EXPONENTIALLY FITTED MODIFIED UPWIND SCHEME FOR SINGULARLY PERTURBED CONVECTION-DIFFUSION PROBLEM

JIMMA UNIVERSITY COLLEGE OF NATURAL SCIENCES

DEPARTMENT OF MATHEMATICS

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(Numerical Analysis)

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Declaration

I undersigned declare that **"Exponentially fitted modified upwind scheme for singularly perturbed convection-diffusion problem"** is my own original work and it has not been submitted for the award of any academic degree or the like in any other institution or university, and that all the sources I have used or quoted have been indicated and acknowledgment as complete references.

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Acronyms

SPPs : Singularly Perturbed Problems.

SPDEs : Singularly perturbed Differential Equation

PPDO : Fitted Finite Difference Operator

EFFDO: Exponentially Fitted Finite Difference Operator

FOM: Fitted Operator Method

VIM: Variation Iteration Method

EFFDS: Exponentially Fitted Finite Difference Scheme

AEM: Asymptotic Expansion Method

Abstract

In this thesis, an exponentially fitted modified upwind difference scheme is presented for solving singularly perturbed convection-diffusion two point boundary value problems whose solution exhibits right boundary layer. A fitting factor is introduced in a modified upwind scheme and is obtained from the theory of singular perturbations. Then, fitted modified upwind scheme is developed and a three term recurrence relation is obtained. A tri-diagonal finite difference scheme is obtained and is solved by using the Thomas algorithm. To validate the applicability of the proposed method two model examples have been considered and solved for different values of perturbation parameters ε and mesh size h. Both theoretical stability and numerical first order of convergence have been established for the method. The numerical results have been presented in tables, graphs and further to examine the effect of fitted parameter on right boundary layer of the solution and oscillatory behavior of the solution. Several linear and nonlinear problems are solved and observed, which show the presented method approximates the exact solutions very well. Concisely, an exponentially fitted modified upwind scheme gives better result than some existing numerical methods reported in the literature.

Key words: *Exponentially fitted scheme, Modified upwind, Convection-diffusion, Boundary layer, Singularly perturbed, Thomas Algorithm, Tri-diagonal.*

Chapter 1

Introduction

1.1 Background of the Study

Due to the difficulties in finding the exact solution or analytic solution of a mathematical problems such as, the exact solution of differential equations, the root of non-linear equation, the evaluation of integration involving complex expression and etc; leads to the development of numerical analysis. Gauche, (2011), defined numerical analysis is a branch of mathematics that provides tools and methods for solving mathematical problems in a numerical form. A differential equation is any equation involving derivatives of one or more dependent variables with respect to one or more independent variables.

Many real life problems are modeled by parameter dependent differential equations whose solution behavior depends on the magnitude of the parameter. Differential equations in which its highest order derivative term is multiplied by small parameter are called singularly perturbed differential equations (SPDEs). Also it defines problems involving differential equations having none smooth solutions with singularities related to the boundary. Such Singularly Perturbation problem (SPPs) are of common occurrence in many branches of applied mathematics and engineering including fluid mechanics, chemical reactors theory, elasticity, gas porous electrodes theory, heat and mass transfer processes in composite materials with small heat conduction or diffusion. Singularly perturbation theory is a vast collection of mathematical methods used to obtain approximate solution to problems that have no closed form analytic solution (Kumar and Parul*, 2011).

A second order singularly perturbed differential equation is said to be convection diffusion type, if the order of the differential equation is reduced by one when the perturbation parameter tends to zero and a second order singularly perturbed differential equation is said to be reaction diffusion type, if the order of differential equation is reduced by two when the perturbation parameter tends to zero. The numerical solution of a singularly perturbed equation exhibit multi-scale character. That is there is a thin layer(s) of the domain where the solution changes rapidly (non-uniform) or jumps suddenly forming boundary layer(s), while away from the layer(s) the solution behaves regularly (uniform) or changes slowly in the outer region.

Therefore, the numerical treatment for singularly perturbed boundary value problems has always been far from trivial. A boundary layer is defined to be a region of the independent variable over which the dependent variables changes rapidly. Also it is small interval near the initial point where the slope of the curve is changing most rapidly is known as boundary layers. An interior layer occurs in the solution to singularly perturbed problems if the coefficient or the source functions are not sufficiently smooth (Tesfaye et al.,2021).

The physical properties associated with a solution of singularly perturbed second order linear two-point boundary value problems in the following form:

$\varepsilon u''(x)+p(x)u'(x)+q(x)u(x)=r(x), a\leq x \leq b; a, b \in \mathbb{R}$ with boundary conditions

$u(a) = \alpha \in \mathbb{R}, u(b) = \beta \in \mathbb{R}$, where $0 < \varepsilon \leq 1$

and $p(x), q(x)$ and $r(x)$ are sufficiently smooth functions. As $\varepsilon \to 0^+$, the order of the differential equation is reduced and the equation that we call reduced equation.

$p(x)u'_0(x) + q(x)u_0(x) = r(x), a \leq x \leq b; a, b \in \mathbb{R}$ is formed.

One can observe that there are two boundary conditions in the original problems but only one of them can be imposed to the reduced equation. Moreover, as ε tend to zero, because of the reduction of the order, rapid changes occur in the solution.

The region in which these rapid changes occur is named as inner layer or boundary layer. The sign of the coefficient function p(x) determines the type of the layer(s). Over the interval [a, b], if $p(x) > 0$ for all x, then a boundary layer occurs at the left-end of the interval, if $p(x) < 0$ for all x, then a boundary layer occurs at the right-end of the interval and p(x) changes sign between (a, b), the interior layer(s) occurs at the zero(s) of p(x).

This thesis main focuses on exponential fitted modified upwind scheme for singularly perturbed of convection diffusion problem that exhibit right boundary layer.

1.2 Statement of the problem

The numerical analysis of singular perturbation problems has always been far from trivial because of the boundary layer behavior of the solution. Such problems undergo rapid changes within very thin layers near the boundary or inside the domain of the problem. Some researchers are tried to develop numerical methods for solving such type of singularly perturbed differential equation problem for different method used.

For instance, Kadalbajoo and Kumar, (2009) presented initial value technique for singularly perturbed two-point boundary value problems using an exponentially fitted finite difference scheme. They present an approximate method for the numerical solution of quasi-linear singularly perturbed two-point boundary value problems in ordinary differential equation having boundary layer at one end (left or right) point. The original problem is reduced to an asymptotically equivalent first order initial value problem by approximating the zeroth order term by outer solution obtained by asymptotic expansion, and then this initial value problem is solved by an exponentially fitted finite difference scheme. It is observed that the presented method approximates the exact solution very well for crude mesh size h.

Mohapatra and Reddy, (2015) presented exponentially fitted finite difference scheme for singularly perturbed two-point boundary value problems. They introduce a simple exponentially fitted finite difference method for solving singularly perturbed two-point boundary value problems with the boundary layer at one end (left or right). Several linear and nonlinear problems are solved to demonstrate the applicability of the method. This method approximates the exact solution very well.

As clearly explained above, most of authors have attempted to obtain different methods to find the solution of singularly perturbed problem. But when the perturbation parameter becomes small the singularly perturbation problems are unstable and fail to give accurate results. So, the issue of accuracy and convergence of the method still needs attention and improvement. In this study exponentially fitted modified upwind scheme is presented for solving singularly perturbed differential equation of convection-diffusion problem. This method is very important to develop an alternative numerical method which may be more improving the accuracy, improving stability and order of convergence for solving singularly perturbed differential equations.

Owning to this, the present study attempts to answer the following questions:

1. How does an exponential fitted modified upwind scheme be described for solving singularly perturbed convection diffusion problems?

2. To what extent the proposed method is convergent?

3. To what extent the proposed method approximates the solutions?

1.3 Objectives of the study

1.3.1 General Objective

The main objective of this study is to develop exponentially fitted modified upwind scheme for singularly perturbed convection diffusion problem.

1.3.2 Specific Objectives

The specific objectives of the present study are:

1. To formulate an exponentially fitted modified upwind scheme for singularly perturbed convection diffusion problems.
2. To establish stability, convergence and consistency of the present scheme.
3. To investigate the accuracy of the present method.

1.4 Significance of the study

The outcomes of this study may help to introduce the application of numerical method in solving problems arising in different field of studies and serve as reference material for scholars who works on this area.

1.5 Delimitation of the study

This study is delimited to exponentially fitted modified upwind scheme for solving the singularly perturbed convection diffusion problem of the form:

$\varepsilon y''(x)+a(x)y'(x)+b(x)y(x)=f(x)$, $x\in (0,1)$,

subjected to the boundary conditions,

$y(0) = A, y(1) = B$,

where $0 < \varepsilon < 1$ is small positive parameter called as perturbation parameter, A and B are given constants, Further we assume that the functions a(x), b(x) and f(x) are sufficiently smooth with $a(x) \leq \alpha < 0$ and $b(x) \leq \beta < 0$ throughout the interval [0,1] for some given constants α and β .

Chapter 2

Review of Related Literature

2.1 Fitted Numerical Methods

The fitted numerical methods are designed to be robust with respect to changes in the singularly perturbation parameter. Secondly the error estimates, are valid at each point of the mesh or domain, and they are measured in the maximum norm. The choice of fitting factor, or the construction of the fitted mesh, requires a priori information about the location and width of the layer that are to be resolved. Fortunately, such information is frequently available from the mathematical literature on the asymptotic analysis of singular perturbation problems. Any scientist or engineer requiring accurate and robust numerical approximation to the solutions.

Early numerical solutions of singularly perturbed differential equations were obtained by using a standard finite difference operator on a uniform mesh. In this method, as the singularly perturbation parameter decreases in magnitude, the mesh is refined sufficiently to capture the boundary or interior layers. Even for problems in one dimension, such methods are inefficient and inaccurate. The best known analysis of convergence for standard finite difference methods involves the concepts of consistency and stability (E.O'Riordan et al., 2021).

When the rate of convergence is required then consistency is replaced by the more stringent condition of accuracy of degree. Then, the fundamental theoretical results are that consistency and order of convergence. It is clear that the standard finite difference methods can be applied immediately to robust finite difference methods, provided that the conditions of accuracy and stability are also robust. Moreover, in many cases, it is not easy to determine if a given finite difference method has robust accuracy of some order, where orders and the local truncation error constant are independent of perturbation parameter. In such cases another method of a proof of the robust convergence may be required.

A finite difference method has two major ingredients: the finite difference operator that is used to approximate the differential operator and the mesh that replaces the continuous domain. Generally, these methods are accurate and hence their solutions converge to the exact solution, as mesh point tends to very large. It turns out however that none of these methods is robust, and so some new attribute is required. There are two approaches to construction of robust methods.

The first approach involves replacing the standard finite difference operator by a finite difference operator which reflects the singular perturbed nature of the differential operator and such finite difference operator is called fitted finite difference operators(FFDO) (Miller,1996). For the linear problems, they may be constructed by choosing their coefficients so that some or all of the exponential functions in the null space of the differential operators, or a part of it, are also in the null space of the finite difference operator. In such case the finite difference operator is called an exponentially fitted finite difference operator(EFFDO). The corresponding numerical method is obtained by applying the fitted finite difference operator to obtain a system of finite difference equations on a standard mesh, which practice is often a uniform mesh. Numerical methods with a fitted finite difference operator and standard mesh are called fitted operator methods(FOM). The fitted operator comprises specially designed finite difference operators on standard mesh. The fitted operator methods, which the mesh remains uniform and the difference reflects the qualitative behavior of the solution(s) inside the layer regions. The fitted operator can be classified in to exponentially fitted parameters, which reflects the implementation of these methods is not straight forward and usually the introduce artificial viscosity and non-standard finite difference is general set of methods in numerical analysis that gives numerical solutions to differential equations by constructing discrete model.

The second approach, to construct robust numerical method involves the use of mesh that is adapted to the singular perturbation. Such methods are called fitted mesh methods(Shishkin, 1990). A fitted mesh can be incorporated into both a finite difference and a finite element method. The fitted mesh methods comprise standard finite difference operators on specially designed meshes. The fitted mesh methods consist in choosing a fine mesh in the layer region(s). The simplest form of fitted mesh is a piecewise uniform mesh with specially chosen transition points separating the coarse and fine meshes. If the boundary conditions happen to be such that no boundary layer is present at the corresponding boundary point, then it is not necessary for the mesh to condense at that boundary point, and consequently, in such cases, the piecewise uniform mesh will be simpler than in the general case. In this approach of numerical schemes, meshes are taken such that are not uniform; highly non-equidistant grids, logarithmic grids. Of course, more complicated meshes may also be used, but the simplicity of piecewise uniform meshes is considered to be one of their major attractions.

The main contribution of this study is developing exponentially fitted modified upwind scheme which converges uniformly in maximum norm; develop the uniform convergence analysis of the scheme.

2.2 Recent Works

The presence of perturbation parameter, lead to bad approximation or oscillation in the computed using standard numerical methods. To avoid this oscillation, an unacceptability

large number of mesh point are required when perturbation parameter is very small.
This is not practical and leads to rounding error. So, to overcome the draw backs associated with standard numerical methods, different authors tries to develop numerical schemes that converges free from oscillation.

Kadalbajoo and Kumar, (2009) presented initial value technique for singularly perturbed two-point boundary value problems using an exponentially fitted finite difference scheme(EFFDS). The authors provide insight to an approximate method (initial value techniques) for the numerical solution of quasilinear singularly perturbed two-point boundary value problems in ordinary differential equation having boundary layer at one end (left or right) point. The original problem is reduced to an asymptotically equivalent first order initial value problem by approximating the zeroth order term by outer solution obtained by asymptotic expansion, and then this initial value problem is solved by an exponentially fitted finite difference scheme. It is observed that the presented method approximates the exact solution very well for crude mesh size h.

Kadalbajoo and Gupta, (2009) presented numerical solution of singularly perturbed convection diffusion problem using parameter uniform B-spline collection method. They concerned with a numerical scheme to solve a singularly perturbed convection problem. The solution of this problem exhibits the boundary layer on the right hand side of the domain due to the presence of singularly perturbed parameter. The scheme involves B-spline collocation method and appropriate piecewise uniform Shishkin mesh. Bounds are established for the derivative of the analytic solution. Moreover, the present method is boundary layer resolving as well as second order uniformly convergent in the maximum norm.

Geng et al., (2014) presented numerical solutions of singularly perturbed convection diffusion problems. The authors developed numerical method based on the asymptotic expansion method(AEM) and the variation iteration method (VIM). First a zeroth order asymptotic expansion for the solution of the given singularly perturbed convection diffusion problem is constructed. Then the reduced terminal value problem is solved by using the VIM. This method can provide very accurate analytic approximation solutions not only in the boundary layer, but it also away from layer.

Mohapatra and Reddy, (2015) presented exponentially fitted finite difference scheme(EFFDS) for singularly perturbed two-point boundary value problems. They introduce a simple exponentially fitted finite difference method for solving singularly perturbed two-point boundary value problems with the boundary layer at one end (left or right). Several linear and nonlinear problems are solved to demonstrate the applicability of the method. It is observed that the present method approximates the exact solution very well.

Mohapatra and Mahalik, (2015) presented an efficient numerical method for singularly perturbed second order ordinary differential equation. They introduce a simple exponentially fitted finite difference method is presented for solving singularly perturbed two-point boundary value problems with the boundary layer(s) at one end (left or right) point. A fitting factor is introduced and the model equation is discretized by a finite difference scheme on the uniform mesh. It is observed that the present method approximates the exact solution very well.

As clearly explained above, the standard numerical have attempted to obtain different methods to find the solution of singularly perturbed problem. But when the perturbation parameter becomes small the singularly perturbation problems are unstable and fail to give accurate results. So it is difficult to find the solutions of problems easily. Know it needs improvement and not to be recommended because of its low accuracy. Therefore, it is important to develop suitable numerical methods for these problems, whose accuracy does not depend on the parameter value, i.e. methods that are convergent robust. So in this thesis we discuss only fitted operator method which develop an alternative numerical method. This method is very important to develop an alternative numerical method which more improving the accuracy and order of convergence for solution of singularly perturbed differential equations.

Chapter 3

Methodology

This chapter consists; study area and period, study design, mathematical procedures, source of information.

3.1 Study area and Period

The study would be conducted at Jimma University, department of mathematics from June 2022 to December 2022 G.C.

3.2 Study Design

This research was conducted by mixed design, i.e. documentary review and experimental design.

3.3 Source of Information

Books, journals, published articles, and internet are some of the source that we have used to perform this research.

3.4 Mathematical Procedure

In this study the researchers follow the procedures:

1. Define the problem
2. Discretization the solution domain
3. Formulating exponentially fitted modified upwind scheme for the defined problem.
4. Establishing the convergence analysis of method,
5. Writing MATLAB code for the formulated method,
6. Provide numerical illustration in order to confirm the theoretical description, and
7. Discussing the results against the previous findings.

Chapter 4

Description of the Method,Result and Discussion

4.1 Description of the Method

Consider the singularly perturbed convection-diffusion problem of the form:

$$Ly(x) = \varepsilon y''(x) + a(x)y'(x) + b(x)y(x) = f(x), x \in (0, 1) \tag{4.1.1}$$

subject to the boundary conditions:

$$y(0) = A, y(1) = B \tag{4.1.2}$$

where $0 < \varepsilon \leq 1$ is a small perturbation parameter, and A and B are given constants. Assume that the functions are sufficiently smooth with the properties of:

$$a(x) \leq \alpha < 0, \forall x \in \bar{\Omega}, b(x) \leq \beta < 0, \forall x \in \bar{\Omega} \tag{4.1.3}$$

From the above conditions, the singularly perturbed convection-diffusion problem under consideration possesses a unique smooth solution with boundary layer on the right side of the solution domain.

Further, let us consider the singularly perturbed homogeneous differential equation:

$$\varepsilon y''(x) + \alpha y' = 0, \forall x \in \Omega := (0,1) \tag{4.1.4}$$

subject to the boundary conditions in Eq. (4.1.2).

The analytical solution for Eq. (4.1.4), is defined by:

$$y(x) = C_1 + C_2 \exp(\frac{-\alpha}{\varepsilon} x) \tag{4.1.5}$$

where C_1, C_2 are constants which can be determined using the conditions in Eq. (4.1.2). dividing the interval [0,1] into N (positive integer)equal subintervals of mesh length h=$\frac{1}{N}$. That is,0=x$_0, x_1, x_2, ..., x_N = 1$ be the mesh points for x$_i = ih, i = 0, 1, 2, ...N$.

Assume that y(x) has continuous higher order derivatives on [0,1] and for suitability let us denote y(x$_i$) $= y_i, y'(x_i) = y'{}_i, y''(x_i) = y''{}_i, ..., y^n(x_{i)} = y^n{}_i$.

From Taylor series expansion, we obtain:

$$y_{i+1} = y_i + hy'_i + \frac{h^2}{2!}y''_i + \frac{h^3}{3!}y''_i + \frac{h^4}{4!}y^4_i + ...$$

$$y_{i-1} = y_i - hy'_i + \frac{h^2}{2!}y''_i - \frac{h^3}{3!}y''_i + \frac{h^4}{4!}y^4_i - ... \tag{4.1.6}$$

Using the expansions in Eq. (4.1.6), we get the approximation for the first and second derivatives respectively as:

$$y_i' = \frac{y_i - y_{i-1}}{h} + \frac{h}{2}y_i'' + O(h^2)$$

$$y_i'' = \frac{y_i - 2y_i + y_{i-1}}{h^2} + O(h^2) \tag{4.1.7}$$

Considering Eq. (4.1.1) at the nodal point and substituting Eq. (4.1.7) in Eq. (4.1.1) after neglecting the O(h^2) from Eq. (4.1.7) yields:

$$\varepsilon\frac{y_i - 2y_i + y_{i-1}}{h^2} + a_i\frac{y_i - y_{i-1}}{h} + a_i\frac{h}{2}y_i'' + b_iy_i = f_i + O(h^2), 1 \le i \le N-1. \tag{4.1.8}$$

Further, Eq. (4.1.8) re-arranged as:

$$\frac{\varepsilon}{h^2}(1 + \frac{a_ih}{2\varepsilon})(y_{i-1} - 2y_i + y_{i+1}) + \frac{a_i}{h}(y_i - y_{i-1}) = f_i - b_iy_i \tag{4.1.9}$$

Let denote $\rho = \frac{h}{\varepsilon}$ and assume that $f_i - b_iy_i$ is bounded, then introduce the fitting parameterσ into Eq. (4.1.9).

Also, multiplying both sides of Eq. (4.1.9) by h and evaluating its limits as h approaches to zero, yields:

$$\frac{\sigma}{\rho} = \frac{\alpha \lim_{h\to 0(y_{i-1} - y_i)}}{\lim_{h\to 0(y_{i+1} - 2y_i + y_{i-1})} } \tag{4.1.10}$$

where $\alpha = \lim_{h\to 0} a_i$ (α is constant as defined in eq.(4.1.3))

In order to obtain the value of the introduced fitted parameter, considering Eq. (4.1.5) on discreet form of $x_i \in [0,1] as$:

$$y(x_i) \equiv y_i = C_1 + C_2 \exp(\frac{-\alpha}{\varepsilon} x)$$

$$= \mathrm{C}_1 + C_2 \exp(\tfrac{-\alpha}{\varepsilon} ix)$$

$$= \mathrm{C}_1 + C_2 \exp(-\alpha \tfrac{ih}{\varepsilon})$$

$$= C_1 + C_2 \exp(-\alpha i\rho) \qquad (4.1.11)$$

Then, allowing Eq.(4.1.11) in to Eq.(4.1.10), we get the value of fitting parameter:

$$\sigma = \frac{\rho\alpha(exp(\alpha\rho) - 1)}{exp(-\alpha\rho) - 2 + exp(\alpha\rho)} \qquad (4.1.12)$$

Thus, the exponentially fitted modified upwind scheme for solving the problem under consideration can be provided as:

$$\frac{\varepsilon\sigma}{h^2}(1 + \frac{a_i\rho}{2})(y_{i+1} - 2y_i + y + i + 1) + \frac{a_i}{h}(y_i - y_{i-1}) + b_i y_i = f_i, 1 \le i \le N - 1. \qquad (4.1.13)$$

Furthermore, Eq. (4.1.13), can be written in three-term recurrence relation of the form:

$$E_i y_{i-1} + F_i y_i + G_i y_{i+1} = H_i, i = 1, 2, 3, ..., N - 1 \qquad (4.1.14)$$

where,

$\mathrm{E}_i = \frac{\varepsilon\sigma}{h^2}(1 + \frac{a_i\rho}{2}) - \frac{a_i}{h}$,

$F_i = -2\frac{\varepsilon\sigma}{h^2}(1 + \frac{a_i\rho}{2}) + \frac{a_i}{h} + b_i$,

$G_i = \frac{\varepsilon\sigma}{h^2}(1 + \frac{a_i\rho}{2})$,

$H_i = f_i$.

This Eq.(1.4.14) gives us the tri-diagonal system which can be solved by Thomas Algorithm since the conditions $| F_i |\geq| E_i | + | G_i |$ satisfied as:

$$| 2\frac{\varepsilon\sigma}{h^2}(1+\frac{a_i\rho}{2}) + \frac{-a_i}{h} - b_i |\geq| \frac{\varepsilon\sigma}{h^2}(1+\frac{a_i\rho}{2}) - \frac{a_i}{h} | + | \frac{\varepsilon\sigma}{h^2}(1+\frac{a_i\rho}{2}) |,$$

$$| \frac{a_i}{h} + b_i |\geq| \frac{a_i}{h} |,$$

$$| b_i |\geq| \beta |> 0, usingEq.(4.1.3) \tag{4.1.15}$$

Further, the obtained system of equation in eq. (4.1.14) can be written in matrix form as:

$$MY = Z \tag{4.1.16}$$

where the matrices

$$M = (m_{ij}) = \begin{bmatrix} F_1 & G_1 & 0 & \cdots & 0 \\ E_2 & F_2 & G_2 & \cdots & 0 \\ 0 & E_3 & \ddots & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & G_{N-2} \\ 0 & \ldots & 0 & E_{N-1} & F_{N-1} \end{bmatrix}$$

$$\text{Y=y}_{i1} = \begin{bmatrix} y_2 \\ y_2 \\ \vdots \\ y_{N-2} \\ y_{N-1} \end{bmatrix} \; and \; \text{Z=Z}_{i1} = \begin{bmatrix} H_1 - E_1 y_0 \\ H_2 \\ \vdots \\ H_{N-2} \\ H_{N-1} - G_{N-1} Y_N \end{bmatrix}$$

Also, a square matrix $M = (m_{ij})$ is said to be strictly diagonally dominant if $\mid m_{ii} \mid > \sum_{j \neq i} \mid m_{ij} \mid, \forall i$ as verified in Eq. (4.1.15) above.

Lemma:(Hermitian Minkowski) Suppose that the square matrix $M = (m_{ij})$ satisfies $m_{ij} \leq 0, \forall i \neq j. Then, M^{-1}$ exists and each entries of M^{-1} are greater or equal to zero for all i,j if M is strictly diagonally dominant with $m_{ii}>0, \forall i$.

A square matrix $M = (m_{ij})$ is said to be an M- matrix if $m_{ij} \leq 0$ for all i$\neq j$ and M^{-1} exists with all its entries are greater than or equal to zero. Hence, difference schemes that employ M- matrices are generally stable (Martin and David, 2018) .

Truncation Error

The local truncation error T(h) between the exact solution $y(x_i)$, and the approximate solution y_i is given by:

$$T(h) \equiv \varepsilon y''(x_i) + a(x_i)y'(x_i) + b(x_i)y(x_i)$$

$$= -\left\{\frac{\varepsilon\sigma}{h^2}(1 + \frac{a_i\rho}{2})(y_{i+1} - 2y_i + y + i + 1) + \frac{a_i}{h}(y_i - y_{i-1}) + b_iy_i\right\} \tag{4.1.17}$$

From Eq. (4.1.7), we have the approximation:

$$\frac{y_i - y_{i-1}}{h} = y_i^{'} - \frac{h}{2}y_i^{''} + O(h^2)$$

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} = y_i^{''} - \frac{h^2}{12}y_i^4 + O(h^4) \tag{4.1.18}$$

Substituting Eq. (4.1.18) in to Eq. (4.1.17) gives:

$$T(h) \equiv \varepsilon y''(x_i) + a(x_i)y'(x_i) + b(x_i)y(x_i)$$

$$= -\left\{\varepsilon\sigma(1+\frac{a_i\rho}{2})(y_i^{''}-\frac{h^2}{12}y_i^4)+a_i(y_i^{'}+\frac{h}{2}y_i^{''})+b_iy_i\right\} \tag{4.1.19}$$

Considering the following approximations at the specified nodal point as:

$$\varepsilon y^{''}(x_i) \simeq \varepsilon y_i^{''}$$

$$a(x_i)y^{'}(x_i) \simeq a_iy_i^{'}, \tag{4.1.20}$$

$$b(x_i)y(x_i) \simeq b_iy_i.$$

By this equality in Eq. (4.1.20), Eq. (4.1.19) becomes:

$$T(h) = (\frac{a_ih}{2}-\frac{a_i\varepsilon\sigma\rho}{2})y_i^{''}+\frac{h^2}{12}(\varepsilon\sigma+\frac{a_i\varepsilon\sigma\rho}{2})y_i^4, \tag{4.1.21}$$

$$T(h) = h(\frac{a_i}{2}-\frac{a_i\sigma}{2})y_i^{''}+h^2(\frac{\varepsilon\sigma}{12}+\frac{a_i\sigma h}{24})y_i^4,$$

since $\rho=\frac{h}{\varepsilon}$.Hence, the norm of truncation error for the formulated scheme is:

$$\mid T \mid \leq CN^{-1}, forh = N^{-1}, \tag{4.1.22}$$

$$where$$

C=$\mid (\frac{a_i}{2}-\frac{a_i}{2})y_i^{''} \mid = \frac{1}{2} \mid a_i - a_i\sigma \mid \left\| y_i^{''} \right\|_\infty$ is arbitrary constant.

Therefore,the described method is first-order convergent. Truncation errors measure how well a finite difference discretization approximates the differential equation. Thus, the described scheme is first-order accurate.

As anyone know, a finite difference scheme is known as consistent if the limit of truncation error is equal to zero as the mesh size goes to zero. Hence, this definition of consistency on the described method with the local truncation error in Eq. (4.1.21) is satisfied. Therefore, using this consistency and stability criteria provided in terms of M- matrix, the proposed scheme is convergent.

4.2 Numerical Illustration

Sample of modeled numerical examples are considered and solve to demonstrate the applicability of proposed method.

Example 1: Consider the singularly perturbed convection-diffusion boundary value problem:

$$\begin{cases} \varepsilon y''(x) - e^{-x}y'(x) - y(x) = f(x), x \in (0,1), \\ y(0) = y(1) = 0. \end{cases}$$

where f(x) is selected such that the exact solution is

$$y(x) = x(\varepsilon + \frac{x}{2}) - \frac{(\varepsilon + \frac{1}{2})(e^{\frac{x-1}{\varepsilon}} - e^{\frac{-1}{\varepsilon}})}{1 - e^{\frac{-1}{\varepsilon}}}$$

Example 2: Consider the singularly perturbed convection-diffusion boundary value problem:

$$\begin{cases} \varepsilon y''(x) - y'(x) - y(x) = 0, x \in (0,1), \\ y(0) = 1 + exp(-\frac{1+\varepsilon}{\varepsilon}). \\ y(1) = 1 + \frac{1}{\varepsilon} \end{cases}$$

The exact solution for this example is $y(x) = exp(-x) + exp(\frac{(x-1)(1+\varepsilon)}{\varepsilon})$

For this considered examples, the maximum absolute error evaluated by formula:

$$E_{\varepsilon}^{N} = max_{\forall x_i \in [0,1]} \left| y(x_i) - y_i \right|$$

where $y(x_i)$ and y$_i$ are exact and approximated solutions respectively.

The corresponding order of convergence is determined by:

$$R_{\varepsilon}^{N} = \frac{\log R_{\varepsilon}^{N} - \log R_{\varepsilon}^{2N}}{\log^{2}}$$

The obtained numerical results are given in following Tables 1-5, and Figures 1and 2

Table 4.1: Comparison of maximum absolute errors for example 1:

ε	N=16	N=32	N=64	N=128	N= 256
$\varepsilon \downarrow N \rightarrow$	16	32	64	128	
Present Method					
10^{-3}	1.21144e-02	5.8313e-03	2.5957e-03	3.8809e-03	
10^{-6}	1.2874e-02	6.5536e-03	3.3062e-03	1.6604e-03	
10^{-9}	1.2874e-02	6.5544e-03	3.3069e-03	1.6611e-03	
Runchang Lin(2009)					
10^{-3}	7.9515893e-02	5.6767294e-02	4.0393762e-02	2.8390349e-02	
10^{-6}	7.9599192e-02	5.6274836e-02	3.9782178e-02	2.8125784e-02	
10^{-9}	7.9598746e-02	5.6274329e-02	3.9781548e-02	2.8124954e-02	

Table 4.2: Effects of introduced fitting parameter on maximum absolute errors for example1:

$\varepsilon \downarrow N \rightarrow$	16	32	64	128
With fitted Parameter				
10^{-3}	1.2144e-02	5.8313e-03	2.5957e-03	3.8809e-03
10^{-6}	1.2874e-02	6.5536e-03	3.3062e-03	1.6604e-03
10^{-9}	1.2874e-02	6.5544e-03	3.3069e-03	1.6611e-03
Without fitted Parameter				
10^{-3}	3.6591e-01	3.2686e-01	2.3444e-01	9.2864e-02
10^{-6}	4.5873e-01	4.9304e-01	5.1138e-01	5.1845e-01
10^{-9}	4.5886e-01	4.9344e-01	5.1268e-01	5.2279e-01

Table 4.3: Comparison of maximum absolute errors for example 2:

$\varepsilon \downarrow N \rightarrow$	16	32	64	128	256	512	1024
Present Method							
10^{-4}	1.1137e-02	5.6340e-03	2.8185e-03	1.3957e-03	6.8063e-04	3.2221e-04	1.4275e-04
10^{-6}	1.1172e-02	5.6699e-03	2.8546e-03	1.4319e-03	7.1697e-04	3.5860e-04	1.7919e-04
10^{-8}	1.1173e-02	5.6702e-03	2.8550e-03	1.4323e-03	7.1734e-04	3.5896e-04	1.7955e-04
10^{-10}	1.1173e-02	5.6702e-03	2.8550e-03	1.4323e-03	7.1734e-04	3.5896e-04	1.7956e-04
10^{-12}	1.1173e-02	5.6702e-03	2.8550e-03	1.4323e-03	7.1734e-04	3.5896e-04	1.7956e04
Kadalbajoo, Gupta(2009)							
10^{-4}	3.2912e-02	1.4195e-02	6.3383e-03	2.9970e-03	1.4339e-03	6.8495e-04	3.2200e-04
10^{-6}	3.2918e-02	1.4156e-02	6.2306e-03	2.8522e-03	1.3590e-03	6.7467e-04	3.4811e-04
10^{-8}	3.2918e-02	1.4156e-02	6.2292e-03	2.8489e-03	1.3518e-03	6.6094e-04	3.2841e-04
10^{-10}	3.2918e-02	1.4156e-02	6.2291e-03	2.8488e-03	1.3517e-03	6.6079e-04	3.2810e-04
10^{-12}	3.2925e-02	1.4175e-02	6.2236e-03	2.8524e-03	1.3544e-03	6.6086e-04	3.2835e-04

Table 4.4: Effects of introduced fitting parameter on maximum absolute errors for example 2:

$\varepsilon \downarrow N \rightarrow$	16	32	64	128	256	512
With FP						
10^{-4}	1.1137e-02	5.6340e-03	2.8185e-03	1.3957e-03	6.8063e-04	3.2221e-04
10^{-6}	1.1172e-02	5.6699e-03	2.8546e-03	1.4319e-03	7.1697e-04	3.5860e-04
10^{-8}	1.1173e-02	5.6702e-03	2.8550e-03	1.4323e-03	7.1734e-04	3.5896e-04
10^{-10}	1.1173e-02	5.6702e-03	2.8550e-03	1.4323e-03	7.1734e-04	3.5896e-04
10^{-12}	1.1173e-02	5.6702e-03	2.8550e-03	1.4323e-03	7.1734e-04	3.5896e-04
Without FP						
10^{-4}	1.2114e+00	1.1532e+00	1.0129e+00	9.4305e-01	8.9907e-01	8.1263e-01
10^{-6}	1.2526+00	1.2808e+00	1.2914e+00	1.2819e+00	1.2274e+00	1.0955e+00
10^{-8}	1.2530e+00	1.2824e+00	1.2975e+00	1.305-e+00	1.3082e+00	1.3073e+00
10^{-10}	1.2530e+00	1.2824e+00	1.2976e+00	1.3053e+00	1.3091e+00	1.3110e+00
10^{-12}	1.2530e+00	1.2824e+00	1.2976e+00	1.3053e+00	1.3091e+00	1.3111e+00

Table 4.5: Rate of convergence for the two considered examples:

$\varepsilon \downarrow N \rightarrow$	32	64	128	256	512	1024
Example 1						
10^{-4}	6.4813e-03	3.2337e-03	1.5878e-03	7.5962e-04	3.4490e-04	1.9178e-04
	1.0031	1.0262	1.0637	1.1391	0.8467	
10^{-6}	6.5536e-03	3.3062e-03	1.6604e-03	8.3170e-04	4.1596e-04	2.0773e-04
	0.9871	0.9936	0.9974	0.9996	1.0017	
10^{-8}	6.5544e-03	3.3069e-03	1.6611e-03	8.3244e-04	4.1669e-04	2.0846e-04
	0.9870	0.9933	0.9967	0.9984	0.9992	
Example 2						
10^{-4}	5.6340e-03	2.8185e-03	1.3957e-03	6.8063e-04	3.2221e-04	1.4275e-04
	0.9992	1.0139	1.0360	1.0789	1.1745	
10^{-6}	5.6699e-03	2.8546e-03	1.4319e-03	7.1697e-04	3.5860e-04	1.7919e-04
	0.9900	0.9956	0.9979	0.9995	1.0009	
10^{-8}	5.6702e-03	2.8550e-03	1.4323e-03	7.1734e-04	3.5896e-04	1.7955e-04
	0.9899	0.9952	0.9976	0.9988	0.9994	

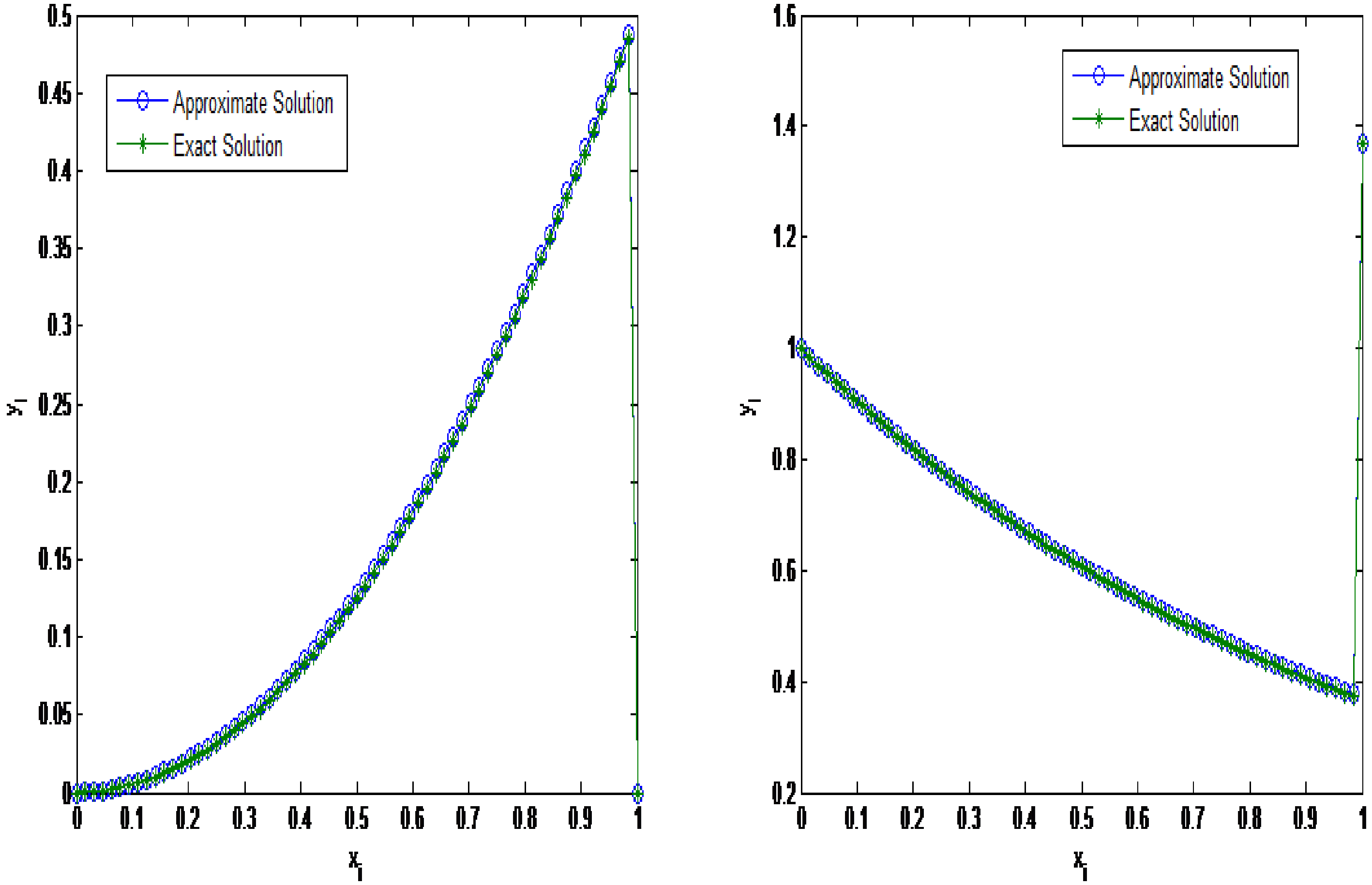

Figure 4.1:
Numerical solution obtained with fitted scheme via exact solution when $\varepsilon = 10^{-4}, N = 64$ for example 1 and 2 respectively.

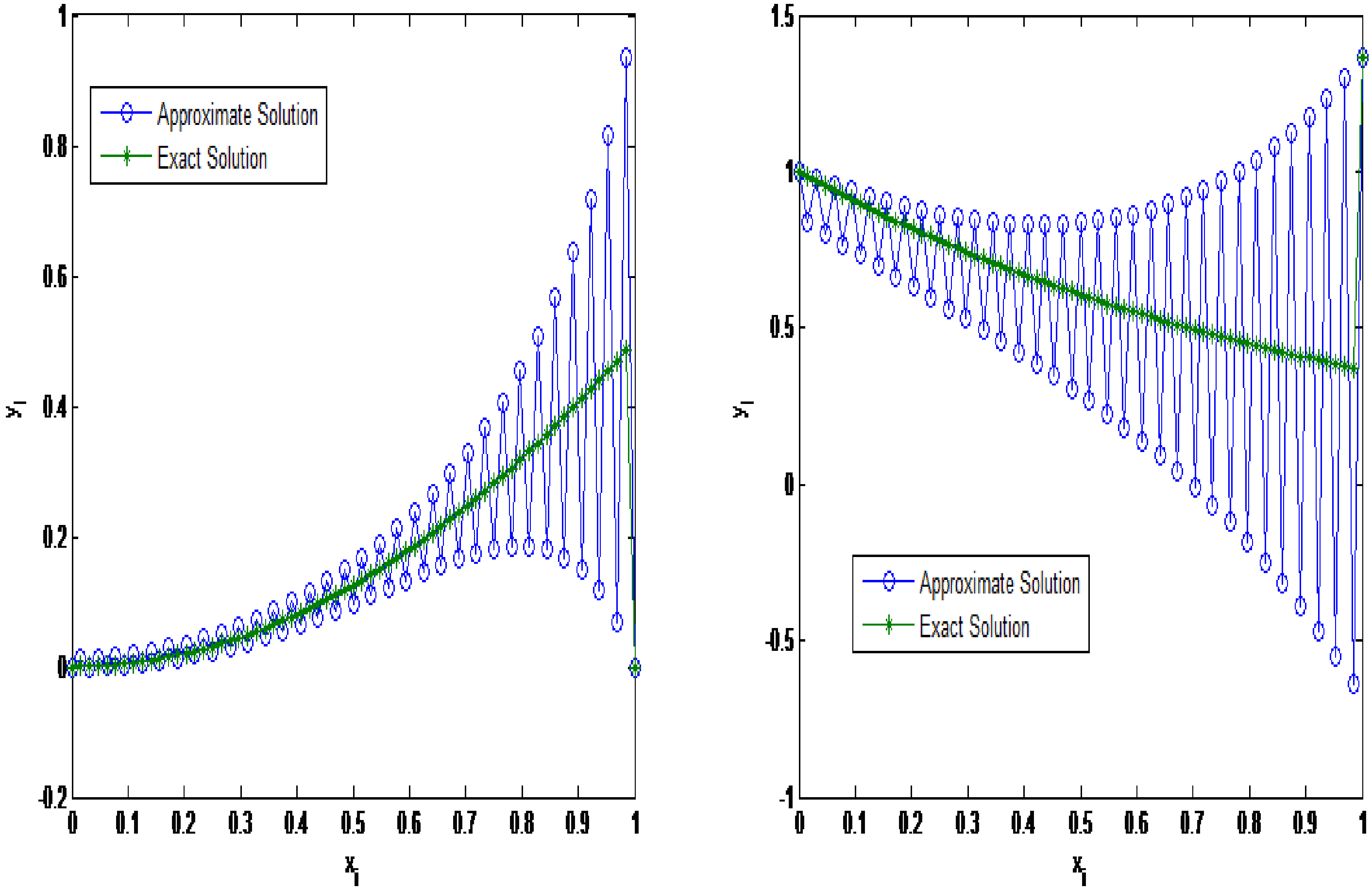

Figure 4.2:
Numerical solution obtained without fitted scheme via exact solution when $\varepsilon = 10^{-4}, N = 64$ for example 1 and 2 respectively.

4.3 Discussion

In this thesis, exponentially fitted modified upwind scheme is presented for solving singularly perturbed convection- diffusion problems. First, the singularly perturbed convection-diffusion equation is replaced by an asymptotically equivalent the modified upwind scheme two-point boundary value problem by using the Taylors series expansion. Then a fitting parameter is introduced into modified upwind scheme to control perturbation parameter and the three term recurrence relation is obtained. The truncation error and convergence analysis of the method have been investigated.The numerical results have been presented in Tables (1) (5) with fitted parameter and without fitted parameter for different values of the perturbation parameter E and mesh points N.

Table 1 and Table 3 show that maximum absolute errors decrease rapidly on the number of mesh interval increases and perturbation parameter become decreases which imply the convergence of the presented method. Table 2 and Table 4 show that effect of introduced fitting parameter on maximum absolute errors decrease rapidly as the number of mesh interval increase and perturbation parameter become decreases which imply the convergence with fitted parameter. We can observe from the Table 5 indicates that the rate of convergence for the present method is almost first order which is in agreement with the theoretical expectation.

To further justify the applicability of the proposed method; graphs have been plotted for the two model examples to compare the exact solutions and approximate solutions at same mesh size h. With fitted scheme plotted in Figure 1 examine the effect of introduced fitting parameter on maximum absolute error are rapidly decreases as mesh size h decreases. So, the present method approximates the exact solution in an excellent manner. But without fitted scheme plotted in Figure 2 indicates the computed solution with uniform mesh oscillates in the right boundary layer regions.

Chapter 5

Conclusion and Scope of Future Work

5.1 Conclusion

This study is implemented on two model examples by taking different values of perturbation parameter, mesh size and the computational results are presented in the Tables and Figures. One can conclude that, the results observed from the Tables demonstrate that the present method approximate the solution very well. A numerical result presented in this thesis shows the betterment of the proposed method over some existing methods reported in the literature. Furthermore, the truncation error and convergence analysis of the method is established well. The results presented (Table 5) confirmed that the computational rate of convergence as well as theoretical estimates indicates that the present method is of first order convergence. The effect of the fitted parameter on the solution of singularly perturbed differential equation is showed by sketching graphs (Figures 1 2). Furthermore, decreases the absolute error also deceases (see Tables (1 4) and Figures(1-2)).In general, the present method is stable, convergent and more accurate for solving singularly perturbed differential equations.

5.2 Scope of the Future Work

In this thesis, the numerical method based on exponential fitted modified scheme method is introduced for solving singularly perturbed convection- diffusion differential equations. Hence, the scheme proposed in this thesis can also be extended to second order finite difference method for solving singularly perturbed convection diffusion differential equations.
The researcher believes that the search for numerical method that involves exponential fitted technique is an interesting and effective work in numerical analysis. so, the forth coming postgraduate and PhD students of mathematics department can exploit this opportunity and conduct their research work in this area.

References

1. Bullo, T. A., Degla, G. A., Duressa, G. F. (2022). Fitted mesh method for singularly perturbed parabolic problems with an interior layer. Mathematics and Computers in Simulation, 193, 371-384.

2. Chakravarthy, P.P., Reddy, Y.N. (2006), Exponential fitted modified upwind scheme for singular perturbation problems. Int.J. Fluid Mech. Res.33(2), 119 135

3. C.G. Lange, and R.M. Miura. (1994), Singular perturbation analysis of boundary value problems for differential equations via small shifts with layer behavior, SIAM Journal on applied mathematics, vol.54, pp.249 272.

4. File, G., Reddy, Y. N. (2012). Fitted-modified upwind finite difference method for solving singularly perturbed differential difference equations. International Journal of Mathematical Models and Methods in Applied Sciences, 6(7), 791-802.

5. Gemechis File, Gashu Gadisa, Tesfaye Aga and Reddy, Y.N. (2017), Numerical solution of singularly perturbed delay reaction diffusion equation with layer or oscillatory behavior. American Journal of Numerical Analysis, 5(1), 1 10.

6. G. M Amiraliyev, and F. Erdogen. (2007), Uniform numerical method for singularly perturbed delay differential equations Computers and Mathematics with applications, vol. 53, no. 8pp. 1251 1259,

7. H. Tian. (2002), The exponential asymptotic stability of singularly perturbed delay differential equations with a bounded lag, J. Math. Anal. Appl.270.

8. Kadalbajoo, M.K.,Kumar., D. (2010), A brief survey on numerical methods for solving singular perturbed problems. Appl. Math. Comput. 217, (3641 3716).

9. Kadalbajoo, M. K., Sharma, K. K. (2005). Numerical treatment of a mathematical model arising from a model of neuronal variability. Journal of mathematical analysis and applications, 307(2), 606-627.

10. Kadalbajoo, M. K., Kumar, D. (2009). Initial value technique for singularly perturbed two point boundary value problems using an exponentially fitted finite difference scheme. Computers Mathematics with Applications, 57(7), 1147-1156.

11. K. Atkinson (1989), An introduction to numerical analysis, 2nd ed., John Wiley Pub.

12. K. W. Mortan. (1996), Numerical solution of convection- diffusion problems, vol. 12 of Applied Mathematics and Mathematical Computation, Chapman and Hall, London, UK,

13. Lak shmiSirisha and Y.N. Reddy (2013), Fitted modified upwind difference scheme for solving singularly perturbed differential difference equations with negative shift, vol. 4, Oct.

14. Mohapatra and S. Natesan. (2010), Parameter uniform numerical method for global solution and global normalized flux of singularly perturbed boundary value problems using grid equidistribution, Computer and Mathematics with Applications, vol. 60,

15. Mohapatra, J., Reddy, N. R. (2015). Exponentially fitted finite difference scheme for singularly perturbed two point boundary value problems. International Journal of Applied and Computational Mathematics, 1(2), 267-278.

16. Mohapatra, J., Mahalik, M. K. (2015). An efficient numerical method for singularly perturbed second order ordinary differential equation. Journal of Mathematical Modeling, 3(1), 33-48.

17. Ranjan, R., Prasad, H. S. (2020). An exponentially fitted scheme for solving singularly perturbed delay problems. Transactions Issue Mathematics, Azerbaijan National Academy of Sciences, 40(1), 161-175.

18. Rakesh Ranjan and H.S.Prasad, (2020), A Fitted finite difference scheme for solving singularly perturbed two point boundary value problems. Information Sciences Letters An International Journal.

19. Rao, R.N., Chakravarty, P.P. (2013), A finite difference method for singularly perturbed differential difference equations with layer and oscillatory behavior. App. Math. Model.

20. Reddy, Y.N., Chakravarty, P.P. (2004), An exponential fitted finite difference method for singular perturbation problem. Appl. Math. Comput. 154, 83 101.

21. R. Nageshwar Rao and P. Pramod Chakravarthy (2014), An exponentially fitted tridiagonal finite difference method for singularly perturbed differential difference equations with small shift. Article in A in Shams Engineering Journal.

22. Stynes, M., Stynes, D. (2018). Convection-diffusion problems (Vol. 196). American Mathematical Soc..