An Iterative Algorithm for Approximating a Common Fixed Point of a Finite Family of Pseudo-pseudocontractive Mapping in Hilbert Space.

A RESEARCH SUBMITTED TO THE DEPARTMENT OF MATHEMATICS IN PARTIAL FULFILLMENT FOR THE REQUIREMENTS OF THE DEGREE OF MASTERS OF SCIENCE IN MATHEMATICS

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Declaration

I, Asefa Abera Ulite, with student ID number RM/0646/13,the undersigned declare that, this thesis paper entitled that An iterative algorithm for approximating a common fixed point of a finite family Pseudo- Pseudo Contractive mapping in Hilbert space is my own original work and it has not been submitted to any institution and University elsewhere for the award of any academic degree or like, where other sources of information that have been used or quoted, they have been indicated and acknowledged.
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Abstract

In this thesis we introduced an iterative algorithm for approximating a common fixed point of a finite family Pseudo-pseudocontractive mappings in Hilbert space and proved a strong convergence of a sequence generated by proposed algorithm to a common fixed point in Hilbert spaces provided that the mappings are uniformly continuous which are sequentially weakly continuous. Finally, we applied our main results to find a common minimum point of a finite family of convex functions in Hilbert spaces. Our results extended and generalized many results in the literature.

Acronym

Throughout this research, we denotes the following.

- H is real Hilbert space.
- C is nonempty close and convex subset of Hilbert space.
- $\|.\|$ is the norm space.
- $\langle .,. \rangle$ is the inner product space.
- ∇f is a gradient of function f.
- $\mathbb{R}$ is the set of real number.

Contents

Chapter 1

Introduction

1.1 Background of the study

Let H be a real Hilbert space with inner product $\langle \cdot\,,\,\cdot \rangle$ and induced norm $||.|| = \sqrt{\langle \cdot\,,\,\cdot \rangle}$. Let C be a nonempty subset of H. Let $T : C \longrightarrow H$ be a nonlinear mapping. Many problems that arises in several branches of applied mathematics such as game theory, variational analysis, optimization and differential equations can be reduced to finding solutions of an equation

$$Tx = x \tag{1.1}$$

(see, e.g., Daman (2012), Dugundji (2003), Zegeye (2007) and Zhang (2008) and the references therein). The solutions to this equation are called fixed points of the mapping T. It has been viewed that many of the most important nonlinear map arising in applied sciences areas can reduced to finding the fixed points of a certain mapping.
In particular, fixed point techniques have been applied in diversified fields, such as science, economics, and engineering. Consequently, many authors concentrate on providing iterative algorithms for approximation of fixed points of mappings when they exists or assuming existence (see, e.g., Mann(1953), Berinde (2007), Browder (1968), Khan (2008) and Krasnoselskii (1955)).

The well known method for approximating a fixed point of contraction mapping is the Picard iterations. However, this iteration method may not always converge to a fixed point of T, when T is nonexpansive mapping.
So, for approximating fixed points of the classes of mappings are more general than the class of contraction mappings. Many iterative schemes, such as Mann iteration, Halpern Iteration, Ishikawa iteration, are introduced by different authors (see, e.g., Mann (1953), Halpern (1964), Ishikawa (1974)).

Many authors have also constructed an iterative algorithms called hybrid Mann and hybrid Ishikawa algorithms to obtained strong convergence of the sequence proposed by their method of converging a fixed point of Lipschitz pseudocontractive mappings (see, e.g., Liu et al. (2011), Marino et al. (2009)).

Zegeye and Wega (2020), introduced a new class of mapping which is more general than the class of pseudocotractive mappings called pseudo-pseudocontractive and established an iterative algorithm which converges strongly to a fixed point of pseudo-pseudocontractive mapping provided that the mapping is T is uniformly continuous which is sequentially weakly continuous.
We also remark that several authors have studied an iterative algorithms for approximating a common fixed point of a finite family of nonlinear mappings (see, e.g. Bauschke (1996), Yao et al. (2007), Zhou (2008), Zegeye and Wega (2020)).

Bauschke (1996), introduced Halpern-type iterative algorithm for approximating a common fixed point for a finite family of nonexpansive self mapping and proved that the sequence generated by his method converges strongly to a common fixed point of a finite family of nonexpansive mappings in Hilbert spaces. Zhou (2008),studied an iterative algorithm and proved the sequence generated by his method converges weakly to a common fixed point of a finite family of pseudocontractive mappings in Hilbert spaces.

Inspired and motivated by the above research works the purpose of this thesis is to introduce a new iterative algorithm for approximating a common fixed point of a finite family of pseudo-pseudo contractive mappings in Hilbert spaces.
Moreover, we give an application to the convex minimization problem and construct a numerical example which supports our main result. Our results extend and generalize many results in the literature.

Now, we recall some definitions that the researcher will need in the following sequel.

Definition 1.1.1 *Let $T : C \longrightarrow H$ be mapping,* *i) T is called $L-$ Lipschitz mapping with Lipschtz constant $L > 0$ if*

$$||Tx - Ty|| \leq L||x - y||$$

for all $x, y \in C$. If $0 \leq L < 1$, then T is called contraction. If $L = 1$, then T is called nonexpansive.

ii) T is called pseudocontractive mapping if for all $x, y \in C$ we have that

$$\langle\, x - y\,,\;\; Tx - Ty\rangle \leq ||x - y||^2.$$

iii) T is called to be $\alpha-$strictly pseudocontractive mapping, if there exists a constant $\alpha > 0$ such that for all $x, y \in C$,

$$\langle\, x - y\,,\;\; Tx - Ty\rangle \leq ||x - y||^2 - \alpha||(x - y) - (Tx - Ty)||^2.$$

iv) T is called $T : C \longrightarrow H$ is said to be pseudo-pseudocontractive mapping provided that for each $x, y \in C$, we have:

$$\langle\, x - Tx, y - x\rangle \geq 0 \;\; \textit{implies} \;\; \langle\, y - Ty, y - x\rangle \geq 0.$$

We remark that the class pseudo-pseudocontractive mappings are more general than the classes of mappings mentioned in (i)-(iii) above.

Definition 1.1.2 *The operator T is called sequentially weakly continuous if for each sequence x_n,we have x_n converges weakly to x implies Tx_n converges to Tx.*

1.2 Statements of the Problem

An Iterative methods for approximating common fixed points of nonexpansive mappings have received vast investigations due to its extensive and wide applications in a variety of applied areas of image recovery, inverse problem, convex feasibility problem, partial differential equations and signal processing (see, Noor (2012), Yao (2007), and the references therein). It is known that strictly pseudocontractive mappings have more powerful applications than nonexpansive mappings in solving inverse problems (see, Scherzer (1995)). Consequently, many researchers have studied iterative methods which converges strongly a common fixed point of a finite family of pseudocontractive mappings in Hilbert spaces (see, Zegeye (2011), Daman and Zegeye (2012), Zegeye and Wega (2020)).

Daman and H. Zegeye (2012), established and proved strong convergence of Halpern-Ishikawa iterative method to a common fixed point of a finite family of Lipschitz pseudocontractive mappings without assuming that the interior point of the set of common fixed points of the mappings is nonempty in Hilbert spaces, either on C or on T.

Recently, Zegeye and Wega in (2020), introduced an iterative scheme for a common fixed point of a finite family of Lipschitz pseudocontractive mappings and proved a sequence generated by their proposed algorithm converges strongly to a common fixed point of the mappings in Hilbert spaces. However, an iterative algorithm which converges to a common fixed point of a finite family of pseudo-pseudocontractive mappings is not yet studied in Hilbert spaces.

Inspired and motivated by the research works of Zegeye (2011), Daman and H. Zegeye (2012) and Zegeye and Wega (2020),now in this thesis the researcher was establish a new iterative algorithm for approximating a common fixed point of a finite family of pseudo-pseudocontractive mappings in Hilbert spaces.

1.3 Objectives of the Study

1.3.1 General Objective

The general objective of this thesis was to study an iterative algorithms for approximating a common fixed point of a finite family of pseudo-pseudocontractive mappings in Hilbert spaces.

1.3.2 Specific Objectives

The specific objectives of this thesis is to:

- investigate an iterative algorithm for approximating a common fixed point of a finite family of pseudo-pseudocontractive mappings in Hilbert spaces.
- prove the sequence generated by the proposed algorithm is bounded in Hilbert spaces.
- apply our main result to solve the minimization problems.

1.4 Significance of the Study

The outcome of this study have the following importance:

- It generalized the study of common fixed point of a finite family of mappings in Hilbert spaces.
- It can be used as a base for any next researcher, who is interested to study the approximation of a common fixed point of a finite family of pseudo-pseudo contractive mapping in Hilbert space.
- It may provide some background information for other researchers who want to conduct a research on related topics.

1.5 Delimitation of the Study

This study was delimited to study an iterative algorithm for approximating a common fixed point of a finite family of pseudo-pseudocontractive mappings in Hilbert spaces.

Chapter 2

Review of Related Literatures

Fixed point results give conditions under mappings of fixed point theory in which the desired iterative method converges to the solution. In the last one century the theory of fixed point has been reached as a powerful and important tool in the study of nonlinear problems. Banach (1922), introduced an iterative algorithm called Picard iteration for the class of contraction mappings and given by:

$$x_0 \in C, x_{n+1} = Tx_n, \quad n \geq 0. \tag{2.1}$$

The sequence generated by algorithm converges strongly to a unique fixed point of contraction mapping. However, this method in general failed to converge if T is not a contraction mapping. For instance, the mapping $T : [0,1] \to [0,1]$ defined $T(x) = 1 - x$ has a unique fixed point $\frac{1}{2} \in [0,1]$, it failed to converge. As a result many researchers introduced different types of algorithms for approximating fixed points mappings in Hilbert spaces (see, e.g., Mann (1953), Halpern (1964), Ishikawa (1974)). For approximating a fixed point of Mann introduced an iterative algorithm called Mann iteration Mann (1954), for approximating fixed points of nonexpansive mappings and it is given by

$$x_0 \in C, x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n, \text{ for } n \geq 0, \tag{2.2}$$

where $\{\alpha_n\}$ is a real sequence in the interval $(0,1)$ satisfying certain conditions. However, it is worth mentioning that the sequence generated by this scheme does not always converge strongly to a fixed point of nonexpansive mapping T. To obtain strong convergence of this method to a fixed point of T one has to impose compactness assumption on C (see, e.g., Chidume (1981), Kirk (1981)). Halpern (1964) introduced an iterative scheme called Halpern-iteration and it is given by

$$u,\ x_0 \in C,\ x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n,\ n \leq 1. \tag{2.3}$$

He proved that the sequence generated by algorithm (2.3) converges to a fixed point of nonexpansive mapping T. Ishikawa (1974), construct an iterative scheme called Ishikawa-iteration for approximating fixed points of the class of pseudocontractive mappings which is more general than the class of nonexpansive mapping and the scheme is given by

$$\begin{aligned} x_0 \in C,\ y_n &= (1-\beta_n)x_n + (1-\beta_n)Tx_n, \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T y_n \text{ for } n \geq 0. \end{aligned} \tag{2.4}$$

Where $\{\beta_n\}$ and $\{\alpha_n\}$ are real sequences in the interval $(0,1)$ satisfying certain conditions. He proved that the sequence generated by algorithm (2.4) converges strongly to a fixed point T provided that T is Lipschitz pseudocontractive mapping and C is a compact convex subset of a Hilbert space H.
Zhou (2008), established an iterative algorithm called hybrid Ishikawa algorithm and proved that the sequence generated by his method converges strongly to a fixed point of Lipschitz- pseudocontractive mapping without imposing the condition that C is compact.

Several authors have also established different schemes for approximating a common fixed point of a finite family of nonlinear mappings (see, e.g., Bauschke (1996), Yao et al. (2007), Zhou (2008), Zegeye et al. (2011) ,Takele and Reddy (2017)).

Zegeye et al. (2011), introduced Ishikawa iterative algorithm and proved that the sequence proposed by their method converges to strong convergence a common fixed point of finite family of Lipschitz pseudocontractive mappings in the setting of Hilbert spaces provided that interior of the set of common fixed points of the mappings is nonempty.

Takele and Reddy (2017), also approximates a common fixed point of a family of nonself and non-expansive mapping in Hilbert spaces and they also proved weak and strong convergence theorems.

2.1 Preliminaries

H is real Hilbert space.

$$||P_c x - x|| = inf_{y\in c}||x-y||, \tag{2.5}$$

hence, P_c satisfies: $\|P_c x - P_c y\|^2 \leq \langle P_c x - P_c y, x-y\rangle$, for all $x,y \in H$.

Lemma 2.1.1 *For all $x,y \in H$, it is known that the following inequality hold.*

i) $\|x+y\| \leq \|x\| + \|y\|$

ii) $\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$

iii) $\|x-y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x-y, y\rangle$

iv) $\|x+y\|^2 = \|x\|^2 + 2\langle x+y, y\rangle$

Lemma 2.1.2 *(Albert,1996). Let C be complex subset of a real Hilbert space H and let $x \in H$, then*

$$x_0 = P_c x \textit{ if and only if } \langle z-x_0, x-x_0\rangle \leq 0, \textit{ for every } z \in C.$$

Lemma 2.1.3 *(Xu, 2002). Let $\{\alpha_n\}$ be a sequence of nonnegative real numbers that satisfying the following relation:*

$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n\gamma_n$ *for* $n \geq n_0$ *where* $\{\alpha_n\} \subseteq (0,1)$ *and* $\{\gamma_n\} \subseteq \mathbb{R}$, *satisfies* $lim_{x\to\infty}\alpha_n = 0$,

$$\sum_{n=1}^{\infty} \alpha_n = \infty, \textit{ and } lim \quad sup_{n\to\infty}\gamma_n \leq 0. \textit{ Then } lim_{n\to\infty}a_n = 0.$$

Lemma 2.1.4 *(Mainge, 2008). Let $\{a_n\}$, be sequence of real numbers such that there exists a subsequence $\{a_{n_i}\}$ of $\{a_n\}$ such that $a_{ni} < a_{ni+1}$, for $i \in N$. Then there exists a non decreasing sequence $\{m_k\} \subset N$,, such that $m_k \to \infty$,and the following properties are satisfied for numbers $k \in N$:*

$$a_{mk} \leq a_{mk+1} \textit{ in fact } m_k = max\{j \leq k : a_j \leq a_{j+1}\}.$$

Lemma 2.1.5 *(Zegeye and Shahzad). Let H be a real Hilbert space. Then for all* $x_i \in H$ *and*$\alpha_i \in [0,1]$ *for* $i = 1,2,3,...n$, *such that* $\alpha_1 + \alpha_2 + \alpha_3 + ... + \alpha_n = 1$, *then the following holds*
$\|\alpha_0 x_0 + \alpha_1 x_1 + ... + \alpha_n x_n\|^2 = \sum_{i=0}^{n} \alpha_i \|x_i\|^2 - \sum_{0 \le i,j \le n} \alpha_i \alpha_j \|x_i - x_j\|^2$.

Lemma 2.1.6 *(He,2006). Let C, be a nonempty, closed and convex subset of H Let* $r(x)$, *be a real valued function on H and defined* $K := \{x \in C : r(x) \le 0\}$. *If K, is nonempty and r is L-Lipshitz continuous with* $L > 0$, *then* $||P_K x - x|| \ge \frac{1}{L} \max\{r(x), 0\}$, *for* $x \in C$.

Chapter 3

Methodology

This chapter contains study site and period, study design, source of information and mathematical procedures.

3.1 Study Area and Period

The study was conducted from September 2021 to June 2022 in Jimma University under the department of mathematics and conceptually the study was focused on studying an iterative algorithm for approximating a common fixed point of a finite family of pseudo-pseudocontractive in real Hilbert spaces.

3.2 Study Design

In order to achieve the objectives of this study was employ analytical methods of designing.

3.3 Sources of Information

The relevant sources of information for this study was published articles, books of different mathematics which related to our research topic.

3.4 Mathematical Procedure

In this thesis, we were follow the standard mathematical procedures given below:

- Establishing an iterative algorithm and constructing theorem for approximating a common fixed point of a finite family of pseudo-pseudomonotone mappings.
- Proving strong convergence of the sequence proposed by the method to a common fixed point of a finite family of pseudo-pseudomonotone mappings.
- Applying our main result to solve minimization problems.

Chapter 4

Main Results

In this section we introduce a common fixed point of a finite family of Pseudo-Pseudo contractive mapping in Hilbert space and we shall using the following assumptions and algorithm for approximation.

Assumption 1:

A1: Let $T_1, T_2 : H \to H$ be uniformly continuous Pseudo-pseudocontractive mappings which are sequentially weakly continuous on bounded subset of H.

A2: $\Omega = F(T_1) \bigcap F(T_2) \neq \emptyset$.

A3: Let $\iota \in (0,1), \mu > 0$ and $\lambda \in [\lambda', \lambda''] \subset (0, \frac{1}{\mu})$

A4: Let $\{\alpha_n\} \subset (0, \varepsilon)$ for some constant real number , $\varepsilon > 0$ be a real sequence such that,

$$lim_{n\to\infty}\alpha_n = 0, \text{ and } \sum_{n=1}^{\infty} \alpha_n = \infty.$$

Remark 4.1 *Note that if we have* $r(x_n) = s(x_n) = 0$ *for some* $n \in N$*, then we get,* $x_n \in \Omega = F(T_1) \cap F(T_2)$*, since we have* $r(x_n) = x_n - z_n = 0$ *implies that,* $x_n = (1 - \lambda)x_n + \lambda T_1 x_n$ *which gives us,* $\lambda(T_1 x_n - x_n) = 0$*, and hence* $T_1 x_n = x_n$*. Similarly,* $s(x_n) - x_n - u_n = 0$*, implies,* $T_2 x_n = x_n$*. Thus,* $T_1 x_n = T_2 x_n = x_n$ *and hence,* $x_n \in \Omega = F(T_1) \cap F(T_2)$*. For the rest of the study we consider only the case that this equality does not hold.*

Lemma 4.0.1 *Suppose that the assumption* A_1 *and* A_2 *hold and* $\{x_n\}, \{y_n\}, \{z_n\}, \{u_n\}$ *and* $\{v_n\}$ *are sequences, generated by Algorithm* 1*.Then the search rules in step two are well defined.*

Proof: Since $\iota \in (0,1)$, T_1 andT_2 are uniformly continuous on H, We have

$$\langle \iota^j (r(x_n) + T_1(x_n - \iota^j r(x_n)) - T_1 x_n, r(x_n) \rangle \to 0 \text{ as } j \to \infty,$$

Algorithm 1: For arbitrary x_0 and $u \in H$, define an iterative algorithm by

Step 1. Compute

$$\begin{cases} z_n = (1-\lambda)x_n + \lambda T_1 x_n \text{ and } r(x_n) = x_n - z_n, \\ u_n = (1-\lambda)x_n + \lambda T_2 x_n, \text{ and } s(x_n) = x_n - u_n. \end{cases} \tag{4.1}$$

Step 2. Compute

$$\begin{cases} y_n = x_n - \Gamma_n r(x_n), \\ v_n = x_n - \Gamma' s(x_n), \end{cases} \tag{4.2}$$

where, $\Gamma_n = \iota^{j_n}$ and j^n is the smallest non negative integer j satisfying;

$$\langle \iota^j (r(x_n)) + T_1(x_n - \iota^j r(x_n)) - T_1 x_n, r(x_n)\rangle \leq \|r(x_n)\|^2,$$

$\Gamma_n' = \iota^{j'n}$ and j'^n is the smallest nonnegative integer and J' satisfying

$$\langle \iota^{j'} s(x_n) + T_2(x_n - \iota^{j'} s(x_n)) - T_2 x_n, s(x_n)\rangle \leq \|s(x_n)\|^2\rangle$$

Step 3. Compute

$$\begin{cases} p_n = p_{C_n} x_n, \\ q_n = p_{D_n} x_n, \\ w_n = \theta_n x_n + \beta_n p_n + \eta_n q_n, \end{cases} \tag{4.3}$$

where $C_n = \{x \in H : \langle y_n - T_1 y_n, x - y_n\rangle \leq 0\}$,
$D_n = \{x \in H : \langle v_n - T_2 v_n, x - v_n\rangle \leq 0\}$ and $\theta_n, \beta_n, \eta_n$ are nonnegative real numbers such that, $\beta_n + \theta_n + \eta_n = 1$ for all $n \geq 0$.

Step 4. Compute

$$x_{n+1} = \alpha_n u + (1-\alpha_n) w_n. \tag{4.4}$$

and

$$\langle \iota^{j'}(s(x_n)+T_2(x_n-\iota^{j'}s(x_n))-T_2x_n, s(x_n)\rangle \to 0 \text{ as } j'\to\infty.$$

Moreover, since $\|r(x_n)\|>0$ and $\|s(x_n)\|>0$ there exist a non-negative integers j_n and j'_n , satisfying the inequalities in Step 2. □

Lemma 4.0.2 *Suppose that the assumption A_1-A_3 hold. If $\{x_n\},\{y_n\},\{z_n\},\{u_n\},\{v_n\}$ are sequences generated by Algorithm 1 then, $\langle x_n-T_1x_n, r(x_n)\rangle=\frac{1}{\lambda}\|r(x_n)\|^2$ and*

$$\langle x_n-T_2x_n, r(x_n)\rangle=\frac{1}{\lambda}\|s(x_n)\|^2 \tag{4.5}$$

Proof: From equations (4.1), we have, $z_n=(1-\lambda)x_n+\lambda T_1x_n$ which gives us, $z_n-x_n=\lambda(T_1x_n-x_n)$ and hence

$$\frac{z_n-x_n}{\lambda}=T_1x_n-x_n \tag{4.6}$$

Thus, from the fact that $r(x_n)=x_n-z_n$, we get

$$\begin{aligned}\langle x_n-T_1x_n, r(x_n)\rangle &= \langle\frac{x_n-z_n}{\lambda}, x_n-z_n\rangle\\ &= \frac{1}{\lambda}\langle x_n-z_n, x_n-z_n\rangle\\ &= \frac{1}{\lambda}\|x_n-z_n\|^2 \end{aligned}\tag{4.7}$$

Similarly, we get

$$\langle x_n-T_2x_n, s(x_n)\rangle=\frac{1}{\lambda}\|x_n-u_n\|^2 \tag{4.8}$$

□

Lemma 4.0.3 *Suppose the assumptions A_1-A_3 holds. Let $x^*\in\Omega$, let $h_n(x_n)=\langle y_n-T_1y_n, x_n-y_n\rangle$ and let $g_n(x_n)=\langle v_n-T_2v_n, x_n-v_n\rangle$. Then, $h_n(x^*)\le 0,\quad g_n(x^*)\le 0,\quad h_n(x_n)\ge\Gamma_n(\frac{1}{\lambda}-\mu)\|r(x_n)\|^2$, and $g_n(x_n)\ge\Gamma'_n(\frac{1}{\lambda}-\mu)\|s(x_n)\|^2$ In particular, if $r(x_n)\ne 0$ and $s(x_n)\ne 0$, then $h(x_n)>0$ and $g(x_n)>0$.*

Proof: For the fact that $x^*\in\Omega$, we have

$$\langle x^*-T_1x^*, y_n-x^*\rangle\ge 0. \tag{4.9}$$

This inequality and the fact that T_1 is Pseudo-pseudocontractive mapping, we obtain

$$h_n(x^*) = \langle y_n - T_1 y_n, y_n - x^* \rangle \geq 0,$$

which gives us,

$$h_n(x^*) = \langle y_n - T_1 y_n, x^* - y_n \rangle \leq .0$$

In addition, from Step 2, of Algorithm 1, we have,

$$\begin{aligned} h_n(x_n) &= \langle y_n - T_1 y_n, x_n - y_n \rangle \\ &= \langle y_n - T_1 y_n, x_n - (x_n - \Gamma_n r(x_n)) \rangle \\ &= \Gamma_n \langle y_n - T_1 y_n, r(x_n) \rangle \end{aligned}$$

Furthermore, from the inequalities in Step 2, we have,

$$\langle x_n - y_n + T_1 y_n - T_1 x_n, r(x_n) \rangle \leq \mu \|r(x_n)\|^2,$$

which implies

$$\langle y_n - T_1 y_n, r(x_n) \rangle \geq \langle x_n - T_1 y_n, r(x_n) \rangle - \mu \|r(x_n)\|^2 \tag{4.10}$$

From Lemma 4.0.2 and inequality above, we obtain

$$\langle y_n - T_1 y_n, r(x_n) \rangle \geq (\frac{1}{\lambda} - \mu) \|r(x_n)\|^2 \tag{4.11}$$

By combining (4.10) and (4.11), we obtain,

$$h_n(x_n) \geq \Gamma_n (\frac{1}{\lambda} - \mu) \|r(x_n)\|^2.$$

Similarly, we obtain,

$$g_n(x_n) \geq \Gamma'_n (\frac{1}{\lambda} - \mu) \|s(x_n)\|^2.$$

□

Theorem 4.0.4 *Suppose the assumptions $A_1 - A_4$ hold. Then, the sequence $\{x_n\}$, generated by the Algorithm* 1 *is bounded in Hilbert space, H.*

Proof: Let $P \in \Omega$ from Lemma 2.1.5 and equation (2.5), we obtain

$$\begin{aligned}
\|X_{n+1} - p\| &= \|\alpha_n u + (1-\alpha_n)w_n - p\| \\
&= \|\alpha_n u + (1-\alpha_n)w_n - \alpha_n p + (1-\alpha_n)p\| \\
&= \|\alpha_n(u-p) + (1-\alpha_n)(w_n - p\| \\
&\leq \alpha_n\|u-p\| + \|(1-\alpha_n)(w_n - p\| \\
&= \alpha_n\|(u-p) + (1-\alpha_n)\|\|(\theta_n x_n + \beta_n p_n + \eta_n q_n) - p\| \\
&= \alpha_n\|u-p\| + (1-\alpha_n)\|(\theta_n(x_n - p) + \beta_n(p_n - p) + \eta_{n(}q_n - p)\| \\
&\leq \alpha_n\|u-p\| + (1-\alpha_n)[\theta_n\|x_n - p\| + \beta_n\|p_n - p\| + \eta_n\|q_n - p\|] \\
&= \alpha_n\|u-p\| + (1-\alpha_n)[\theta_n\|x_n - p\| + \beta_n\|(p_{C_n}x_n - p)\| + \eta_n\|p_{D_n}x_n - p\|] \\
&\leq \alpha_n\|u-p\| + (1-\alpha_n)[\theta_n\|x_n - p\| + \beta_n\|x_n - p\| + \eta_n\|x_n - p\|] \\
&= \alpha_n\|u-p\| + (1-\alpha_n)[\theta_n + \beta_n + \eta_n)\|x_n - p\|] \\
&= \alpha_n\|u-p\| + (1-\alpha_n)\|x_n - p\|]
\end{aligned}$$

Hence, by induction

$$\|x_{n+1} - p\| \leq max\{\|u-p\|, \|x_0 - p\|\}.$$

Thus, the sequence $\{x_n\}$ is bounded and hence the sequences $\{z_n\}, \{y_n\}, \{u_n\}, \{T_1 x_n\}$ and $\{T_2 x_n\}$ are bounded. □

Theorem 4.0.5 *Suppose the assumption $A_1 - A_4$ hold. Then the sequence $\{x_n\}$, generated by the algorithm 1, converges strongly to $p = P_\Omega(u)$.*

Proof: Now, let $P = P_\Omega(u)$. From equation 2.5, we have,

$$\|p - p_n\|^2 \leq \|p - x_n\|^2 - \|x_n - p_n\|^2.$$

Similarly, we get

$$\|p - q_n\|^2 \leq \|p - x_n\|^2 - \|x_n - q_n\|^2 \tag{4.12}$$

Since T_1 is bounded on bounded subset of H ,Then their exists $L > 0$, such that

$$\|T_1 y_n - y_n\| \leq L,$$

for all $n \geq 0$. Thus,

$$\begin{aligned}|h_n(z) - h_n(w)| &= |\langle y_n - T_1 y_n, z - y_n\rangle - \langle y_n - T_1 y_n, z - y_n\rangle| \\ &= |\langle y_n - T_1 y_n, z - w\rangle \\ &\leq |y_n - T_1 y_n\| \|z - w\| \\ &\leq L\|z - w\|,\end{aligned}$$

which gives us that h_n is L- Lipschitz continuous on H. Thus, from Lemma 2.1.6 and Lemma 4.0.3, we obtain

$$\|x_n - p_n\|^2 \geq \frac{h_n x_n}{2L^2} \geq \Gamma_n^2 (\frac{1}{\lambda} - \mu)^2 \|r(x_n)\|^2 \tag{4.13}$$

Thus, from (4.12) and (4.13), we get

$$\|p - p_n\|^2 \leq \|p - x_n\|^2 - \Gamma_n^2 (\frac{1}{\lambda} - \mu)^2 \|r(x_n)\|^4 \tag{4.14}$$

Similarly, we get

$$\|p - q_n\|^2 \leq \|p - x_n\|^2 - \Gamma_n'^2 (\frac{1}{\lambda} - \mu)^2 \|s(x_n)\|^4 \tag{4.15}$$

By Lemma 2.1.1 and Lemma 2.1.5, we get

$$\begin{aligned}\|x_{n+1} - p\|^2 &= \|\alpha_n u + (1 - \alpha_n) w(n) - p\|^2 \\ &\leq (1 - \alpha_n)^2 \|w_n - p\|^2 + 2\alpha_n \langle u - p, x_{n+1} - p\rangle \\ &= (1 - \alpha_n)^2 \|w_n - p\|^2 + 2\alpha_n \langle u - p, x_{n+1} - x_n + x_n - p\rangle \\ &= (1 - \alpha_n)^2 \|w_n - p\|^2 + 2\alpha_n \langle u - p, x_{n+1} - x_n\rangle + 2\alpha_n \langle u - p, x_n - p\rangle \\ &\leq (1 - \alpha_n)^2 \|w_n - p\|^2 \\ &\quad + 2\alpha_n + \|u - p\| \|x_{n+1} - x_n\| + 2\alpha_n \langle u - p, x_n - p\rangle\end{aligned}$$

Since from Lemma 2.1.5, we have

$$\begin{aligned}
\|w_n - p\|^2 &= \|\theta_n x_n + \beta_n p_n + \eta_n q_n - p\|^2 \\
&= \|\theta_n(x_n - p) + \beta_n(p_n - p) + \beta_n(q_n - p)\|^2 \\
&\leq \theta_n\|x_n - p\|^2 + \beta_n\|p_n - p\|^2 + \eta_n\|q_n - p\|^2 \\
&\quad -\theta_n\beta_n\|x_n - p_n\|^2 - \theta_n\eta_n\|x_n - q_n\|^2 - \theta_n\beta_n\|p_n - q_n\|^2
\end{aligned}$$

By setting

$$R_n = \theta_n\beta_n\|x_n - p_n\|_n^2 + \theta_n\eta_n\|x_n - q_n\|^2 + \theta_n\beta_n\|p_n - q_n^2\|,$$

from ((4.14)), and ((4.15)) we get

$$\begin{aligned}
&\|xn+1-p\|^2 \leq (1-\alpha_n)^2[\theta_n\|x_n - p\|^2 + \beta_n\|p_n - p\|^2 \\
&+\eta_n\|q_n - p\|^2](1-\alpha_n)^2 R_n + 2\alpha_n\|u-p\|\|x_{n+1} - x_n\| \qquad (4.16)\\
&+2\alpha_n\langle u-p, x_n - p\rangle \\
&\leq (1-\alpha_n)^2[\theta_n\|x_n - p\|^2 + \beta_n\|p_n - p\|^2 + \eta_n\|q_n - p\|^2] \\
&-(1-\alpha_n)^2 R_n + 2\alpha_n\|u-p\|\|x_{n+1} - x_n\| + 2\alpha_n\langle u-p, x_n - p\rangle \\
&-\beta_n\Gamma_n^2(\tfrac{1}{\lambda} - \mu)^2\|r(x_n)\|^4 - \eta_n\Gamma_n'^2(\tfrac{1}{\lambda} - \mu)^2\|s(x_n)\|^4 \\
&= (1-\alpha_n)^2\|x_n - p\|^2 - (1-\alpha_n)^2 R_n + 2\alpha_n\|u-p\|\|x_{n+1} - x_n\| \\
&+2\alpha_n\langle u-p, x_n - p\rangle - (\tfrac{1}{\lambda} - \mu)^2[\Gamma_n^n\beta_n\|r(x_n)\|^4 + \eta_n\Gamma_n'^2\|s(x_n)\|^4, \qquad (4.17)
\end{aligned}$$

which gives us

$$\begin{aligned}
&(1-\alpha_n)^2 R_n + (\frac{1}{\lambda} - \mu)^2[\Gamma_n^2\beta_n\|r(x_n)\|^4 + \eta_n\Gamma_n'^2\|s(x_n)\|^4] \\
\leq\ &(1-\alpha_n)^2\|x_n - p\|^2 - \|x_{n+1} - p\|^2 + 2\alpha_n\|u-p\|\|x_{n+1} - x_n\| + 2\alpha_n\langle u-p, x_n - p\rangle \\
\leq\ &\|x_n - p\|^2 - \|x_{n+1} - p\|^2 + 2\alpha_n\|u-p\|\|x_{n+1} - x_n\| + 2\alpha_n\langle u-p, x_n - p\rangle. \qquad (4.18)
\end{aligned}$$

Now, we that the sequence $\{\|x_n - p\|\}$ converges strongly to zero. For this we consider two cases as follows:

Case 1: Assume that there exist $n_0 \in N$, such that the sequence of real numbers $\{\|x_n - p\|^2\}$ is decreasing for all $n \geq n_0$. Thus, the sequence $\{\|x_n - p\|^2\}$ convergent and hence from (4.18) and the fact that $\alpha_n \to 0$, we obtain $lim_{n\to\infty} R_n = 0$, which implies that

$$lim_{n\to\infty}\|x_n - p\|^2 = lim_{n\to\infty}\|x_n - q_n\|^2 = lim_{n\to\infty}\|p_n - q_n\|^2 = 0.$$

In addition, we have

$$lim_{n\to\infty}\Gamma_n^2\|r(x_n)\|^4 = lim_{n\to\infty}\Gamma_n'^2\|s(x_n)\|^4 = 0.$$

Then, from this we obtain that

$$lim_{n\to\infty}\Gamma_n\|r(x_n)\|^2 = lim_{n\to\infty}\Gamma_n'\|s(x_n)\|^2 = 0. \tag{4.19}$$

Since the sequence $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$, of $\{x_n\}$ which converges weakly to $q \in H$ and

$$\lim_{n\to\infty}\langle u-p, x_n - q\rangle = \lim\ \sup_{n\to\infty}\langle u-p, x_{n_k} - q\rangle \tag{4.20}$$

Now, we prove that

$$\lim k\to\infty\|x_{n_k} - z_{n_k}\| = 0, \lim k\to\infty\|x_{n_k} - u_{n_k}\| = 0 \tag{4.21}$$

First consider the case, when $\liminf_{k\to\infty}\Gamma_{n_k} > 0$ In this case there is $\Gamma > 0$ such that $\Gamma_{n_k} > \Gamma > 0$, for all $k \in N$. Thus, we have

$$\|x_{n_k} - z_{n_k}\|^2 = \frac{1}{\Gamma_{n_k}}\Gamma_{n_k}\|x_{n_k} - z_{n_k}\|^2 \leq \frac{1}{\Gamma}\Gamma_{n_k}\|x_{n_k} - z_{n_k}\|^2.$$

From this inequality and ((4.19)), we obtain

$$\lim k\to\infty\|x_{n_k} - z_{n_k}\|^2 = 0$$

and hence

$$\lim k\to\infty\|x_{n_k} - z_{n_k}\|$$

Second consider, when $liminf_{k\to\infty}\Gamma_{n_k} = 0$. In this case

$$\lim\ _{k\to\infty}\Gamma_{n_k} = 0 \text{ and } \lim\ _{k\to\infty}\|x_{n_k} - z_{n_k}\|^2 = c > 0 \tag{4.22}$$

Consider, $y'_{n_k} = \frac{1}{\iota}\Gamma_{n_k} z_{n_k} + (1 - \frac{1}{\iota}\Gamma_{n_k})x_{n_k}$
Thus, from (4.22), we have

$$\lim_{k\to\infty} \|y'_{n_k} - z_{n_k}\| = \lim_{k\to\infty} \frac{1}{\iota}\Gamma_{n_k}\|.x_{n_k} - z_{n_k}\| = 0 \tag{4.23}$$

From inequality in Step 2 and definition of y'_{n_k}, we obtain

$$\begin{aligned} \mu\|x_{n_k} - z_{n_k}\|^2 &< \langle x_{n_k} - y'_{n_k} + T_1 y'_{n_k} - T_1 x_{n_k}, x_{n_k} - z_{n_k}\rangle \\ &\leq \langle x_{n_k} - y'_{n_k}, x_{n_k} - z_{n_k}\rangle + \langle T_1 y'_{n_k} - T_1 x_{n_k}, x_{n_k} - z_{n_k}\rangle \\ &\leq \|x_{n_k} - y'_{n_k}\|\|x_{n_k} - z_{n_k}\| + \|T_1 y'_{n_k} - T_1 x_{n_k}\|\|x_{n_k} - z_{n_k}\|. \end{aligned} \tag{4.24}$$

From (4.23), (4.24) and the fact that T_1 is uniformly continuous, we get $\lim_{n\to\infty}\|x_{n_k} - z_{n_k}\| = 0$, which contradict (4.22). Thus, from this fact the equation (4.21) holds. Furthermore, from Step 1 of the Algorithm 1, we have

$$z_{n_k} = (1-\lambda)x_{n_k} + \lambda T_1 x_{n_k},$$

which gives as

$$\|z_{n_k} - x_{n_k}\| = \lambda\|x_{n_k} - T_1 x_{n_k}\|. \tag{4.25}$$

Hence, from equation (4.21), we obtain,

$$\lim_{k\to\infty}\|x_{n_k} - T_1 x_{n_k}\| = 0. \tag{4.26}$$

Similarly, we get

$$\lim_{k\to\infty}\|x_{n_k} - T_2 x_{n_k}\| = 0. \tag{4.27}$$

From (4.26) and the fact that T_1 is sequentially weakly continuous, we get $q \in F(T_1)$. Similarly, $q \in F(T_2)$. Therefore, $q \in \Omega$. From the definition of x_{n+1}, we have $\|x_{n+1} - w_n\| = \alpha_n\|u - w_n\| \to 0$, as $n \to \infty$, since $\alpha_n \to \infty$, as $n \to \infty$ and from (4.18), we get

$$\begin{aligned} \|x_{n+1} - w_n\| &\leq \|x_{n+1} - w_n\| + \|w_n - x_n\| \\ &\leq \|x_{n+1} - w_n\| + \|\theta_n x_n + \beta_n p_n + \eta_n q_n - x_n\| \\ &\leq \|x_{n+1} - w_n\| + \theta_n\|x_n - x_n\| \qquad (4.28) \\ &\quad + \beta_n\|p_n - x_n\| + \eta_n\|q_n - x_n\| \to 0 \text{ as } n \to 0. \qquad (4.29) \end{aligned}$$

Moreover,

$$\|x_n - w_n\| \le \theta_n \|x_n - x_n\| + \beta_n \|p_n - x_n\| + \eta_n \|q_n - x_n\| \to 0, \text{ as } n \to 0. \quad (4.30)$$

Thus, from (4.28) and (4.30), we obtain

$$\begin{aligned} \|x_{n+1} - x_n\| &= \|x_{n+1} - w_n + w_n - x_n\| \\ &\le \|x_{n+1} - w_n\| + \|w_n - x_n\| \to 0, \text{ as } n \to 0. \quad (4.31) \end{aligned}$$

From (4.20), we have

$$\begin{aligned} \text{lim sup}_{n\to\infty} \langle u - p, x_n - p \rangle &\le \text{lim sup}_{k\to\infty} \langle u - q, x_{n_k} - q \rangle \\ &= \langle u - p, q - p \rangle \\ &\le 0. \quad (4.32) \end{aligned}$$

Finally, from (4.16), (4.31), (4.32) and Lemma 2.1.3, we get $\|x_n - p\|^2 \to 0$, as $n \to \infty$ and hence $x_n \to p$, as $n \to \infty$.

Case 2: Suppose that there exists a subsequence $\{\|x_{n_i} - p\|^2\}$ of $\{\|x_n - p\|^2\}$ such that

$$\|x_{n_i} - p\|^2 < \|x_{n_i+1} - p\|^2, \text{ for } i \ge 0. \quad (4.33)$$

Thus by Lemma 2.1.4 there exists a non-decreasing sequence $\{m_k\}$,of the set of positive integer of numbers such that $m_k \to 0$, as $k \to \infty$,
$\|x_{m_k} - p\|^2 \le \|x_{m_k+1} - p\|^2$ and

$$\|x_{m_k} - p\|^2 \le \|x_{m_k+1} - p\|^2, \text{ for all } k \ge 1. \quad (4.34)$$

Hence, by following the method of Case 1, from the inequality (4.20), we obtain

$$lim_{k\to\infty} R_{m_k} = 0,$$

and hence,

$$lim_{k\to\infty} \|x_{m_k} - p_{m_k}\| = lim_{k\to\infty} \|x_{n_k} - q_{m_k}\| = 0.$$

In addition,

$$lim_{k\to\infty} \|x_{m_k} - z_{m_k}\| = 0,$$

$$lim_{k\to\infty}\|x_{m_k+1}-x_{m_k}\|=0,$$

and

$$limsup_{n\to\infty}\langle u-p,x_{m_k}-p\rangle\leq 0.$$

Thus, from Lemma 2.1.5, we have

$$\begin{aligned}\|x_{m_k+1}-p\|^2 &\leq (1-\alpha_{m_k})\|x_{m_k+1}-p\|^2 \\ &\quad +\alpha_{m_k}\|u-p\|\|x_{m_k+1}-x_{m_k}\|+\alpha_{m_k}\langle u-p,x_{m_k}-p\rangle. \end{aligned} \tag{4.35}$$

Now, from (4.34) and (4.35) we get

$$\begin{aligned}\|x_k-p\|^2 &\leq \|x_{m_k+1}-p\|^2 \\ &\leq \langle u-p,x_{m_k}-p\rangle+\|u-p\|\|x_{m_k+1}-x_{m_k}\| \end{aligned} \tag{4.36}$$

which implies, $\lim_{k\to\infty}\|x_k-p\|^2=0$ and hence $x_k\to p$ as, $k\to\infty$. □

Algorithm 2: For arbitrary x_0 and $u\in H$, define an iterative algorithm by

---

Step 1. Compute
$z_{n,i}=(1-\lambda)x_n+\lambda T_ix_n$ and $r_i=x_n-z_{n,i}$, for $i=1,2,...m$

Step 2. Compute

$$y_{n,i}=x_n-\Gamma_{n,i}r_n(x_n),$$

for $i=1,2,3,...m$, where,$\Gamma_{n,i}=\iota^{jn,i}$ and j^n is the smallest non negative integer j^isatisfying

$$\langle\iota^{j,i}r_i(x_n)+T_i(x_n-\iota^{j,i}r_i(x_n))-T_ix_n,r_i(x_n)-T_ix_n,r_i(x_n)\rangle\leq\mu\|r_i(x_n)\|^2$$

Step 3. Compute

$$w_n=\beta_{n,1}u_{n,1}+\beta_{n,2}u_{n,2}+...+\beta_{n,m}u_{n,m},$$

where, $u_{n,i}=P_{C_{n,i}}x_n$, $C_{n,i}=\{x\in H:\langle y_{n,i}-T_iy_{n,i},x_n-y_{n,i}\rangle\leq 0\}$ and $\beta_{n,1}+\beta_{n,2}+...+\beta_{n,m}=1$

Step 4. Compute

$$x_{n+1}=\alpha_nu+(1-\alpha_n)w_n.$$

We remark that the method of proof of Theorem 4.0.5, provides the following theorem for approximating a common fixed point of a finite family of uniformly continuous pseudo-pseudocontractive mappings in Hilbert spaces.

Theorem 4.0.6 *Let H be a Hilbert space, suppose A_1 and A_2 hold. Let $T_i : H \to H$ be uniformly continuous pseudo-pseudocontractive mappings which are sequentially weakly continuous on bounded subset of H , such that $\Omega = \cap_{i=1}^{m} F(T_i) \neq \varnothing$, for $i = 1,2,...m$. Then, the sequence $\{x_n\}$, generated by the Algorithm 2, converges strongly to $p = P_\Omega(u)$.*

We remark that from Theorem 4.0.5 we obtain the following the following result for two pseudocontractive mappings which are sequentially weakly continuous on bounded subset of H.

Corollary 4.0.7 *Suppose the assumption $A_2 - A_4$ hold. Let $T_1, T_2 : H \to H$ be uniformly continuous pseudocontractive mappings which are sequentially weakly continuous on bounded subset of H; Then the sequence $\{x_n\}$ generated by the Algorithm 1, converges strongly to $p = P_\Omega(u)$.*

We remark that from Theorem 4.0.6 we obtain the following the following result for a finite family of pseudocontractive mappings which are sequentially weakly continuous on bounded subset of H.

Corollary 4.0.8 *Let H be a real Hilbert space, suppose A_3 and A_4 hold. Let T_2 : $H \to H$, be uniformly continuous pseudo contractive mappings which are sequentially weakly continuous on bounded subset of H , such that $\Omega = \cap_{i=1}^{m} F(T_i) \neq \varnothing$, for $i = 1,2,...m$. Then, the sequence $\{x_n\}$, generated by the Algorithm 2, converges strongly to $p = P_\Omega(u)$.*

4.1 Application to Convex Minimization Problem

In this section, we apply Corollary 4.0.7 to find a common minimum point of a finite family of convex functions in Hilbert Spaces.

Let $f : H \to \mathbb{R}$ be a convex smooth function. We consider the problem of finding a point $z \in E$ such that

$$f(z) = \min_{x \in E} \{f(x)\}. \tag{4.37}$$

According to Fermat's rule, this problem is equivalent to the problem of finding $z \in H$ such that

$$\nabla fz = 0, \tag{4.38}$$

where ∇f is a gradient of f. We note that ∇f is monotone mapping (see, e.g., , Peypouquet, J. (2015)) and hence pseudomonotone mapping.
We note that T is pseudocontractive if and only if $A := I - T$, where I is the identity mapping on H, is monotone and hence the set of fixed points of T, $F(T) : \{x \in D(T) : Tx = x\}$, is the set of zero points of A, $N(A) := \{x \in D(A) : Ax = 0\}$. Now, if in Algorithm 2, we assume $\nabla f_i = I - T_i$, then we obtain the following Algorithm 3 for a common minimum point problem of a finite family convex functions in Hilbert spaces.

Algorithm 3: For arbitrary x_0 and $u \in H$, define an iterative algorithm by

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1. **Step 1.** Compute
   $z_{n,i} = x_n - \lambda \nabla f_i x_n x_n$ and $r_i = x_n - z_{n,i}$, for $i = 1, 2, ...m$

2. **Step 2.** Compute
   $$y_{n,i} = x_n - \Gamma_{n,i} r_n(x_n),$$
   for $i = 1, 2, 3, ...m$, where,$\Gamma_{n,i} = \iota^{jn,i}$ and j^n is the smallest non negative integer j^isatisfying
   $$\langle \iota^{j,i} r_i(x_n) + T_i(x_n - \iota^{j,i} r_i(x_n)) - T_i x_n, r_i(x_n) - T_i x_n, r_i(x_n) \rangle \leq \mu \|r_i(x_n)\|^2$$

3. **Step 3.** Compute
   $$w_n = \beta_{n,1} u_{n,1} + \beta_{n,2} u_{n,2} + ... + \beta_{n,m} u_{n,m},$$
   where, $u_{n,i} = P_{C_{n,i}} x_n$, $C_{n,i} = \{x \in H : \langle y_{n,i} - T_i y_{n,i}, x_n - y_{n,i} \rangle \leq 0\}$ and $\beta_{n,1} + \beta_{n,2} + ... + \beta_{n,m} = 1$

4. **Step 4.** Compute
   $$x_{n+1} = \alpha_n u + (1 - \alpha_n) w_n.$$

The method of proof Theorem 4.0.4 provides the proof of the following theorem of finding a common minimum point of a finite family convex functions in Hilbert spaces.

Theorem 4.1.1 *Suppose the Assumptions (A1) and (A2) hold. Let* $f_i : E \to \mathbb{R}$*for* $i = 1, 2, \ldots m$ *be a finite family of convex smooth functions with* ∇f_i *are sequentially weakly continuous on bounded subset of* H *and* $\cap_{i=1}^{m} \Omega_i \neq \emptyset$*, where* $\Omega_i = \{z : f_i(z) = \min_{x \in H} f(x)\}$ *for* $i = 1, 2, \ldots m$*. Then, the sequens* $\{x_n\}$ *generated by Algorithm 3 converges strongly to an element* $x^* = P_\Omega(u)$*.*

Chapter 5

Conclusion and Future scope

5.1 Conclusion

In this thesis, we established an iterative algorithm for approximating a common fixed point of a finite family pseudo- pseudocontractive mappings in real Hilbert space.
In addition, we also proved a strong convergence of a sequence generated by the proposed algorithm to a common fixed point provided that the mappings are uniformly continuous which are sequentially weakly continuous. Our result generalizes the results of many Authors such as (Zegeye (2011), Daman and $H.$ Zegeye (2012) and Zegeye and Wega (2020)).

5.2 Future Scope

In this thesis we obtained the common fixed point result for a finite family pseudo-pseudocontractive mapping on real Hilbert space. However, extending this result to a Banach spaces more general Hilbert spaces is an open problem. So, any interested researchers can use this opportunity to conduct their research work in this area.

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