JIMMA UNIVERSITY COLLEGE OF NATURAL SCIENCES

DEPARTMENT OF MATHEMATICS

EXPONENTIALLY FITTED NUMERICAL METHOD FOR SINGULARLY PERTURBED DIFFERENTIAL EQUATION INVOLVING BOTH SMALL AND LARGE DELAY

A Thesis Submitted to the Department of Mathematics, College of Natural Sciences in Partial Fulfillment for the Requirements of the Degree of Masters of Science in Mathematics.

(Numerical Analysis)

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February 2022

Jimma, Ethiopia

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We, the undersigned , member of the board of examiners of the final open defense by Nagasa Ruda have read and evaluated his /her thesis entitled ”exponentially fitted numerical method for singularly perturbed differential equation involving small and large delay” and examined the candidate . This is therefore to cerify that the thesis has been accepted inpartial fulfilement of the requirements for the degree masters of science in mathematics (Numerical analysis)

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Declaration

I , undersigned declare that this thesis entitled ”Exponentially Fitted numerical method for singularly perturbed differential equations involving both large and small delay” is my own original work and it has not been submitted for the award of any academic degree or the like in any other institution or University , and that all the sources I have used or quoted have been indicated and acknowledged as complete references

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Acknowledgment

First of all, I would like to thank my God for the reason that he gives me long life ,strength and helped me to reach this precise time .Secondly, I would like to express my gratitude and appreciation to my advisor Habtamu Garoma(Ph.D) and my co-advisor Gemechis File(PROF.) for their limitless support in guiding and advising during this Thesis.My gratitude also goes to my colleagues for their motivation , and providing me supportive ideas throughout the process of my Thesis.

Acronyms

SPPs- Singularly Perturbed Problems.

SPDE Singularly perturbed differential equation

SPDDE - Singularly perturbed delay differential equation

FDM- Finite Difference Method.

FOM- Fitted Operator Method.

ODEs- Ordinary Differential Equations.

Abstract

The aim of this thesis is to present exponentially fitted numerical method for singularly perturbed differential equations involving both large and small delay.The stability and parameter uniform convergence of the proposed method are proved. To validate the applicability of the scheme,one models problems are considered for numerical experimentation and solved for different values of the perturbation parameter and mesh size. The numerical results are tabulated in terms of maximum absolute errors and rate of convergence and it observed that the present method is accurate and ε-uniformly convergent

Chapter 1

INTRODUCTION

1.1 Background of the Study

Numerical analysis is a branch of mathematics concerned with theoretical foundations of numerical algorithms for the solution of problems arising in scientific applications,(Wasow,1942). An equation involving one dependent variable and its derivative with respect to one or more independent variable is called differential equation. There are two classes of differential equations. These are ordinary differential equations and partial differential equations .In real life , we often encounter many problems which are described by parameter dependent differential equations .

Any differential equation in which the highest order derivative is multiplied by a small positive parameters is called singularly perturbed problem and the parameter is known as the perturbation parameter (Fredrics and Wasow.1946)). If the solution of the reduced problem (i.e., the problem which is obtained by putting $\varepsilon = 0$ in the original problem) as the perturbation parameter tends to zero, the problem is known as regularly perturbed otherwise , it is known as singularly perturbed problem. Singularly perturbed problem often have very thin boundary and internal layers where the solution varies rapidly change, whereas away from the layer , solution behaves regularly and Varies slowly, so that the numerical treatment of singularly perturbed problems faces major difficulties (Miller, 1974. Riordan, 2003). Due to the variation in the width of the layer with respect to small perturbation parameters several difficulties are experienced in solving the singularly perturbed problems using the standard numerical Methods with uniform mesh (kadalbajoo,2005). Singular perturbation problem (SPPs) model convection diffusion process in applied mathematics that arise in diverse area, in diverse area, including linearized Navier- stokes equation at high Reynolds number and the drift diffusion equation of semiconductor device modeling, heat and mass transfer at high peelet number etc(Roose et al..1996, Doolan et al.1980).

A delay differential equation is an equation where the evolution of the system at a certain time depends on the state of the system at early time . A differential equations is said to be singularly perturbed delay differential equations , if it includes at least one delay term, involving unknown functions occurring with different arguments and also the highest derivatives is multiplied by a small parameter. Such type of delay differential equations play very important role in the mathematical modeling of various practical phenomena and also widely applicable in the various fields, like micro- scale heat transfer (Tzou ,1997) ,the hydrodynamics of liquid helium (Joseph and Preziosi ,1989) ,second -sound theory (Joseph and Preziosi , 1990) ,optically,bi-stable devices (Derstine et al.,1982) ,diffusion in polymers (Liu et al.,(2005) and a models of the red cell system (Wazeweska-Czyzeweska and Lasota,1976). Finding the solution of singularly perturbed delay differential equations is a challenging problems. In response to these , in recent years there had been arrowing interest in numerical methods on singularly perturbed delay differential equations.

In mid-eighties to mid-nineties, Lange and Miura(1982) studied a class of boundary value problems for second order differential difference equations in which the highest order derivative is multiplied by a small parameter and proposed some asymptotic method to approximate the solution of this class of differential equations. Amiraliyev and Cimen (2010) proposed a first order uniform convergent fitted finite difference scheme for singularly perturbed boundary value problems for a linear second order delay differential equations with large delay in a reaction term. Subburayan and Rama ujam(2013)and Chakravarthy et al .,(2017) were solved singularly perturbed boundary value proplems for second order delay differential equations of convection-diffusion problems with large delay. Recently ,Debela and Duressa,(2019) considered numerical solution of the governing problems under consideration with exponential fitted operator method with integral boundary conditions. Duressa, (2021). Novel approach to solve singularly perturbed boundary value problems with negative shift parameter. In the present paper , motivated by the work of Subburayan and Ramabujam, (2012) and Duressa, (2021) we developed exponential fitted numerical scheme on uniform mesh for the numerical solution of second order singularly perturbed convection -diffusion equations with large delay and small delay. We tried to develop more accurate, stable and ε-uniformly convergent numerical method for solving singularly perturbed differential equations involving small and large delay

1.2 Statement of the problem

Chakravarthy et al .,(2017) deals with singularly perturbed boundary value problems for a linear second order delay differential equation. It is known that the classical numerical methods are not satisfactory when applied to solve singularly perturbed problems in a delay differential equations. This author presented an exponentially fitted finite difference scheme to overcome the draws backs of the corresponding classical counter parts. The stability of the scheme is investigated. Debela and Duressa, (2019) consider exponentially fitted finite difference method for solving singularly perturbed delay differential equations with integral boundary condition. Authors applied Simpson's rule to treat the intgral boundary condition. The stability and parameter uniform convergence of the proposed method are proved. Subburayan and Ramabujam,(2012) suggested a numerical method as initial value technique(IVT) to solve the singularly perturbed boundary value problems for the second order ordinary dif-

ferential equations of convection-diffusion type with large delay. But, still there is a room to increase the accuracy. Besides, as far as the researchers' knowledge is concerned the problem under consideration via exponential fitted operator method is not yet considered.
Hence, the aim of this project is to formulate uniformly convergent exponential fitted operator method to solve singularly perturbed problem having both large and small delay.
Therefore, the main objective of this study is to develop more accurate, and ε-uniformly convergent method for the problem under consideration. Owing to this, the present study will attempt to answer the following questions:

- How does this study was describe the numerical method for singularly perturbed differential equation involving both large and small delay ?
- To what extent the proposed method converges?
- To what extent the present method approximate the exact solution?

1.3 Objectives of the study

1.3.1 General Objective

The general objective of this study is to develop exponentially fitted numerical method for singularly perturbed differential equations involving both large and small delay.

1.3.2 Specific Objective

The specific objectives of the present study are:

- To describe exponentially fitted numerical method for solving singularly perturbed differential equation involving both large and small delay .
- To establish the stability and convergence of the scheme.
- To investigate the accuracy of the scheme

1.4 Significance of the study

The results obtained in this research may

- Serve as a reference material for scholars who works on this area.
- Give an idea about the application of numerical methods in different field of studies
- ,Help the graduate students to acquire research skills and scientific procedure.

1.5 Delimitation of the study

The singularly perturbed delay differential equations perhaps arise in variety of applied mathematics that contributes for the advancement of science and technology. Though , singularly perturbed delay differential equations are vast topics and have many applications in the real world , this study is delimited to singularly perturbed delay convection-diffusion equation of the form

$$Ly(x) = -\varepsilon y''(x) + a(x)y'(x) + b(x)y(x) + c(x)y(x-1) + d(x)y'(x-\delta) = f(x), x \in \Omega = (0, 2),$$

$$y(x) = \phi(x), x \in [-1, 0], y(2) = \ell, \ell \in (0, 2).$$

where δ is small, $0 < \varepsilon << 1$, $\phi(x)$ is sufficiently smooth on $[-1, 0]$. For all $x \in \Omega$, it is assumed that the sufficient smooth functions $a(x), b(x)$, $c(x)$ and $d(x)$ satisfy $a(x) \geq a_1 > a > 0, b(x) \geq b \geq 0, c(x) \leq \gamma < 0, d(x) \geq \zeta > 0$, and $2a + 5b + 5\gamma \geq \eta > 0, a(a_1 - a) > -2\gamma$. We assume that, $\overline{\Omega} = [0, 2]$, $\Omega = (0, 2)$, $\Omega_1 = (0, 1]$, $\Omega_2 = (1, 2)$. $\Omega^* = \Omega_1 \cup \Omega_2$, $y \in X = C^0(\bar{\Omega}) \cap C^1(\Omega) \cap C^2(\Omega*)$.

Chapter 2

RIVIEW OF RELATED LITERATURE

2.1 Singular perturbation Theory

Science and technology develops many practical problems, such as the mathematical boundary layer theory or approximation of solution of various problems described by differential equations involving small parameters have become increasingly complex and therefore require the use of asymptotic methods. The term singular perturbation was introduced in 1940s by(Wasow, 1942). Singularly perturbed problem arise frequently in application including geophysical fluid dynamics ,oceanic and atmospheric circulation ,chemical reaction, civil engineering , optimal control, etc. the classification of singularly perturbed higher order problems depend on how the order of the original equation is affected if one ε=0 where is a small positive parameter multiplying the highest derivative occurring in the differential equation. If the order is reduced by one, we say that the problem is of convection diffusion type and of reaction-diffusion boundary value problems is described by slowly and rapidly varying parts. so there are thin transition layers where the solution can jump suddenly while away from the solution varies slowly and behaves regularly (Akram and Afia, (2013) This leads to boundary and /or interior layers in the solution of the problems. Classical numerical methods fail to produce good approximation for these problems, therefore it is important to develop numerical methods for these problems, whose accuracy does not depend upon the perturbation parameter(s) which are called as parameter-uniform numerical methods, there are two important approaches a widely used in the literature for the development of uniformly convergent numerical methods. Namely , fitted operators method (FOM)and fitted mesh method (FMM). In FOM because of the uniform mesh, the layers well be resolved automatically without having to decompose the solution, but FMMs use standard classical finite difference schemes on specially designed piece wise uniform mesh. Many scholars have studied the analytical and numerical solution of these problems. Abrahamson et al. (1974) solved singularly perturbed ordinary differential equations using Difference approximations Numerical treatment of Singularly perturbed boundary value problems for higher-order non linear ordinary differential equation has a great role in fluid, dynamics. The development of numerical methods for solving singularly perturbed problems, stated with methods aimed at solving ordinary differential equations, an account of which can be found in the first monograph on this subeject by Doolan et al.(1980).

2.2 Singularly perturbed Delay Differential Equations

A singularly perturbed delay differential equation is an ordinary differential equation in which the highest derivative is multiplied by small parameter and involving at least one delay term. In the past , less attention had been paid for the numerical solution of the singularly perturbed delay differential equations. But in recent years, delay differential equations have attached the attention of many researchers because of their applications in many scientific and technical fields . For example, first -exist proplems in neurobiology , in the study of of bistable devices(Derstine et Al,1982), evolutionary biology (Wazewaska Czyzeweska and Lasota, 1976), in a variety of models for physiological processes or diseases(Mackey and

Glass,1977), to describe the human pupil-light reflex(Longtin and Milton, 1988) and so on.

2.3 Finite difference method

The finite difference method (FDM) is one of the most used techniques to approximate solution of the differential equation . This method is mainly based on the replacement of the continuous variables in the differential equation by a model including discrete variables. In fact this is procedure for constructing approximate values of the exact solution at the mesh points an extract finite difference scheme is one for which the solution to the difference equation has the same general solution as the associated differential equation. For linear problems , such operator may be obtained by choosing the coefficients of the difference operator so that , some or all the exponential functions in the null space of the differential operator are also in the null-space of the finite difference operator. Such fitted operators have been developed by many authors and usually work with uniform meshes. The implementation of these methods is not straight forward and they usually introduced artificial diffusion . We note that the methods can be applied with out apriori knowledge of the breadth and position of the boundary or interior layers

2.4 Recent developments

Subburayan and Ramabujam,(2013) suggested a numerical method as initial value technique(IVT) to solve the singularly perturbed boundary value problems for the second order ordinary differential equations of convection-diffusion type with large delay. In this technique ,the singularly perturbed problems is solved by the second order hybrid finite difference scheme, where as the delay problems is solved by the fourth order Runge-Kutta method with hermite interpolation. Chakravarthy et al .,(2015) deals with the singularly perturbed boundary value problems for the second order delay differential equations. Similar boundary value problems are associated with expected first-exist times of the membrane potential in models of neurons.An exponentially fitted difference scheme on a uniform mesh is accomplished by the method based on cubic spline in compression. The difference scheme is shown to converge to the continuous solution uniformly with respect to the perturbation parameter. Geng and Qian,.(2015) presented a numerical method for singularly perturbed convection-diffusion problems with a delay.

The method is a combination of the asymptotic expansion technique and producing kernel method(RKM) . First an asymptotic expansion for the solution of the given singularly perturbed delayed boundary value problems is constructed. Chakravarthy et al .,(2017) deals with singularly perturbed boundary value problems for a linear second order delay differential equation. It is known that the classical numerical methods are not satisfactory when applied to solve singularly perturbed problems in a delay differential equations this author presented an exponentially fitted finite difference scheme to overcome the draws backs of the corresponding classical counter parts. The stability of the scheme is investigated. Debela and Duressa, (2018) consider exponentially fitted finite difference method for solving singularly perturbed delay differential equations with integral boundary condition.Here,authors applied Simpson's rule to treat the intgral boundary condition. The stability and parameter uniform convergence of the proposed method are proved. Kumar and Rao,(2020) presented a stabilized central difference method for the boundary value problem of singularly perturbed differential equations with a large negative shift. The central difference approximations for the derivatives are modified by re-approximating the error terms,leading to a stabilizing effect. The method is found to be second order convergent.

As we have discussed in the above literature , most researchers have been tried to find numerical solution for singularly perturbed differential difference problem. The researchers studied on one end boundary layer. In this thesis , we presents a more accurate and convergent numerical method for singularly perturbed differential difference equations with large and small delay by using exponentially fitted numerical method.

Chapter 3

METHODOLOGY

3.1 Study Area and Period

The study was conducted in Jimma University, department of Mathematics from September 2020 to November2021.

3.2 Study Design

The study was employed mixed-design (documentary review and experiment).

3.3 Source of Information

The relevant sources of information for this study were books, published articles on reputable journal and related studies from internet services.

3.4 Mathematical Procedures of the study

In order to achieve the stated objectives , the study procedures followed were:

1. Defining the problem.
2. Approximating the delay parameter using Taylors series approximations.
3. Formulating numerical schemes for the problem.
4. Establishing the stability and convergence of the formulated scheme .

5. Writing a matlab code for the formulated scheme.

6. Validating the schemes by using numerical examples.

7. Presenting the results using tables and graphs.

8.Discussing and providing conclusions.

Chapter 4

FORMULATION OF THE METHOD, RESULTS AND DISCUSSION

4.1 Formulation of the method

Consider the following singularly perturbed problem

$$Ly(x) = -\varepsilon y''(x)+a(x)y'(x)+b(x)y(x)+c(x)y(x-1)+d(x)y'(x-\delta) = f(x), x \in \Omega, \quad (4.1.1)$$

$$y(x) = \phi(x), x \in [-1, 0], y(2) = l, l \in \Omega. \quad (4.1.2)$$

Expand $y'(x-\delta)$ around x using the Taylor's expansion and discard higher order terms. Then, Eqs. (4.1.1)-(4.1.2) can be approximated by

$$Ky(x) = c_\varepsilon(x)y''(x) + p(x)y'(x) + b(x)y(x) + c(x)y(x-1) = f(x), \quad (4.1.3)$$

$$y(x) = \phi(x), x \in [-1, 0], y(2) = l. \quad (4.1.4)$$

where $c_\varepsilon = -(\varepsilon + \delta d(x))$ and $p(x) = a(x) + d(x)$.

As we observed from Eqs. (4.1.3) and (4.1.4), the values of $y(x-1)$ are known for the domain Ω_1 and unknown for the domain Ω_2 due to the large delay at $x = 1$. So, it impossible to treat the problem throughout the domain $(\bar{\Omega})$. Thus, we have to treat the problem at Ω_1 and Ω_2 separately.

Eqs. (4.1.3)–(4.1.4) are equivalent to

$$Ky(x) = S(x), \tag{4.1.5}$$

where

$$Ky(x) = \begin{cases} K_1 y(x) = c_\varepsilon y''(x) + p(x)y'(x) + b(x)y(x), x \in \Omega_1, \\ K_2 y(x) = c_\varepsilon y''(x) + p(x)y'(x) + b(x)y(x) + c(x)y(x-1), x \in \Omega_2. \end{cases} \tag{4.1.6}$$

$$R(x) = \begin{cases} f(x) - c(x)\phi(x-1), x \in \Omega_1, \\ f(x), x \in \Omega_2. \end{cases} \tag{4.1.7}$$

with boundary conditions

$$\begin{cases} y(x) = \phi(x), x \in [-1, 0], \\ y(1^-) = y(1^+), y'(1^-) = y'(1^+), \\ y(2) = l. \end{cases} \tag{4.1.8}$$

where $y(1^-)$ and $y(1^+)$ denote the left and right limits of y at $x = 1$, respectively.

4.2 Properties of Continuous solution

Lemma 4.2.1 *(Minimum Principle) Let $\psi(x)$ be any function in X such that $\psi(0) \geq 0, \psi(2) \geq 0, K_1\psi(x) \geq 0, \forall x \in \Omega_1, K_2\psi(x) \geq 0, \forall x \in \Omega_2$ and $[\psi'](1) \leq 0$ then $\psi(x) \leq 0, \forall x \in \bar{\Omega}$.*

Proof 4.2.2 *Define a test function*

$$s(x) = \begin{cases} \frac{1}{8} + \frac{x}{2}, x \in [0,1], \\ \frac{3}{8} + \frac{x}{4}, x \in [1,2]. \end{cases}$$

Note that $s(x) > 0, \forall x \in \bar{\Omega}, Ls(x) > 0, \forall x \in \Omega_1 \cup \Omega_2, s(0) > 0, s(2) > 0$ *and* $[s'](1) < 0$. *Let* $\mu = \max\{\frac{-\psi(x)}{s(x)} : x \in \bar{\Omega}\}$. *Then, there exists* $x_0 \in \bar{\Omega}$ *such that* $\psi(x_0) + \mu s(x_0) = 0$ *and* $\psi(x) + \mu s(x) \geq 0, \forall x \in \bar{\Omega}$. *Therefore, the function* $(\psi + \mu s)$ *attains its minimum at* $x = x_0$. *Suppose the lemma does not hold true, then* $\mu > 0$.

Case (i): $x_0 = 0$

$$0 < (\psi + \mu s)(0) = \psi(0) + \mu s(0) = 0,$$

it is a contradiction.

Case (ii): $x_0 \in \Omega_1$

$$0 < L(\psi + \mu s)(x_0) = c_\varepsilon(\psi + \mu s)''(x_0) + p(x_0)(\psi + \mu s)'(x_0) + b(x_0)(\psi + \mu s)(x_0) \geq 0,$$

it is a contradiction.

Case (iii): $x_0 = 1$

$$0 \leq [(\psi + \mu s)'](1) = [\psi'](1) + \mu[s'](1) < 0,$$

it is a contradiction.

Case (iv): $x_0 \in \Omega_2$

$$0 < L(\psi + \mu s)(x_0) = c_\varepsilon(\psi + \mu s)''(x_0) + p(x_0)(\psi + \mu s)'(x_0) + b(x_0)(\psi + \mu s)(x_0)$$
$$+ c(x_0)(\psi + \mu s)(x_0 - 1) \geq 0,$$

it is a contradiction.

Case (v): $x_0 = 2$

$$0 < (\psi + \mu s)(2) = (\psi + \mu s)(2) \leq 0,$$

it is a contradiction. Hence, the proof of the Lemma.

Lemma 4.2.3 *(Stability Result) The solution* $y(x)$ *of Eqs.* (4.1.1)-(4.1.2)*, satisfies the bound*

$$|y(x)| \leq C \max\{|y(0)|, |y(2)|, \sup_{x \in \Omega*} |Ly(x)|\}, \quad x \in \overline{\Omega}.$$

Proof 4.2.4 *This Lemma can be proved by using Lemma 4.2.1 and the barrier functions* $\theta^{\pm}(x) = CMs(x) \pm y(x), \quad x \in \overline{\Omega}$, *where* $M = \max\left\{|y(0)|, |y(2)|, \sup_{x\in\Omega*}|Ly(x)|\right\}$ *and* $s(x)$ *is the test function as in Lemma 4.2.1.*

Lemma 4.2.5 *Let* y_ε *be the solution of* (P_ε)*. Then, for* $k = 0, 1, 2, 3, 4,$

$$| y_\varepsilon^{(k)}(x) | \leq C(1 + c_\varepsilon^{-k} \exp(\frac{-px}{c_\varepsilon})), \forall x \in [0, l].$$

Proof 4.2.6 *For the proof refer Bansal and Sharma (2017).*

4.3 Numerical Scheme Formulation

The linear ordinary differential equation in Eq. (4.1.1) cannot, in general, be solved analytically because of the dependence of $p(x)$, $b(x)$ and $c(x)$ on the spatial coordinate x. We divide the interval $[0, 2]$ into $2N$ equal parts with constant mesh length h. If we consider the interval $x \in (0, 1)$, the domain [0,1] is discretized into N equal number of subintervals, each of length h. Let $0 = x_0 < x_1 < x_2 < ... < x_N = 1$ be the points such that $x_i = ih, \quad i = 1, 2, 3, ..., N$. We apply an exponentially fitted operator finite difference method (FOFDM).

From Eq. (4.1.6) and Eq. (4.1.7), we have

$$\begin{cases} c_\varepsilon y''(x) + p(x)y'(x) + b(x)y(x) = S(x), \quad x \in \Omega_1, \\ y(0) = \phi(0), \quad y(1) = \theta, \end{cases} \tag{4.3.9}$$

where $S(x) = f(x) - c(x)\phi(x-1)$.

To find the numerical solution of Eq. (4.3.9), we use the theory applied in asymptotic method for solving singularly perturbed BVPs. In the considered case, the boundary layer is in the left side of the domain i.e. near $x = 0$. From the theory of singular perturbations given by O'Malley(1991), we get the asymptotic solution up to first order approximation as

$$y(x) = y_0(x) + \frac{p(0)}{p(x)}(\mu_0 - y_0(0)) \exp\left(- \int_0^x \left(\frac{p(x)}{c_\varepsilon} - \frac{b(x)}{p(x)} \right) dx \right) + O(c_\varepsilon).$$

By using Taylor series at $x = 0$ for $p(x)$ and $b(x)$ and simplifying, we obtain

$$y(x) = y_0(x) + (\mu_0 - y_0(0)) \exp\left(\frac{-p(0)x}{c_\varepsilon} \right), \tag{4.3.10}$$

where $y_0(x)$ is the solution of the reduced problem (obtained by setting $c_\varepsilon = 0$ of Eq. (4.3.9) which is given by

$$p(x)y'(x) + b(x)y(x) = r(x), \quad \text{with} \quad y_0(1) = \theta_1, \tag{4.3.11}$$

where $\theta_1 = l$.

Considering h small enough, the discretized form of Eq. (4.3.10) becomes

$$y(ih) = y_0(ih) + (\phi_0 - y_0(0)) \exp\left(\frac{-p(0)}{c_\varepsilon} ih \right), \tag{4.3.12}$$

which simplifies to

$$y(ih) = y_0(ih) + (\phi - y_0(0)) \exp(-i\rho p(0)), \tag{4.3.13}$$

where $\rho = \frac{h}{c_\varepsilon}$, $h = \frac{1}{N}$.

To handle the effect of the perturbation parameter artificial viscosity (exponentially fitting factor $\sigma(\rho)$ is multiplied on the term containing the perturbation parameter as

$$c_\varepsilon \sigma(\rho) y''(x) + p(x) y'(x) + b(x) y(x) = S(x), \tag{4.3.14}$$

with boundary conditions $y_0(0) = \phi_0$ and $y(1) = \theta_1$.

Next, on a uniform mesh point $\overline{\Omega}^N = \{x\}_{i=0}^N$ and denote $h = x_{i+1} - x_i$.

$$\begin{cases} D^+ Y_i = \frac{Y_{i+1} - Y_i}{h}, \\ D^- Y_i = \frac{Y_i - Y_{i-1}}{h}, \\ D^0 Y_i = \frac{Y_{i+1} - Y_{i-1}}{2h}, \\ D^+ D^- Y_i = \frac{Y_{i+1} - 2Y_i + Y_{i-1}}{h^2}, \end{cases} \tag{4.3.15}$$

By applying the central finite difference formula in Eq. (4.3.14) takes the form

$$c_\varepsilon \sigma(\rho) (D^+ D^- y(x_i)) + p(x_i)(D^0 y(x_i)) + b(x_i) y(x_i) = S(x_i). \tag{4.3.16}$$

Using operator, Eq. (4.3.16) is rewritten as

$$L^N_{c_\varepsilon} Y_i = S_i, \tag{4.3.17}$$

with boundary conditions $y(0) = y_0$ and $y(1) = \theta_1$.

From Eq. (4.3.16), we have

$$c_\varepsilon \sigma(\rho) \left(\frac{Y_{i+1} - 2Y_i + Y_{i-1}}{h^2} \right) + p(x_i) \left(\frac{Y_{i+1} - Y_{i-1}}{2h} \right) + b(x_i) Y_i = S_i. \tag{4.3.18}$$

Multiplying Eq. (4.3.18) by h and considering h small and truncating the term $(S_i - b(x_i) Y_i) h$

results

$$\frac{\sigma(\rho)}{\rho}\left(Y_{i+1}-2Y_i+Y_{i-1}\right)+\frac{p(x_i)}{2}\left(Y_{i+1}-Y_{i-1}\right)=0. \tag{4.3.19}$$

Now, using Taylor's series for Y_{i-1} and Y_{i+1} up to first term and substituting the results in Eq. (4.3.19) into Eq. (4.3.16) and simplifying, the exponential fitting factor is obtained as

$$\sigma(\rho)=\frac{\rho p(0)}{2}\coth\left(\frac{\rho p(0)}{2}\right). \tag{4.3.20}$$

Assume that $\overline{\Omega}^{2N}$ denotes the partition of [0,2] into $2N$ subintervals such that $0=x_0,x_1,x_2,...,x_N=1$ and $x_{N+1},x_{N+2},...,x_{2N}=2$ with $x_i=ih,\ h=\frac{2}{2N}=\frac{1}{N},\ \ i=0,1,2,...,2N.$

Case (1): Consider Eqs. (4.1.6) and (4.1.7) on the domain Ω_1 which is given by

$$c_\varepsilon y''(x)+p(x)y'(x)+b(x)y(x)=f(x)-c(x)\phi(x-1), \tag{4.3.21}$$

Hence, the required finite difference scheme becomes

$$\begin{aligned}\left(\frac{c_\varepsilon\sigma(\rho)}{h^2}-\frac{p(x_i)}{2h}\right)Y_{i-1}+\left(\frac{-2c_\varepsilon\sigma(\rho)}{h^2}+b(x_i)\right)Y_i+\left(\frac{c_\varepsilon\sigma(\rho)}{h^2}+\frac{p(x_i)}{2h}\right)Y_{i+1}\\ =f_i-c_i\phi(x_{i-N}),\end{aligned} \tag{4.3.22}$$

The numerical scheme in Eq. (4.3.22) can be written in recurrence relation as

$$E_iY_{i-1}+F_iY_i+G_iY_{i+1}=H_i,\quad i=1,2,...,N, \tag{4.3.23}$$

where $E_i=\frac{c_\varepsilon\sigma(\rho)}{h^2}-\frac{p_i}{2h},\ \ F_i=\frac{-2c_\varepsilon\sigma(\rho)}{h^2}+b_i,\ \ G_i=\frac{c_\varepsilon\sigma(\rho)}{h^2}+\frac{p_i}{2h},\ \ H_i=f_i-c_i\phi(x_i-N).$

Case (2): Consider Eqs. (4.1.6) and (4.1.7) on the domain Ω_2 using exponentially fitted finite difference method, which is given by

$$c_\varepsilon\sigma(\rho)\left(\frac{Y_{i+1}-2Y_i+Y_{i-1}}{h^2}\right)+p_i\left(\frac{Y_{i+1}-Y_{i-1}}{2h}\right)+b_iY_i+c_iY(x_i-1)=f_i. \tag{4.3.24}$$

Similarly, this equation can be written as

$$E_iY_{i-1} + F_iY_i + G_iY_{i+1} + C_i = H_i, \quad i = N+1, N+2, ..., 2N-1, \tag{4.3.25}$$

where $E_i = \frac{c_\varepsilon \sigma(\rho)}{h^2} - \frac{p_i}{2h}$, $F_i = \frac{-2c_\varepsilon \sigma(\rho)}{h^2} + b_i$, $G_i = \frac{c_\varepsilon \sigma(\rho)}{h^2} + \frac{p_i}{2h}$, $C_i = c_i y(x_i - 1)$ and $H_i = f_i$. Therefore, on the whole domain $\overline{\Omega} = [0, 2]$, the basic schemes to solve Eqs. (4.1.1)-(4.1.3) are the schemes given in Eqs. (4.3.23) and (4.3.25).

Uniform Convergence Analysis

The discrete scheme corresponding to the original Eqs. (4.1.6)-(4.1.7) is as follows

For $i = 1, 2, 3, ..., N$

$$K_1^N Y_i = f_i - c_i \phi_{i-N} \tag{4.3.26}$$

For $i = N+1, N+2, ..., 2N-1$

$$K_2^N Y_i = f_i \tag{4.3.27}$$

subject to the boundary conditions:

$$Y_i = \phi_i, \quad i = -N, -N+1, ..., 0 \tag{4.3.28}$$

$$Y_{2N} = l \tag{4.3.29}$$

and where

$$\begin{cases} K_1^N Y_i = c_\varepsilon D^+ D^- Y_i + p(x_i) D^0 Y_i + b(x_i) Y_i \\ K_2^N Y_i = c_\varepsilon D^+ D^- Y_i + p(x_i) D^0 Y_i + b(x_i) Y_i + c(x_i) Y_{i-N} \end{cases} \tag{4.3.30}$$

Lemma 4.3.1 : *(Discrete Minimum Principle) Assume that the mesh function* $\psi(x_i)$ *satisfies* $\psi(x_0) \geq 0$ *and* $\psi(x_{2N}) \geq 0$. *Then* $K_1^N \psi(x_i) \geq 0, \quad \forall x_i \in \Omega_1^{2N}$, $K_2^N \psi(x_i) \geq 0$, $\forall x_i \in \Omega_2^{2N}$ *and* $\psi'(1^+) - \psi'(1^-) = [\psi'](1) \leq 0$. *Then* $\psi(x_i) \leq 0$, $\forall x_i \in \overline{\Omega}^{2N}$.

Proof 4.3.2 *Let us define*

$$s(x_i) = \begin{cases} \frac{1}{8} + \frac{x_i}{2}, x_i \in [0,1] \cap \overline{\Omega}^{2N} \\ \frac{3}{8} + \frac{x_i}{4}, x_i \in [1,2] \cap \overline{\Omega}^{2N} \end{cases}$$

Note that $s(x_i) > 0, \forall x_i \in \overline{\Omega}^{2N}$, $K^N s(x_i) > 0, \forall x_i \in \Omega_1^{2N} \cup \Omega_2^{2N}$ *and* $[s'](x_N) < 0$.
Let use the notation $\mu = \max\left(\frac{-\psi(x_i)}{s(x_i)} : x_i \in \overline{\Omega}^{2N}\right)$. *Then there exists* $x_i \in \overline{\Omega}^{2N}$ *such that* $\psi(x_k) + \mu s(x_k) = 0$ *and* $\psi(x_k) + \mu s(x_k) \geq 0$, $\forall x_i \in \overline{\Omega}^{2N}$. *Therefore, the function* $\psi + \mu s$ *attains its minimum at* $x = x_k$. *Suppose the theorem does not hold true, then* $\mu > 0$.

Case (i): $x_k = x_0$

$$0 < (\psi + \mu s)(x_0) = 0, \text{ it is a contradiction.}$$

Case (ii): $x_k \in \Omega_1^{2N}$

$$0 < K_1^N(\psi + \mu s)(x_k) = c_\varepsilon(\psi + \mu s)''(x_k) + p(x_k)(\psi + \mu s)'(x_k) + b(x_k)(\psi + \mu s)(x_k) \leq 0,$$

it is a contradiction.

Case (iii): $x_k = x_N$

$$0 < [(\psi + \mu s)'](x_N) = [\psi'](x_N) + [s'](x_N) < 0, \text{ it is a contradiction.}$$

Case (iv): $x_k \in \Omega_2^{2N}$

$$0 < K_2^N(\psi+\mu s)(x_k) = c_\varepsilon(\psi+\mu s)''(x_k) + p(x_k)(\psi+\mu s)'(x_k) + b(x_k)(\psi+\mu s)(x_k) + c(x_k)(\psi+\mu s)(x_k-1) \leq 0,$$

it is a contradiction.

Case (v): $x_k = x_{2N}$

$$0 < (\psi + \mu s)x_{2N} \leq 0, \text{ it is a contradiction.} \tag{4.3.31}$$

Hence, the proof of the lemma is finished.

Lemma 4.3.3 *Let $\psi(x)$ be any mesh function. Then, for $0 < i < 2N$*

$$|\psi(x_i)| \leq C \max\{|\psi(x_0)|, |\psi(x_{2N})|, \max_{i \in \Omega_1^{2N} \cup \Omega_2^{2N}} |K^N \psi(x_i)|\}$$

Proof 4.3.4 *: Consider the barrier functions*

$$\theta^{\pm}(x_i) = CM \pm \psi(x_i), \quad \forall x_i \in \overline{\Omega}^{2N} \tag{4.3.32}$$

where $M = \max\{|\psi(x_0)|, |\psi(x_{2N})|, \max_{i \in \Omega_1^{2N} \cup \Omega_2^{2N}} |L^N \psi(x_i)|\}$.
From Eq. (4.3.32) it is clear that $\theta^{\pm}(x_0) \geq 0$ and $\theta^{\pm}(x_{2N}) \geq 0$

$$K_1^N \theta^{\pm}(x_i) \geq 0, \quad \forall x_i \in \Omega_1^{2N}$$
$$K_2^N \theta^{\pm}(x_i) \geq 0, \quad \forall x_i \in \Omega_2^{2N}$$
$$[\theta^{\pm\prime}](x_N) \leq 0$$

Using Lemma 4.3.1, $\theta^{\pm}(x_i) \geq 0, \quad \forall x_i \in \overline{\Omega}^{2N}$.

We proved above that the discrete operator K^N satisfies the maximum principle. Next, we analyze the uniform convergence of the method.

Theorem 4.3.5 *Let $y(x_i)$ and Y_i be the exact solution of Eqs.* (4.1.1)-(4.1.3) *and numerical solutions of Eq.* (4.3.17) *respectively. Then, for a sufficiently large N, the following* *parameter uniform error estimate holds*

$$|L^N(y(x_i) - Y_i)| \leq \frac{CN^{-2}}{N^{-1} + c_\varepsilon}\left(1 + c_\varepsilon^{-3}\exp\left(-\frac{px_i}{c_\varepsilon}\right)\right). \tag{4.3.33}$$

Proof 4.3.6 *Let us consider the local truncation error defined as*

$$\begin{aligned} L^N(y(x_i) - Y_i) &= c_\varepsilon\sigma(\rho)(y''(x_i) - D^+D^-y(x_i)) + p(x_i)(y'(x_i) - D^0y(x_i)), \\ &= c_\varepsilon\left[\frac{\rho p(1)}{2}\coth\left(\frac{\rho p(1)}{2}\right) - 1\right]D^+D^-y(x_i) \\ &+ c_\varepsilon(y''(x_i) - D^+D^-y(x_i)) + p(x_i)(y'(x_i) - D^0y(x_i)), \end{aligned} \tag{4.3.34}$$

where $\sigma(\rho) = p(1)\frac{\rho}{2}\coth(p(1)\frac{\rho}{2})$, *and* $\rho = \frac{N^{-1}}{c_\varepsilon}$.
since $|z\coth(z) - 1| \leq z^2$ *holds if* $z \neq 0$ *and also* $|z\coth(z) - 1| \leq z$ *if* $z > 0$ *values, Now, for* $z > 0$, C_1 *and* C_2 *are constants, and we have* $|z\coth(z) - 1| \leq C_1z^2, \quad z \leq 1$. *Similarly, for* $z \longrightarrow \infty$, *since* $\lim_{z\longrightarrow\infty}\coth(z) = 1$, $|z\coth(z) - 1| \leq C_1z$ *is given.*
In general, for all $z > 0$, *we write*

$$C_1\frac{z^2}{z+1} \leq z\coth(z) - 1 \leq C_2\frac{z^2}{z+1} \tag{4.3.35}$$

implying that

$$c_\varepsilon[p(1)\frac{\rho}{2}\coth(p(1)\frac{\rho}{2}) - 1] \leq c_\varepsilon\left(\frac{(N^{-1}/c_\varepsilon)^2}{(N^{-1}/c_\varepsilon) + 1}\right) = \frac{N^{-2}}{N^{-1} + c_\varepsilon}. \tag{4.3.36}$$

Using Taylor series expansion, we can rewrite $y(x_{i-1})$ *and* $y(x_{i+1})$ *in terms of the values and derivatives of* $y(x_i)$ *as*

$$\begin{cases} y(x_{i-1}) = y(x_i) - hy'(x_i) + \frac{h^2}{2!}y''(x_i) - \frac{h^3}{3!}y^{(3)}(x_i) + \frac{h^4}{4!}y^{(4)}(x_i) + O(h^5), \\ y(x_{i+1}) = y(x_i) + hy'(x_i) + \frac{h^2}{2!}y''(x_i) + \frac{h^3}{3!}y^{(3)}(x_i) + \frac{h^4}{4!}y^{(4)}(x_i) + O(h^5). \end{cases}$$

We obtain the bound for the second order derivatives as

$$\begin{cases} |D^+D^-y(x_i)| \leq C|y''(x_i)|, \\ |y''(x_i) - D^+D^-y(x_i)| \leq CN^{-2}|y^{(4)}(x_i)|. \end{cases} \tag{4.3.37}$$

Similarly, for the first derivative term

$$|y'(x_i) - D^0y(x_i)| \leq CN^{-2}|y^{(3)}(x_i)|, \tag{4.3.38}$$

where $|y^{(k)}(x_i)| = \sup_{x_i \in (x_0, x_N)} |y^{(k)}(x_i)|, \quad k = 2, 3, 4.$

Using the bounds in Eq.(4.3.37) and Eq.(4.3.38), we obtain

$$\begin{aligned} |L^N(y(x_i) - Y_i)| &\leq C\frac{N^{-2}}{N^{-1} + c_\varepsilon}|y''(x_i)| + c_\varepsilon CN^{-2}|y^{(4)}(x_i)| + CN^{-2}|y^{(3)}(x_i)|, \\ &\leq C\frac{N^{-2}}{N^{-1} + c_\varepsilon}|y''(x_i)| + CN^{-2}[c_\varepsilon|y^{(4)}(x_i)| + |y^{(3)}(x_i)|]. \end{aligned}$$

Now, using the bounds for the derivatives of the solution in lemma (4.2.5), *we have*

$$\begin{aligned} |L^N(y(x_i) - Y_i)| \leq\ & \frac{CN^{-2}}{N^{-1} + c_\varepsilon}\left(1 + c_\varepsilon^{-2}\exp\left(\frac{-px_j}{c_\varepsilon}\right)\right) \\ & + CN^{-2}\left[c_\varepsilon\left(1 + c_\varepsilon^{-4}\exp\left(\frac{-px_j}{c_\varepsilon}\right)\right) + \left(1 + c_\varepsilon^{-3}\exp\left(\frac{-px_j}{c_\varepsilon}\right)\right)\right] \\ \leq\ & \frac{CN^{-2}}{N^{-1} + c_\varepsilon}\left(1 + c_\varepsilon^{-2}\exp\left(\frac{-px_j}{c_\varepsilon}\right)\right) \\ & + CN^{-2}\left[\left(c_\varepsilon + c_\varepsilon^{-3}\exp\left(\frac{-px_j}{c_\varepsilon}\right)\right) + \left(1 + c_\varepsilon^{-3}\exp\left(\frac{-px_j}{c_\varepsilon}\right)\right)\right], \end{aligned}$$

which simplifies to

$$|L^N(y(x_i) - Y_i)| \leq \frac{CN^{-2}}{N^{-1} + c_\varepsilon}\left(1 + c_\varepsilon^{-3}\exp\left(\frac{-px_j}{c_\varepsilon}\right)\right), \quad \textit{since} \quad c_\varepsilon^{-3} \geq c_\varepsilon^{-2}. \tag{4.3.39}$$

Lemma 4.3.7 *For a fixed mesh and for $c_\varepsilon \to 0$, the following holds:*

$$\lim_{c_\varepsilon \to 0} \max_{1\le j\le N-1} \frac{\exp\left(\frac{-px_j}{c_\varepsilon}\right)}{c_\varepsilon^m} = 0, \quad m = 1, 2, 3, ...$$

Proof 4.3.8 *For the proof refer Debela and Duressa (2020).*

Theorem 4.3.9 *Let $y(x_i)$ and Y_i be the exact solution of Eqs.* (4.1.1)-(4.1.3) *and numerical solutions of Eq.* (4.3.17) *respectively. Then, the following error bound holds*

$$\sup_{0<c_\varepsilon<<1} |(y(x_i) - Y_i)|| \le \frac{CN^{-2}}{N^{-1} + c_\varepsilon} \le CN^{-1}. \tag{4.3.40}$$

Proof 4.3.10 *By substituting the results of lemma 4.3.7 in to Theorem 4.3.5 and applying the discrete maximum principle, we obtain the required bound.*

For the case $c_\varepsilon > N^{-1}$ the scheme secures second order convergence and we expect to lose an order of convergence for $c_\varepsilon \le N^{-1}$, and in fact it turns out that the scheme guarantees second order uniformly convergent.

4.4 Numerical Examples and Results

In this section, one example is given to illustrate the numerical method discussed above. The exact solutions of the test problem is not known. Therefore, we use the double mesh principle to estimate the error and compute the experimental rate of convergence to the computed solution. For this we put

$$E_\varepsilon^N = \max_{0\le i\le 2N} |Y_i^N - Y_{2i}^{2N}|, \tag{4.4.41}$$

where Y_i^N and Y_{2i}^{2N} are the i^{th} and $2i^{th}$ components of the numerical solutions on meshes of N and $2N$ respectively. We compute the uniform error and the rate of convergence as

$$E^N = \max_{\varepsilon} E_{\varepsilon}^N, \text{and} R^N = \log_2 \left(\frac{E^N}{E^{2N}} \right). \qquad (4.4.42)$$

The numerical results are presented for the values of the perturbation parameter $\varepsilon \in \{ 10^{-4}, 10^{-8}, ..., 10^{-20}\}$.

Example 4.4.1 *Consider the model singularly perturbed boundary value problem:*

$$\varepsilon y''(x) - 10y'(x) + y(x-1) - y'(x-\varepsilon) = -x \quad x \in (0,1) \cup (1,2),$$

subject to the boundary conditions

$$y(x) = 1, \;\; x \in [-1, 0], \quad y(2) = 2.$$

Table 4.1: Maximum absolute errors for Example 4.4.1 at number of mesh points $2N$.

ε	N=32	N=64	N=128	N=256	N=512
10^{-4}	1.9799e-04	1.0004e-04	5.0281e-05	2.5206e-05	1.2619e-05
10^{-8}	1.9799e-04	1.0004e-04	5.0281e-05	2.5206e-05	1.2619e-05
10^{-12}	1.9799e-04	1.0004e-04	5.0281e-05	2.5206e-05	1.2619e-05
10^{-16}	1.9799e-04	1.0004e-04	5.0281e-05	2.5206e-05	1.2619e-05
10^{-20}	1.9799e-04	1.0004e-04	5.0281e-05	2.5206e-05	1.2619e-05
E^N	1.9799e-04	1.0004e-04	5.0281e-05	2.5206e-05	1.2619e-05
R^N	0.9849	0.9925	0.9962	0.9982	

4.5 Discussion and Conclusion

This thesis introduces an exponential fitted numerical method for singularly perturbed differential equations having both small and large delay. The numerical scheme is developed on uniform mesh using exponential fitted operator in the given differential equation. The

Table 4.2: Comparison of Maximum absolute errors and order of convergence for Example 1 at number of mesh points N.

	N=64	N=128	N=256	N=64	N=128	N=256
	Present M			Subbu and Rama (2012)		
E^N	1.0004e-04	5.0281e-05	2.5206e-05	7.7350e-04	2.4335e-04	7.5353e-05
R^N	0.9925	0.9962	0.9982	1.6684	1.6913	1.6849

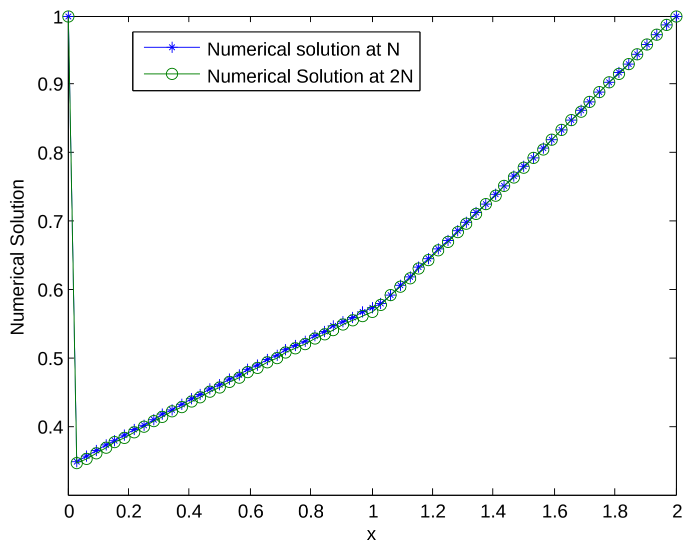

Figure 4.1: The behavior of the Numerical Solution for Example 4.4.1 at $\varepsilon = 10^{-12}$ and $N = 32$.

stability of the developed numerical method is established and its uniform convergence is proved. To validate the applicability of the method, one model problem is considered for numerical experimentation for different values of the perturbation parameter and mesh points. The numerical results are tabulated in terms of maximum absolute errors, numerical rate of convergence and uniform errors (see Table 4.1). Further, behavior of the numerical solution (Figure 4.1), point-wise absolute error (Figure **??**) and the ε -uniform convergence of the method is shown by the log-log plot (Figure 4.2). The method is shown to be ε-uniformly convergent with order of convergence $O(h)$. The proposed method gives an accurate, stable and ε-uniform numerical result (see Table 4.2)

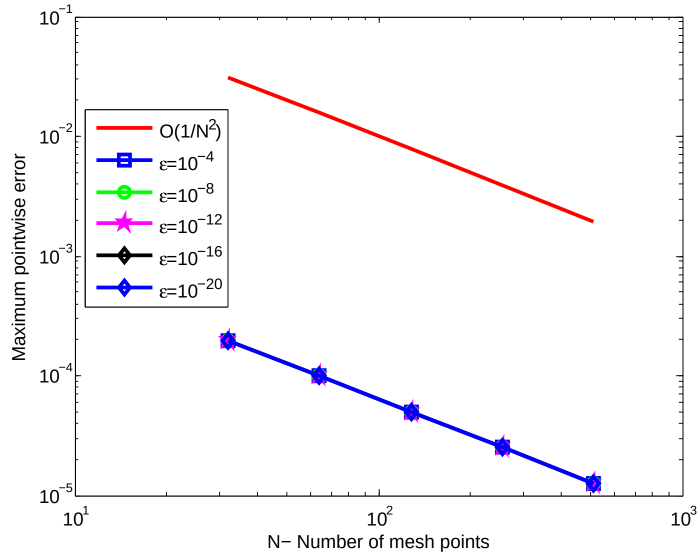

Figure 4.2: ε-uniform convergence with fitted operator in log-log scale for Example 4.4.1.

4.6 The scope of the future work

In this thesis , an exponentially fitted numerical methods were constructed for solving singularly perturbed differential equations . Hence , the scheme proposed in this thesis can also be extended to solve singularly perturbed differential equation involving both large and small delay .

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