EXPONENTIAL FITTED OPERATOR METHOD FOR SINGULARLY PERTURBED DELAY DIFFERENTIAL EQUATION WITH DISCONTINUOUS SOURCE TERM

JIMMA UNIVERSITY COLLEGE OF NATURAL SCIENCES

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Declaration

I , undersigned declare that this thesis entitled "Exponential Fitted operator method for singularly perturbed delay differential equation with discontinuous source term " is my own original work and it has not been submitted for the award of any academic degree or the like in any other institution or University , and that all the sources I have used or quoted have been indicated and acknowledged as complete references

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Acronyms

SPPs : Singularly Perturbed Problems.

DE : Differential Equation

DDE : Delay Differential Equation

ODE: Ordinary Differential Equation

SPDE : Singularly perturbed differential equation

SPDDE : Singularly perturbed delay differential equation

EFOM : Exponential Fitted Operator Method.

FOFDM : Fitted Operator Finite Difference Method

Abstract

The aim of this thesis is to present exponential fitted operator method for singularly perturbed delay differential equation with discontinuous source term.Due to the discontinuity an interior layer appears in the solution . The method uses exponential fitted operator and a uniform mesh length, which is fitted to the interior layer. To validate the applicability of the scheme,one model problem is considered for numerical experimentation and solved for different values of the perturbation parameter and mesh size. The numerical results are tabulated in terms of maximum absolute errors and rate of convergence and it observed that the present method is more accurate and ε-uniformly convergent.

keywords: Singular perturbation; Delay; Discontinuous source term; Exponential fitted operator

Chapter 1

Introduction

1.1 Background of the Study

Consider the following singularly perturbed problem with a discontinuous source term:

$$-\varepsilon y''(x) + a(x)y'(x) + b(x)y(x) + c(x)y(x-1) = f(x), x \in \Omega^* \tag{1.1.1}$$

with interval and boundary conditions,

$$y(x) = \phi(x), x \in [-1, 0], y(2) = \ell, \ell \in \Omega. \tag{1.1.2}$$

where Ω =(0,2) , $\overline{\Omega}$ = [0,2], Ω^- = (0,1], Ω^+ = (1,2).$\Omega^* = \Omega^- \cup \Omega^+$, $\phi(x)$ is smooth on [-1,0], *and* ℓ is a constant independent of ε.

Delay differential equations (DDE) play an important role in the mathematical modeling of various practical phenomena in the bioscience and control theory. Any system involving feedback control will almost always involve time delays. These arise because a finite time is required to sense the information and then react to it. A subclass of these equations consists of singularly perturbed ordinary differential equations with a delay, that is, a DE in which the highest derivative is multiplied by a small parameter and involves at least one delay term. Such types of equations arise frequently in the mathematical modeling of various practical phenomena. For example, in the modeling of the human pupil light reflex (Longtin,1988), the study of bi-stable devices (Derstine et al.,1982), micro-scale heat transfer (Tzou,1997), the hydrodynamics of liquid helium (Joseph et al.,1989), and variational problems in control theory (Glizer,2003).

It is well known that standard discretization methods for solving SPDEs are sometimes unstable and fail to give accurate results when the perturbation parameter is small. Therefore, it is important to develop suitable numerical methods to solve this type of equation, whose accuracy does not depend on the perturbation parameter, that is, the methods are uniformly convergent with respect to the parameter. For more details, one may refer to Doolan(1980) ,Farrel et al.,(2000,2004) and Roos(2008). In the past, only very few people had worked in the area of Numerical Methods for SPDDE. But in recent years, there has been growing interest in this area.

In fact, Erdogan(2009) proposed an exponential fitted operator method for singularly perturbed first order delay differential equation, Kadalbajoo and Sharma (2005 , 2006 and 2008), Kadalbajoo and Kumar (2008), and Mohapatra and Natesan(2010) proposed some numerical methods for SPDDEs with a small delay. Amiraliyev and Cimen (2010) derived a first-order convergence fitted finite difference scheme for approximating the singularly perturbed convection-diffusion problem with a delay (negative shift). It may be noted that Lange and Miura (1982) gave an asymptotic approximation to solve singularly perturbed second-order delay differential equations. In this research, a numerical method named the exponential fitted operator method is suggested to solve the problem under consideration.

1.2 Statement of the problem

The novel aspect of the problem under consideration is that we take a source term in the DE that has a jump discontinuity at one point in the interior of the domain. This gives rise to an interior layer in the exact solution of the problem, in addition to the boundary layer at the boundary point. Problems with discontinuous data were treated theoretically, in the case of the solution of the convection-diffusion with the Dirichlet case problem(Farrel et,al., 2004). Chandru(2005),Farrel et al.,(1998),Roos and Zarin(2002) discussed a self-adjoint Dirichlet type problem with a discontinuous source term. Roos and Zarin(2002),Shanthi and Ramanujam(2002) , Shanti and Natesan(2006) have examined two parameter singularly perturbed boundary value problems for second-order ordinary differential equations with discontinuous source terms. Debela and Duresa(2020) and Abegero et al.,(2021) discussed fitted nonstandard finite difference methods for singularly perturbed second-order ordinary differential equations. A singularly perturbed delay DE was examined by Mohapatra and Natesan(2010) on an adaptively generated grid. Chandru and Shanthi(2015) and Abegero et al.,(2021) have presented a fitted mesh and fitted operator method to solve singularly perturbed Robin-type boundary value problems with discontinuous source terms. Subbu-rayanin (2016), has presented SPDDEs having discontinuous source terms using the fitted mesh method.

Indeed, still, there is room to increase the accuracy and show the parameter uniform convergence because the treatment of SPP is not trivial distribution and the solution depends on perturbation parameter ε and mesh size, Doolan,(1980).Due to this, the numerical treatment of singularly perturbed interval boundary layer problems needs improvement. Therefore, it is important to develop a more accurate and convergent numerical method for solving singularly perturbed interval boundary layer problems under consideration.

1.3 Objectives of the study

The motivation of this thesis revolves around developing, analyzing and improving the $\varepsilon-$uniform convergent numerical method for singularly perturbed delay differential equations with discontinuous source term.

1.3.1 General Objective

The main objective of this study is to develop an exponential fitted operator method for singularly perturbed delay differential equation with discontinuous source term.

1.3.2 Specific Objectives

The specific objectives of this study are:

1. To Formulate exponential fitted operator method for singularly perturbed delay differential equation with discontinuous source term.

2. Analyze the convergence of the present scheme.

3. To Investigate the accuracy of the proposed method.

1.4 Significance of the study

The out came of this study may have :

•Provide a numerical method for the numerical solution of the considered problem.

1.5 Delimitation of the study

The singularly perturbed delay differential equations perhaps arise in variety of applied mathematics that contributes for the advancement of science and technology. Though , singularly perturbed delay differential equations are vast topics and have many applications in the real world.

This study is delimited by the exponential fitted operator method for Singularly Perturbed Delay Differential Equation with discontinuous source term :

$$-\varepsilon y''(x) + a(x)y'(x) + b(x)y(x) + c(x)y(x-1) = f(x), x \in \Omega^* \tag{1.5.3}$$

with interval and boundary conditions,

$$y(x) = \phi(x), x \in [-1, 0], y(2) = \ell, \ell \in \Omega. \tag{1.5.4}$$

where , $0 < \varepsilon \ll 1$, a(x)$\geq a_1 > a > 0, b(x) \geq \beta_0 \geq 0, \gamma_0 \leq$c(x) < 0 , $2\alpha + 5\beta_0 + 5\gamma_0 \geq \eta > 0$, $a(a_1 - a) > -2\gamma_0$. a(x) , b(x) , c(x) are sufficiently smooth functions on Ω and f(x) has a discontinuity at x=1(Subburayan and Ramanujam,2013)

Chapter 2

Review Of Related Literature

The solution to singular perturbation problems typically contains layers. Singular perturbation problem now is a maturing mathematical subject with a fairly long history and a strong promise for continued important applications throughout science and engineering.
A singularly perturbed delay differential equation is an ODE in which the highest derivative is multiplied by a small parameter and involves at least one delay term. In the past, less attention had been paid to the numerical solution of singularly perturbed delay differential equations. But in recent years, delay differential equations have attracted the attention of many researchers because of their applications in many scientific and technical fields.

Subburayan and Ramanujam (2013), suggested a numerical method named as Initial Value Technique (IVT) to solve the singularly perturbed boundary value problem for the second order ordinary differential equations of convection-diffusion type with a large delay. In this technique, the singularly perturbed problem is solved by the second-order hybrid finite difference scheme, whereas the delay problem is solved by the fourth-order RungeKutta method with Hermite interpolation. Chakravarthy et al. in (2015) deal with the singularly perturbed boundary value problem for the second order delay differential equation. Similar boundary value problems are associated with expected first-exit times of the membrane potential in models of neurons. An exponentially fitted difference scheme on a uniform mesh is accomplished by the method based on a cubic spline in compression. The difference scheme is shown to converge to the continuous solution uniformly with respect to the perturbation parameter. Geng and Qian (2015) presented a numerical method for singularly perturbed convection-diffusion problems with a delay. The method is a combination of the asymptotic expansion technique and there producing kernel method (RKM). First an asymptotic expansion for the solution of the given singularly perturbed delayed boundary value problem is constructed. Then the reduced regular delayed differential equation is solved analytically using the RKM. Chakravarthy et al.(2017) deal with a singularly perturbed boundary value problem for a linear second order delay differential equation. It is known that the classical numerical methods are not satisfactory when applied to solve singularly perturbed problems in delay differential equations and they presented an exponentially fitted finite difference scheme to overcome the drawbacks of the corresponding classical counterparts.

Debela and Duressa (2020) consider an accelerated fitted operator finite difference method for singularly perturbed delay differential equations with nonlocal boundary conditions. Here, the authors applied Simpsons rule to treat the integral boundary condition. The stability and parameter uniform convergence of the proposed method is proved. Kumar and Rao (2020) presented a stabilized central difference method for the boundary value problem of singularly perturbed differential equations with a large negative shift. The central difference approximations for the derivatives are modified by approximating the error terms, leading to a stabilizing effect. The method is found to be second-order convergent. As introduced in the literature, most researchers have tried to find an approximate solution for singularly perturbed differential equations with a large delay, but mainly focuses on continuous source term, and some others who have done for discontinuous source term were unable to generate more accurate solutions.

An exponentially fitted difference scheme is constructed in an equidistant mesh (Debela and Duressa, 2020). The finite difference operator is referred to as an exponentially fitted finite difference operator. The corresponding numerical method is then obtained by applying the fitted finite difference operator to obtain a system of finite difference equations on a standard mesh (a uniform mesh).

The researchers studied boundary layer in the given domain.In this thesis, we presents a more accurate and convergent numerical method for singularly perturbed differential equations with a large delay with discontinuous source terms using the exponential fitted operator method.

Chapter 3

Methodology

3.1 Material and Methods

This research was developed and analyzed ε-uniformly convergent numerical scheme using fitted operator finite difference method for solving the problem under consideration.

3.2 Study Site and Period

This study was conducted at Jimma University College of Natural Sciences Department of Mathematics from June 2022 up to Dec. 2022

3.3 Study Design

This study was employed mixed design: an intensive document review and numerical experimentation approach.

3.4 Source of Information

The relevant sources of information for this study are books, published articles in reputable journals, related studies, and so on.

3.5 Mathematical Procedure

A mathematical procedure is the designed of the skeleton of the framework we follow in this thesis. To achieve the stated objectives, the study was followed the following mathematical procedure:

1. Describing the problem
2. Analyzing the properties of the continuous solution.
3. Discretizing the solution domain.
4. Developing a numerical scheme for the problem.
5. Establishing the stability and convergence analysis of the developed scheme.
6. Developing an algorithm and writing code for the presented scheme.
7. Validating the schemes using numerical example.

Chapter 4

Formulation Of The Method,Result and Discussion

4.1 Formulation of the Method

Consider the following singularly perturbed problem

$$-\varepsilon y''(x) + a(x)y'(x) + b(x)y(x) + c(x)y(x-1) = f(x), x \in \Omega^* \tag{4.1.1}$$

with interval and boundary conditions,

$$y(x) = \phi(x), x \in [-1, 0], y(2) = \ell, \ell \in \Omega. \tag{4.1.2}$$

This BVP exhibits strong boundary layer at x = 2 and interior layer at x = 1. So, the solution $y(x)$ of Eqs(4.1.1) and (4.1.2) has a boundary layer near $x = 2$ due to the perturbation parameter, ε and interior layer due to the fact that the function $f(x)$ is discontinuous at x=1.

As we observed from Eqs(4.1.1) and (4.1.2), the values of $y(x-1)$ are known for the domain

Ω^- and unknown for the domain Ω^+ due to the large delay at $x = 1$. So, it impossible to treat the problem throughout the domain ($\bar{\Omega}$). Thus, we have to treat the problem at Ω^- and Ω^+ separately.

Eqs. (4.1.1) and (4.1.2) are becomes

$$\begin{cases} -\varepsilon y''(x) + a(x)y'(x) + b(x)y(x) = f(x) - c(x)y(x-1), x \in \Omega^- \\ -\varepsilon y''(x) + a(x)y'(x) + b(x)y(x) + c(x)y(x-1) = f(x), x \in \Omega^+. \end{cases} \tag{4.1.3}$$

$$y(x) = \phi(x), x \in [-1, 0], y(2) = \ell, \ell \in \Omega. \tag{4.1.4}$$

Eqs. (4.1.3) and (4.1.4) are equivalent to

$$Ly(x) = R(x), \tag{4.1.5}$$

where

$$Ly(x) = \begin{cases} L_1 y(x) = -\varepsilon y''(x) + a(x)y'(x) + b(x)y(x), x \in \Omega^-, \\ L_2 y(x) = -\varepsilon y''(x) + a(x)y'(x) + b(x)y(x) + c(x)y(x-1), x \in \Omega^+. \end{cases} \tag{4.1.6}$$

and

$$R(x) = \begin{cases} f(x) - c(x)\phi(x-1), x \in \Omega^-, \\ f(x), x \in \Omega^+. \end{cases} \tag{4.1.7}$$

with boundary conditions

$$\begin{cases} y(x) = \phi(x), x \in [-1, 0], \\ y(1^-) = y(1^+), y'(1^-) = y'(1^+), \\ y(2) = l. \end{cases} \tag{4.1.8}$$

where $y(1^-)$ and $y(1^+)$ denote the left and right limits of y at $x = 1$, respectively.

4.2 Properties of Continuous solution

The following Lemmas are necessary for existence and uniqueness of the solution for the problem to well-posed.

Lemma 4.2.1 *(Maximum Principle) Let* $\psi(x)$ *be any function in* X *such that* $\psi(0) \geq 0$, $\psi(2) \geq 0, L_1\psi(x) \geq 0, \forall x \in \Omega^-$, $L_2\psi(x) \geq 0, \forall x \in \Omega^+$ *and* $[\psi'](1) = [\psi'](1^+) - [\psi'](1^-) \leq 0$ *then* $\psi(x) \geq 0$, $\forall x \in \bar{\Omega}$

Proof: Define a test function s(x) given by

$$s(x) = \begin{cases} \frac{1}{8} + \frac{x}{2}, x \in [0,1], \\ \frac{3}{8} + \frac{x}{4}, x \in [1,2]. \end{cases} \tag{4.2.9}$$

Note that $s(x) > 0, \forall x \in \bar{\Omega}$, $\mathrm{L}_1 s(x) > 0, \forall x \in \Omega^-, L_2 s(x) > 0, \forall x \in \Omega^+, s(0) > 0, s(2) > 0$ and $[s'](1) < 0$. Let $\mu = \max\{\frac{-\psi(x)}{s(x)} : x \in \bar{\Omega}\}$. Then, there exists $x_0 \in \bar{\Omega}$ such that $\psi(x_0) + \mu s(x_0) = 0$ and $\psi(x) + \mu s(x) \geq 0, \forall x \in \bar{\Omega}$.Therefore, the function $(\psi + \mu s)$ attains its minimum at $x = x_0$. Suppose the lemma does not hold true, then $\mu = \max\{\frac{-\psi(x)}{s(x)} : x \in \bar{\Omega}\} > 0$ or $(\mu > 0)$

Case (i): $x_0 = 0$

$$0 < (\psi + \mu s)(0) = \psi(0) + \mu s(0) = 0,$$

it is a contradiction.

Case (ii): $x_0 \in \Omega^-$ =(0,1)

$$0 < L(\psi + \mu s)(x_0) = -\varepsilon(\psi + \mu s)''(x_0) + a(x_0)(\psi + \mu s)'(x_0) + b(x_0)(\psi + \mu s)(x_0) \geq 0,$$

it is a contradiction.

Case (iii): $x_0 = 1$

$$0 \leq [(\psi + \mu s)'](1) = [\psi'](1) + \mu[s'](1) < 0,$$

it is a contradiction.

Case (iv): $x_0 \in \Omega^+ = (1,2)$

$$0 < L(\psi + \mu s)(x_0) = -\varepsilon(\psi + \mu s)''(x_0) + a(x_0)(\psi + \mu s)'(x_0) + b(x_0)(\psi + \mu s)(x_0)$$
$$+c(x_0)(\psi + \mu s)(x_0 - 1) \geq 0,$$

it is a contradiction.

Case (v): $x_0 = 2$

$$0 < (\psi + \mu s)(2) = (\psi + \mu s)(2) = \psi(2) + \mu s(2) \leq 0,$$

it is a contradiction. Hence, the proof of the Lemma.

Lemma 4.2.2 *(Stability Result) The solution y(x) of Eqs.* (4.1.1) − (4.1.2)*, satisfies the bound*

$$|y(x)| \leq C \max\{\big|y(0)\big|, \big|y(2)\big|, \sup_{x\in\Omega*} \big|Ly(x)\big|\}, \quad x \in \overline{\Omega}.$$

Proof: For the proof refer Gemechis and Habtamu (2021).

Lemma 4.2.3 *The bound for derivative of the solution $y(x)$ of (*4.1.1*)- (*4.1.2*) when $x \in \Omega^- = (0,1)$ is given by*

$$|y^{(k)}(x)| \leq C\big(1 + \varepsilon^{-k} exp(\frac{-a(1-x_j)}{\varepsilon}), 0 \leq k \leq 4, j = 1, 2, ..., N-1$$

proof*: Refer Bansal and Sharma(2017)*

4.3 Numerical Scheme Formulation

The linear ordinary differential equation in Eq. (4.1.6) *cannot, in general, be solved analytically because of the dependence of $a(x)$, $b(x)$ and $c(x)$ on the spatial coordinate x. We divide the interval $[0,2]$ into $2N$ equal parts with constant mesh length h. Let $0 = x_0 < x_1 <$*

$x_2 < ... < x_N = 1 < x_{N+1} < x_{N+2} < ... < x_{2N} = 2$ be the mesh points. Then, we have $x_i = ih, \quad i = 1, 2, 3, ..., 2N$.

If we consider, the interval $x \in (0,1)$, the domain [0,1] is discretized into N equal number of subintervals, each of length h. Let $0 = x_0 < x_1 < x_2 < ... < x_N = 1$ be the points such that $x_i = ih, \ i = 1, 2, 3, ..., N$. We apply an exponentially fitted operator finite difference method (FOFDM).

From Eq. (4.1.6) and Eq. (4.1.7), we have

$$\begin{cases} -\varepsilon y''(x) + a(x)y'(x) + b(x)y(x) = R(x), \quad x \in \Omega^-, \\ y_0 = \phi(0), \quad y(1) = \theta, \end{cases} \tag{4.3.10}$$

where $R(x) = f(x) - c(x)\phi(x-1)$.

To find the numerical solution of Eq. (4.3.10), we use the theory used in asymptotic method for solving singularly perturbed BVPs. In the considered case, the boundary layer is in the right side of the domain i.e. near $x = 1$. From the theory of singular perturbations given in O'Malley(1991) we get the asymptotic solution up to first order approximation as

$$y(x) = y_0(x) + \frac{a(1)}{a(x)}(\theta - y_0(1))exp\bigg(- \int_x^1 \left(\frac{a(x)}{\varepsilon} - \frac{b(x)}{a(x)}\right)dx\bigg) + O(\varepsilon),$$

by using Taylor series about $x = 1$ for $a(x)$ and $b(x)$ and simplifying we obtain

$$y(x) = y_0(x) + (\theta - y_0(1))exp\bigg(- \frac{a^2(1) - \varepsilon b(1)}{\varepsilon a(1)}(1-x)\bigg) + O(\varepsilon), \tag{4.3.11}$$

where $y_0(x)$ is the solution of the reduced problem (obtained by setting $\varepsilon = 0$) of Eq. (4.3.10) which is given by

$$a(x)y'(x) + b(x)y(x) = R(x), \quad y_0 = \phi(0). \tag{4.3.12}$$

Considering h small enough, the discretized form of Eq. (4.3.11) *becomes*

$$y(ih) = y_0(ih) + (\theta - y_0(1)) \exp\bigg(- \frac{a^2(1) - \varepsilon b(1)}{a(1)} (1/\varepsilon - i\rho)\bigg), \tag{4.3.13}$$

where $\rho = \frac{h}{\varepsilon}$, $h = \frac{1}{N}$. *Similarly, we write*

$$y_{i\pm1} = y_0((i \pm 1)h) + (\theta - y_0(1)) \exp\bigg(- \frac{a^2(1) - \varepsilon b(1)}{a(1)} (1/\varepsilon - (i \pm 1)\rho)\bigg).$$

Using Taylors series approximation for $y_0((i + 1)h)$ *and* $y_0((i - 1)h)$ *up to first order, we obtain*

$$\begin{cases} y_{i+1} = y_0(ih) + (\theta - y_0(1)) \exp\bigg(- \frac{a^2(1)-\varepsilon b(1)}{a(1)} (1/\varepsilon - (i + 1)\rho)\bigg), \\ y_{i-1} = y_0(ih) + (\theta - y_0(1)) \exp\bigg(- \frac{a^2(1)-\varepsilon b(1)}{a(1)} (1/\varepsilon - (i - 1)\rho)\bigg). \end{cases} \tag{4.3.14}$$

To handle the effect of the perturbation parameter rtificial viscosity (exponentially fitting factor $\sigma(\rho)$*) is multiplied on the term containing the perturbation parameter as*

$$-\varepsilon\sigma(\rho)y''(x) + a(x)y'(x) + b(x)y(x) = R(x), \tag{4.3.15}$$

with boundary conditions $y_0 = \phi(0)$ *and* $y(1) = \theta$, *where* $y(1)$ *is evaluated by Runge-Kutta method from the reduced solution of Eq.* (4.3.11).

Next, we consider the difference approximation of Eq. (4.3.10) *on a uniform grid* $\overline{\Omega}^N = \{x\}_{i=0}^N$ *and denote* $h = x_{i+1} - x_i$. *When we apply central finite difference formula on Eq.* (4.4.29) *takes the form*

$$-\varepsilon\sigma(\rho)\Big(D^+D^-y(x_i)\Big) + a(x_i)\Big(D^0y(x_i)\Big) + b(x_i)y(x_i) = R(x_i). \tag{4.3.16}$$

Using operator, Eq. (4.3.10) *in* Ω^- *is rewritten as*

$$L_1^N Y_i = R_i, \tag{4.3.17}$$

with boundary conditions $Y_0 = \phi(0)$ *and* $Y(1) = \theta$, *where*

$$L_1^N Y_i = -\varepsilon\sigma_1(\rho)\left(\frac{Y_{i+1} - 2Y_i + Y_{i-1}}{h^2}\right) + a(x_i)\left(\frac{Y_{i+1} - Y_{i-1}}{2h}\right) + b(x_i)Y_i = R_i. \tag{4.3.18}$$

Multiplying Eq. (4.3.18) *by* h *and considering* h *small and truncating the term* $(R_i - b(x_i)Y_i)h$, *results to*

$$\frac{-\sigma_1(\rho)}{\rho}\left(Y_{i+1} - 2Y_i + Y_{i-1}\right) + \frac{a(x_i)}{2}\left(Y_{i+1} - Y_{i-1}\right) = 0. \tag{4.3.19}$$

Substituting the results in Eq.(4.3.13) *and Eq.*(4.3.14) *into Eq.* (4.3.19) *and simplifying, the exponential fitting factor is obtained as*

$$\sigma_1(\rho) = \frac{\rho a(1)}{2}\coth\left(\frac{\rho a(1)}{2}\right). \tag{4.3.20}$$

In similar manner, we obtained the fitting factor for Ω^+ *as*

$$\sigma_2(\rho) = \frac{\rho a(2)}{2}\coth\left(\frac{\rho a(2)}{2}\right). \tag{4.3.21}$$

Assume that $\overline{\Omega}^{2N}$ *denote partition of [0,2] into* $2N$ *subintervals such that* $0 = x_0, x_1, x_2, ..., x_N = 1$ *and* $x_{N+1}, x_{N+2}, ..., x_{2N} = 2$ *with* $x_i = ih,\ h = \frac{2}{2N} = \frac{1}{N},\ \ i = 0, 1, 2, ..., 2N$.

Case (1): *Consider Eq.* (4.1.5) *on the domain* $\Omega^- = (0, 1)$ *which is given by*

$$-\sigma_1(\rho)\varepsilon y''(x) + a(x)y'(x) + b(x)y(x) = f(x) - c(x)\phi(x-1), \tag{4.3.22}$$

Hence, the required finite difference scheme becomes

$$\left(\frac{-\varepsilon\sigma_1(\rho)}{h^2} - \frac{a(x_i)}{2h}\right)Y_{i-1} + \left(\frac{2\varepsilon\sigma_1(\rho)}{h^2} + b(x_i)\right)Y_i + \left(\frac{-\varepsilon\sigma_1(\rho)}{h^2} + \frac{a(x_i)}{2h}\right)Y_{i+1} = f_i - c_i\phi(x_i - N). \tag{4.3.23}$$

The numerical scheme in Eq. (4.3.23) *can be written in three term recurrence relation as*

$$E_iY_{i-1} + F_iY_i + G_iY_{i+1} = H_i, \quad i = 1, 2, ..., N, \tag{4.3.24}$$

where $E_i = \frac{-\varepsilon\sigma_1(\rho)}{h^2} - \frac{a_i}{2h}$, $F_i = \frac{2\varepsilon\sigma_1(\rho)}{h^2} + b_i$, $G_i = \frac{-\varepsilon\sigma_1(\rho)}{h^2} + \frac{a_i}{2h}$, $H_i = f_i - c_i\phi(x_i - N)$.

Case (2)*: Consider Eq.* (4.1.5) *on the domain* $\Omega^+ = (1, 2)$ *using exponentially fitted finite difference method, which is given by*

$$-\varepsilon\sigma_2(\rho)\left(\frac{Y_{i+1} - 2Y_i + Y_{i-1}}{h^2}\right) + a_i\left(\frac{Y_{i+1} - Y_{i-1}}{2h}\right) + b_iY_i + c_iY(x_i - 1) = f_i. \tag{4.3.25}$$

Similarly, this equation can be written as

$$E_iY_{i-1} + F_iY_i + G_iY_{i+1} + C_i = H_i, \quad i = N + 1, N + 2, ..., 2N - 1, \tag{4.3.26}$$

where $E_i = \frac{-\varepsilon\sigma_2(\rho)}{h^2} - \frac{a_i}{2h}$, $F_i = \frac{2\varepsilon\sigma_2(\rho)}{h^2} + b_i$, $G_i = \frac{-\varepsilon\sigma_2(\rho)}{h^2} + \frac{a_i}{2h}$, $C_i = c_iy(x_i - 1)$ *and* $H_i = f_i$. *Therefore, on the whole domain* $\overline{\Omega} = [0, 2]$, *the basic schemes to solve Eqs.* (4.1.1)-(4.1.2) *are the schemes given in Eqs.* (4.3.24) *and* (4.3.26) *together with the local truncation error .*

4.4 Uniform Convergence Analysis

The discrete scheme corresponding to the original Eqs. $(4.1.7) - (4.1.6)$ *is as follows*

For $i = 1, 2, 3, ..., N$

$$L_1^N Y_i = f_i - c_i\phi_{i-N} \tag{4.4.27}$$

For $i = N+1, N+2, ..., 2N-1$

$$L_2^N Y_i = f_i \tag{4.4.28}$$

subject to the boundary conditions:

$$Y_i = \phi_i, \quad i = -N, -N+1, ..., 0 \tag{4.4.29}$$

$$Y_{2N} = l \tag{4.4.30}$$

and where

$$\begin{cases} L_1^N Y_i = -\varepsilon D^+ D^- Y_i + a(x_i) D^0 Y_i + b(x_i) Y_i \\ L_2^N Y_i = -\varepsilon D^+ D^- Y_i + a(x_i) D^0 Y_i + b(x_i) Y_i + c(x_i) Y_{i-N} \end{cases} \tag{4.4.31}$$

Lemma 4.4.1 *: (Discrete maximum Principle) Assume that the mesh function* $\psi(x_i)$ *satisfies* $\psi(x_0) \geq 0$ *and* $\psi(x_{2N}) \geq 0$ *. Then* $L_1^N \psi(x_i) \geq 0, \quad \forall x_i \in \Omega_1^{2N}$, $L_2^N \psi(x_i) \geq 0$, $\forall x_i \in \Omega_2^{2N}$ *and* $\psi'(1^+) - \psi'(1^-) = [\psi'](1) \leq 0$*. Then* $\psi(x_i) \geq 0$, $\forall x_i \in \overline{\Omega}^{2N}$.

Proof : *Let us define s(x) :*

$$s(x_i) = \begin{cases} \frac{1}{8} + \frac{x_i}{2}, x_i \in [0,1] \cap \overline{\Omega}^{2N} \\ \frac{3}{8} + \frac{x_i}{4}, x_i \in [1,2] \cap \overline{\Omega}^{2N} \end{cases}$$

Note that $s(x_i) > 0, \forall x_i \in \overline{\Omega}^{2N}$, $L^N s(x_i) > 0, \forall x_i \in \Omega_1^{2N} \cup \Omega_2^{2N}$ *and* $[s'](x_N) < 0$.
Let use the notation $\mu = \max\left(\frac{-\psi(x_i)}{s(x_i)} : x_i \in \overline{\Omega}^{2N}\right)$. *Then there exists* $x_i \in \overline{\Omega}^{2N}$ *such that* $\psi(x_k) + \mu s(x_k) = 0$ *and* $\psi(x_k) + \mu s(x_k) \geq 0$, $\forall x_i \in \overline{\Omega}^{2N}$. *Therefore, the function* $\psi + \mu s$ *attains its minimum at* $x = x_k$*. Suppose the theorem does not hold true, then* $\mu > \max\left(\frac{-\psi(x_i)}{s(x_i)} : x_i \in \overline{\Omega}^{2N}\right) > 0$.
Case (i): $x_k = x_0$

$$0 < (\psi + \mu s)(x_0) = 0, \text{ it is a contradiction.}$$

Case (ii): $x_k \in \Omega_1^{2N}$

$$0 < L_1^N(\psi + \mu s)(x_k) = -\varepsilon(\psi + \mu s)''(x_k) + a(x_k)(\psi + \mu s)'(x_k) + b(x_k)(\psi + \mu s)(x_k) \leq 0,$$

it is a contradiction.

Case (iii): $x_k = x_N$

$$0 < [(\psi + \mu s)'](x_N) = [\psi'](x_N) + [s'](x_N) < 0, \ \textit{it is a contradiction.}$$

Case (iv): $x_k \in \Omega_2^{2N}$

$$0 < L_2^N(\psi+\mu s)(x_k) = -\varepsilon(\psi+\mu s)''(x_k)+a(x_k)(\psi+\mu s)'(x_k)+b(x_k)(\psi+\mu s)(x_k)+c(x_k)(\psi+\mu s)(x_k-1) \leq 0,$$

it is a contradiction.

Case (v): $x_k = x_{2N}$

$$0 < (\psi + \mu s)x_{2N} \leq 0, \ \textit{it is a contradiction.} \tag{4.4.32}$$

Hence, the proof of the lemma is finished.

Lemma 4.4.2 *Let $\psi(x)$ be any mesh function. Then, for $0 < i < 2N$*

$$|\psi(x_i)| \leq C \max\{|\psi(x_0)|, |\psi(x_{2N})|, \max_{i \in \Omega_1^{2N} \cup \Omega_2^{2N}} |L^N \psi(x_i)|\}$$

Proof : Consider the barrier functions

$$\theta^{\pm}(x_i) = CM \pm \psi(x_i), \quad \forall x_i \in \overline{\Omega}^{2N} \tag{4.4.33}$$

where $M = \max\{|\psi(x_0)|, |\psi(x_{2N})|, \max_{i\in\Omega_1^{2N}\cup\Omega_2^{2N}} |L^N\psi(x_i)|\}$.

From Eq.(4.4.33) it is clear that $\theta^{\pm}(x_0) \geq 0$ and $\theta^{\pm}(x_{2N}) \geq 0$

$$
\begin{aligned}
&L_1^N\theta^{\pm}(x_i) \geq 0, \quad \forall x_i \in \Omega_1^{2N}\\
&L_2^N\theta^{\pm}(x_i) \geq 0, \quad \forall x_i \in \Omega_2^{2N}\\
&[\theta^{\pm\prime}](x_N) \leq 0
\end{aligned}
$$

Using Lemma 4.4.1, $\theta^{\pm}(x_i) \geq 0, \quad \forall x_i \in \overline{\Omega}^{2N}$. We proved above that the discrete operator L^N satisfies the maximum principle. Next, we analyze the uniform convergence of the method.

Theorem 4.4.3 *Let $y(x_i)$ and Y_i be the exact solution of Eqs.* (4.1.1)-(4.1.2) *and numerical solutions of eqs.(4.1.5) respectively. Then, for a sufficiently large N, the following parameter uniform error estimate holds*

$$|L^N(y(x_i) - Y_i)| \leq \frac{CN^{-2}}{\varepsilon + N^{-1}}\left(1 + \varepsilon^{-3}e^{\left(-a\frac{(1-x_i)}{\varepsilon}\right)}\right). \tag{4.4.34}$$

proof: Let us consider the local truncation error defined as

$$
\begin{aligned}
L^N(y(x_i) - Y_i) &= -\varepsilon\sigma(y''(x_i) - D^+D^-y(x_i)) + a(x_i)(y'(x_i) - D^0y(x_i)),\\
&= -\varepsilon\left[\frac{\rho a(1)}{2}\coth\left(\frac{\rho a(1)}{2}\right) - 1\right]D^+D^-y(x_i)\\
&+ \varepsilon(y''(x_i) - D^+D^-y(x_i)) + a(x_i)(y'(x_i) - D^0y(x_i)),
\end{aligned}
\tag{4.4.35}
$$

where Now, for $z > 0$, C_1 and C_2 are constants, and we have $|z\coth(z) - 1| \leq C_1z^2, \quad z \leq 1$. Similarly, for $z \longrightarrow \infty$, since $\lim\limits_{z\longrightarrow\infty} \coth(z) = 1$, $|z\coth(z) - 1| \leq C_1z$ is given.

In general, for all $z > 0$, we write

$$C_1\frac{z^2}{z+1} \leq z\coth(z) - 1 \leq C_2\frac{z^2}{z+1} \tag{4.4.36}$$

implying that

$$\varepsilon[a(1)\frac{\rho}{2}\coth(a(1)\frac{\rho}{2})-1]\leq\varepsilon\left(\frac{(N^{-1}/\varepsilon)^2}{(N^{-1}/\varepsilon)+1}\right)=\frac{N^{-2}}{\varepsilon+N^{-1}}. \tag{4.4.37}$$

We obtain the bound for the second order derivatives as

$$\begin{cases} |D^+D^-y(x_i)|\leq C|y''(x_i)|, \\ |y''(x_i)-D^+D^-y(x_i)|\leq CN^{-2}|y^{(4)}(x_i)|. \end{cases} \tag{4.4.38}$$

Similarly, for the first derivative term

$$|y'(x_i)-D^0y(x_i)|\leq CN^{-2}|y^{(3)}(x_i)|, \tag{4.4.39}$$

where $|y^{(k)}(x_i)|=\sup_{x_i\in(x_0,x_N)}|y^{(k)}(x_i)|,\quad k=2,3,4.$

Using the bounds in Eq.(4.4.38) and Eq.(4.4.39), we obtain

$$\begin{aligned} |L^N(y(x_i)-Y_i)| &\leq C\frac{N^{-2}}{\varepsilon+N^{-1}}|y''(x_i)|+\varepsilon CN^{-2}|y^{(4)}(x_i)|+CN^{-2}|y^{(3)}(x_i)|, \\ &\leq C\frac{N^{-2}}{\varepsilon+N^{-1}}|y''(x_i)|+CN^{-2}[\varepsilon|y^{(4)}(x_i)|+|y^{(3)}(x_i)|]. \end{aligned}$$

Now, using the bounds for the derivatives of the solution in lemma (4.2.3) and the assumption $\varepsilon\leq N^{-1}$,we have

$$\begin{aligned} |L^N(y(x_i)-Y_i)| &\leq \frac{CN^{-2}}{\varepsilon+N^{-1}}\left(1+\varepsilon^{-2}e^{(\frac{-a(1-x_j)}{\varepsilon})}\right)+CN^{-2}\left[\varepsilon\left(1+\varepsilon^{-4}e^{(\frac{-a(1-x_j)}{\varepsilon})}\right)+\left(1+\varepsilon^{-3}e^{(\frac{-a(1-x_j)}{\varepsilon})}\right)\right] \\ &\leq \frac{CN^{-2}}{N^{-1}+\varepsilon}\left(1+\varepsilon^{-2}e^{(\frac{-a(1-x_j)}{\varepsilon})}\right)+CN^{-2}\left[\left(\varepsilon+\varepsilon^{-3}e^{(\frac{-a(1-x_j)}{\varepsilon})}\right)+\left(1+\varepsilon^{-3}e^{(\frac{-a(1-x_j)}{\varepsilon})}\right)\right], \end{aligned}$$

which simplifies to

$$|L^N(y(x_i)-Y_i)|\leq\frac{CN^{-2}}{\varepsilon+N^{-1}}\left(1+\varepsilon^{-3}e^{(\frac{-a(1-x_j)}{\varepsilon})}\right),\quad \textit{since}\quad \varepsilon^{-3}\geq\varepsilon^{-2}. \tag{4.4.40}$$

Lemma 4.4.4 *For a fixed mesh and for $\varepsilon \to 0$, the following holds:*

$$\lim_{\varepsilon\to 0}\max_{1\le j\le N-1}\frac{e^{\left(\frac{-ax_j}{\varepsilon}\right)}}{\varepsilon^m}=0, \quad m=1,2,3,....$$

$$\lim_{\varepsilon\to 0}\max_{1\le j\le N-1}\frac{\exp\left(\frac{-a(1-x_j)}{\varepsilon}\right)}{\varepsilon^m}=0, \quad m=1,2,3,....$$

Proof : For the proof refer Debela and Duressa (2020).

Theorem 4.4.5 *Let $y(x_i)$ and Y_i be the exact solution of Eqs. (4.1.1)$-$(4.1.2) and numerical solutions of Eq.(4.1.5) respectively. Then, the following error bound holds*

$$\sup_{0<\varepsilon<<1}|(y(x_i)-Y_i)||\le \frac{CN^{-2}}{\varepsilon+N^{-1}}\le CN^{-1}. \tag{4.4.41}$$

Proof : By applying the discrete maximum principle, we obtain the required bound. For the case $\varepsilon > N^{-1}$ the scheme secures second order convergence and we expect to lose an order of convergence for $\varepsilon \le N^{-1}$, and in fact it turns out that the scheme guarantees second order uniformly convergent.

4.5 Numerical Example and Result

To validate the established theoretical results, we perform numerical experiments using the model problem of the form in (4.1.1) $-$ (4.1.2):

Example1: The model singularly perturbed boundary value problem:

$$-\varepsilon y''(x)+5y'(x)-\frac{1}{2}y(x-1)=\begin{cases}1, x\in\Omega^-\\ -1, x\in\Omega^+\end{cases}$$

subject to interval and boundary condition

$$y(x) = 1, x \in [-1, 0], y(2) = 2$$

Having $y_j \equiv y_j^N$ (the approx. solution obtained via exponential fitted operator method) for different values of N and ε , the maximum errors. Since the exact solution is not available, the maximum errors (denoted by E_ε^N) are evaluated using the double mesh principle (Farrel et al. (2004)) for FOFDMs using the formula

$$E_\varepsilon^N := \max_{0 \le j \le N} |y_j^N - y_{2j}^{2N}|, \tag{4.5.42}$$

Further we will tabulate the errors

$$E^N = \max_{0 \le \varepsilon \le 1} E_\varepsilon^N \tag{4.5.43}$$

The numerical rates of convergence are computed using the formula:

$$r_\varepsilon^N := \log_2^{\left(\frac{E_\varepsilon^N}{E_\varepsilon^{\frac{N}{2}}}\right)} \tag{4.5.44}$$

and the numerical rate of "ε -uniform convergence" is computed using

$$R^N = \log_2^{\left(\frac{E^N}{E^{\frac{N}{2}}}\right)} \tag{4.5.45}$$

Table 4.1: Maximum absolute error and rate of convergence for different values of N and ε with exponential fitted operator method for the given Example

ε	N=16	N=32	N=64	N=128	N= 256
10^{-4}	2.1973e-04	1.1353e-04	5.7678e-04	2.9068e-05	1.4591e-05
10^{-8}	2.1973e-04	1.1353e-04	5.7678e-04	2.9068e-05	1.4591e-05
10^{-12}	2.1973e-04	1.1353e-04	5.7678e-04	2.9068e-05	1.4591e-05
10^{-16}	2.1973e-04	1.1353e-04	5.7678e-04	2.9068e-05	1.4591e-05
10^{-20}	2.1973e-04	1.1353e-04	5.7678e-04	2.9068e-05	1.4591e-05
E^N	2.1973e-04	1.1353e-04	5.7678e-04	2.9068e-05	1.4591e-05
R^N	0.9527	0.9770	0.9886	0.9944	

Table 4.2: Comparision of Maximum absolute error for different values of N and ε with exponential fitted operator method

ε	N=64	N=128	N= 256
present method			
E^N	1.6924e-04	9.1097e-05	4.7185e-05
R^N	0.8936	0.9491	
Method in Subb.and Raman(2013)			
E^N	6.2066e-04	1.95256e-04	6.6439e-05
R^N	1.6685	1.6918	

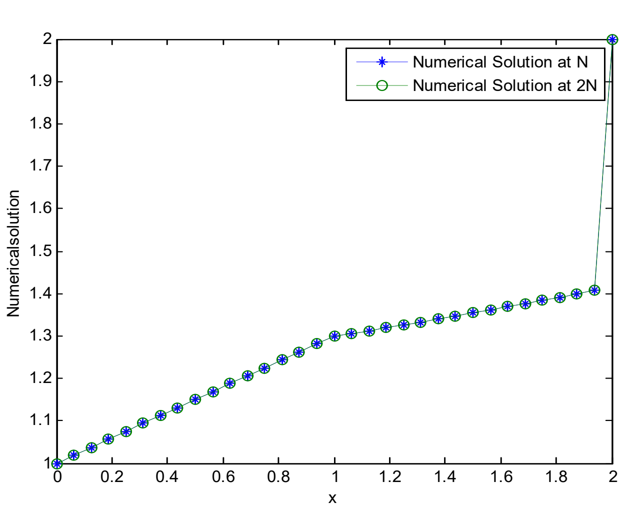

Figure 4.1

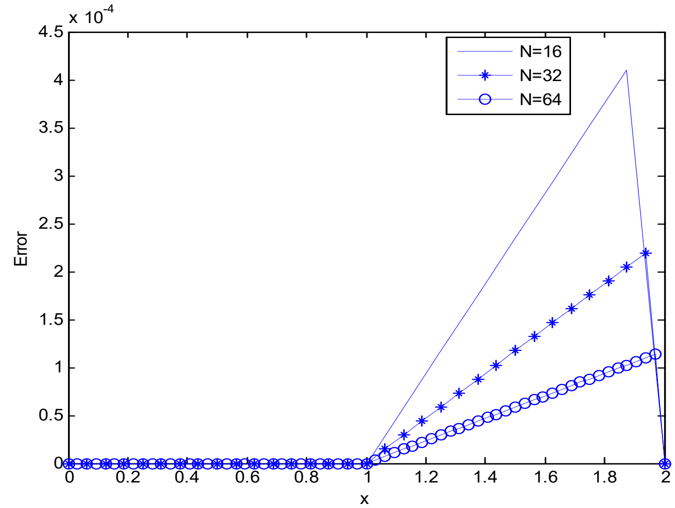

Figure 4.2

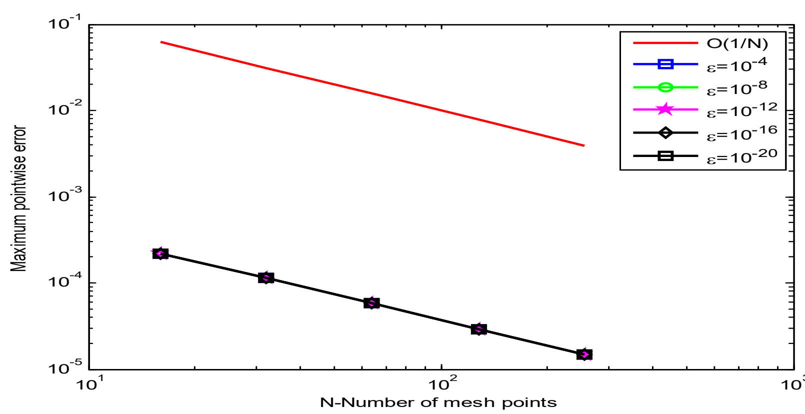

Figure 4.3

Figure 4.1: The behavior of the numerical solution for the given example at $\varepsilon = 10^{-8}$ and N

Figure 4.2: point wise absolute error of the given example at ε =10^{-8} with different mesh point

Figure 4.3: ε-uniform convergence with fitted operator in log-log scale for the given example

Chapter 5

Conclusion and scope for Future work

5.1 Conclusion

This study introduces exponential fitted Operator method for singularly perturbed delay differential equation with discontinuous source term .Due to the discontinuity an interior layer appears in the solution.

To validate the applicability of the method , one model problem is considered for numerical experimentation for different values of the perturbation parameter and mesh points .The numerical results are tabulated in terms of maximum absolute errors and rate of convergence for different values of ε and N with exponential fitted operator method(see table 4.1). The performance of the proposed scheme is investigated by comparing with prior study (see table 4.2).

Further, The behavior of the numerical solution (fig 4.1) ,point-wise absolute error (fig 4.2) and the ε-uniform convergence of the method shown the log-log scale(fig 4.3) .As the number of mesh points increases, the maximum point wise absolute errors decreases(see fig 4.2) .The method shown be ε-uniformly convergent with order of convergent O(h) .The proposed method gives more accurate and ε-uniform convergent numerical result.

5.2 The Scope Of The Future Work

In this thesis , an exponential fitted Operator method were constructed for solving singularly perturbed delay differential equation with discontinuous source term.

Hence , the scheme proposed in this thesis can also be extended to solve higher order singularly perturbed delay differential equation with discontinuous source term .

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