Mathematical Modeling and Analysis of Coffee Fruit Disease

A Thesis Submitted to the Department of Mathematics, Jimma University in Partial Fulfillment for the Requirements of the Degree of Masters of Science (M.Sc.) in Mathematics

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Declaration

Here, I submit a thesis entitled "**Mathematical Modeling and Analysis of Coffee Fruit Disease"** for the award of degree of Master of Science in Mathematics. I, the undersigned declare that, this study is original and it has not been submitted to any institution elsewhere for the award of any academic degree or the like, where other sources of information have been used, they have been acknowledged.

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Abstract

In this thesis, mathematical model for coffee fruit disease was developed based on compartmental approach. The thesis encompasses the following fruitful findings. Boundedness and positivity of the model were proved. The model was linearized at equilibrium point. Basic reproduction number was also calculated by using next generation matrix. The local and global stability conditions of the model were also well investigated for both disease free and endemic equilibrium points. Furthermore, sensitivity analysis of the model parameters was also carried out. Finally, in order to verify the applicability of the result MATLAB simulation was implemented and agrees with the analytical result.

Key words: Mathematical model, equilibrium point, basic reproduction number,

Local stability, global stability, sensitivity analysis.

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CHAPTER ONE

1. INTRODUCTION

1.1 Background of the Study

Mathematical model plays indispensable role in different field of disciplines such as physics, biology, and electrical engineering and also in the social sciences (such as economics, sociology and political science). Physicist, Engineers, Computer scientist and Economists use mathematical models most extensively. Mathematical modeling can play a significant role in the efficient and sustainable management of renewable resources. It is mainly used to describe the real phenomena leading to design better prediction, prevention, management and control techniques. Several well documented mathematical models regarding real life problems can be found by (Biswas *et al.,* 2017; Biswas *et al.,* 2016; Chaudhary, 1986; Clark, 1979; Dubey *et al.,* 2003 and Mondal *et al.,* 2017).

Plants play a very important role in almost every ecosystem on the planet, mainly food for the Human and Animals. Sometimes plants may become infected with a virus. These infections can be devastating to not only the plants themselves but also the ecosystem that depends on them. Plant viruses are important constraints to crop production worldwide and the most limiting factors to modern agriculture, especially in developing countries (Toledo and Moguel, 2012).

Coffee is one of the most widely consumed beverages in the world; its trade satisfies the regular consumption of more than two billion people and exceeds $10 billion worldwide (Toledo and Moguel, 2012). Its cultivation is an important factor of social stability as it sustains the living of not less than twenty-five million small producers and their families worldwide. The total production of all exporting countries in 2018 was more than 172 million 60-kilogram bags (Djuikem *et al.,* 2021).

Coffee exports worldwide are worth around $24 billion, and coffee is a key part of the economies and cultures of many of the more than 80 countries that produce it (Rodríguez, 2014). Infestation of coffee by the coffee berry borer (CBB) causes about $500 million in losses each year (Vega *et al.,* 2019).

The CBB is now found in almost all coffee-producing countries and is very difficult to control. Since its detection in Puerto Rico in 2007, the CBB has colonized the entire coffee-growing region. Infestation reaches to 95% in some farms, but varies greatly from year to year (Marino *et al.*, 2017). Therefore, there is a great need for and interest in new tools for CBB infestation prediction and management.

Mathematical modeling has become an important tool in understanding the dynamics of disease transmission and in decision making processes regarding intervention programs for disease control. For instance, to reduce the losses caused by the maize stroke disease (MSD) different alternative tactics are used such as cultural control, biological control and chemical control (Magenya *et al.,* 2008; Karavina, 2014) and stakeholders are encouraged to combine at least two strategies in dealing with the disease (Jeger *et al.,* 2004; Karavina, 2014). Currently, vector-borne plant diseases have attracted the interest of many mathematical modeling researchers (Shi *et al.,* 2014). Ordinary differential equations (ODEs) have been used to model plants infected with viruses (Jeger *et al.,* 2004; Shi *et al.,* 2014 and Alemneh *et al.,* 2020). For instance, the authors in (Shi *et al.,* 2014), formulated and analyzed the dynamics of a vector-borne plant disease model. The study in (Alemneh *et al.,* 2020) developed and analyzed a mathematical model for MSV pathogen interaction with pest invasion on maize plant. A mathematical model of plant disease with the effect of fungicide and obtained that the optimal control can reduce the number of infected hosts compared to that of without control formulated and analyzed by (Anggriani *et al.,* 2018). The study in (Meng & Li, 2010) developed a model to combat plant viruses by continuously removing infected plants and replacing them with healthy plants.

On the other hand, fewer studies have investigated Mathematical models of the interactions between coffee fruits and coffee fruit disease. For example, Vandermeer *et al.* (2014) study the interaction between the regional and local dynamics of CLR model by representing the evolution of the proportion of infected bushes and farms. Bebber *et al.* (2016) determine the germination and infection risk depending on the climate in Colombia and neighbouring countries. Vandermeer *et al.* (2018) also represent the CLR dynamics in a coffee farm in Chiapas using an SI epidemiological model of the host.

In spite of aforementioned studies there is no research conducted in Ethiopia with regard to coffee fruit disease by using mathematical modeling and analysis. Therefore, the central goal of this study is to propose a new mathematical model of coffee fruit disease by incorporating different assumption that holds in the context of our country.

1.2. Statement of the problem

Mathematical modelling is the practice of transforming problems from an implementation field into tractable mathematical symbols, equations, and formulas whose theoretical and numerical analysis offers inspiration, solutions, and guidelines for the originating application. Coffee plays an important role in the economic growth of many developing countries such as Brazil, Cameroon, Ethiopia, Ivory Coast, Mexico, Viet Nam and many others. Coffee production throughout the world is affected by several pests and diseases. Among these pests, the coffee berry borer (CBB) is considered as one of the pest which bring lose economically (Aristizábal *et al.,* 2016; Vega *et al.,* 2009). It causes direct loss such as a reduction of coffee production and indirect losses such as a lowering of the quality of the coffee berries. In order to tackle or reduce such problems, a few scholars conduct research on coffee fruit disease using mathematical model and analysis approach. However, those researches were not conducted in Ethiopia in the existing literature. Therefore, it sounds to propose a new mathematical model which represents the dynamics of coffee fruit disease in our context followed by some rigorous mathematical analysis. Bearing this in mind, this research focuses on the following points.

- Formulation of mathematical model for coffee fruit disease,
- Boundedness and positivity of the solution of model,
- Computing basic reproduction number,
- Local stability condition of model given,
- Global stability condition of model,
- Sensitivity analysis of the model parameters,

1.3 Objectives of the Study

1.3.1 General Objective

The general objective of this study is to investigate mathematical model and analysis of coffee fruit disease.

1.3.2 Specific Objectives

The specific objectives of the study are:

- To formulation of mathematical model for coffee fruit disease,
- To prove boundedness and positivity of the solution of model,
- To computing basic reproduction number,
- To determine local stability condition of model given,
- To determine global stability condition of model,
- To carryout sensitivity analysis of the model parameters,

1.4 Significance of the Study

It is well known that coffee provides enormous advantages to sustainability of human life. However, coffee fruit disease leads to decrement coffee production and eventually affect economic development of one country. Therefore, this study contributes the way in which farmers/ coffee producers will have more production by recommending appropriate strategies.

1.5 Delimitation of the Study

The study is delimited to mathematical modeling and analysis of coffee fruit disease.

CHAPTER TWO

2. LITERATURE REVIEW

Mathematical modeling can play a significant role in the efficient and sustainable management of renewable resources. It is mainly used to describe the real phenomena leading to design better prediction, prevention, management and control techniques. Mathematical models in harvesting of fisheries were studied first by Clark in 1979. In 2015, knife and Koya studied stability of dynamics of harvesting fishery. Their result shows that fish population converges to a linear asymptote while the predator either converges to a lower positive asymptote, converges to zero early or diverges to positive infinity early. In 2016, Walters *et al.* studied about predation from simple predator prey theory about impacts of harvesting forage fishes. They considered harvesting for both prey and predator species. Then they described complex dynamics of the proposed model system including positivity and uniform boundedness of the system, and existence and stability criteria of various equilibrium points.

Plant diseases cause economic devastation especially in developing countries by severely affecting production of staple food crops due to yield losses. Cassava mosaic disease (CMD) and Maize lethal necrosis (MLN) are some of the most damaging crop diseases in the world (Redinbaugh and Stewart, 2018; Rey and Vanderschuren, 2017). CMD occurs in many regions across Africa, India, and Sri Lanka, areas in which cassava is considered a primary food crop (Alabi *et al.,* 2021). As CMD significantly decreases tuber production, it is a major constraint to cassava production (vanden Bosch *et al.,* 2006).

The spread of crop diseases, in particular airborne pathogens such as fungi, has received a lot of attention from researchers. Among models that represent the pathogen spatial dispersal, one can cite the DDAL framework that focuses on the deployment of susceptible and resistant crop hosts in an agricultural landscape. Fewer models represent the pathogen spread by a diffusion term in partial differential equations (PDE).

For instance, Sapoukhina *et al.* (20120) also study susceptible and resistant crop mixtures for a fungal disease propagated by airborne spores in a field, while Burie *et al.* (2008) explore the dynamical behaviour of mildew in a vineyard. These disease dynamics are relevant for CLR modelling: they include a latency period, a sporulation period, spore dispersal and germination. CLR models in the literature represent different scales, from the individual coffee bush to the country or even the continent. Avelino *et al.* (2006) investigate the factors (coffee tree characteristics, crop management patterns, environment that affect CLR intensity in several plots in Honduras. Bebber *et al.* (2016) determine the germination and infection risk depending on the climate in Colombia and neighbouring countries. In contrast to these static approaches, Vandermeer *et al.* (2014) study the interaction between the regional and local dynamics of CLR model by representing the evolution of the proportion of infected bushes and farms. Vandermeer *et al.* (2014) also represent the CLR dynamics in a coffee farm in Chiapas using an SI epidemiological model of the host. In these two latter studies, the fungus life cycle is not represented. Some other models investigate CLR control. Vandermeer *et al.* (2014) look at the interaction between *H. vastatrix* and a mycoparasite *Lecanicillium lecanii,* while Arroyo *et al.* (2019) consider interactions with antifungal bacteria. However, no existing CLR model considers *H. vastatrix* dynamics together with its interaction with the coffee host. In particular none considered the impact of CLR on berry production, which is the variable of agronomic interest.

The capacity to accurately model the time evolution of coffee plantations allows not only to effectively predict trends in the process, but also to act on them. In this context, optimal control theory has proven to be a powerful tool for investigating potential control strategies in pest treatment (Abbasi *et al.*, 2020 and Belbas and Schmidt, 2009). Such approaches are commonly based on the well-known PMP (Pontryagin's maximum principle) (Pontryagin, 2018): for a given cost function to maximize, this theory can provide necessary (and often sufficient) conditions for optimality of control strategies in systems of ordinary differential equations, partial differential equations and hybrid systems with given constraints (Dmitruk and Kaganovich, 2008). The present contribution will offers an impulsive perspective to the modeling of the propagation of coffee fruit disease in a coffee plantation.

CHAPTER THREE

METHODOLOGY

3.1. Study Area and Period

The study was conducted in Jimma University under the department of Mathematics from January, 2021 to January, 2022 G.C.

3.2. Study Design

This study employed mixed-design (documentary review design and experimental design).

3.3. Source of Information

The relevant sources of information for this study were books, published articles and related studies from internet.

3.4. Mathematical Procedures

In order to achieve the stated objectives, the following procedures were followed.

1. Formulating mathematical model of coffee fruit disease,
2. Proving boundedness and positivity of solution for model,
3. Determining equilibrium points of the model,
4. Computing basic reproduction number via next generation matrix,
5. Linearizing the model about equilibrium points,
6. Determining eigenvalues of Jacobian matrix,
7. Determining local stability conditions of the model,
8. Constructing appropriate Lyapunov functions,
9. Determining global stability conditions of the model,
10. Carrying out sensitivity analysis of the model parameters,
11. Verifying the applicability of the result using MATLAB simulation.

CHAPTER FOUR

RESULT AND DISCUSSION

4.1 Preliminaries

Definition 4.1: Consider non-linear system $\frac{dx}{dt} = f(x)$, where $f : R^n \to R^n$. A point $x^* \in R^n$ is an equilibrium point if $\frac{dx}{dt}(x^*) = f(x^*) = 0$

Definition 4.2: For a linear system $\frac{dx}{dt} = AX$ the stability of equilibrium point can be completely determined by location of eigenvalues of *A*. This is expressed as follows;

I. If the all eigenvalues of the Jacobian matrix have real parts less than zero, then the linear system is locally asymptotically stable and

II. If at least one of the eigenvalue of Jacobian matrix has real part greater than zero, then the system is unstable (Khalil, 2002).

Definition 4.3: Let x^* is an equilibrium point and a scalar function $V : D \to R$ is said to be:

1. Positive definite function if $V(x^*) = 0$ and $V(x) > 0$ for all $x \in D - \{x^*\}$
2. Negative definite function if $V(x^*) = 0$ and $V(x) < 0$ for all $x \in D - \{x^*\}$

Theorem 4.1: Lyapunov Stability Theorem (Khalil, 2002)

Let $x = x^*$ be an equilibrium point of non-linear system of $\frac{dx}{dt} = f(x)$, $f : D \to R^n$.

Suppose $V : D \to R$ be continuously differentiable function such that:-

I. $V(x^*) = 0$

II. $V(x) > 0$ for all $x \in D - \{x^*\}$

III. $\frac{dV(x)}{dt} \le 0$ for all $x \in D - \{x^*\}$ (Domain D excluding x^*).Then $x = x^*$ is stable.

Theorem 4.2: (Globally asymptotically stable)

Let $x = x^*$ be an equilibrium point of non-linear system of $\frac{dx}{dt} = f(x)$, $f : D \to R^n$.

Let $V : D \to R$ be continuously differentiable function such that:-

1. $V(x^*) = 0$
2. $V(x) > 0$ for all $x \in D - \{x^*\}$ (Domain D excluding x^*)
3. $\frac{dV(x)}{dt} < 0$ for all $x \in D - \{x^*\}$ (Domain D excluding x^*)

Then $x = x^*$ is globally asymptotically stable.

Theorem 4.3: (Gronwall Inequality) (Perko, 2013)

Let $x(t)$ be a function that is satisfying the following differential inequality

$\frac{dx}{dt} \le ax + b;\ x(0) = x_0$ where a, b are constants. Then, for all $t \ge 0$ we have:

$x(t) \le x_0 e^{at} + \frac{b}{a}(e^{at} - 1);\ a \ne 0$ and $x(t) \le x_0 + bt;\ a = 0$

Theorem 4.4: *LaSalle invariance principle (*LaSalle, 1976)

Suppose that $x^* = 0$ is an equilibrium point of system an autonomous dynamical system, and V is a Lyapunov function on some neighborhood U of $x^* = 0$. If $x_0 \in U$ has its forward trajectory bounded with limit points in U, and M is the largest invariant set of $E = \{x^* \in U : V(x^*) = 0\}$, then the solution $\phi(x_0, t) \to M$ as $t \to \infty$.

4.2 Formulation of Mathematical Model

The total coffee fruit is divided into five sub-populations: Susceptible individuals (denoted by S) are those which are not infected by the disease pathogen but there is a possibility to be infected. Exposed individuals (denoted by E) are those which are in the silent period after being infected and have no visible signs. After some period of time those exposed coffee fruit passes to the infected compartment or recovered compartment; infected individuals (denoted by I) are those which developed the symptom of the disease. Recovered individuals (denoted by R) are those

which recovered from the disease. Collected coffee fruit denoted by C which are prepared for export or usage. The following is flow chart diagram for coffee fruit disease.

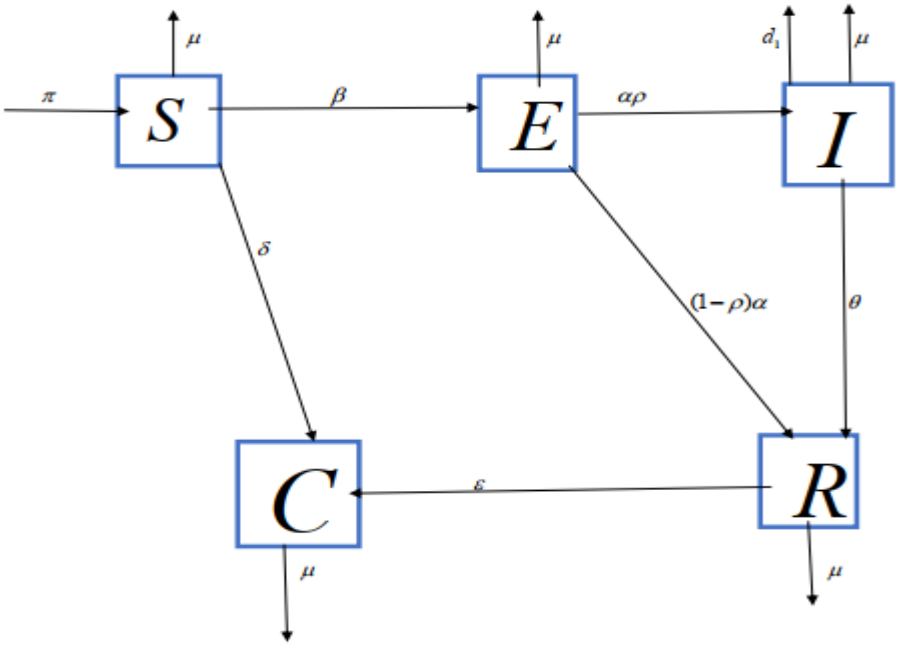

Figure 1: Schematic Flow Diagram for Coffee Fruit Disease

From schematic flow diagram in Figure (1)

$$\left.\begin{aligned}
\frac{dS}{dt} &= \pi - \beta ES - (\delta + \mu)S \\
\frac{dE}{dt} &= \beta ES - (\alpha + \mu)E \\
\frac{dI}{dt} &= \alpha\rho E - (\theta + d_1 + \mu)I \\
\frac{dR}{dt} &= (1-\rho)\alpha E + \theta I - (\varepsilon + \mu)R \\
\frac{dC}{dt} &= \delta S + \varepsilon R - \mu C
\end{aligned}\right\} \qquad (4.1)$$

Subjected to initial conditions

$$S(0) = S_0 > 0\,,\quad E(0) = E_0 > 0\,,\quad I(0) = I_0 > 0\,,\quad R(0) = R_0 > 0\quad,\ C(0) = C_0 > 0 \qquad (4.2)$$

Table 1: Description of parameters of the model (4.1)

Parameters	Description
π	Influx rate
ε	The rate at which recovered coffee fruit becomes collected coffee fruit
β	Contact rate of susceptible individuals
α	Transmission rate from exposed to infected or recovered
ρ	Proportion of exposed individuals which joins infected individuals
μ	Natural mortality rate
θ	The rate at which infected coffee fruit becomes recovered coffee fruit
d_1	disease induced death rate of infected individuals
δ	The rate at which susceptible coffee fruit becomes collected coffee fruit

4.3 Positivity of the Solutions of the Model

Since the state variables involved in model (4.1) represents coffee fruit, it is needful to show that all the state variables are also positive for all time $t \geq 0$.

Theorem 1: All the state variables $S(t), E(t), I(t), R(t), C(t)$ of model (4.1) subjected to initial conditions (4.2) remain positive for all time $t \geq 0$.

Proof: From the first equation of Eq. (4.1),

$$\frac{dS}{dt} = \pi - \beta ES - (\delta + \mu)S$$

$$\frac{dS}{dt} \geq -(\beta E + (\delta + \mu))S$$

$$S(t) \geq S_0 e^{\int_0^t -(\beta E + (\delta + \mu))dt} > 0 \qquad (4.3)$$

From the second equation of Eq. (4.1),

$$\frac{dE}{dt} = \beta ES - (\alpha + \mu)E$$

$$\frac{dE}{dt} \geq -(\alpha + \mu)E$$

$$E(t) \geq E_0 e^{-(\alpha+\mu)t} > 0 \qquad (4.4)$$

From the third equation of Eq. (4.1),

$$\frac{dI}{dt} = \alpha\rho E - (\theta + d_1 + \mu)I$$

$$\frac{dI}{dt} \geq -(\theta + d_1 + \mu)I$$

$$I(t) \geq I_0 e^{-(\theta+d_1+\mu)t} > 0 \qquad (4.5)$$

From the fourth equation of Eq. (4.1),

$$\frac{dR}{dt} = (1-\rho)\alpha E + \theta I - (\varepsilon + \mu)R$$

$$\frac{dR}{dt} \geq -(\varepsilon + \mu)R$$

$$R(t) \geq R_0 e^{-(\varepsilon+\mu)t} > 0 \qquad (4.6)$$

From the last equation of Eq. (4.1),

$$\frac{dC}{dt} = \delta S + \varepsilon R - \mu C$$

$$\frac{dC}{dt} \geq -\mu C$$

$$C(t) \geq C_0 e^{-\mu t} > 0 \qquad (4.7)$$

From Eqs. (4.3)- (4.7), all the state variables are positive and hence the theorem is proved.

4.4 Boundedness of the Solutions of the Model

Theorem 2: All the solution of $S(t), E(t), I(t), R(t)$ and $C(t)$ of model (4.1) which initiate in R_+^5 are uniformly bounded.

Proof: Let $\left(S(t), E(t), I(t), R(t), C(t)\right)$ be positive solution of the model (4.1) with initial condition (4.2).

Let $W(t) = S(t) + E(t) + I(t) + R(t) + C(t)$,

$$\frac{dW}{dt} = \frac{dS}{dt} + \frac{dE}{dt} + \frac{dI}{dt} + \frac{dR}{dt} + \frac{dC}{dt}$$

After some mathematical simplification,

$$\frac{dW}{dt} < \pi - \mu W(t) \tag{4.8}$$

Applying Gronwall's inequality on Eq. (4.8),

$$0 < W(t) \leq \frac{\pi}{\mu}\left(1 - e^{-\mu t}\right) + W(0)\, e^{-\mu t}$$

Apply limit sup as $t \to \infty$, $\quad \lim_{t\to\infty} S\,\mathrm{up}\, W(t) < \lim_{t\to\infty} S\,\mathrm{up}\, \frac{\pi}{\mu}\left(1 - e^{-\mu t}\right) + \lim_{t\to\infty} S\,\mathrm{up}\, w(0)\, e^{-\mu t}$

$$0 < W(t) < \frac{\pi}{\mu} \tag{4.9}$$

As a result, all the solutions of model (4.1) that initiated in R_+^5 are attracted to the region

$$\Omega = \left\{ (S(t), E(t), I(t), R(t), C(t)) \in R_+^5 : W(t) = S(t) + E(t) + I(t) + R(t) + C(t) < \frac{\pi}{\mu} \right\}$$

Which is the feasible solution set for the model (4.1) and all the solution set is uniformly bounded in it and hence the theorem is proved.

4.5 Existence and uniqueness Solution of the Model

Theorem 3: With initial conditions given by Eq. (4.2), the solution of $\big(S(t), E(t), I(t), R(t), C(t)\big)$ exists in R_+^5.

Proof: Mathematical model given by Eq. (4.1) can be expressed as $\dot{x} = f(x)$ where

$$\dot{x} = \begin{pmatrix} S(t) \\ E(t) \\ I(t) \\ R(t) \\ C(t) \end{pmatrix} \quad , \quad f(x) = \begin{pmatrix} \pi - \beta ES - (\delta + \mu)S \\ \beta ES - (\alpha + \mu)E \\ \alpha\rho E - (\theta + d_1 + \mu)I \\ (1-\rho)\alpha E + \theta I - (\varepsilon + \mu)R \\ \delta S + \varepsilon R - \mu C \end{pmatrix}$$

Since f has a continuous first derivative in R_+^5, it is then locally Lips*chitz.* As a result, by the well known fundamental existence and uniqueness theorem (Perko, 2013) and theorem 1 and 2 proved above, there exists a unique, positive and bounded solution for the system of differential equation given by Eq. (4.1) in R_+^5**.**

4.6 Equilibrium points of the Model

To find equilibrium points,

$$\frac{ds}{dt} = \frac{dE}{dt} = \frac{dI}{dt} = \frac{dR}{dt} = \frac{dC}{dt} = 0$$

$$\left.\begin{aligned} &\pi - \beta ES - (\delta + \mu)S = 0 \\ &\beta ES - (\alpha + \mu)E = 0 \\ &\alpha\rho E - (\theta + d_1 + \mu)I = 0 \\ &(1-\rho)\alpha E + \theta I - (\varepsilon + \mu)R = 0 \\ &\delta S + \varepsilon R - \mu C = 0 \end{aligned}\right\} \qquad (4.10)$$

From the second equation of Eq. (4.10),

$$E = 0 \text{ or } S = \frac{\alpha + \mu}{\beta}$$

For $E = 0$, the following equilibrium point is obtained

$$E_0 = \left(\frac{\pi}{\delta + \mu}, 0, 0, 0, \frac{\delta\pi}{\mu(\delta + \mu)} \right)$$

This equilibrium point is called disease free equilibrium point (DFEP).

For $S = \frac{\alpha + \mu}{\beta}$, after some mathematical manipulation the following are obtained.

$$E = \frac{\beta\pi - (\delta + \mu)(\alpha + \mu)}{\beta(\alpha + \mu)}$$

$$I = \frac{\alpha\rho\beta\pi - \alpha\rho(\delta + \mu)(\alpha + \mu)}{\beta(\alpha + \mu)(\theta + d_1 + \mu)}$$

$$R = \frac{(1-\rho)\alpha\beta\pi(\theta + d_1 + \mu) + \alpha\rho\beta\theta\pi - \alpha(1-\rho)(\delta + \mu)(\alpha + \mu)(\theta + d_1 + \mu) - \alpha\theta\rho(\delta + \mu)(\alpha + \mu)}{\beta(\alpha + \mu)(\varepsilon + \mu)(\theta + d_1 + \mu)}$$

$$C = \frac{\varepsilon}{\mu}\left(\frac{(1-\rho)\alpha\beta\pi(\theta + d_1 + \mu) + \alpha\rho\beta\theta\pi - \alpha(1-\rho)(\delta + \mu)(\alpha + \mu)(\theta + d_1 + \mu) - \alpha\theta\rho(\delta + \mu)(\alpha + \mu)}{\beta(\alpha + \mu)(\varepsilon + \mu)(\theta + d_1 + \mu)} \right)$$

$$+ \frac{\delta(\alpha + \mu)}{\mu\beta}$$

Endemic equilibrium point: $E_1 = \left(S^*, E^*, I^*, R^*, C^* \right)$,

where $S^* = \frac{\alpha + \mu}{\beta}$, $\quad E^* = \frac{\beta\pi - (\delta + \mu)(\alpha + \mu)}{\beta(\alpha + \mu)}$, $I^* = \frac{\alpha\rho\beta\pi - \alpha\rho(\delta + \mu)(\alpha + \mu)}{\beta(\alpha + \mu)(\theta + d_1 + \mu)}$

$$R^* = \frac{(1-\rho)\alpha\beta\pi(\theta + d_1 + \mu) + \alpha\rho\beta\theta\pi - \alpha(1-\rho)(\delta + \mu)(\alpha + \mu)(\theta + d_1 + \mu) - \alpha\theta\rho(\delta + \mu)(\alpha + \mu)}{\beta(\alpha + \mu)(\varepsilon + \mu)(\theta + d_1 + \mu)}$$

$$C^* = \frac{\varepsilon}{\mu}\left(\frac{(1-\rho)\alpha\beta\pi(\theta + d_1 + \mu) + \alpha\rho\beta\theta\pi - \alpha(1-\rho)(\delta+\mu)(\alpha+\mu)(\theta + d_1 + \mu) - \alpha\theta\rho(\delta+\mu)(\alpha+\mu)}{\beta(\alpha+\mu)(\varepsilon+\mu)(\theta + d_1 + \mu)} \right)$$

$$+ \frac{\delta(\alpha+\mu)}{\mu\beta}$$

Theorem 3: The endemic equilibrium point is positive if the following condition is satisfied.

$$E = \beta\pi > (\delta+\mu)(\alpha+\mu) \tag{4.11}$$

Proof: Since all parameters of the model are positive, $S^* = \frac{\alpha+\mu}{\beta}$, $S^* > 0$

$$E^* > 0, \quad \frac{\beta\pi - (\delta+\mu)(\alpha+\mu)}{\beta(\alpha+\mu)} > 0 \quad , \quad \beta\pi - (\delta+\mu)(\alpha+\mu) > 0$$

$$\beta\pi > (\delta+\mu)(\alpha+\mu)$$

If E^* is positive, then I^*, R^* and C^* are positive. Hence, the proof is completed.

4.7 Basic Reproduction Number

The basic reproduction number, denoted by R_0, is defined as number of secondary infections appears from one infected individual. The basic reproduction number of the system is calculated by applying the next generation matrix method. The next-generation matrix is used to derive the basic reproduction number, for a compartmental model of the spread of infectious diseases. The infected subsystem of the model is:

$$\left.\begin{aligned} \frac{dE}{dt} &= \beta ES - (\alpha+\mu)E \\ \frac{dI}{dt} &= \alpha\rho E - (\theta + d_1 + \mu)I \end{aligned}\right\} \tag{4.12}$$

$$T = \begin{pmatrix} \dfrac{\beta\pi}{\delta+\mu} & 0 \\ 0 & 0 \end{pmatrix}, \qquad V = \begin{pmatrix} \alpha+\mu & 0 \\ -\alpha\rho & \theta+d_1+\mu \end{pmatrix}$$

T represents transmission matrix and V represents transition matrix.

$$V^{-1} = \begin{pmatrix} \dfrac{1}{\alpha+\mu} & 0 \\ \dfrac{\alpha\rho}{(\alpha+\mu)(\theta+d_1+\mu)} & \dfrac{1}{\theta+d_1+\mu} \end{pmatrix}$$

$$TV^{-1} = \begin{pmatrix} \dfrac{\beta\pi}{(\alpha+\mu)(\delta+\mu)} & 0 \\ 0 & 0 \end{pmatrix}$$

The spectral radius of TV^{-1} is $\dfrac{\beta\pi}{(\alpha+\mu)(\delta+\mu)}$.

This spectral radius is called basic reproduction number. As a result,

$$R_0 = \frac{\beta\pi}{(\alpha+\mu)(\delta+\mu)}$$

4.8 Local Stability Analysis DFEP

Theorem 4: The disease free equilibrium point $E_0 = \left(\dfrac{\pi}{\delta+\mu}, 0, 0, 0, \dfrac{\delta\pi}{\mu(\delta+\mu)}\right)$ is locally asymptotically stable if $R_0 < 1$.

Proof: The Jacobian matrix of the model evaluated at disease free equilibrium point is:

$$J=\begin{pmatrix} -(\delta+\mu) & \dfrac{-\beta\pi}{\delta+\mu} & 0 & 0 & 0 \\ 0 & \dfrac{\beta\pi-(\alpha+\mu)(\delta+\mu)}{(\delta+\mu)} & 0 & 0 & 0 \\ 0 & \alpha\rho & -(\theta+d_1+\mu) & 0 & 0 \\ 0 & (1-\rho)\alpha & \theta & -(\varepsilon+\mu) & 0 \\ \delta & 0 & 0 & \varepsilon & -\mu \end{pmatrix}$$

The eigenvalues of J are:

$$\lambda_1=\frac{\beta\pi-(\alpha+\mu)(\delta+\mu)}{(\delta+\mu)} \quad , \quad \lambda_2=-(\delta+\mu) \quad , \quad \lambda_3=-(\theta+d_1+\mu) \quad ,$$
$$\lambda_4=-(\varepsilon+\mu) \quad , \quad \lambda_5=-\mu$$

Since all parameters of the model are positive, $\lambda_2,\lambda_3,\lambda_4,\lambda_5$ are all negative.

For $\lambda_1<0\Rightarrow\dfrac{\beta\pi-(\alpha+\mu)(\delta+\mu)}{(\delta+\mu)}<0\Rightarrow R_0<1$. Hence, the proof completed.

4.9 Local Stability Analysis EEP

Theorem 5: The endemic equilibrium point $E_1=\left(S^*,E^*,I^*,R^*,C^*\right)$ is locally asymptotically stable if $R_0>1$.

Proof: The Jacobian matrix of the model evaluated at endemic equilibrium point is:

$$J=\begin{pmatrix} \dfrac{-\beta\pi}{(\alpha+\mu)} & -(\alpha+\mu) & 0 & 0 & 0 \\ \dfrac{\beta\pi-(\alpha+\mu)(\delta+\mu)}{(\alpha+\mu)} & 0 & 0 & 0 & 0 \\ 0 & \alpha\rho & -(\theta+d_1+\mu) & 0 & 0 \\ 0 & (1-\rho)\alpha & \theta & -(\varepsilon+\mu) & 0 \\ \delta & 0 & 0 & \varepsilon & -\mu \end{pmatrix}$$

The eigenvalues of J are:

$$\lambda_1 = \frac{-\beta\pi}{2(\alpha+\mu)} - \frac{1}{2}\sqrt{\left(\frac{\beta\pi}{\alpha+\mu}\right)^2 - 4\left(\beta\pi - (\alpha+\mu)(\delta+\mu)\right)}\ ,$$

$$\lambda_2 = \frac{-\beta\pi}{2(\alpha+\mu)} + \frac{1}{2}\sqrt{\left(\frac{\beta\pi}{\alpha+\mu}\right)^2 - 4\left(\beta\pi - (\alpha+\mu)(\delta+\mu)\right)}\ ,$$

$$\lambda_3 = -(\theta + d_1 + \mu) \quad , \lambda_4 = -(\varepsilon+\mu) \quad , \quad \lambda_5 = -\mu$$

Since all parameters of the model are positive, $\lambda_3, \lambda_4, \lambda_5$ are all negative.

λ_1 & λ_2 have negative real part due to condition given by Eq. (4.11). More specifically, condition given by Eq. (4.11) gives the following.

$$\beta\pi > (\delta+\mu)(\alpha+\mu)$$

$$\frac{\beta\pi}{(\delta+\mu)(\alpha+\mu)} > 1 \quad ; \ R_0 > 1. \text{ Hence, the proof completed.}$$

4.10 Global Stability Analysis DFEP

Theorem 6: The DFEP given by $E_0 = \left(\frac{\pi}{\delta+\mu}, 0, 0, 0, \frac{\delta\pi}{\mu(\delta+\mu)}\right)$ is globally asymptotically stable provided that the following two conditions are satisfied.

$$\begin{cases} (i)\ \delta \le 2\mu \\ (ii)\ R_o \le \dfrac{\mu}{\alpha+\mu} \end{cases} \tag{4.13}$$

Proof: Consider the following Lyapunov function,

$$V(S,E,I,R,C) = \frac{\left(S-S^*\right)^2}{2S^*} + E + I + R + \left(C - C^* - C^* \ln\frac{C}{C^*}\right)$$

$(1)\ V(S,E,I,R,C) > 0$

$(2)\ V\left(S^*,0,0,0,C^*\right) = 0$

Differentiating V with respect to t along the solution trajectories of Eq. (4.1) gives:

$$\frac{dV}{dt} = \frac{\left(S - S^*\right)}{S^*}\frac{dS}{dt} + \frac{dE}{dt} + \frac{dI}{dt} + \frac{dR}{dt} + \left(\frac{C - C^*}{C}\right)\frac{dC}{dt}$$

$$\frac{dV}{dt} = \frac{\left(S - S^*\right)}{S^*}\left[\pi - (\delta + \mu)S - \beta ES\right] + \left[\beta ES - (\alpha + \mu)E\right] + \left[\alpha PE - (\theta + d_1 + \mu)I\right] + (1-\rho)\alpha E + \theta I - (\varepsilon + \mu)R + \left(\frac{C - C^*}{C}\right)(\delta S + \varepsilon R - \mu C)$$

$$\frac{dV}{dt} = \frac{\left(S - S^*\right)}{S^*}\left[\pi - (\delta + \mu)\left(S - S^*\right) - (\delta + \mu)S^* - \beta E\left(S - S^*\right) - \beta ES^*\right] + \left[\beta ES - (\alpha + \mu)E\right] + \left[\alpha\rho E - (\theta + d_1 + \mu)I\right] + (1-\rho)\alpha E + \theta I - (\varepsilon + \mu)R + \left(\frac{C - C^*}{C}\right)\left(\delta\left(S - S^*\right) + \varepsilon R - \mu\left(C - C^*\right)\right)$$

$$\frac{dV}{dt} = \frac{-(\delta + \mu)}{S^*}\left(S - S^*\right)^2 - \frac{\beta E}{S^*}\left(S - S^*\right)^2 - \beta E\left(S - S^*\right) + \beta ES - (\alpha + \mu)E + \alpha\rho E - (\theta + d_1 + \mu)I + \alpha E - \alpha\rho E + \theta I - \varepsilon R - \mu R + \frac{\delta}{C}\left(C - C^*\right)\left(S - S^*\right) - \varepsilon R - C^*R - \frac{\mu}{C}\left(C - C^*\right)^2$$

$$\frac{dV}{dt} < \frac{-(\delta + \mu)}{S^*}\left(S - S^*\right)^2 + \beta ES^* - \mu E + \frac{\delta}{C}\left(C - C^*\right)\left(S - S^*\right) - \frac{\mu}{C}\left(C - C^*\right)^2$$

$$\frac{dV}{dt} < \frac{-(\delta + \mu)}{S^*}\left(S - S^*\right)^2 + \left(\frac{\beta\pi}{\delta + \mu} - \mu\right)E + \frac{\delta}{2C}\left(C - C^*\right)^2 + \frac{\delta}{2C}\left(S - S^*\right)^2 - \frac{\mu}{C}\left(C - C^*\right)^2$$

$$\frac{dV}{dt} < \left(\frac{\delta}{2C} - \frac{(\delta + \mu)}{S^*}\right)\left(S - S^*\right)^2 + \left(\frac{\beta\pi}{\delta + \mu} - \mu\right)E + \left(\frac{\delta}{2C} - \frac{\mu}{C}\right)\left(C - C^*\right)^2$$

When the above two conditions given by Eq. (4.13) are satisfied,

$$\frac{dV}{dt} < 0$$

$$\frac{dV}{dt} = 0 \text{ when } \left(S,\ E,\ I,\ R,\ C\right) = \left(S^*,\ 0,\ 0,\ 0,\ C^*\right)$$

This fact indicates that the largest invariant set where $\frac{dV}{dt}=0$ the singleton

is $E_0=\left(\frac{\pi}{\delta+\mu},0,0,0,\frac{\delta\pi}{\mu(\delta+\mu)}\right)$. Thus, by LaSalle's invariance principle, the DFEP is globally asymptotically stable.

Note: $R_0<\frac{\mu}{\delta+\mu}\Rightarrow R_0<1$

This fact revealed that global stability implies local stability.

4.11 Global Stability Analysis EEP

Theorem 7: The EEP given by $E_1=\left(S^*,E^*,I^*,R^*,C^*\right)$ is globally asymptotically stable provided that the following conditions are satisfied

$$\begin{cases}(i)\,\delta+\varepsilon<2\mu\\(ii)\,\varepsilon R<2(\varepsilon+\mu)C\\(iii)\,2\left(\beta E^*+\delta+\mu\right)C>\delta S\\(iv)\left(\alpha\rho\left(E-E^*\right)\right)^2R<4(\theta+d_1+\mu)(1-\rho)\alpha\left(E-E^*\right)\left(R-R^*\right)I\end{cases}\tag{4.14}$$

Proof: Consider the following Lyapunov function,

$$V(S,E,I,R,C)=\left(S-S^*-S^*\ln\frac{S}{S^*}\right)+\left(E-E^*-E^*\ln\frac{E}{E^*}\right)+\left(I-I^*-I^*\ln\frac{I}{I^*}\right)$$
$$+\left(R-R^*-R^*\ln\frac{R}{R^*}\right)+\left(C-C^*-C^*\ln\frac{C}{C^*}\right)$$

$(1)V(S,E,I,R,C)>0$

$(2)V\left(S^*,E^*,I^*,R^*,C^*\right)=0$

Differentiating V with respect to t along the solution of Eq.(4.1),

$$\frac{dV}{dt}=\frac{\left(S-S^*\right)}{S}\frac{ds}{dt}+\frac{\left(E-E^*\right)}{E}\frac{dE}{dt}+\frac{\left(I-I^*\right)}{I}\frac{dI}{dt}+\frac{\left(R-R^*\right)}{R}\frac{dR}{dt}+\frac{\left(C-C^*\right)}{C}\frac{dC}{dt}$$

$$\frac{dV}{dt}=\frac{\left(S-S^*\right)}{S}\left(\pi-\beta ES-\left(\delta+\mu\right)S\right)+\frac{\left(E-E^*\right)}{E}\left(\beta ES-\left(\alpha+\mu\right)E\right)+\frac{\left(I-I^*\right)}{I}\left(\alpha PE-\left(\theta+d_1+\mu\right)I\right)$$
$$+\frac{\left(R-R^*\right)}{R}\left(\left(1-\rho\right)\alpha E-\left(\varepsilon+\mu\right)R\right)+\frac{\left(C-C^*\right)}{C}\left(\delta S+\varepsilon R-\mu C\right)$$

$$\frac{dV}{dt}=\frac{\left(S-S^*\right)}{S}\left(\pi-\beta E\left(S-S^*\right)-\beta ES^*-\left(\delta+\mu\right)\left(S-S^*\right)-\left(\delta+\mu\right)S^*\right)+$$
$$\frac{\left(E-E^*\right)}{E}\left(\beta E\left(S-S^*\right)+\beta ES^*-\left(\alpha+\mu\right)\left(E-E^*\right)-\left(\alpha+\mu\right)E^*\right)+$$
$$\frac{\left(I-I^*\right)}{I}\left(\alpha P\left(E-E^*\right)+\alpha PE^*-\left(\theta+d_1+\mu\right)\left(I-I^*\right)-\left(\theta+d_1+\mu\right)I^*\right)+$$
$$\frac{\left(R-R^*\right)}{R}\left(\left(1-\rho\right)\alpha\left(E-E^*\right)+\left(1-\rho\right)\alpha E^*-\left(\varepsilon+\mu\right)\left(R-R^*\right)-\left(\varepsilon+\mu\right)R^*\right)+$$
$$\frac{\left(C-C^*\right)}{C}\left(\delta\left(S-S^*\right)+\delta S^*+\varepsilon\left(R-R^*\right)+\varepsilon R^*-\mu\left(C-C^*\right)-\mu C^*\right)$$

$$\frac{dV}{dt}=\frac{\left(S-S^*\right)}{S}\left(\pi-\beta E\left(S-S^*\right)-\beta S^*\left(E-E^*\right)-\beta E^*S^*-\left(\delta+\mu\right)\left(S-S^*\right)-\left(\delta+\mu\right)E^*\right)+$$
$$\frac{\left(E-E^*\right)}{E}\left(\beta E\left(S-S^*\right)+\beta S^*\left(E-E^*\right)+\beta E^*S^*-\left(\alpha+\mu\right)\left(E-E^*\right)-\left(\alpha+\mu\right)E^*\right)+$$
$$\frac{\left(I-I^*\right)}{I}\left(\alpha\rho\left(E-E^*\right)-\left(\theta+d_1+\mu\right)\left(I-I^*\right)\right)+$$
$$\frac{\left(R-R^*\right)}{R}\left(\left(1-\rho\right)\alpha\left(E-E^*\right)-\left(\varepsilon+\mu\right)\left(R-R^*\right)\right)+$$
$$\frac{\left(C-C^*\right)}{C}\left(\delta\left(S-S^*\right)+\varepsilon\left(R-R^*\right)-\mu\left(C-C^*\right)\right)$$

$$\frac{dV}{dt}=\frac{\left(S-S^*\right)}{S}\left(-\beta E\left(S-S^*\right)-\beta S^*\left(E-E^*\right)-\left(\delta+\mu\right)\left(S-S^*\right)\right)$$
$$+\frac{\left(E-E^*\right)}{E}\left(\beta E\left(S-S^*\right)+\beta S^*\left(E-E^*\right)-\left(\alpha+\mu\right)\left(E-E^*\right)\right)$$
$$+\frac{\left(I-I^*\right)}{I}\left(\alpha\rho\left(E-E^*\right)-\left(\theta+d_1+\mu\right)\left(I-I^*\right)\right)$$
$$+\frac{\left(R-R^*\right)}{R}\left(\left(1-\rho\right)\alpha\left(E-E^*\right)-\left(\varepsilon+\mu\right)\left(R-R^*\right)\right)$$
$$+\frac{\left(C-C^*\right)}{C}\left(\delta\left(S-S^*\right)+\varepsilon\left(R-R^*\right)-\mu\left(C-C^*\right)\right)$$

$$\frac{dV}{dt}=\frac{-\beta E}{S}\left(S-S^*\right)^2-\frac{\beta S^*}{S}\left(S-S^*\right)\left(E-E^*\right)-\left(\frac{\delta+\mu}{S}\right)\left(S-S^*\right)^2$$
$$+\beta\left(S-S^*\right)\left(E-E^*\right)+\frac{\beta S^*}{E}\left(E-E^*\right)^2-\left(\frac{\alpha+\mu}{E}\right)\left(E-E^*\right)^2$$
$$+\frac{\alpha\rho}{I}\left(E-E^*\right)\left(I-I^*\right)-\left(\frac{\theta+d_1+\mu}{I}\right)\left(I-I^*\right)^2$$
$$+\frac{\left(1-\rho\right)}{R}\alpha\left(E-E^*\right)\left(R-R^*\right)-\frac{\left(\varepsilon+\mu\right)}{R}\left(R-R^*\right)^2$$
$$+\frac{\delta}{C}\left(S-S^*\right)\left(C-C^*\right)+\frac{\varepsilon}{C}\left(R-R^*\right)\left(C-C^*\right)-\frac{\mu}{C}\left(C-C^*\right)^2$$

$$\frac{dV}{dt}=\frac{-\left(\beta E+\delta+\mu\right)}{S}\left(S-S^*\right)^2+\frac{\alpha\rho}{I}\left(E-E^*\right)\left(I-I^*\right)-\left(\frac{\theta+d_1+\mu}{I}\right)\left(I-I^*\right)^2$$
$$+\frac{\left(1-\rho\right)\alpha}{R}\left(E-E^*\right)\left(R-R^*\right)-\frac{\left(\varepsilon+\mu\right)}{R}\left(R-R^*\right)^2+\frac{\delta}{C}\left(S-S^*\right)\left(C-C^*\right)$$
$$+\frac{\varepsilon}{C}\left(R-R^*\right)\left(C-C^*\right)-\frac{\mu}{C}\left(C-C^*\right)^2$$

$$\frac{dV}{dt}\le\frac{-1}{2CS}\left(2\left(\beta E+\delta+\mu\right)C-\delta S\right)\left(S-S^*\right)^2-\frac{1}{2C}\left(2\mu-\left(\delta+\varepsilon\right)\right)\left(C-C^*\right)^2$$
$$-\frac{1}{2CR}\left(2\left(\varepsilon+\mu\right)C-\varepsilon R\right)\left(R-R^*\right)^2-\frac{1}{IR}\left(\theta+d_1+\mu\right)R\left(I-I^*\right)^2-$$
$$2\rho R\left(E-E^*\right)\left(I-I^*\right)-\left(1-\rho\right)\alpha I\left(E-E^*\right)\left(R-R^*\right)$$

When the conditions given by Eq. (4.14) are satisfied $\frac{dV}{dt}\le 0$

As a result, $\frac{dV}{dt} < 0$ & $\frac{dV}{dt} = 0$ when $(S, E, I, R, C) = (S^*, E^*, I^*, R^*, C^*)$

Hence, by LaSalle's invariance principle, the EEP is globally asymptotically stable.

4.12 Sensitivity Analysis

Sensitivity analysis is carried out on the basic parameters, to identify their effect to the transmission of the coffee disease fruit. To go through sensitivity analysis, we applied the normalized forward sensitivity index definition. The Normalized forward sensitivity index of a variable, p, that depends differentiable on a parameter, p, is defined as: $\Lambda_p^{R_0} = \frac{\partial R_0}{p} \times \frac{p}{R_0}$ *for* p represents all the basic parameters.

$$\Lambda_\pi^{R_0} = \frac{\partial R_0}{\partial \pi} \times \frac{\pi}{R_0} = 1 > 0$$

$$\Lambda_\beta^{R_0} = \frac{\partial R_0}{\partial \beta} \times \frac{\beta}{R_0} = 1 > 0$$

$$\Lambda_\mu^{R_0} = \frac{\partial R_0}{\partial \mu} \times \frac{\mu}{R_0} = \frac{-\mu(\alpha + 2\mu + \delta)}{(\alpha + \mu)(\delta + \mu)} < 0$$

$$\Lambda_\alpha^{R_0} = \frac{\partial R_0}{\partial \alpha} \times \frac{\alpha}{R_0} = \frac{-\alpha}{\alpha + \mu} < 0$$

$$\Lambda_\delta^{R_0} = \frac{\partial R_0}{\partial \delta} \times \frac{\delta}{R_0} = \frac{-\delta}{\delta + \mu} < 0$$

The sensitivity indices of the basic reproductive number with respect to main parameters revealed that, those parameters that have positive indices are (π, β) show that they have great impact on expanding the disease if their values are increasing. Also those parameters in which their sensitivity indices are negative (μ, α, δ) have an effect of minimizing the disease as their values increase. Therefore, this study recommends for coffee producers to work on decreasing the positive indices and increasing negative indices parameters.

4.13 MATLAB Simulation

MATLAB simulation was implemented by using the following parameters value subjected to initial conditions. Some of the parameters values were taken from the literature and others were assumed as the model was developed for the first time by this study.

Table 2: Parameters value used for MATLAB Simulation

Parameters	Value	Parameters	Value
π	85	δ	0.07
β	0.00034	ρ	0.2
		Initial Conditions	
α	0.1	$S(0)$	5,000
ε	0.35	$E(0)$	2000
μ	0.05	$I(0)$	1000
θ	0.09	$R(0)$	400
ε	0.045	$C(0)$	100
d_1	0.003		

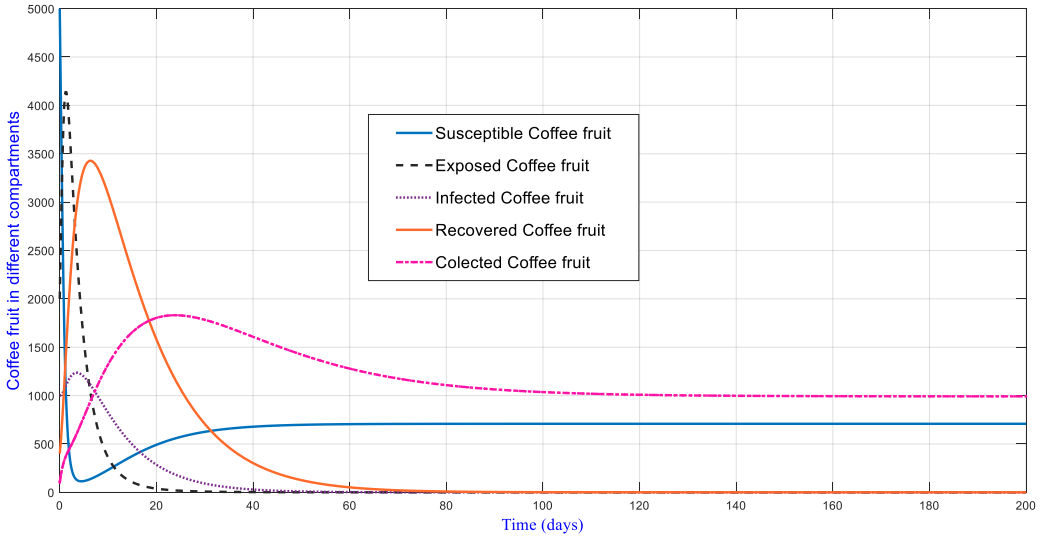

Figure 2: Graph of coffee fruit verses time for parameters value given in table 2.

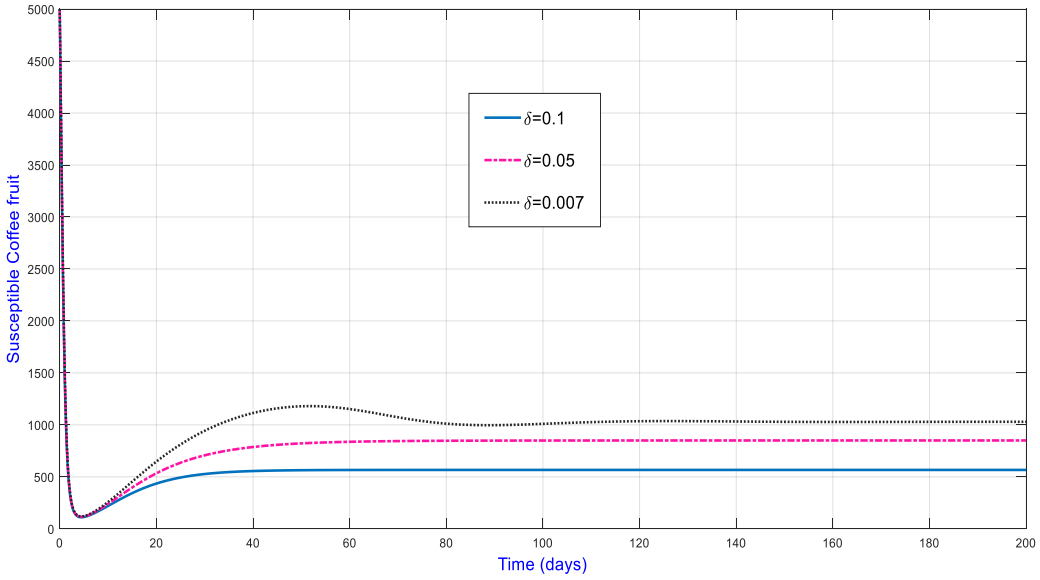

Figure 3: Graph of Susceptible Coffee fruit for different values of δ and keeping others parameters value constant.

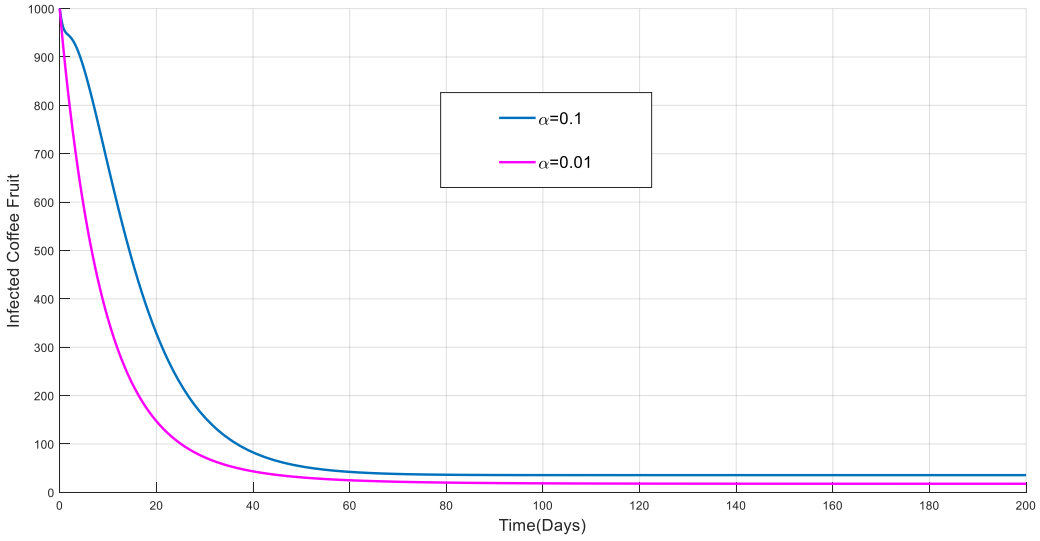

Figure 4: Graph of infected Coffee fruit for different values of α keeping others parameters value constant.

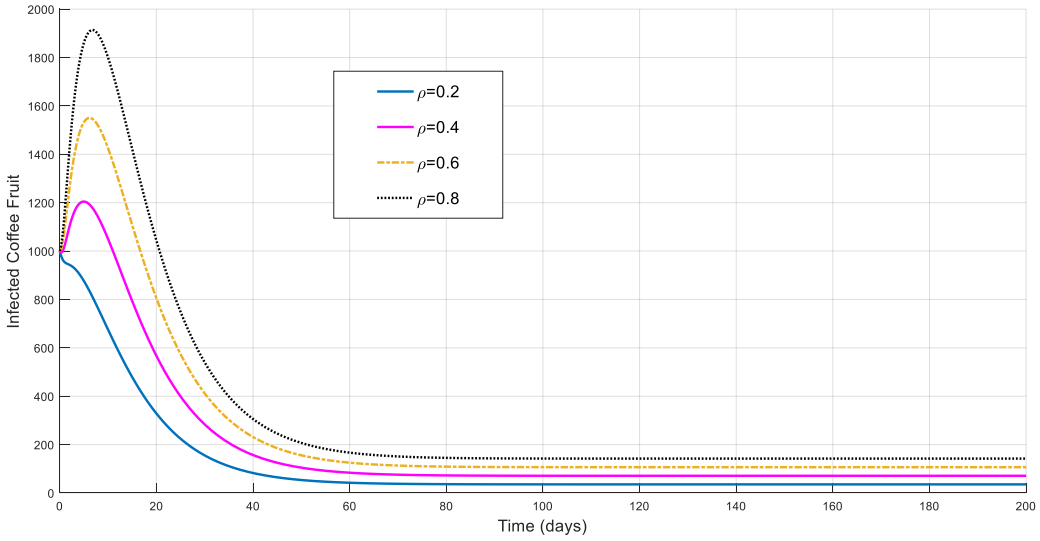

Figure 5: Graph of infected Coffee fruit for different values of ρ keeping others parameters value constant.

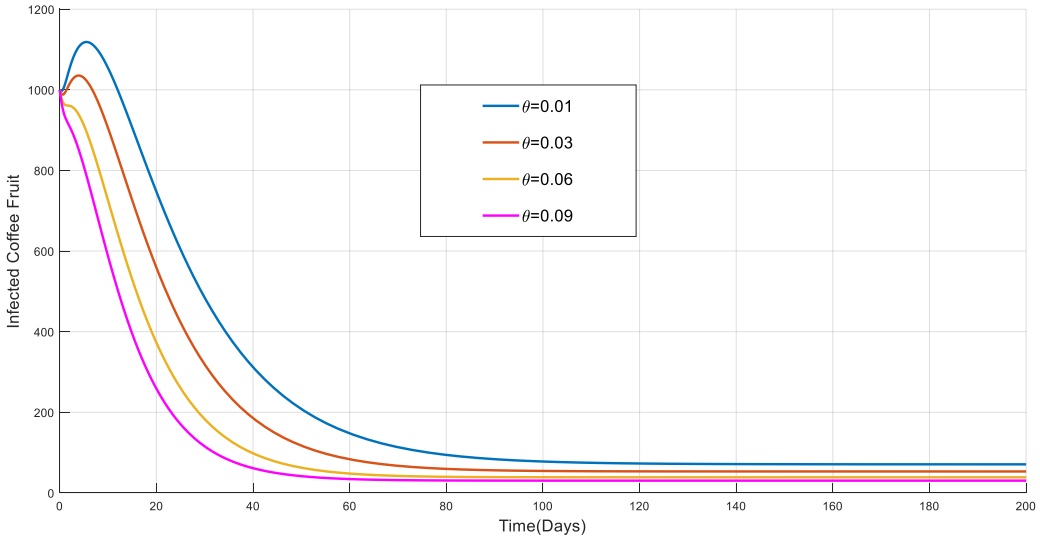

Figure 6: Graph of infected Coffee fruit for different values of θ keeping others parameters value constant.

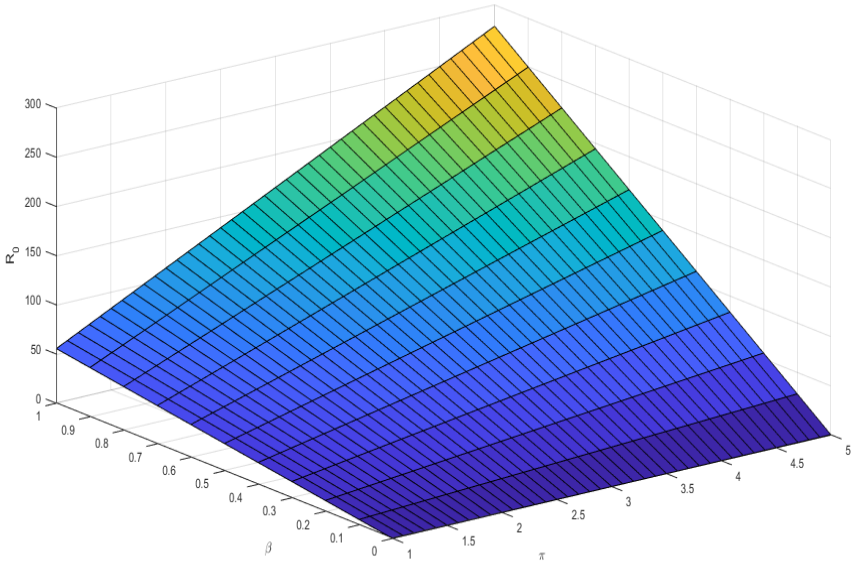

Figure 7: Graph of Basic Reproduction Number verses π and β

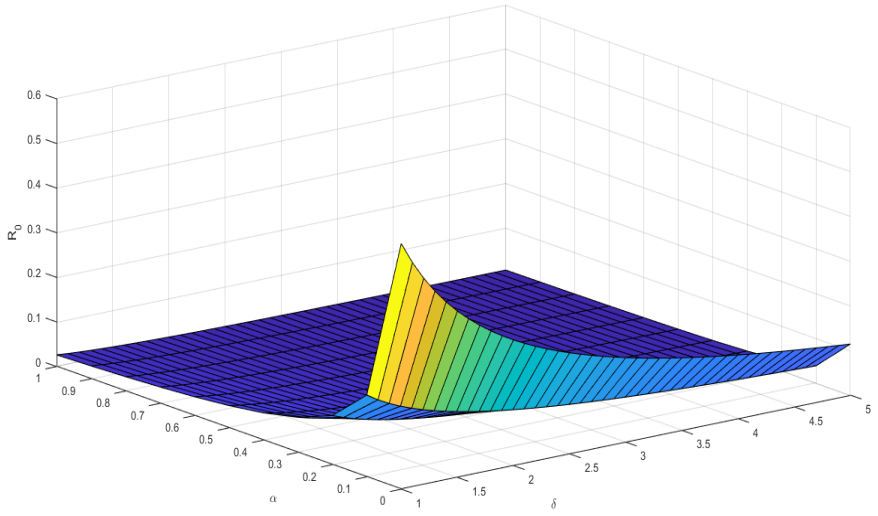

Figure 8: Graph of Basic Reproduction Number verses α and δ

4.14 Discussions

Figure 2 indicates that the graph of coffee fruit in different compartment verses time. It revealed the fact that equilibrium point is locally asymptotically stable when the basic reproduction number is less than one. Figure 3 indicates graph Susceptible Coffee fruit for different values of rate at which susceptible coffee fruit becomes collected coffee fruit. As rate at which susceptible coffee fruit becomes collected coffee fruit increases the Susceptible Coffee fruit decreases. Figure 7 depicts that graph of Basic Reproduction Number verses influx rate and contact rate. It revealed the fact that increasing influx rate and contact rate has the capacity to increase the basic reproduction number which in turn lowers coffee production. Figure 8 depicts that graph of Basic Reproduction Number verses rate at which susceptible coffee fruit becomes collected coffee fruit and transmission rate from exposed to infected or recovered. It revealed the fact that increasing rate at which susceptible coffee fruit becomes collected coffee fruit and transmission rate from exposed to infected or recovered has the capacity to decrease the basic reproduction number which in turn maximizes coffee production.

CHAPTER FIVE

5. CONCLUSION AND FUTURE SCOPE

5.1 Conclusion

The findings of this thesis are concluded as follows.

- New mathematical model for coffee fruit disease were developed,
- Qualitative analysis like boundedness and positivity of the model were proved,
- Equilibrium points of the model (disease free and endemic equilibrium points) were calculated ,
- Basic reproduction number was calculated by using next generation matrix,
- The local and global stability conditions of the model were also well investigated for both disease free and endemic equilibrium points,
- Furthermore, sensitivity analysis of the model parameters was also carried out,
- Finally, in order to verify the applicability of the result, MATLAB simulation was implemented and agrees with the analytical result.

5.2 Future Scope

One can conduct the following further investigation on this area of study.

- Refinement of the mathematical model by incorporating other important factor,
- Optimal control analysis of the model is also further investigation,
- It is also possible to extend the model to fractional order derivative to make new analysis with new result,
- Furthermore, introducing time delay into the model and conducting qualitative analysis like bifurcation, global stability, existence of periodic solution and limit cycle is also future scope of the study.

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