Higher-Order Numerical Scheme for Semi-linear Singularly Perturbed Reaction-Diffusion Problem

A Research Submitted to the Department of Mathematics in Partial Fulfillment for the Requirements of the Degree of Masters of Science in Mathematics

(Numerical Analysis)

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Declaration

I, undersigned declare that this thesis entitled **Higher-Order Numerical Scheme for Semilinear Singularly Perturbed Reaction-Diffusion Problem** is my own original work and it has not been submitted for the award of any academic degree or the like in any other institution or university, and that all the sources I have used or quoted have been indicated and acknowledged.

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Abstract

In this paper, a higher-order numerical method presented for solving semi-linear singularly perturbed reaction-diffusion problems. The quasilinearization technique is used to linearize the semi-linear term. After uniform discertization of the solution domain the finite difference scheme is formulated. The stability and consistency are investigated to guarantee the convergence of the method. Further, numerical illustrations are provided to validate the applicability and efficiency of the proposed method. Moreover, the obtained results shows the betterment than some existing results in the literature.

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Chapter 1

Introduction

1.1 Background of the study

Numerical analysis is the branch of mathematics that deals with the computational methods which helps to find approximate solutions for difficult problems such as finding the roots of non-linear equations, integration involving complex expressions and solving differential equations for which analytical solutions do not exist. Numerical analysis plays a significant role when difficulties encountered in finding the exact solution of an equation using a direct method and when it becomes very difficult or impossible to apply theoretical methods to find the exact solution.

In real life, we often encounter many problems which are described by parameter dependent differential equations. The behavior of the solutions of these types of differential equations depend on the magnitude of the parameters. Any differential equation in which the highest order derivative is multiplied by a small positive parameter is called Singular Perturbation Problem ε (SPP) and the parameter is known as the perturbation parameter. Singular perturbation problems have always played prominent role in the theory of differential equation. If the perturbation parameter is present at other places other than highest derivatives, then the problem is called regular perturbation problem(RPPs). Singular perturbation problem ε (SPP) arise very frequently in diversified fields of applied mathematics and engineering, for instance, in fluid mechanics, elasticity, aerodynamics, quantum mechanics, chemical reactor theory, oceanography, meteorology, modeling of semiconductor devices and many others area.

A boundary layer is a small part of the region in which solutions change very rapidly to satisfy the given condition.

Semi-linear singularly perturbed reaction-diffusion problem is a differentail equation which is a half part of nonlinear differentail equation problems and it has degree two or more than two, but not less than two.

Kerem and Erdogan (2020), presented a numerical scheme for semi-linear singularly perturbed reaction-diffusion problem. They have introduced a basic and computational approach scheme based on Numerov's type on uniform mesh. They indicated that the method is uniformly convergence, according to the discrete maximum norm, independently of the parameter of ε . The proposed method was supported by numerical example.

Mariappan and Tamilselvan (2021), presented 'higher order numerical method for a semi-linear system of singularly perturbed differential equations'. To solve this problem, a parameter-uniform numerical method is constructed, which consists of a classical finite difference scheme and a piecewise uniform Shishkin mesh. It is proved that the convergence of the proposed numerical method is essentially second order in the maximum norm. A numerical illustration presented here supports the proved theoretical results. However, some of these methods are not uniformly convergent, simple and more accurate when the perturbation parameter ε is small and the number of mesh points, N increases.

On the above authors presented presented methods have not been sufficiently developed for a good result when we take different values of the perturbation parameter ε, and different number of mesh point, N in maximum absolute errors, rate of convergence and accuracy, that is, the previous numerical method is not more effective. Because, as perturbation parameters,ε decreasing, the errors in the previous numerical result is increasing but, I am going to find a good result for maximum absolute errors, rate of convergence and accuracy for solving higher-order numerical scheme for semi-linear singularly perturbed reaction-diffusion problem by using fourth order Numerov's finite difference method.

1.2 Statement of the Problem

The numerical analysis of singular perturbation problems has always been far from trivial because of the boundary layer behavior of the solution. Such problems undergo rapid changes within very thin layers near the boundary or inside the domain of the problem. However, the computation of its solution has been a great challenge and has been of great importance due to the versatility of such equations in the mathematical modeling of processes in various application fields, where they provide the best simulation of observed phenomena and hence the numerical approximation of such equations has been growing more and more.

The increasing desire for the numerical solutions to such mathematical problems, which are more difficult or impossible to solve analytically, has become the present-day scientific research.

Kerem and Erdogan (2020) stated that, the method is uniformly convergence, according to the discrete maximum norm, independently of the parameter of ε. It is well known that when the parameter of perturbation is small, conventional numerical methods to solve the problem do not work well. Therefore should be develop approprate numerical methods for such problems, whose convergence does not depend on the perturbation parameter,ε.

Owing to this, the main gap for the above presented methods was better results when we take different values of the perturbation parameter ε, and different number of mesh point, N in maximum absolute errors, rate of convergence and accuracy that is the previous numerical method is not more effective. Because, as perturbation parameters, ε decreasing, the errors in the previous numerical result is increasing but, I am going to find a good result for maximum absolute errors, rate of convergence and accuracy by using higher-order numerical scheme for for solving semi-linear singularly perturbed reaction-diffusion problem.

Owing to this, the present study will attempt to answer the following questions:

1. How does this study describe the higher-order numerical scheme for semilinear singularly perturbed reaction-diffusion problem?

2. To what extent these methods converge?

3. To what extent the present methods approximate the exact?

1.3 Objectives of the study

1.3.1 General Objectives

The general objective of the study is to present higher-order numerical scheme for semilinear singularly perturbed reaction-diffusion problem.

1.3.2 Specific objectives

The specific objectives of the present study are:

1. To formulate a higher-order numerical scheme for semi-linear singularly perturbed reaction-diffusion problem.

2. To establish the convergence of the described method.

3. To investigate the accuracy of the proposed method.

1.4 Significance of the study

The results obtained in this research may:

- Provide some background information for other researchers who work on this area.

- Give an idea about the application of numerical methods in different field of studies.

- Help the graduate students to acquire research skills and scientific procedures.

1.5 Delimitation of the Study

This study was delimited to the higher-order numerical scheme for solving the semilinear singularly perturbed reaction-diffusion problem of the form:

$$-\varepsilon y''(x) = f(x, y(x)), \quad x \in (0,1), \tag{1.1}$$

subject to the boundary conditions,

$$y(0) = \alpha, y(1) = \beta, \tag{1.2}$$

where ε is a small positive singular perturbation parameter such that, $0 < \varepsilon < 1$, f is assumed to be sufficiently smooth functions and continuously differentiable functions and α, β are given constants.

Chapter 2

Literature Review

2.1 Singular Perturbation problem

Singular perturbation problem was first introduced by Prandtl (1905) during his talk on fluid motion with small friction in a seven-page report presented at the Third International Congress of Mathematicians in Heidelberg in 1904 in which he demonstrated that fluid flow past over a body can be divide in two regions, a boundary layer and outer region. However, the term 'Singular perturbations' was first used by Friedrichs et al (1946) in a paper presented at a seminar on nonlinear vibrations at New York University. The solutions of singular perturbation problems typically contain layers. Prandtl (1905), originally introduced the term 'boundary layer', but this term came into more general following the work of Wasow (1942).

In Mathematics, more precisely in perturbation problem, a singular perturbation problem is a problem containing a small parameter that cannot be approximated by setting the parameter value to zero. This is in contrast to regular perturbation problems, for which an approximation can be obtained by simply setting the small parameter to zero.

More precisely, the solution cannot be uniformly approximated an asymptotic expansion as. Here ε is the small parameter of the problem. This is in contrast to regular perturbation problems, for which a uniform approximation of this form can be obtained. The problems in which the highest order derivative term is multiplied by a small parameter are known to be perturbed problems and the parameter is known as the perturbation parameter.

Most often in applications, an acceptable approximation to a regularly perturbed problem is found by simply replacing the small parameter ε by zero everywhere in the problem statement. This corresponds to taking only the first term of the expansion, yielding an approximation that converges, perhaps slowly, to the true solution as ε decreases. The solution to a singularly perturbed problem cannot be approximated in this way. Thus, naively taking the parameter to be zero changes the nature of the problem. In the case of differential equations, boundary conditions cannot be satisfied; in algebraic equations, the possible number of solutions is decreased. Exact solutions are rare in many branches of fluid mechanics, solid mechanics, motion,and physics because of nonlinearities, in homogeneities, and general boundary conditions.

During the last few years much, progress has been made in the theory and in the computer implementation of the numerical treatment of singular perturbation problems. Typically, these problems arise very frequently in fluid mechanics, fluid dynamics, elasticity, aero dynamics, plasma dynamics, magneto hydrodynamics, rarefied gas dynamics, oceanography, and other domains of the great world of fluid motion. A few notable examples are boundary layer problems, Wentzel-Kramers-Brillouin (WKB) problems, the modeling of steady and unsteady viscous flow problems with large Reynolds numbers, convective heat transport problems with large Peclet numbers, magneto-hydrodynamics duct problems at high Hartman numbers, etc. These problems depend on a small positive parameter in such a way that the solution varies rapidly in some parts and varies slowly in some other parts.

In this thesis, higher order numerical scheme have been presented for solving semi-linear singularly perturbed reaction-diffusion problem.

2.2 Recent Development Methods

Kerem and Erdogan (2020), presented a numerical scheme for semi-linear singularly perturbed reaction-diffusion problem. They have introduced a basic and computational approach scheme based on Numerov's type on uniform mesh. They indicated that the method is uniformly convergence, according to the discrete maximum norm, independently of the parameter ε of the proposed method was supported by numerical example.

Mariappan and Tamilselvan (2021), presented 'higher order numerical method for a semi-linear system of singularly perturbed differential equations'. To solve this problem, a parameter-uniform numerical method is constructed, which consists of a classical finite difference scheme and a piecewise uniform Shishkin mesh. It is proved that the convergence of the proposed numerical method is essentially second order in the maximum norm. A numerical illustration presented here supports the proved theoretical results.

Owing to this, the main gap for the above presented methods was no good result when we take different values of the perturbation parameter ε, and different number of mesh point, N in maximum absolute errors, rate of convergence and accuracy but, I am going to find a good result for maximum absolute errors, rate of convergence and accuracy for solving higher-order numerical scheme for semi-linear singularly perturbed reaction-diffusion problem by using fourth order Numerov's finite difference method.

2.3 Quasilinearizaion Process

The quasilinearization technique (1965) has been used to reduce the given semi-linear singularly perturbed reaction-diffusion problem (1.1) - (1.2) into a sequence of linear singularly perturbed reaction-diffusion problem. A higher order numerical method is presented for solving sequence of linear singularly perturbed reaction-diffusion problem. We use the quasi-linearization procedure for solving semi-linear singularly perturbed reaction-diffusion problem subject to the boundary conditions involving the following steps. First, we linearize the semi-linear ordinary differential equation which satisfies the specified boundary conditions. Second, we solve a sequence of two-point boundary value problems in which the solution of the linear two-point boundary-value problem satisfies the specified boundary conditions.

The main advantages of this method are as follows:

- The method approximates the solution of semi-linear differential equations by treating the semi-linear terms as a perturbation about the linear ones, and is not based, unlike perturbation theories, on the existence of some kind of small parameter.
- Once the quasi-linear iteration sequence at some interval starts to converge, it will always continue to do so. Unlike an asymptotic perturbation series, the quasi-linearization method yield the required precision once a successful initial guess generates convergence after a few steps.
- If the procedure converges, it converges quadratically to the solution of the original problem. The linear equation is obtained by using the first and second terms of the Taylor's series expansion of the original semi-linear differential equation.

Chapter 3

Methodology

3.1 Study Area and Period

The study was conducted in Jimma University, Department of Mathematics from September 2021 to June 2022.Conceptually the study focuses on numerical methods particularly on higher-order numerical scheme for semilinear singularly perturbed reaction-diffusion problems.

3.2 Study Design

This study was employed mixed-design (documentary review design and experimental design) on higher-order numerical scheme for semilinear singularly perturbed reaction-diffusion problems.

3.3 Source of Information

The relevant source of information for this study are books, published articles and related studies from internet services and the experimental results obtained by writing MATLAB code for the present numerical methods.

3.4 Mathematical Procedures

In order to achieve the above-mentioned objectives, the study was followed the following steps:

1. Defining the semilinear singularly perturbed reaction-diffusion problems.
2. Linearize the semilinear parts of the defined problem.
3. Discuss the properties of the exact solution for the linearized problem.
4. Discretize the solution domain.

5. Describe the higher-order numerical scheme for the linearized problem.
6. Establish the convergence analysis of the described scheme.
7. Writing MATLAB code for the scheme.
8. Provide numerical illustrations.

3.5 Ethical Considerations

Ethical clearance was obtained from Research and Post Graduate Program Coordinator Office of College of Natural Sciences, Jimma University and any concerned body were informed about the purpose of the study.

Chapter 4

Description of the method and Results

4.1 Description of the Method

Consider the semilinear singularly perturbed reaction-diffusion problem of the form:

$$-\varepsilon y''(x) = f(x, y(x)), \quad x \in (0, 1), \tag{4.1}$$

subject to the boundary conditions:

$$y(0) = \alpha, y(1) = \beta, \tag{4.2}$$

where ε is a small positive singular perturbation parameter such that, $0 < \varepsilon < 1$, f is assumed to be sufficiently smooth functions and continuously differentiable functions and α, β are given constants.

4.1.1 Linearization

To apply the quasi-linearization technique on the semi-linear part of Eq. (4.1), let us take the initial approximation:

$$y_0(x) = mx + b \tag{4.3}$$

where m and b are constant real numbers to be determined using the boundary conditions in Eq. (4.2).

Hence, using the defined boundary conditions in Eq. (4.2), we have:

$$y_0(0) = \alpha = m(0) + b = b$$
$$y_0(1) = \beta = m(1) + b$$

Thus, both constants m and b are defined:

$$\begin{cases} b = \alpha \\ m = \beta - \alpha \end{cases} \tag{4.4}$$

Therefore, the considered initial approximation in Eq. (4.3)can be re-written as:

$$y_0(x) = (\beta - \alpha)x + \alpha \tag{4.5}$$

Now using the quasi-linearization technique at the first iteration, the semi-linear part of Eq. (4.1) can be linearized as:

$$f(x, y(x)) \approx f(x, y_0(x)) + (y(x) - y_0(x)) \frac{\partial f}{\partial y(x)}\bigg|_{(x,y_0(x))} \tag{4.6}$$

Substituting Eq. (4.6) into Eq. (4.1) and after algebraic re-arrargement, we get:

$$-\varepsilon y''(x) + b(x)y(x) = f(x) \tag{4.7}$$

where

$$b(x) = -\frac{\partial f}{\partial y(x)}\bigg|_{(x,y_0(x))}$$

$$f(x) = f(x, y_0(x)) - y_0(x)) \frac{\partial f}{\partial y(x)}\bigg|_{(x,y_0(x))}$$

Thus, the semi-linear problems in Eq. (4.1) and the boundary condition in Eq. (4.2) asymptotically equivalent to the linear boundary value problem of the form:

$$\begin{cases} -\varepsilon y''(x) + b(x)y(x) = f(x), \ 0 < x < 1, \\ y(0) = \alpha, \\ y(1) = \beta \end{cases} \tag{4.8}$$

where $b(x)$ and $f(x)$ are given sufficiently smooth functions. Further, the coefficient function of the reaction term satisfy $b(x) \geq b_0 > 0$, for constant b_0.

4.1.2 Properties of the Analytical Solution

Some important properties of the solution of Eq. (4.8) which are required in later sections for the analysis of the numerical solution.

Lemma 1: (Maximum Principle)

Let L be the operator as Eq. (4.8) such that $y(0) = \alpha$, $y(1) = \beta$. Suppose $\phi(x)$ is any smooth function satisfying $y(0) \geq 0$ and $y(1) \geq 0$ and $L\phi(x) \geq 0$, for all $0 < x < 1$, then $\phi(x) \geq 0$, for all $0 \leq x \leq 1$.
proof: The proof is by contradiction. Let x^* be such that $\phi(x^*) = \min_{x \in [0,1]} \phi(x)$ and assume that $\phi(x^*) < 0$. Clearly, $x^* \notin \{0, 1\}$ and therefore $\phi''(x^*) \geq 0$.

Further, $L\phi(x^*) = -\varepsilon\phi''(x^*) + b(x^*)\phi(x^*) \leq 0$, which is a contradiction. It follows that $\phi(x^*) \geq 0$

and thus $\phi(x) \geq 0$, for all $x \in [0,1]$.

Lemma 2: (Stability). For any $v(x)$ function, let $v(x) \in C[0,1] \cap C^2(0,1)$. The following estimate is true.

$$|v(x)| \leq |v(0)| + |v(1)| + \alpha^{-1} \max_{1 \leq i \leq N-1} |Lv(x)|, 0 \leq x \leq 1. \tag{4.9}$$

proof. Let us define the $\phi(x)$ function as follows:

$$\phi(x) = \pm v(x) + |v(0)| + |v(1)| + \alpha^{-1} \max_{1 \leq i \leq N-1} |Lv(x)|, 0 \leq x \leq 1. \tag{4.10}$$

Then the following inequalities are satisfied.

$$\phi(0) \geq 0, \phi(1) \geq 0, and\ L\phi(x) \geq 0. \tag{4.11}$$

The maximum principle gives $\phi(x) \geq 0$, for all $0 \leq x \leq 1$, and so the inequality (4.9) holds.
Lemma 3: (Boundedness). Let $b(x)$, $f(x)$ are given sufficiently smooth functions and $y(x)$ be the solution of the problem (4.8). Then the following estimates hold.

$$||y(x)||_\infty \leq C, 0 \leq x \leq 1 \tag{4.12}$$

$$|y'(x)| \leq C\left(1 + \frac{1}{\sqrt{\varepsilon}}\left(e^{-\frac{\sqrt{\alpha}(x)}{\sqrt{\varepsilon}}} + e^{-\frac{\sqrt{\alpha}(1-x)}{\sqrt{\varepsilon}}}\right)\right) \tag{4.13}$$

proof: Applying lemma 1 to Eq. (4.8), we have Eq. (4.12)

$$Lv(x) = \psi(x) \tag{4.14}$$

$$v(0) = O(\frac{1}{\sqrt{\varepsilon}}), v(1) = O(\frac{1}{\sqrt{\varepsilon}}) \tag{4.15}$$

where,

$$Lv(x) = y'(x), \psi(x) = f'(x) - b'(x)y(x) \tag{4.16}$$

The solutions of the problem (4.14)-(4.15) has the following forms:

$$v(x) = v_0(x) + v_1(x) \tag{4.17}$$

The functions $v_0(x)$ and $v_1(x)$ are the solutions of the following problems:

$$\begin{cases} Lv_0(x) = \psi(x), 0 < x < 1 \\ v_0(0) = v_0(1) = 0 \end{cases} \tag{4.18}$$

$$\begin{cases} Lv_1(x) = 0, 0 < x < 1 \\ Lv_1(0) = v_1(1) = 0 \end{cases} \tag{4.19}$$

from lemma 1, for the solution of the problem (4.18), we have:

$$|v_0(x)| \leq \alpha^{-1} \max_{1 \leq i \leq N-1} |\psi(x)|$$

Thus, we obtain:

$$|v_0(x)| \leq C, 0 \leq x \leq 1 \tag{4.20}$$

Applying maximum principle to the problem (4.19), we get:

$$|v_1(x)| \leq u(x) \tag{4.21}$$

where $u(x)$ is the solution of the following problems:

$$\begin{cases} -\epsilon u'' + \alpha u' = 0, 0 < x < 1 \\ u(0) = |v_1(0)|, u(1) = |v_1(1)| \end{cases} \tag{4.22}$$

The solution of problem (4.22) has the form:

$$u(x) = \frac{1}{\sinh(\frac{\sqrt{\alpha}}{\sqrt{\varepsilon}})}\left(|v_1(0)| \sinh\left(\frac{\sqrt{\alpha}(1-x)}{\sqrt{\varepsilon}}\right) + |v_1(1)| \sinh\left(\frac{\sqrt{\alpha}x}{\sqrt{\varepsilon}}\right)\right) \tag{4.23}$$

and it is from that is hold.

$$u(x) \leq \frac{C}{\sqrt{\varepsilon}}\left(e^{-\frac{\sqrt{\alpha}(x)}{\sqrt{\varepsilon}}} + e^{-\frac{\sqrt{\alpha}(1-x)}{\sqrt{\varepsilon}}}\right) \tag{4.24}$$

The combining Eqs. (4.20), (4.21) and (4.24) in the following inequality, it can be obtained:

$$|u'(x)| \leq |v_0(x)| + |v_1(x)|.$$

Thus, the solutions of the defined problem in Eq. (4.8) is exist and unique.

4.1.3 Formulation of the Numerical Scheme

In this section, we construct a numerical scheme for solving Eq. (4.8) on a uniform mesh by sub-dividing the interval [0, 1] into N equal partition, which defined the nodal points as:

$$\begin{cases} x_i = ih, \quad i = 0, 1, 2, ..., N \\ h = \frac{1}{N} \end{cases} \tag{4.25}$$

Let us denote the approximate solution $y(x_i) \approx y_i$ at the nodal point x_i, $i = 1, 2, ..., N-1$. Then, a general k-step method for the solution of Eq. (4.8) can be written as:

$$y_{j+1} = \sum_{i=1}^{k=2} a_i y_{j-i+1} + h^2 \sum_{i=0}^{k=2} b_i y''_{j-i+1} \tag{4.26}$$

where a_i and b_i are arbitrary constant.
From Eq. (4.26), we obtain:

$$y_{i+1} - a_1 y_i - a_2 y_{i-1} - h^2[b_0 y''_{i+1} + b_1 y''_i + b_2 y''_{i-1}] = T_1 \tag{4.27}$$

By using Taylor series expansion, we obtain:

$$y_{i+1} = y_i + hy'_i + \frac{h^2}{2!}y''_i + \frac{h^3}{3!}y'''_i + \frac{h^4}{4!}y_i^4 + \frac{h^5}{5!}y_i^5 + \frac{h^6}{6!}y_i^6 + O(h^7) \tag{4.28}$$

$$y_{i-1} = y_i - hy'_i + \frac{h^2}{2!}y''_i - \frac{h^3}{3!}y'''_i + \frac{h^4}{4!}y_i^4 - \frac{h^5}{5!}y_i^5 + \frac{h^6}{6!}y_i^6 + O(h^7) \tag{4.29}$$

$$y''_{i+1} = y''_i + hy'''_i + \frac{h^2}{2}y_i^4 + \frac{h^3}{3!}y_i^5 + \frac{h^4}{4!}y_i^6 + O(h^7) \tag{4.30}$$

$$y''_{i-1} = y''_i - hy'''_i + \frac{h^2}{2}y_i^4 - \frac{h^3}{3!}y_i^5 + \frac{h^4}{4!}y_i^6 + O(h^7) \tag{4.31}$$

Substituting Eqs. (4.28)-(4.31) into Eq. (4.27), we get:

$$\begin{cases} y_i + hy'_i + \frac{h^2}{2}y''_i + \frac{h^3}{3!}y'''_i + \frac{h^4}{4!}y_i^4 + \frac{h^5}{5!}y_i^5 + \frac{h^6}{6!}y_i^6 + O(h^7) \\ -a_1 y_i \\ -a_2[y_i - hy'_i + \frac{h^2}{2}y''_i - \frac{h^3}{3!}y'''_i + \frac{h^4}{4!}y_i^4 - \frac{h^5}{5!}y_i^5 + \frac{h^6}{6!}y_i^6 + O(h^7)] \\ -h^2[b_0(y''_i + hy'''_i + \frac{h^2}{2}y_i^4 + \frac{h^3}{3!}y_i^5 + \frac{h^4}{4!}y_i^6 + O(h^7)) + b_1 y''_i \\ +b_2(y''_i - hy'''_i + \frac{h^2}{2}y_i^4 - \frac{h^3}{3!}y_i^5 + \frac{h^4}{4!}y_i^6 + O(h^7))] = T_1 \end{cases}$$

By comparing the coefficients of various powers of h on both sides, we get:

$$\begin{cases} [1 - a_1 - a_2]y_i + h[1 + a_2]y'_i \\ +h^2[\frac{1}{2} - \frac{a_2}{2} - b_0 - b_1 - b_2]y''_i \\ +h^3[\frac{1}{6} + \frac{a_2}{6} - b_0 + b_2]y'''_i \\ +h^4[\frac{1}{24} - \frac{a_2}{24} - \frac{b_0}{2} - \frac{b_2}{2}]y_i^4 \\ +h^5[\frac{1}{5!} + \frac{a_2}{5!} - \frac{b_0}{6} + \frac{b_2}{6}]y_i^5 \\ +h^6[\frac{1}{6!} - \frac{a_2}{6!} - \frac{b_0}{4!} - \frac{b_2}{4!}]y_i^6 = T_1 \end{cases} \tag{4.32}$$

By definition of a linear (k- step) method from the above equation is said to be an order in the truncation error in Eq. (4.32) is order of p+2 that is, p=2, p+2=4, then the order is 4. The unknown coefficients are so determined and by taking a coefficient equals to zero, we get:

$$\begin{cases} 1 - a_1 - a_2 = 0, \\ 1 + a_2 = 0, \\ \frac{1}{2} - \frac{a_2}{2} - b_0 - b_1 - b_2 = 0, \\ \frac{1}{6} + \frac{a_2}{6} - b_0 + b_2 = 0, \\ \frac{1}{24} - \frac{a_2}{24} - \frac{b_0}{2} - \frac{b_2}{2} = 0, \\ \frac{1}{120} + \frac{a_2}{120} - \frac{b_0}{6} + \frac{b_2}{6} = 0, \\ \frac{1}{720} - \frac{a_2}{720} - \frac{b_0}{24} - \frac{b_2}{24} \neq 0 \end{cases}$$

Then, the values of arbitrary constants:

$$\begin{cases} a_1 = 2, \\ a_2 = -1, \\ b_0 = \frac{1}{12}, \\ b_1 = \frac{10}{12} = \frac{5}{6}, \\ b_2 = b_0 = \frac{1}{12} \end{cases} \tag{4.33}$$

By substituting the values a_1, a_2, b_0, b_1, b_2 of Eq. (4.33) into Eq. (4.27), we get:

$$y_{i+1} - 2y_i + y_{i-1} = \frac{h^2}{12}[y''_{i+1} + 10y''_i + y''_{i-1}] \tag{4.34}$$

Now, writing Eq. (4.8), we obtain:

$$-\varepsilon y''_i = f_i - b_i y_i \tag{4.35}$$

From Eq. (4.35), we get:

$$\begin{cases} y''_i = \frac{1}{\varepsilon}[b_i y_i - f_i], \\ y''_{i+1} = \frac{1}{\varepsilon}[b_{i+1} y_{i+1} - f_{i+1}], \\ y''_{i-1} = \frac{1}{\varepsilon}[b_{i-1} y_{i-1} - f_{i-1}] \end{cases} \tag{4.36}$$

Substituting Eq. (4.36) into Eq. (4.34), we get:

$$\frac{12\varepsilon}{h^2}(y_{i+1} - 2y_i + y_{i-1}) = b_{i+1}y_{i+1} - f_{i+1} + 10[b_i y_i - f_i] + b_{i-1}y_{i-1} - f_{i-1} \tag{4.37}$$

Re-arranging Eq. (4.37), we obtain:

$$\frac{-12\varepsilon}{h^2}(y_{i+1} - 2y_i + y_{i-1}) + b_{i+1}y_{i+1} + 10b_i y_i + b_{i-1}y_{i-1} = f_{i-1} + 10f_i + f_{i+1} \tag{4.38}$$

By using coefficients of y_{i-1}, y_i, y_{i+1}, we get:

$$(\frac{-12\varepsilon}{h^2} + b_{i-1})y_{i-1} + (\frac{24\varepsilon}{h^2} + 10b_i)y_i + (-\frac{12\varepsilon}{h^2} + b_{i+1})y_{i+1} = f_{i-1} + 10f_i + f_{i+1} \tag{4.39}$$

This Eq. (4.39) can be written as the three term recurrence relation of the form:

$$E_i y_{i-1} + F_i y_i + G_i y_{i+1} = H_i, \; for \; i = 1, 2, ..., N-1. \tag{4.40}$$

where

$$E_i = b_{i-1} - \frac{12\varepsilon}{h^2}$$

$$F_i = 10b_i + \frac{24\varepsilon}{h^2}$$

$$G_i = b_{i+1} - \frac{12\varepsilon}{h^2}$$

$$H_i = f_{i-1} + 10f_i + f_{i+1}$$

4.1.4 Stability Analysis

The matrix form of the stability analysis obtained scheme can be written as:

$$KY = L \tag{4.41}$$

where

$$K = \begin{bmatrix} F_1 & G_1 & o & 0 & 0 & \cdots & 0 \\ \\ E_2 & F_1 & G_2 & 0 & 0 & \cdots & 0 \\ 0 & E_2 & F_3 & G_3 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & \cdots & 0 \\ \vdots & \vdots & 0 & E_{N-2} & F_{N-2} & & G_{N-2} \\ 0 & \cdots & \cdots & \cdots & E_{N-1} & & F_{N-1} \end{bmatrix}$$

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{N-2} \\ y_{N-1} \end{bmatrix}$$

and

$$L = \begin{bmatrix} H_1 - E_1 y_0 \\ H_2 \\ H_2 \\ \vdots \\ 0 \\ H_{N-1} - G_{N-1} y_N \end{bmatrix}$$

Definition: (M-Matrices)(from the book by Martin and David, 2018): A given square matrix $M = (m_{ij})$ is said to be $M-$matrix if $m_{ij} \leq 0$, for all $i \neq j, m_{ii} > 0$, for all i and then the inverse, M^{-1} exists with each entries are greater or equal to zero. Then, the difference schemes whose coefficient matrices satisfy M-matrices are generally stable. Further, a square matrix $M = (m_{ij})$ is said to be strictly diagonally dominant if

$$F_i > |E_i + G_i|$$

In our case, from system of Eq. (4.41), we have:

$$\begin{aligned} m_{ii} &= F_i, \\ m_{ij} &= E_i, i + 1 = j, \\ m_{ij} &= G_i, i = j + 1, \\ m_{ij} &= 0, otherwise \end{aligned}$$

Thus, the diagonal dominance defined above can be verified as:

$m_{ii} > |m_{i+1j} + m_{ij+1}|$
$F_i > |E_i + G_i|$
$\frac{24\varepsilon}{h^2} + 10b_i > |b_{i-1} - \frac{12\varepsilon}{h^2} + b_{i+1} - \frac{12\varepsilon}{h^2}|$ which can be satisfied by: $b_i \geq \gamma > 0$.

Hence, the difference schemes in Eq. (4.41) that employ matrix is stable. Thus, the formulated scheme in Eq. (4.40) satisfies the definition of M-matrix that consequences stability of scheme.

Truncation Error

The truncation error for the described method will be investigated. To achieve this investigation, the local truncation error $T(h)$ between the exact solution $Y(x_i)$, and the approximate solution Y_i is given by:

$$\begin{cases} \tau_i(h) = -\varepsilon y_i'' + b_i y_i \\ -[y_{i+1} - 2y_i + y_{i-1} - \frac{h^2}{12}(b_{i+1}y_{i+1} + 10b_i y_i + b_{i-1}y_{i-1})] \end{cases} \tag{4.42}$$

By using Taylor series expansion, we obtain:

$$y_{i+1} = y_i + hy_i' + \frac{h^2}{2!}y_i'' + \frac{h^3}{3!}y_i''' + \frac{h^4}{4!}y_i^4 + \frac{h^5}{5!}y_i^5 + \frac{h^6}{6!}y_i^6 + O(h^7) \tag{4.43}$$

$$y_{i-1} = y_i - hy_i' + \frac{h^2}{2!}y_i'' - \frac{h^3}{3!}y_i''' + \frac{h^4}{4!}y_i^4 - \frac{h^5}{5!}y_i^5 + \frac{h^6}{6!}y_i^6 + O(h^7) \tag{4.44}$$

$$y_{i+1}'' = y_i'' + hy_i''' + \frac{h^2}{2}y_i^4 + \frac{h^3}{3!}y_i^5 + \frac{h^4}{4!}y_i^6 + O(h^7) \tag{4.45}$$

$$y_{i-1}'' = y_i'' - hy_i''' + \frac{h^2}{2}y_i^4 - \frac{h^3}{3!}y_i^5 + \frac{h^4}{4!}y_i^6 + O(h^7) \tag{4.46}$$

$$y'' = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} + T(h) \tag{4.47}$$

From these four basic Eqs. (4.43)-(4.46), we obtain the following:

$$y_{i+1} + y_{i-1} = 2y_i + h^2 y_i'' + \frac{h^4}{12}y_i^4 + \frac{h^6}{360}y_i^6 + O(h^7) \tag{4.48}$$

$$y_{i+1}'' + y_{i-1}'' = 2y_i'' + h^2 y_i'' + \frac{h^4}{12}y_i^6 + O(h^7) \tag{4.49}$$

Recall at the nodal point x_i: $y''(x) = y_i''$ and $b(x)y(x) = b_i y_i$

From the above Eq. (4.27) and Eq.(4.47), we obtain:

$$h^2 y_i'' = y_{i+1} - 2y_i + y_{i-1} - \frac{h^2}{12}[y_{i+1}'' + 10y_i'' + y_{i-1}''] + T(h) \tag{4.50}$$

By substituting Eqs.(4.48) and (4.49) into Eq. (4.50), we obtain:

$$h^2 y_i'' = y_{i+1} + y_{i-1} - 2y_i - \frac{h^2}{12}[y_{i+1}'' + y_{i-1}'' + 10y_i''] + T(h) \tag{4.51}$$

$$h^2y_i'' = 2y_i - 2y_i + h^2y_i'' + \frac{h^4}{12}y_i^4 + \frac{h^6}{360}y_i^6 - \frac{h^2}{12}[2y_i'' + h^2y_i^4 + \frac{h^4}{12}y_i^6 + 10y_i''] + T(h) \quad (4.52)$$

$$h^2y_i'' = h^2y_i'' + \frac{h^4}{12}y_i^4 + \frac{h^6}{360}y_i^6 - \frac{h^2}{12}[2y_i'' + h^2y_i^4 + \frac{h^4}{12}y_i^6 + 10y_i''] + T(h) \quad (4.53)$$

Re-arranging the term in the bracket and multiplying by $\frac{-h^2}{12}$ of Eq. (4.53), we get:

$$h^2y_i'' = h^2y_i'' + \frac{h^4}{12}y_i^4 + \frac{h^6}{360}y_i^6 - [h^2y_i'' + \frac{h^4}{12}y_i^4 + \frac{h^6}{144}y_i^6] + T(h) \quad (4.54)$$

Removing the bracket and collecting like terms of Eq. (4.54), we get:

$$h^2y_i'' = h^2y_i'' - h^2y_i'' + \frac{h^4}{12}y_i^4 - \frac{h^4}{12}y_i^4 + \frac{h^6}{360}y_i^6 - \frac{h^6}{144}y_i^6 + T(h) \quad (4.55)$$

From Eq. (4.55), we have:

$$h^2y_i'' = \frac{h^6}{360}y_i^6 - \frac{h^6}{144}y_i^6 + T(h) \quad (4.56)$$

$$h^2y_i'' = (\frac{1}{360} - \frac{1}{144})h^6y_i^6 + T(h) \quad (4.57)$$

$$h^2y_i'' = (\frac{-3}{720})h^6y_i^6 + T(h) \quad (4.58)$$

By dividing h^2 and simplifying Eq, (4.58),from both sides, we get:

$$y_i'' = (\frac{-h^4}{240})y_i^6 + T(h) \quad (4.59)$$

Thus, the local trancation error $T(h))$ is:

$$T(h) = \frac{-h^4}{240}y_i^6 \quad (4.60)$$

Richardson extrapolation

The basic idea behanid extrapolation is that whenever the leading term in the error for an approximation formula is known, we can combine two approximations obtained from that formula using different values of parameter mesh size h to obtain a higher-order approximation and the technique is is known as Richardson extrapolation. The two linear combination of convergence acceleration is turn out to be a better approximation. Hence, we have

$$|y(x_i) - Y_N| \approx C(h^4), x \in \Omega = (0,1) \quad (4.61)$$

where $y(x_i)$ and Y_N are exact and approximation solutions respectively, C is constant independent of mesh size h.

Ω^{2N} be the mesh obtained by bisecting each mesh interval in Ω^N and denote the approximation of the solution on Ω^{2N} by Y_{2N}. Consider Eq. (4.60) works for any $h \neq 0$, which implies:

$$y(x_i) - Y_N \approx C(h^4), x_i \in \Omega^N \tag{4.62}$$

So that, it works for any $\frac{h}{2} \neq 0$ yields:

$$y(x_i) - Y_{2N} \approx C((\frac{h}{2})^4) + R^{2N}, x_i \in \Omega^{2N} \tag{4.63}$$

where the remainders, R^N and R^{2N} are $O(h^4)$. A combination of inequalities in Eqs. (4.61) and (4.62) leads to $15y(x_i) - (16Y_{2N} - Y_N) = O(h^4)$, which suggests that

$$(Y_N)^{ext} = \frac{1}{15}(16Y_{2N} - Y_N) \tag{4.64}$$

is also an approximation of $y(x_i)$. Using this approximation to evaluate the truncation error, we obtain:

$$|y(x_i) - (Y_N)^{ext}| \approx Ch^4 \tag{4.65}$$

Now, using these two different solutions which are obtained by the same scheme given by Eq. (4.60), we get another third solution in terms of the two equation (4.65). This is Richardson extrapolation method to accelerate the rate of convergence to fourth order.

4.2 Numerical Example and Disussion

To demonstrate the applicability of the method, we have solve the following example: a non-homogenous SPP and a SPP with variable coefficients. For each ε and N, the maximum absolute errors at nodal points are approximated by the formula below for $i = 0, 1, ..., N$ and write y_i^N and y_{2i}^{2N} are the exact and computed solution of the given problem and nodal points x_i.

Example 4.1: Consider the following semi-linear singularly perturbed reaction-diffusion problem in (2020):

$$\begin{cases} -\varepsilon y''(x) - e^{-(x^2+y)} = 0 \\ y(0) = 0 \\ y(1) = 1 \end{cases}$$

The exact solution is not known.

Table 4.1: Comparision of maximum absolute error obtained for Example 4.1

$\varepsilon\downarrow N\rightarrow$	32	64	128	256
After				
2^{-5}	1.0929e-10	1.7549e-12	1.5055e-13	5.3624e-13
2^{-6}	7.4998e-10	1.1859e-11	1.5099e-13	7.0255e-13
2^{-7}	5.0741e-09	8.1276e-11	1.3035e-12	4.1145e-13
2^{-8}	3.4467e-08	5.6564e-10	8.9415e-12	1.5654e-13
2^{-9}	2.1980e-07	3.9747e-09	6.3025e-11	1.0191e-12
Before				
2^{-5}	1.2025e-06	7.5262e-08	4.7055e-09	2.9415e-10
2^{-6}	4.2806e-06	2.6824e-07	1.6776e-08	1.0486e-09
2^{-7}	1.5334e-05	9.6312e-07	6.0373e-08	3.7747e-09
2^{-8}	5.6159e-05	3.5423e-06	2.2192e-07	1.3879e-08
2^{-9}	1.9857e-04	1.3224e-05	8.3025e-07	5.2072e-08
Kerem and Erdogan (2020)				
2^{-5}	2.3e-4	5.9e-05	1.5e-05	3.7e-06
2^{-6}	3.8e-04	9.5e-05	2.4e-05	6.2e-06
2^{-7}	6.4e-04	1.6e-04	4.0e-05	1.0e-05
2^{-8}	1.1e-03	2.4e-04	6.9e-05	1.7e-05
2^{-9}	1.9e-03	4.9e-04	1.2e-04	3.1e-05
R^N	2.013	2.003	2.002	1.998
R_ε^N	2.009	2.014	2.019	2.020

Table 4.2: Rate of convergence before extrapolation obtained for Example 4.1

$\varepsilon\downarrow N\rightarrow$	32	64	128
2^{-5}	3.9980	3.9995	3.9997
2^{-6}	3.9962	3.9991	3.9999
2^{-7}	3.9929	3.9957	3.9995
2^{-8}	3.9868	3.9966	3.9991
2^{-9}	3.9083	3.9935	3.9950

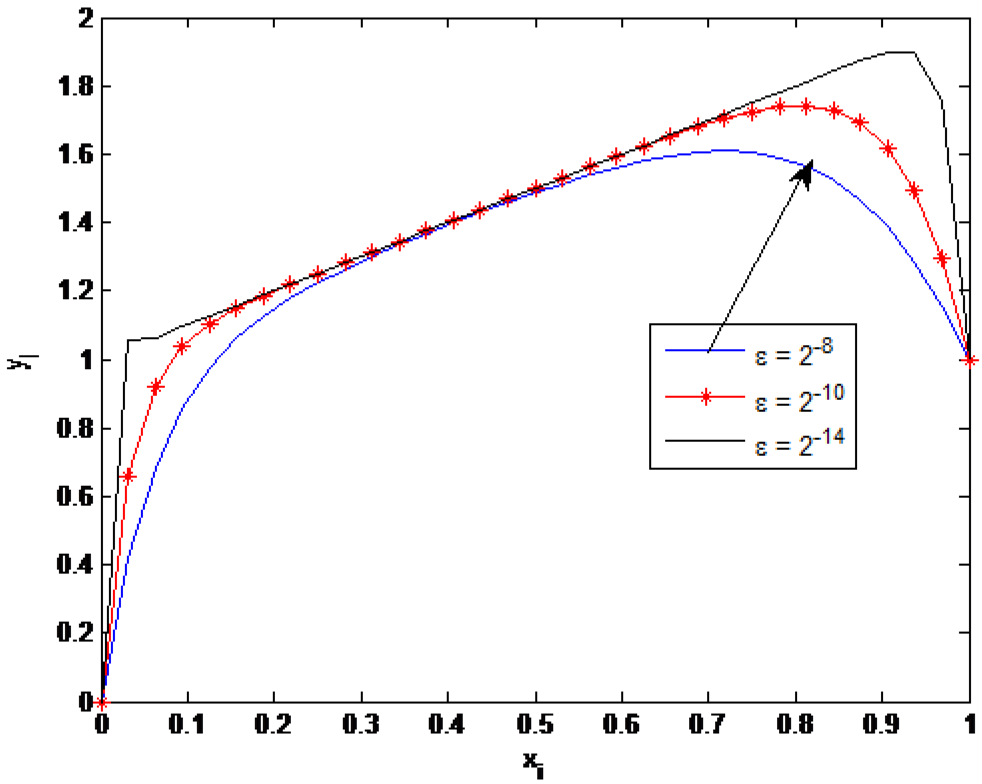

Figure 4.1: Numerical Solution versus exact solution of Example 4.1, as for different values of perturbation parameter $\varepsilon = 2^{-8}, \varepsilon = 2^{-10}, \varepsilon = 2^{-14}$ and mesh point N=32, to show the behavior of the layer is very large in both ends at $x = 0$ and $x = 1$.

Discussion

In this thesis, the higher-order numerical scheme is presented for solving semi-linear singularly perturbed reaction-diffusion problem. First we linearized semi-linear singularly perturbed reaction-diffusion problem by using quasilinearization technique. The numerical results have been presented in Tables(4.1)-(4.2), and graph for different values of the perturbation parameters, ε and different number of mesh points, N. From the above table, we observed that the present method (scheme) is more convergent and accurate result than above author's numerical method. As it can be observed that the numerical results presented in tables and graph, the present method approximates the exact solution very well when we compared with previous finding of other methods of author's method reported in the literature. The maximum absolute errors are presented in table is to show more accurate of the scheme. As perturbation parameters, ε decreasing, the maximum absolute errors also decreasing. This implies that the accuracy of the presented method (scheme) is very increasing when we compared with the some previous finding of author's method reported in the literature and as different number of mesh points, N is increasing, then the maximum absolute error of the presented method when we compared with numerical method reported in literature. This implies that the presented method (scheme) is more convergent than methods reported in the literature. So, the present method which is given more convergent and accurate numerical solutions for semi-linear singularly perturbed reaction-diffusion problem.

Therefore from the above tables, the previous numerical method is not more effective when we compared with the presented method. Because, as perturbation parameters, ε decreasing, the errors in the previous numerical result is increasing. But, in the present method the errors of numerical results are decreasing.

We have also presented graphical representation of the numerical and exact solution for the problems in figure, to show the behavior of the layer is very large in both ends at x=0 and x=1. From the graph, we observed that numerical solution approximate exact solution very well in the boundary layers and the extrapolation is shows the fourth order convergent.

Chapter 5

Conclusion and Future Work

5.1 Conclusion

In this thesis, we have presented a higher-order numerical method to solve a semilinear singularly perturbed reaction-diffusion problem. We linearized a semilinear singularly perturbed reaction-diffusion problem by using quasilinearization techniques. To validate the applicability of the present method model example of semi-linear singularly perturbed reaction-diffusion problem have been considered and the maximum absolute errors of the numerical results are presented in tables and graph. The numerical results have been presented in Tables for different values of the perturbation parameters, ε and different number of mesh points, N. The numerical results presented in Tables to show that the present method (scheme) is approximated to the exact solution very well when we compared with author's method reported in the literature which show that the present method (scheme) is given more convergent and accurate numerical results than that of its author's in the literature.

5.2 Scope for future work

The order of the numerical method is increasing the convergence and accuracy of the method is also increasing. Therefore, the present method (scheme) in this thesis one can also be extended to higher-order numerical scheme for semi-linear singularly perturbed reaction-diffusion problem.

References

Angel, E. and Bellman, R. (1972). Dynamic programming and partial differential equations, Academic press, NewYork.

Bawa, R. K. (2007). A parallel approach for self-adjoint singular perturbation problems using Numerov's scheme. International Journal of Computer Mathematics, 84(3), 317-323.

Elsgolt's, L, E. and Norkin, S.B.(1973). Introduction to the theory and applications of Differential equations with deviating arguments academic press. New York.

Friedrichs, K. O., and Wasow, W. (1946), "Singular Perturbations of Nonlinear Oscillations," DukeMathematicalJournal,13: 367-381.

Jain M, K, Iyengar S.R.K. and Jain K.K.J.(2007). Numerical methods for scientific and Engineering computation, 5th edition, New age international (p) limited, publishers.

Natesan S, Bawa K. Rajesh, Claveroc (2007). Uniformly convergent compact numerical scheme for normalized flux of singularly perturbed reaction-diffusion problems, Int.J. Uniform. Syst. Sci, 3(2): 207-221.

Kerem Yamac and Fevzi Erdogan (2020). A numerical scheme for semi-linear singularly Perturbed reaction-diffusion problems, Applied Mathematics and Nonlinear Sciences, 5(1): 405-412.

K.phaneendera, K.Madhulatha, Y, N. Reddy (2013). A finite difference technique for Singularly perturbed two-point boundary value problem using deviating arguments, International journal of scientific and engineering research, 4: 1253-1258.

K.K. Sharma, P. Rai, C.P. Kailash (2013). A review on singularly perturbed differential Equations with turning points and interior layers, Appl. Math. Comp. 219(22): 10575– 10609.

Manikandan Mariappan and Ayyadurai Tamilselvan (2021). Higher order numerical method For a semi-linear system of singularly perturbed differential equations, mathematical Communications Math. Commun, 26: 41-52.

Patidar K. C.(2007). High order uniform numerical method for singularly perturbed Problems, Appl. Math. Comput, 188: 720-733.

Prandtl L. Uberfluss bewegungbeikleinerreibung,Verhandlungen, (1905) International Mathematical Kongresses, Tuebner, Leipzig, 484-491.

R.E. Bellman, R.E.and Kalaba (1965). Quasililinearization and nonlinear boundary value Problems. Modern analytic and computational methods in science mathematics American Elsevier pub.Co. New York.

Roos, H.G. Stynes, M. and Tobiska, L.(1996). Numerical methods for singularly perturbed Differential equations: Convection-Diffusion and Flow Problems, Springer Verlag.

S. S. Motsa and P. Sibanda (2013). "Some modification of the quasilinearization method With higher-order convergence for solving nonlinear BVPs," Numerical Algorithms, 63: 399–417.

Siraj, M.K., G.F., and Bullo, T. A. (2019). Fourth-Order stable central diffference with Richardson extrapolation method for second-order self-adjoint singularly perturbed boundary value problems, Journal of the Egyptian Mathematics Society , 27(1), 1-14. Stynes, M., and styes .D. (2018). Convection-diffusion problems (vol.196). American Mathematician Society.

Samarskii A. A.(2001). Theory of difference schemes. Monographs and textbooks in pure And applied mathematics Marcel Dekker, New York, v (240-761).
Stynes, M., and styes .D. (2018). Convection-diffusion problems (vol.196). American Mathematician Society.

G. D. Smith(1985). "Numerical solution of partial differential equations, Finite difference methods", Third Edition, Oxford University Press.

Yuan-Ming Wang, Wen-Jia, and Massimo Scalia (2012). Numerov's method for a class of Nonlinear multipoint boundary value problems, department of mathematics, East China Normal university, 13: 1-29.

Wasow.W. (1942), Boundary layer problems in the theory of ordinary differential Equations Doctoral Dissertation, New York University.