{d(u,Tu)\ d(u,gv) + d\ (v,gv)\ d(v,Tu)}{Max\{d(u,gv), d(Tu,v)\}} \end{matrix}\right\} = d(u, v).$$

Thus, we have

$$d(u,v) \leq \varphi\big(d(u,v)\big)d(u,v) \leq \frac{1}{\delta}\, d(u,v) < d(u,v),$$

which is a contradiction.

Hence, the common fixed point is unique.

Corollary 4.2.1: Let (X,d) be a b-metric space with coefficient $s \geq 1$ and $T: X \to X$ is self-map which satisfies the following condition: There exist $\delta \in (s,\infty)$, $k \in (0,\infty)$ and a function $\varphi\colon \mathbb{R}^+ \to [0, \frac{1}{\delta})$ such that, for all $x, y \in X$ with $x \neq y$

$$\frac{1}{2s}\, d(x,Tx) < d(x,y) \Rightarrow d(Tx,Ty) \leq \varphi\big(M(x,y)\big)M(x,y)$$

$$\text{where, } M(x,y) = \left\{ \begin{matrix} d(x,y), \quad d(x,Tx), \quad d(y,Ty) \\ \dfrac{d(x,Tx)\, d(y,Ty)}{Max\,\{k, d(x,y)\}}, \dfrac{d(x,Tx)\ d(y,Ty)}{Max\,\{k, d(Tx\,,Ty)\}} \end{matrix} \right\}.$$

Then T has a unique fixed point $u \in X$ and for every $x \in X$ the sequence $(T^n x)$ converges to converges to u.

Proof: The result can be obtained by taking $g = T$ in Theorem 4.2.1.

Remark: The result of the work of (Rahrovi and Piri, 2019) follows as a corollary to Corollary 4.2.1.

Corollary 4.2.2: Let (X,d) be a complete b-metric space with parameter $s \geq 1$ and $T: X \to X$ be self-map on X which satisfy the following condition: There exist $\delta \epsilon (s,\infty)$,$k \in (0,\infty)$ and the function $\varphi: \mathbb{R}^+ \to [0,\frac{1}{\delta})$ such that

$$d(Tx,Ty) \leq \begin{cases} \varphi\big(M(x,y)\big)M(x,y) & \text{if} \quad Max\{d(x,Ty), d(Tx,y)\} \neq 0 \\ 0 & \text{if} \quad Max\{d(x,y), d(Tx,y)\} = 0 \end{cases},$$

where

$$M(x,y) = Max\left\{\begin{matrix} d(x,y), \quad d(x,Tx), \quad d(y,Ty) \\ \dfrac{d(x,Tx)\, d(x,Ty) + d(y,Ty)\, d(y,Tx)}{Max\{k, S[d(x,Tx) + d(y,Ty)]\}} \\ \dfrac{d(x,Tx)\, d(x,Ty) + d(y,Ty)\, d(y,Tx)}{Max\{d(x,Ty),\ d(Tx,y)\}} \end{matrix}\right\}.$$

Then T has a unique fixed point $u \in X$ and a sequence (x_n) in X converges to u.

Proof : The result follows by taking $g = T$ in the theorem 4.2.1.

In the following we give an example in support of theorem 4.2.1.

Example 1: Let $X = \mathbb{R}^+$ and a function $d : X \times X \rightarrow \mathbb{R}^+$ defined by

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ (x+y)^2 & \text{if } x \neq y; \end{cases}$$

Clearly, (X,d) is a complete b-metric space with $s = 2$. We define $T, g : X \rightarrow X$ by

$$T(x) = \begin{cases} \frac{x}{5} & \text{if } x \in [0,1) \\ \frac{1}{5} & \text{if } x \in [1,\infty) \end{cases} \quad \text{and} \quad g(x) = \begin{cases} \frac{x}{6} & \text{if } x \in [0,1) \\ \frac{1}{3} & \text{if } x \in [1,\infty). \end{cases}$$

Let $k \in (0,\infty)$, $\delta = \frac{21}{10}$ and define $\varphi: \mathbb{R}^+ \rightarrow [0, \frac{1}{\delta})$ by $\varphi(t) = \frac{1}{t+3}$, $t \geq 0$.

Without loss of generality, we assume that $x \geq y$.

Case I: $x, y \in [0,1)$.

In this case $Tx = \frac{x}{5}$, $gx = \frac{y}{6}$, $d(x,y) = (x+y)^2$, $d(x,Tx) = (x + \frac{x}{5})^2$

$d(y,gy) = \left(y + \frac{y}{6}\right)^2$ $d(Tx,gy) = (\frac{x}{5} + \frac{y}{6})^2$, $\frac{1}{2s} d(x,Tx) = \frac{1}{4}\left(x + \frac{x}{5}\right)^2 < (x+y)^2 = d(x,y)$.

Consider: $d(Tx,gy) \leq \varphi(M(x,y))M(x,y)$,

$$\text{Where,} \quad M(x,y) = \left\{\begin{matrix} d(x,y), \quad d(x,Tx), \quad d(y,gy) \\ \dfrac{d(x,Tx)\, d(y,gy)}{Max\{k, d(x,y)\}}, \ \dfrac{d(x,Tx)\, d(y,gy)}{Max\{k, d(Tx,gy)\}} \end{matrix}\right\}$$

$$M(x,y) = Max\left\{ \begin{matrix} (x+y)^2, & \left(x+\frac{x}{5}\right)^2, \ \left(y+\frac{y}{6}\right)^2 \\ \frac{\left(x+\frac{x}{5}\right)^2\left(y+\frac{y}{6}\right)^2}{Max\,\{k,\ (x+y)^2\}}, & \frac{\left(x+\frac{x}{5}\right)^2\left(y+\frac{y}{6}\right)^2}{Max\,\left\{k, \left(\frac{x}{5}+\frac{y}{6}\right)^2\right\}} \end{matrix} \right\} = (x+y)^2.$$

Thus,

$$\varphi\big(M(x\,,y)\big)M(x\,,y) = \frac{(x+y)^2}{3+\,(x+y)^2}.$$

Now, $d(Tx,gy) = \left(\frac{x}{5}+\ \frac{y}{6}\right)^2 \leq \ \frac{(\,x\ +\ y)^2}{3+(\,x\ +\ y\,)^2} = \varphi\big(M(x,y)\big)M(x,y).$

Sub-case (a): If $y \leq \frac{x}{5}$, $\quad d(x,y) = (x+y)^2.$

|Observe that

$$d\big(Tx,\ gy\big) = (\frac{x}{5}+\frac{y}{6}\ )^2 \leq\ \frac{(\,x+y\,)^2}{3+\ (x\ +\ y)^2}(x+y)^2 = \varphi(x,y))M_T(x,y).$$

Sub-case (b): $y\ > \frac{x}{5}$, $\qquad M(x,y) = (x+y)^2.$

Then, we have

$$d(Tx,gy) = (\ \frac{x}{5}+\frac{y}{6}\ )^2\ \leq\ \frac{(x\ +\ y\,)^2}{3+(\,x+\,y\,)^2}\ (x+y)^2 = \varphi(M(x,y))M(x,y).$$

Case II: $x,y\ \in [1,\infty)$. In this case

$Tx =\ \frac{1}{5}\,,$ $\qquad gx =\ \frac{1}{3}\,,$

$d(x,y)= (x+y)^2,$ $\qquad \frac{1}{2s}d(x,Tx)\ =\ \frac{1}{4}\left(x+\frac{1}{5}\right)^2$

$d(y,g\,y) = \left(y+\frac{1}{3}\right)^2,$ $\qquad d(Tx,gy) = \left(\frac{1}{5}+\frac{1}{3}\right)^2 =\ \frac{64}{225}.$

Then, $\quad \frac{1}{2s}d(x,Tx) =\ \frac{1}{4}\left(x+\frac{1}{5}\right)^2 < (x+y)^2 = d(x,y).$

$$M(x,y) = Max\left\{\begin{matrix} (x+y)^2, & \left(x+\frac{1}{5}\right)^2, \left(y+\frac{1}{3}\right)^2 \\ \frac{\left(x+\frac{1}{5}\right)^2\left(y+\frac{1}{3}\right)^2}{Max\,\{k,(x+y)^2\}}, & \frac{\left(x+\frac{1}{5}\right)^2\left(y+\frac{1}{3}\right)^2}{Max\,\left\{k,\left(\frac{1}{5}+\frac{1}{3}\right)^2\right\}} \end{matrix}\right\} = (x+y)^2.$$

$$\varphi\big(M(x,y)\big)M(x,y) = \varphi((x+y)^2)(x+y)^2 = \frac{(x+y)^2}{3+(x+y)^2}.$$

Also, $d(Tx,gy) = \left(\frac{1}{5}+\frac{1}{3}\right)^2 \leq \frac{(x+y)^2}{3+(x+y)^2} = \varphi(M(x,y))M(x,y).$

Case III: $x \in [1,\infty),\ \ y \in [0,1).$

In this case $Tx = \frac{1}{5}$, $gy = \frac{y}{6}$

$$d(x,Tx) = \left(x+\frac{1}{5}\right)^2 d(x,y) = (x+y)^2$$

$$d(y,g\,y) = \left(y+\frac{y}{6}\right)^2 d(Tx,gy) = \left(\frac{1}{5}+\frac{y}{6}\right)^2.$$

Observe that, $\frac{1}{2s}d(x,Tx) = \frac{1}{4}\left(x+\frac{1}{5}\right)^2 < (x+y)^2 = d(x,y).$

Sub-case (a): $y \leq \frac{1}{5}$. Then

$M_T(x,y) = (x+y)^2.$

Consider $d(Tx,gy) = (\frac{1}{5} + \frac{y}{6})^2 \leq \frac{(x+y)^2}{3+(x+y)^2}(x+y)^2 = \varphi(M(x,y))M(x,y)$

Sub-case (b): $y > \frac{1}{5}$, $M(x,y) = (x+y)^2.$

Now; $d(Tx,gy) = (\frac{1}{5} + \frac{y}{6})^2 \leq \frac{(x+y)^2}{3+(x+y)^2}(x+y)^2 = \varphi\big(M(x,y)\big)M(x,y).$

In all cases; $\frac{1}{2s}\ d(x,Tx) < M(x,y)$ and $d(Tx,gy) \leq \varphi\big(M(x,y)\big)M(x,y)$. Therefore, T and g satisfy all the assumptions of Theorem 4.2.1 with common fixed $0 \in X$. That is $T0 = 0 = g0$, moreover it is unique.

In the following we give an example in support of Theorem 4.2.2.

Example 2: Let $X = \mathbb{R}^+$ and $d: X \times X \to \mathbb{R}^+$ be defined by

$$d(x,y) = \begin{cases} 0 & \text{if} \quad x = y; \\ 3 & \text{if} \quad x, y \in [o, 1); \\ 5 + \frac{1}{x+y} & \text{if} \quad x, y \in [1, \infty); \\ \frac{25}{66} & \text{otherwise.} \end{cases} .$$

Then, (X, d) is a complete b-metric space with $s = \frac{25}{24}$.

Observe that, when $x = \frac{3}{2}$, $z = 2 \in [1,\infty)$ and $y \in [0, 1)$ we have

$$d(x,z) = 5 + \frac{1}{x+z} = 5 + \frac{2}{7} = \frac{37}{7} \text{ and } (x,z) + d(y,z) = \frac{25}{66} + \frac{25}{66} = \frac{50}{66},$$

$$d\,(x,z) \nleq d(x,y) + d(y,z).$$

Hence, the given d is b-metric with $s = \frac{25}{24} > 1$ but it is not a metric.

Now, we define $T, g: X \to X$ by

$$Tx = \begin{cases} \frac{x}{4} + 2 & \text{if} \quad x \in [0, 1); \\ 3x - 2 & \text{if} \quad x \in [1, \infty) \end{cases} \qquad \text{and} \qquad gy = \begin{cases} x & \text{if} \quad x \in [0, 1); \\ \frac{1}{x} & \text{if} \quad x \in [1, \infty). \end{cases}$$

Clearly T and g are b-continuous functions. Now, we define

$$\varphi: \mathbb{R}^+ \to [0, \tfrac{1}{\delta}) \qquad \text{and} \qquad \varphi(t) = \frac{1}{t+2}, \; t \geq 0,$$

Let $k \in [0, \infty)$ and take $\delta = \frac{13}{12}$, then we have the following possible cases.

Case I: $x, y \in [0, 1)$.

In this case $\quad Tx = \frac{x}{4} + 2 \in [0,1) \qquad gy = y \in [0,1),$

$$d(x, Tx) = \frac{25}{66}, \qquad d(y, gy) = d(x,y) = 3,$$

$$d(y, Tx) = \frac{25}{66}, \qquad\qquad d(Tx, gy) = \frac{25}{66}.$$

Observe that $\frac{1}{2s}\, d(x, Tx) = \frac{12}{25}\left(\frac{25}{66}\right) = \frac{12}{66} < 3 = d(x,\ y).$

Consider $(Tx, gy) \ \le \begin{Bmatrix} \varphi\big(M_A(x,y)\big)M_A(x,y) \text{ if } & Max\,\{d(x,gy),\ d(Tx,y)\} \neq 0 \\ 0 \qquad\qquad\qquad \text{ if } & Max\,\{d(x,gy),\ d(Tx,y)\} = 0 \end{Bmatrix}.$

$Max\{d(x,y),\ d(Tx,y)\} = Max\left\{3,\ \frac{25}{66}\right\} = 3 \neq 0.$

So $\quad d(Tx, gy) \le \ \varphi(M_A(x,y))M_A(x,y),$

where, $M_A(x,y) = Max \begin{Bmatrix} d(x,y), d(x,Tx), d(y,gy) \\ \dfrac{d(x,Tx)\ d(x,gy)\ +\ d(y\,,gy)\, d(y,Tx)}{Max\,\{k\,,s[\,d(x,Tx)\ +\ d(y,gy)]\}} \\ \dfrac{d(x,Tx)\, d(x,gy)\ \ +\ d(y,gy)\, d(y,Tx)}{Max\,\{\,d(x,gy),\ \ d(y,Tx)\}} \end{Bmatrix}$

$$= Max\left\{3\,,\ \frac{25}{66}, 3\,, \frac{18}{k}, 6\right\} = \ 6.$$

Then $\quad \varphi\big(M_A(x,y)\big)M_A(x,y) = \varphi(6)(6),$

where, $\varphi(t) = \frac{1}{t+2} \Rightarrow \varphi(6)(6) = \frac{(1)(6)}{8} = \frac{3}{4},$

$$d(Tx, gy) = \frac{25}{66} \ \le \ \frac{3}{4} = \varphi\big(M_A(x,y)\big)M_A(x,y).$$

Case II: $x,\ y \in\ [1,\infty).$

In this case, $\quad Tx = 3x - 2\ \in [1,\infty), \qquad gy = \frac{1}{y}\ \in [0,1), \qquad d(x, gy) = 3$

$$d(x,y) = 5 + \frac{1}{x+y}\ \ge\ 5, \qquad d(y, gy) = \frac{25}{66}\,,$$

$$d(y, Tx) = \frac{25}{66} \qquad d(x, Tx) = 5 + \frac{1}{x+y}\ = 5 + \frac{1}{4x-2}\ \ge 5.$$

Consider $\ Max\{d(x,gy), d(Tx,y)\} = Max\left\{3, \frac{25}{66}\right\} = 3 \neq 0.$

$$d(Tx, gy)\ \le\ \varphi\big(M_A(x,y)\big)M_A(x,y)$$

Where $M_A(x,y) = \text{Max}\left\{\begin{matrix} 5+\frac{1}{x+y}, & 5+\frac{1}{4x-2}, & \frac{25}{66} \\ \frac{5+\frac{1}{4x-2}(3)+\left(\frac{25}{66}\right)\left(\frac{25}{66}\right)}{k}, & \frac{\left(5+\frac{1}{4x-2}\right)(3)+\left(\frac{25}{66}\right)\left(\frac{25}{66}\right)}{3} \end{matrix}\right\} = 5+\frac{1}{x+y} > 5.$

Then $\quad d(Tx,gy) = \frac{25}{66} \leq 5+\frac{1}{x+y} = \varphi\big(M_A(x,y)\big)M_A(x,y).$

Case III: $x \in [0,1),\ y \in [1,\infty).$

In this case, $Tx = \frac{x}{4}+2 \in [1,\infty),$ $\quad gy = \frac{1}{y} \in [0,1),$

$d(x,y) = \frac{25}{66},$ $\quad d(x,Tx) = \frac{25}{66},$

$d(x,gy) = 3, d(Tx,gy) = \frac{25}{66},$

$d(y,gy) = \frac{25}{66} d(y,Tx) = 5+\frac{1}{y+\frac{x}{4}+2} \geq 5.$

Consider $\quad Max\{d(x,gy),\ d(Tx,y)\} = \max\left\{3,\ 5+\frac{1}{y+\frac{x}{4}+2}\right\} = 5+\frac{1}{y+\frac{x}{4}+2} \neq 0.$

Then, $\quad d(Tx,gy) \leq \varphi\big(M_A(x,y)\big)M_A(x,y)$

where,

$$M_A(x,y)$$

$$= \text{Max}\left\{\begin{matrix} \frac{25}{66}, & \frac{25}{66}, & \frac{25}{66} \\ \frac{\frac{25}{66}\left[3+\left(5+\frac{1}{y+\frac{x}{4}+2}\right)\right]}{Max\left\{k,\ \frac{13}{12}\left[\frac{25}{66}+\frac{25}{66}\right]\right\}}, & \frac{\frac{25}{66}\left[3+\left(5+\frac{1}{y+\frac{x}{4}+2}\right)\right]}{Max\left\{3,\ 5+\frac{1}{y+\frac{x}{4}+2}\right\}} \end{matrix}\right\}$$

$$= \text{Max}\left\{\begin{matrix} \frac{25}{66}, & \frac{25}{66}, & \frac{25}{66}, \\ \frac{\frac{25}{66}\left[8+\frac{4}{4y+x+8}\right]}{k}, & \frac{\frac{25}{66}\left[8+\frac{4}{4y+x+8}\right]}{5+\frac{4}{4y+x+8}} \end{matrix}\right\}$$

$$= \frac{\frac{25}{66}\left[8+\frac{4}{4y+x+8}\right]}{5+\frac{4}{4y+x+8}}.$$

$$\varphi\big(M_A(x,y)\big)M_A(x,y)=\varphi\left(\frac{\frac{25}{66}\left[8+\frac{4}{4y+x+8}\right]}{5+\frac{4}{4y+x+8}}\right)M_A\left(\frac{\frac{25}{66}\left[8+\frac{4}{4y+x+8}\right]}{5+\frac{4}{4y+x+8}}\right)=\frac{\frac{\frac{25}{66}\left[8+\frac{4}{4y+x+8}\right]}{5+\frac{4}{4y+x+8}}}{2+\frac{\frac{25}{66}\left[8+\frac{4}{4y+x+8}\right]}{5+\frac{4}{4y+x+8}}}.$$

$$d(Tx,gy)\ \leq \varphi\big(M_A(x,y)\big)M_A(x,y).$$

$$d(Tx\,,gy)=\frac{25}{66}\ \leq\ 4\left[\frac{25}{66}\left(3+5\ +\ \frac{1}{y\ +\ \frac{x}{4}\ +\ 2}\right)\right]\frac{25}{66}\left[3\ +\left(5+\ \frac{1}{y\ +\ \frac{x}{4}\ +\ 2}\right)\right].$$

$$=\ \varphi\big(M_A(x,y)\big)M_A(x,y)$$

Case IV: $x\in[1\,,\infty)$and $y\in[0\,,1)$. In this case,

$Tx=3x\ -2\in[1\,,\infty)$, $gy=y\in[0\,,1)$,

$d(x,y)=\ \frac{25}{66}$, $d(x,Tx)=5+\frac{1}{4x-2}$,

$d(y,Tx)=\frac{25}{66}$, $d(y,gy)=3$,

$d(Tx,gy)=\frac{25}{66}$, $d(x,gy)=\frac{25}{66}$.

Consider $Max\{d(x,gy),d(Tx,y)\}=\ Max\left\{\frac{25}{66}\ ,\ \frac{25}{66}\right\}=\frac{25}{66}\neq 0.$

So, $d(Tx,gy)\leq\ \varphi\big(M_A(x,y)\big)M_A(x,y),$

where

$$M_A(x,y)=\text{Max}\left\{\begin{matrix}\frac{25}{66}, & 5+\frac{1}{4x-2}, & 3\\ \frac{\left(5+\frac{1}{4x-2}\right)\left(\frac{25}{66}\right)\ +(3)\left(\frac{25}{66}\right)}{Max\left\{k,\ s\left[5+\frac{1}{4x-2}\ ,\ 3\right]\right\}}, & \frac{\left(5+\frac{1}{4x-2}\right)\left(\frac{25}{66}\right)+(3)\left(\frac{25}{66}\right)}{Max\left\{\frac{25}{66},\ \frac{25}{66}\right\}} & \end{matrix}\right\}$$

$$=\text{Max}\left\{\begin{matrix}\frac{25}{66}, & 5+\frac{1}{4x-2}, & 3\\ \frac{\frac{25}{66}\left[\left(5+\frac{1}{4x-2}\right)\ +3\ \right]}{k}, & \frac{\frac{25}{66}\left[\left(5+\frac{1}{4x-2}\right)\ +3\ \right]}{\frac{25}{66}} & \end{matrix}\right\}$$

$$= \text{Max}\left\{\frac{25}{66}, \quad 5+\frac{1}{4x\ -\ 2}, \ 3, \ \frac{\frac{25}{66}\left[\left(5\ +\ \frac{1}{4x-2}\right)\ +\ 3\ \right]}{k}, \ 8+\ \frac{1}{4x\ -\ 2}\right\}$$

$$= \ 8+\ \frac{1}{4x-2}.$$

$$\varphi\big(M_A(x,y)\big)M_A(x,y) = \varphi\left(8+\ \frac{1}{4x\ -\ 2}\right)\left(8+\ \frac{1}{4x\ -\ 2}\right) = \frac{\left(8\ +\ \frac{1}{4x-2}\right)}{\left(8\ +\ \frac{1}{4x-2}\right)+2}.$$

Therefore,

$$d(Tx,gy) = \frac{25}{66} \leq \frac{8+\frac{1}{4x-2}}{\left(8+\frac{1}{4x-2}\right)+2} = \varphi\big(M_A(x,y)\big)M_A(x,y).$$

In all the cases considered above, T and g satisfy the Inequality (12) and all the assumptions of Theorem 4.2.2 with unique common fixed point $u = 1 \in X$.

CHAPTER FIVE

CONCLUSION AND RECOMMENDATION

5.1 Conclusion

(Rahrovi and Piri, 2019), established fixed point theorems for Suzuki-rational Geraghty contractive mappings in complete b-metric spaces and proved the existence and uniqueness of fixed points. In this research work, we introduced common fixed point result for a pair of mappings satisfying Suzuki-rational Geraghty contractive condition in the setting b-metric spaces. We proved the existence and uniqueness of common fixed points for the mappings introduced. Our results extend and generalize related fixed point results in the literature in particular that of (Rahrovi and Piri, 2019). We have also supported the main results of this thesis work by providing examples.

5. 2 Future scope

There are some published results related to the existence of common fixed point theorems of mappings defined on b-metric spaces. The researcher believe that the search for the existence and uniqueness of common fixed points for a pair of self-mappings satisfying Suzuki-rational Geraghty contraction conditions in b-metric spaces is an active area of study. So, any interested researchers can use this opportunity and conduct their research work in this line of research.

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