EXISTENCE OF MULTIPLE POSITIVE SOLUTIONS FOR FOURTH ORDER INTEGRAL BOUNDARY VALUE PROBLEMS WITH TWO PARAMETERS

A RESEARCH SUBMITTED TO THE DEPARTMENT OF MATHEMATICS IN PARTIAL FULFILLMENT FOR THE REQUIREMENTS OF THE DEGREE OF MASTERS OF SCIENCE IN MATHEMATICS

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Declaration

I, the undersigned declare that, this research paper entitled "**Existence of multiple positive solutions for fourth order integral boundary value problems with two parameters"** is my own original work and it has not been submitted for the award of any academic degree or the like in any other institution or university, and that all the sources I have used or quoted have been indicated and acknowledged.

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Contents

Acronyms

* BVP- Boundary Value Problem.

* NLS-Normed linear space

* ODE -Ordinary Differential Equation.

Abstract

In this research we established existence of multiple positive solutions fo fourth order integral boundary value problems with two parameters by applying Leggett William fixed point theorem for operators on a cone.In this study ,based on the work of Longfei Lin,Yansheng Liu and Daliang Zhao (2020) are concerned with multiple solutions for a class of nonlinear fourth order boundary value problem with two parameters will be dealt.the research is sequenced in the following manner.The first chapter introduces positive solutions for boundary value problems.this study was mostly use secondary source of data such as journals,books and internet sevice.

Chapter 1

Introduction

1.1 Background of the study

In mathematics, particularly in differential equations, a boundary value problem is a differential equation together with a set of additional constraints, called the boundary conditions. A solution to a boundary value problem is a solution to the differential equation which also satisfies the boundary conditions. A large classes of important boundary value problems are the strum-Leouville problems. The boundary value problems with integral boundary conditions for ordinary differential equations arise in different areas of applied mathematics and physics. For example, heat conduction, chemical engineering, underground water flow, thermo elasticity, and plasma physics can all be reduced to nonlocal problems with integral boundary conditions. Some theories such as the Krasnoselskii's fixed point theorem, the Leggett-Williams fixed point theorem, Avery's generalization of the Leggett-Williams fixed point theorem and Avery-Henderson fixed point theorem have given a decisive impetus for the development of the modern theory of differential equations. The advantage of these techniques lies in that they do not demand the knowledge of solution, but have great power in application, in finding positive solutions, multiple positive solutions for which there exists one or more positive solutions. In analyzing nonlinear phenomena many mathematical models give rise to problems for which only positive solutions make sense. Therefore, since the publication of the monograph positive solutions of Operator Equations in the year 1964 by academician M.A. Krasnoselkii, hundreds of research articles on the theory of positive solutions of nonlinear problems have appeared. The existence of positive solutions of boundary value problems was studied by many researchers: In the past few years, there has been increasing interest in studying certain Multiple Solutions for a Class of Nonlinear fourth-Order Boundary Value Problems; to identify a few:
Xinan Hao,Luyao Zhanga, Lishan Liua, (2019) they studied a higher order fractional differential equation with integral boundary conditions and a parameter. Under different conditions of nonlinearity, existence and nonexistence results for positive solutions are derived in terms of different intervals of parameter by applying on the Guo–Krasnoselskii

fixed point theorem on cones.

XingFang Feng, Hanying Feng (2022) they studied the existence of positive solutions for a singular third-order three-point boundary value problem with a parameter. By using fixed point index theory.

$$\begin{cases} u'''(t) = \lambda q(t) f(t, u(t)), t \in (0,1), \\ u(0) = \alpha u(\mu), \\ u'(\mu) = 0, \\ u''(1) = 0. \end{cases} \tag{1.1}$$

where $\alpha \in (0,1), \mu \in [\frac{1}{2}, 1)$, are constants, λ is a positive parameter.

q: $(0,1) \to [0,\infty), f : [0,1]x[0,\infty) \to [0,\infty)$ are continuous and q(t) may be singular at $t = 0$ and 1.

Xingqiu Zhang ,et al, (2022) investigated the properties of the Green function, together with a fixed point theorem due to Avery-Peterson, height functions on special bounded sets are constructed to obtain the existence of triple positive solutions for higher-order fractional derivatives of integral boundary value problems.

In this thesis, we study the following fourth-order integral boundary value problem (BVP) with two parameters:

$$u^{(4)}(t) + \beta u''(t) - \alpha u(t) = f(t, u), t \in (0,1), \tag{1.2}$$

$$\begin{aligned} u(0) = u(1) = 0, \\ u''(0) = \int_0^1 u(s)\varphi_1(s)ds, \\ u''(1) = \int_0^1 u(s)\varphi_2(s)ds), \end{aligned} \tag{1.3}$$

where nonlinear term function f is allowed to change sign, and also $\alpha, \beta \in R$ and on the other hand, owing to the occurrence of parameter α, β in this boundary value problem including integral boundary conditions, it is not easy to transform the BVP (1.2)-(1.3) into an integral equation directly. To overcome these difficulties, we first introduce operator spectrum method combined with some analysis technique, and establish existence of positive solution to BVP applying Leggett William fixed point theorem on Banach cone. By the positive solution of (1.2)-(1.3) we understand a function u(t)which is positive on $0 < t < 1$ and satisfies the differential equation (1.2) for $0 < t < 1$ and the boundary conditions (1.3). We use Leggett William fixed point theorem to obtain the existence of multiple positive solution of fourth order integral boundary value problem (1.2)-(1.3).

1.2 Statements of the problem

This study was focused on establishing the existence of multiple positive solutions for fourth order integral boundary value problems with two parameters (1.2)-(1.3).
This study will try to answer the questions related to the following:

- How can we construct a Green's function for the corresponding homogeneous equation to boundary value problem with two parameters (1.2)-(1.3)?
- How can we formulate operator for the corresponding problem.
- How can we establish existence of multiple positive solutions by using Leggett William fixed point theorem (1.2)-(1.3) ?

1.3 Objectives

1.3.1 General objective

The general objective of this study is to establish the existence of multiple positive solutions for a fourth order integral boundary value problems with two parameters given in (1.2)-(1.3).

1.3.2 Specific objectives

The specific objectives of this study are:

- To construct the Green's function for the corresponding homogeneous equation.
- To formulate operator for the corresponding problem.
- To establish existence of multiple positive solutions by using Leggett William fixed point theorem.

1.4 Significance of the study

The result of this thesis may have the following importance.

- The outcome of this thesis was give a better understanding of constructing a study research for the researcher.
- It may contribute to research activities on the study area.
- It may provide some background information for other researchers who want to conduct a research on related topics. Furthermore, this thesis would be useful for graduate program of the department and built the research skill and scientific communication of the researcher.

1.5 Delimitation of the Study

This study was delimited to finding the existence of multiple positive solutions for fourth order integral boundary value problem with two parameters of the form (1.2)-(1.3) by applying Leggett William fixed point theorem.

Chapter 2

Review of Related Literatures

Positive solution is very important in diverse disciplines of mathematics since it can be applied for solving various problems and it is one of the most dynamic research subjects in nonlinear analysis. In this area the first important result was proved by Erbe and Wang in 1994 (the existence of positive solutions for second-order two-point boundary value problems). Following Erbe and Wang, the existence of three positive solutions of boundary value problems was studied by many researchers. We list down few of them related to our particular problem. Erbe, et al,(1994),study the existence of multiple positive solution of some boundary value problems.
The purpose of this study is to consider the existence of positive solutions for the following system of fourth-order differential equations:

$$u^{(4)}(t) = \lambda f(t,u(t)),v(t)), \qquad t \in [0,1],$$

$$v^{(4)}(t) = \mu g(t,u(t)),v(t)), \qquad t \in [0,1],$$

subject to the coupled integral boundary conditions

$$\begin{aligned} u(0) &= u''(1) = u''(1) = 0, \\ u''(0) &= \int_0^1 h_1(s)v''(s)ds, \\ v(0) &= v''(1) = v'''(1) = 0, \\ v''(0) &= \int_0^1 h_2(s)u''(s)ds, \end{aligned} \tag{2.1}$$

where λ and μ are two positive parameters and $h_1, h_2 \in C[0,1]$, where f is continuous and $f(t,u) \geq 0$for $t \in [0,1]$ and $u \geq 0, \alpha, \beta, \gamma, \sigma \geq 0$, $p = \gamma\beta + \alpha\gamma + \alpha\sigma > 0$. Henderson, (2000),showed the existence of multiple solution for second- order boundary value problem.

$$y'' + f(y) = 0, 0 \leq t < 1$$

$$y(0) = 0 = y(1)$$

Xiaoming He and Weigao Ge. (2002), studied existence of triple solution for second-order three- point boundary value problem,

$$u''(t) = \lambda h(t) f(u) = 0, t \in c(0,1),$$

$$u(0) = 0,$$

$$\alpha u(\eta) = u(1),$$

(A) where λ is a positive parameter; $\eta \in c(0,1)$ and $0 < \alpha\eta < 1$.

(B) $a : [0,1] \to [0,\infty)$ is continuous and does not vanish identically on any subset of positive measure.

(C) $f : [0,\infty] \to [0,\infty)$ is continuous,

(D) $f_\infty = \lim_{u\to\infty} \frac{f(u)}{u} = \infty$

Specifically, Li, [2003] studied existence of positive solution for fourth-order boundary value problem,

$$u^{(4)}(t) + \beta u''(t) - \alpha u(t) = f(t,u), t \in (0,1)$$
$$u(0) = u(1) = u''(0) = u''(1) = 0$$

by the using fixed point index theorem, where $f \in C((0,\infty),[0,\infty))$ Ge, (2006) constructed the iterative solutions for some fourth-order p-Laplace equation boundary value problems,
Dehong ji, Weigao Ge.(2008). Established the existence of at least three positive solutions for sturm leouville like four point boundary value problem with P- laplacian.

$$-x''(t) + f(x(t)) = 0,$$

$$x(0) = x'(t_2) = x''(1) = 0,$$
$$\frac{1}{2} \le t_2 < 1.$$

Chai,(2007) studied Existence of positive solutions for second-order boundary value problem with one parameter.

Wei and Pang,(2007) investigated Positive solutions and multiplicity of fourth-order m-point boundary value problems with two parameters,
Dehong ji, Weigao Ge.(2008). Established the existence of at least three positive solutions for sturm leouville like four point boundary value problem with P- laplacian.

$$(\phi_p(u)'')(t) + f(t,u(t)) = 0, \qquad t \in (0,1)$$

$$u(0) - \alpha u'(\xi) = 0$$

$$u(1) + \beta u'(\eta) = 0$$

Ruyun Ma and Yulian, (2009), Global structure of positive solutions for nonlocal boundary value problems involving integral conditions.

$$u''(t) + \mu h(t) f(t,u(t)) = 0, 0 < t < 1,$$

$$u(0) = 0, u(1) = \int_0^1 u(s) dA(s),$$

by using global bifurcation techniques, where $f \in C((0,\infty),[0,\infty)), h \in C((0,1),[0,\infty))$
In Jiang and Liu (2009) established the existence of positive solution for second-order singular Sturm-Liouville integral boundary value problems.

$$u''(t) = \mu h(t) f(t,u(t)), 0 < t < 1,$$

$$\alpha u(0) - \beta u'(0) = \int_0^1 a(s) u(s) ds,$$

$$\gamma u(1) + \delta u'(1) = \int_0^1 b(s) u(s) ds$$

by using the fixed point theory in cones, where $f \in C([0,1]x(0,\infty),[0,\infty)$
He , et al, (2018) investigated the existence of positive solutions for a class of high order fractional differential equation integral boundary value problems with changing sign nonlinearity.By applying cone expansion and cone compression fixed point theorem.

$$D_0^\alpha + u(t) + \lambda f(t,u(t)) = 0, t \in (0,1)$$

$$u(0) = u\prime(0)u... = u(n-2)(0) = 0$$

$$D_0^{\beta} + u(1) = \int_0^1 D_0^{\beta} + u(t)dA(t)$$

where D_0^{α} is the Riemann-Leouville fractional derivative of order.
$\alpha, n-1 < \alpha \leq n, n \geq 3, 0 < \beta \leq 1, \lambda > 0$, and $\int_0^1 D_0^{\beta} u(t)dA(t)$ Denotes the Riemann-Stieltjes integrals with respect to A, in which $A(t)$ is a monotone increasing function and $f : [0,1] \times R+ \to R$ may change sign, $R+ = [0,\infty)$.
Longfei Lin, Yansheng Liu and Daliang Zhao (2020) are concerned with multiple solutions for a class of nonlinear fourth order boundary value problems with parameters by constructing a special cone and applying fixed point index theory:

$$u^{(4)}(t) + \beta u''(t) - \alpha u(t) = f(t,u), t \in (0,1)$$

$$u(0) = u(1) = 0$$

where $f \in C([0,1] \times (\mathbb{R}_+, R_+))$, $\alpha, \beta \in R$ and satisfy $\beta < 2\pi^2, \alpha \geq -\frac{\beta^4}{4}\ \frac{\alpha}{\pi^4} + \frac{\beta}{\pi^2} < 1$.
Kovács , (2021) investigated the existence of positive solutions for the singular fourth-order differential system

$$u^{(4)}(t) = \varphi u(t) + f(t,u,u'',\phi), 0 < t < 1,$$

$$-\varphi = \mu g(t,u,u''), 0 < t < 1,$$

$$u(0) = u(1) = u''(o) = u''(1) = 0,$$

$$\varphi(0) = \phi(1) = 0,$$

where $\mu > 0$ is a constant, and the nonlinear terms f, g, may be singular with respect to both the time and space variables by using fixed point theorem of cone expansion and compression.

2.1 Preliminaries

In this section, we provide some definitions, basic concepts on Green's function, definition of existence of positive solutions and statements of few standard fixed point theorems, which are frequently used in thesis.

Definition 2.1.1 *Let X be a non-empty set. A map $T : X \to X$ is said to be a self-map with domain of $T = D(T) = X$ and range of $T = R(T) = T(X) \subseteq X$.*

Definition 2.1.2 *Let $T : X \to X$ be self-map. A point $x \in X$ is called fixed point of T if $Tx = x$.*

Consider the second-order linear differential equation:

$$p_0(x)y'' + p_1(x)y' + p_2(x)y = r(x), \qquad x \in J = [\alpha, \beta] \tag{2.2}$$

where the functions $p_0(x), p_1(x), p_2(x)$ and $r(x)$ are continuous in J and boundary conditions of the form

$$\begin{aligned} l_1[y] &= a_0 y(\alpha) + a_1 y\prime(\alpha) + b_0(\beta) + b_1 y\prime(\beta) = A, \\ l_2[y] &= c_0 y(\alpha) + c_1 y\prime(\alpha) + d_0(\beta) + d_1 y\prime(\beta) = B, \end{aligned} \tag{2.3}$$

where $a_i, b_i, c_i, d_i, i = 0, 1$ and A, B are given constants. The boundary value problems 2.2-2.3 are called a non-homogeneous two-point linear boundary value problem, whereas the homogeneous differential equation.

$$p_0(x)y'' + p_1(x)y' + p_2(x)y = 0, \qquad x \in J = [\alpha, \beta] \tag{2.4}$$

together with the homogeneous boundary conditions

$$l_1[y] = 0, l_2[y] = 0, \tag{2.5}$$

be called a homogeneous two point linear boundary value problem. The function called a Green's function G(x,t) for the homogeneous boundary value problem (2.4)-(2.5)and the solution of the non homogeneous boundary value problem (2.2)-(2.3) can be explicit expressed in terms of G(x,t).
Obviously ,for the homogeneous problem (2.4)-(2.5) is defined in the square $[\alpha, \beta] \times [\alpha, \beta]$ and possesses the following fundamental properties:

(E_1) $G(x,t)$ is continuous in $[\alpha, \beta] \times [\alpha, \beta]$,

(E_2) $\partial \frac{G(x,t)}{\partial x}$ is continuous in each of the triangles $\alpha \le x \le t \le \beta$ and $\alpha \le t \le x \le \beta$;moreover,

(E_3) $\frac{\partial G(t^+,t)}{\partial x} - \frac{\partial G(t^-,t)}{\partial x} = -\frac{1}{p_0(x)}$, where

$$\frac{\partial G(t^+,t)}{\partial x} = \lim_{x \to t_x > t} \frac{\partial G(x,t)}{\partial x}, \frac{\partial G(t^-,t)}{\partial x} = \lim_{x \to t_x < t} \frac{\partial G(x,t)}{\partial x}$$

,

E_4 for every $t \in [\alpha,\beta], z(x) = G(x,t)$ is a solution of the differential equation (2.5) in each of the intervals $\alpha,t)$ and $(t,\beta]$

E_5 for every $t \in [\alpha,\beta], z(x) = G(x,t)$ satisfies the boundary conditions(2.6),these properties completely characterize Green's function $G(x,t)$.

Definition 2.1.3 *A normed linear space X in which each vector x there corresponds a real number,denoted by $\|.\|$ called the norm of x and has the following properties:*

1. $\|x\| \geq 0$ *for all* $x \in X$ *and* $\|x\|$ *if and only if* $x = 0$,
2. $\|x+y\| \leq \|x\| + \|y\|$*for all* $x,y \in X$
3. $\|\alpha x\| = |\ \alpha\ |\ \|x\|$, *for all* $x \in X$ *and* α *being scalar.*

Definition 2.1.4 *A normed linear space X is said to be complete ,if every Cauchy sequence in X converges to a point in X.*

Definition 2.1.5 *Let X be a normed linear space with norm denoted by $\|.\|$. A sequence of elements $\{x_n\}$ of X is cauchy sequence,if for every $\varepsilon > 0$ there exists an integer N such that $\|x_n - x_m\| < \varepsilon, \forall m,n \leq N$.*

Definition 2.1.6 *A map $f : [0,1]xR \to R$ is said to be L^1-Caratheodory if*

- i) $t \to f(t,u)$ *is measurable for each* $u \in R$,
- ii) $u \to f(t,u)$ *is continuous for almost each* $t \in [0,1]$,
- iii) *For every* $r > o$ *there exists* $h_r \in L^1([0,1],R)$ *such that* $|\ f(t,u)\ | \leq h_r(t)$ *a.e.t* $\in [0,1]$ *and* $|\ u\ | \leq r$.

Definition 2.1.7 *A Banach space is a complete normed linear space.*

Definition 2.1.8 *Let E be a Banach space over R. A nonempty,closed set $P \subset E$ is said to be a cone provided that,*

(a) $\alpha u + \beta v \in P$ *for all* u,v *and all* $\alpha,\beta \geq 0$ *and,*

(b) $u,-u \in P$ *implies* $u = 0$.

Definition 2.1.9 *Let X and Y be a Banach space and $T : X \to Y$. The operator T is said to be completely continuous, if T is continuous and for each bounded sequence $\{x_n\}$ has a convergent subsequence.*

Definition 2.1.10 *The map ω is said to be non negative continuous concave functional on P provided that $\omega : P \to [0,\infty)$ is continuous and*

$$\omega(tx+(1-t))y \geq t\omega(x)+(1-t)\omega(y), \forall x,y \in P$$

$$and, 0 \leq t \leq 1.$$

Definition 2.1.11 ***Closed.***
We call A a closed subset of X if, for any convergent sequence $\{f_n\} \geq_1 \subset A$, the limit point is also in A. Open. We call A an open subset if, $\forall x \in A$, there exists ,$\delta > 0$ such that $y \in x$,

$$ky - xk < \delta \Longrightarrow y \in A$$

Definition 2.1.12 ***Bounded***
We call A a bounded subset if there exist $M > 0$ such that, $\forall x \in A, ||x|| < M$.

Theorem 2.1.1 *(Leggett-Williams)*
Suppose $T : \overline{K_c} \to \overline{K_c}$ is completely continuous and suppose there exists a concave positive functional α on K such that $\alpha(u) \leq \|u\|$ for $u \in \overline{K_c}$.
Suppose there exist constants $0 < a < b < d \leq c$ such that:

(B_1) $\{u \in K(\alpha,b,d) : \alpha(u) > b\} \neq \phi$ *and* $\alpha(T_u) > b$ *for* $u \in K(\alpha,b,d)$,

(B_2) $\|T_u\| < u$ *if* $u \in K_a$, *and*

(B_3) $\alpha(T_u) > b$ *for* $u \in K(\alpha,b,c)$ *with* $\|T_u\| > d$. *then T has at least three fixed points* u_1, u_2, *and* u_3 *such that* $\|u_1\| < a, b < \alpha(u_2)$ *and* $\|u_3\| > a$ *with* $\alpha(u_3) < b$.

Theorem 2.1.2 ***Arzela-Ascoli Theorem***
For $A \subset C[0,1], A$ is compact if and only if A is closed, bounded, and equicontinuous.

Theorem 2.1.3 Bounded Norm Theorem
Consider a sequence $\{f_n\} \geq_1$ *in* $C[0,1]$ *and a positive constant K such that* $\|f_n\| \leq K \forall n$*.*
Then the sequence $\{F_n\} \geq 1$ *defined by*

$$F_n(x) = \int_0^x f_n(x)dx,$$

for $X \in [0,1]$*is in* $C[0,1]$ *and has a uniformly convergent subsequence.*

Chapter 3

Methodology

3.1 Study area and period

The study was conducted at Jimma University under the department of mathematics from June, 2022 G.C. to December, 2022 G.C.

3.2 Study Design

In order to achieve the objective of the study was employed analytical method of design.

3.3 Source of Information

The relevant source of information for this study are books, published articles on reputable journals and related study from Internet.

3.4 Mathematical Procedure of the Study

In order to achieve the stated objectives, the study we followed the following mathematical procedures.

1. explain how to solve fourth order integral boundary value problems with two parameters.
2. Constructing a Green's function for the corresponding homogeneous equation.
3 Formulating the equivalent operator equation for the boundary value problem (1.2)-(1.3).
4. Determining the existence of multiple positive solution for the problem(1.2)-(1.3).

Chapter 4

Main Result and Discussion

4.1 Main Results

Let us begin with listing the following assumption conditions, which will be used in the sequel.

Let $I = [0,1], \Re = (-\infty,\infty), \Re = (-\infty,0], \Re = [0,\infty)$.

$(H1) f \in C[I \times \Re+, \Re]$ and exists $N \in L^1(0,1) \cap C[(0,1), \Re+]$ such that

$$f(t,u) + N(t) \geq 0, (t,u) \in [(0,1) \times \Re+] \tag{4.1}$$

(H2) $\alpha, \beta \in \Re, \beta < 2\pi^2, \alpha \geq \frac{-\beta^2}{4}, \frac{\alpha}{\pi^4} + \frac{\beta}{\pi^2} < 1.$

Let λ_1, λ_2 be the roots of the polynomial $p(\lambda)$ $=\lambda^2 + \beta\lambda - \alpha$;namely, $\lambda_1, \lambda_2 = \frac{-\beta \pm \sqrt{\beta^2 + 4\alpha}}{2}$ by (H2) ,it is to see that $\lambda_1 \geq \lambda_2 > -\pi^2$.

Let $\Gamma_0 = \pi^4 - \beta\pi^2 - \alpha$,then (H2) implies $\Gamma_0 > 0$, let $X = C[0,1]$ be the real Banach space equipped with the norm $\|u\| = \max_{0 \leq t \leq_1} |u(t)|$.

Denoted by $P = \{u \in X : u(t) \geq 0, t \in I\}$ in X. In this section, we construct Green's function for corresponding homogeneous boundary value problem to 1.2 and 1.3. Let $G_i(t,s)(i = 1,2)$ be Green's function for the homogeneous problem

Lemma 4.1.1 *(Li,(2003),Wei and Pang (2007)).Suppose that* (H_2) *holds,then there exist unique* $\varphi_i, \psi_i, i = 1,2$ *satisfying*

$$\begin{cases} -\varphi_i''(t) + \lambda_i \varphi_i(t) = 0, t \in [0,1], \\ \varphi_i(0) = 0, \varphi_i(1) = 1, \\ -\psi_i''(t) + \lambda_i \psi_i(t) = 0, t \in [0,1], \\ \psi_i(0) = 1, \psi_i(1) = 0 \end{cases} \tag{4.2}$$

respectively,and $\phi_i \geq 0, \varphi_i \geq 0$ *on* $[0,1]$ *,where* λ_i *is as in* $\lambda_1, \lambda_2 = \frac{-\beta \pm \sqrt{\beta^2 + 4\alpha}}{2}$.

Moreover, ψ_i, φ_i *have the expression*

$$\phi_i(t) = \begin{cases} \frac{sinh\omega_i(t)}{sinh\omega_i}, \lambda_i > 0, \\ t, \lambda_i = 0, \\ \frac{sin\omega_t}{sin\omega_i}, -\pi^2 < \lambda_i, \end{cases} \qquad \varphi_i(t) = \begin{cases} \frac{sinh\omega_i(1-t)}{sinh\omega_i}, \lambda_i > 0, \\ 1-t, \lambda_i = 0, \\ \frac{sin\omega_i(1-t)}{sin\omega_i}, -\pi^2 < \lambda_i < 0 \end{cases} \tag{4.3}$$

where $\omega_i = \sqrt{|\lambda_i|}$ *,*$i = 1,2$*,*

Let $G_i(t,s)(i = 1,2)$ be the Green function of the linear boundary value problem

$$-u''(t) + \lambda_i u(t) = 0, t \in [0,1], u(0) = u(1) = 0, \tag{4.4}$$

by (Li,(2003),Wei and Pang ,(2007)), $G_i(t,s)$ can be expressed by the formula

$$G_i(t,s) = \frac{1}{\sigma_i} \begin{cases} \varphi_i(t)\psi_i(s), & o \le t \le s \le 1 \\ \psi_i(t)\varphi_i(s), & 0 \le s \le t \le 1 \end{cases} \tag{4.5}$$

where,

$$\sigma_i = \begin{cases} \frac{\omega_i}{sinh\omega_i}, if\lambda_i > 0, \\ 1, \quad if\lambda_i = 0, i = 1,2, \\ \frac{\omega_i}{sin\omega_i}, if - \pi^2 < \lambda_i < 0. \end{cases} \tag{4.6}$$

Lemma 4.1.2 *(Li,(2003),Wei and Pang (2007))* $G_i = G_i(t,s)(i = 1,2)$ *have the following properties:*

i. $G_i(t,s) > 0, \qquad \forall t,s \in (0,1)$

ii. $G_i(t,s) \le C_i G_i(s,s), \forall t,s \in [0,1], \phi_i \le C_i, \psi_i \le C_i, t \in [0,1].$

iii. $G_i(t,s) \ge \delta_i G_i(t,t) G_i(s,s), \forall t,s \in [0,1], \phi_i(t) \ge \delta_i G_i(t,t), \psi_i(t) \ge \delta_i G_i(t,t), t \in [0,1],$

where

$$C_i = \begin{cases} 1, if\lambda_i \ge 0, \\ \frac{\omega_i}{sin\omega_i}, if - \pi^2 < \lambda_i < 0, \end{cases} \qquad \delta_i = \begin{cases} \frac{\omega_i}{sinh\omega_i}, if\lambda_i \ge 0, \\ 1, if\lambda_i \ge 0, \\ \omega_i \sin\omega_i, if - \pi^2 < \lambda_i < 0, \end{cases} \tag{4.7}$$

Put

$$D_i = \max_{t\in I} \int_0^1 G_i(t,s)ds, i = 1,2.$$

Set $E_{2,1} = D_2C_1, \quad E_{1,2} = D_1C_2$, where C_i described as before we need also the following assumptions.

(H3) functions blank satisfy $D = E_{1,2} \int_0^1 |\varphi_1(s)|ds + E_{2,1} \int_0^1 |\varphi_2(s)|ds < 1$.

Let $h \in C(0,1) \cap L^1(0,1)$, consider the following BVP:

$$\begin{aligned} u^{(4)}(t) + \beta u''(t) - \alpha u(t) = h(t), t \in (0,1), \\ u(0) = u(1) = u''(0) = u''(1) = 0 \end{aligned} \tag{4.8}$$

By papers Li,(2003),(Wei and Pang ,(2007)) ,BVP (4.8) has a unique solution u=kv expressed by

$$kv(t) = \int_0^1 \int_0^1 G_1(t,s)G_2(s,\tau)v(\tau)d\tau ds = \int_0^1 \int_o^1 G_2(t,s)G_1(s,\tau)v(\tau)d\tau ds, t \in [0,1] \tag{4.9}$$

Let $w = kN$.Since $N \in L^1(0,1) \cap C[[0,1], R_+]$, by Lemma 4.1.2 it is easy to verify that $\omega \in P$. Let

$$g_1(t) = -\int_o^1 G_2(t,s)\phi_1(s)ds, \qquad t \in [0,1]$$

where ϕ_i is as in (4.2) By Lemmas 4.1.1 and 4.1.2,we have $g_1 \in C^2([0,1], \Re_-)$ and

$$\begin{cases} -g_1''(t) + \lambda_2 g_1(t) = -\phi_1(t), t \in [0,1] \\ g_1(0) = g_1(1) = 0. \end{cases} \tag{4.10}$$

On the other hand ,ϕ_1 satisfies the following relation:

$$\phi_1''(t) + \lambda_1\phi_1(t) = 0, t \in [0,1], \phi_1(0) = \phi_1(1) = 0. \tag{4.11}$$

S0,from (4.11)-(4.12),it follows that

$$g_1''(0) = \lambda_2 g_1(0) + \phi_1(0) = 0, g_1''(1) = \lambda_2 g_1(1) + \phi_1(1) = 1 \tag{4.12}$$

Now,we make the following decomposition:

$$
\begin{aligned}
g_1^{(4)}+\beta g_1''-\alpha g_1 &= (-\frac{d^2}{dt^2}+\lambda_1)(-\frac{d^2}{dt^2}+\lambda_2)g_1 \\
&= (\frac{-d^2}{dt^2}+\lambda_1)(-g_1''+\lambda_2 g_1) \\
&= \frac{d^2\phi_1}{dt^2}-\lambda_1\frac{d\phi_1}{dt}=0.
\end{aligned} \tag{4.13}
$$

So,by (4.11),(4.13)-(4.14), it follows that

$$
g_1^{(4)}+\beta g_1''(0)-\alpha g_1=0, t\in[0,1], g_1(0)=g_1(1)=0, g_1''(0)=0, g_1''(1)=1, \tag{4.14}
$$

similarly,by setting

$$
g_2(t)=-\int_0^1 G_1(t,s)\phi_2(s)ds, t\in[0,1], \tag{4.15}
$$

We have

$$
\begin{cases}
g_2^{(4)}(t)+\beta g_2''(t)-\alpha g_2(t)=0, & t\in[0,1],\\
g_2(0)=g_2(1)=0, g_2''(0)=g_2''(1)=0, \\
g_2(t)\le 0, & t\in[0,1].
\end{cases} \tag{4.16}
$$

For any $u\in X$,define u^* as

$$
u^*(t)=\begin{cases}
u(t), if u(t)\ge 0,\\
0, if u(t)<0.
\end{cases} \tag{4.17}
$$

Obviously,$u^*\in P$ for any $u\in X$. Let $h\in L^1(0,1)\cap C(0,1)$; consider the BVP with integral boundary conditions

$$
\begin{cases}
u^{(4)}(t)+\beta u''(t)-\alpha u(t)=h(t), & t\in(0,1),\\
u(0)=u(1)=0, \\
u''(0)=\int_0^1[u-w]^*(s)\varphi_1(s)ds, \\
u''(1)=\int_0^1[u-w]^*(s)\varphi_2(s)ds.
\end{cases} \tag{4.18}
$$

Denoted operator B on $C[0,1]$ by

$$Bu(t) = g_2(t)\int_0^1 [u-w]^*(s)\varphi_1(s)ds + g_1(t)\int_0^1 [u-w]^*(s)\varphi_2(s)ds \qquad (4.19)$$

It is easy to see that B maps $C[0,1]$ in to $C[0,1]$.
Define operator $L : C^4(0,1) \to C(0,1)$ as follows:

$$Lu = u^{(4)} + \beta u'' - \alpha u. \qquad (4.20)$$

We desire(need) the following Lemma.

Lemma 4.1.3 *Let (H2) holds.Assume that* $h \in L^1(0,1) \cap C(0,1)$ *and* $\varphi_i \in [I, \Re_-], i = 1,2.$
Then $\overline{u} \in C^4(0,1) \cap C^2[0,1]$ *is a solution of (4.1) if and only if* $\overline{u}$ *is a solution of operator equation u=*$kv + Bu \in C[0,1]$.

Proof: Assume $\overline{u} \in C^4(0,1) \cap [0,1]$ is a solution of (4.19).By (4.15)-(4.21),we have

$$(B\overline{u})(0) = (B\overline{u})(1) = (B\overline{u}'')(0) = \int_0^1 [u-w]^*(s)\varphi_1(s)ds, (B\overline{u})''(1) = \int_0^1 [\overline{u}-w]^*(s)\varphi_2(s)ds, \qquad (4.21)$$

$L(B\overline{u})$=$(Lg_2)\int_0^1 [\overline{u}-w]^*\varphi_1(s)ds$ +$(Lg_1)\int_0^1 [\overline{u}-w]^*(s)\varphi_2(s)ds = 0$
Let $\overline{v}(t) = \overline{u} - B\overline{u}$. Then

$$\begin{aligned} L\overline{v}(t) = L\overline{u}(t) - LB\overline{u}(t) = L\overline{u}(t) = h(t), t \in (0,1);\\ \overline{v}(0) = \overline{u}''(0) - (B\overline{u})''(0) = 0,\\ \overline{v}''(1) - (B\overline{u})''(1) = 0.\\ \overline{v}(1) = \overline{u}(1) - (B\overline{u})(1) = 0;\\ \overline{v}''(0) = \overline{u}''(0) - (B\overline{u}''(0) = 0,\\ \overline{v}''(1) = \overline{u}''(1) - (B\overline{u}''(1)) = 0. \end{aligned}$$

Thus,by(4.8)-(4.9),we have $\overline{v} = Kh, \overline{v} \in [0,1]$, and so $\overline{u} = Kh + B\overline{u}, \overline{u} \in [0,1]$.
(2) Inversely,assume $\overline{u} \in C[0,1]$ satisfies $\overline{u} = Kh + B\overline{u}$.Then $\overline{u} \in C^4(0,1) \cap C^2[0,1]$.By

(4.8),(4.9),(4.15)-(4.21) , we have

$$LKh = h, LB\overline{u} = 0, (Kh)(0) = (Kh)(1) = (Kh)''(1) = 0,$$
$$(B\overline{u})''(0) = (B\overline{u})''(1) = 0, (B\overline{u})''(0) = \int_0^1 [u-w]^*(s)\varphi_1(s)ds, \tag{4.22}$$

$(B\overline{u})''(1) = \int_0^1 [\overline{u}-w]^*(s)\varphi_2(s)ds$. Consequently,

$$\begin{cases} L\overline{u} = LKh + LB\overline{u} = h, \\ \overline{u}(0) = (Kh)(0) + (B\overline{u})(0) = 0, \overline{u}(1) = (Kh)(1) + (B\overline{u})(1) = 0, \\ \overline{u}''(0) = \int_0^1 [\overline{u}-w]^*(s)\varphi_1(s)ds, \\ \overline{u}''(1) = (Kh'')(1) + B\overline{u}''(1) = \int_0^1 [\overline{u}-w]^*(s)\varphi_2(s)ds \end{cases} \tag{4.23}$$

Thus, $\overline{u}$ is a solution of (4.19).The proof is complete.
□ We have also the following Lemma.

Lemma 4.1.4 *Suppose (H3) holds,Then* $B : X \to X$ *is bounded operator with* $||B|| \leq D(< 1)$ *and* $BX \subset P$.

Proof: In view of Lemma 4.1.2 (ii),by (4.10) ,(4,16),(4.20) and (H3), noticing that $w \in P$, for any $u \in X$ and $t \in I$,we have

$$|g_2(s)| \int_0^1 |[u-w]^* \varphi_1(s)|ds + |g_1(t)| \int_0^1 |[u-w]^* \varphi_2(s)|ds$$
$$|Bu| \leq |g_2(t)| \leq E1,2 \int_0^1 |u(s)||\varphi_1(s)|ds + E_{2,1} \int_0^1 |u(s)||\varphi_2(s)|ds$$
$$\leq D||u||. \tag{4.24}$$

Thus,$||Bu|| \leq D||u||$, and so, $||B|| \leq D(< 1)$.
On the other hand ,from $g_1(t) \leq 0, \varphi_i(t) \leq 0, t \in I, i = 1,2$, we have $BX \subset P$.
So,Lemma 4.1.4 is true. By (4.8),(4.9), it follows from w=KN that

$$w^{(4)}(t) + \beta w''(t) - \alpha w(t) = N(t), t \in (0,1),$$
$$w(0) = w(1) = w''(0) = w''(1) = 0. \tag{4.25}$$

For any $u \in X$,

let $\overline{f}u(t) = f(t,[u-w]^*(t)), t \in [0,1]$ and $Gu(t) = \overline{u}(t) + N(t), t \in (0,1)$.
under conditions (H1)-(H3), consider the following auxiliary BVP:

$$u^{(4)}(t) + \beta u''(t) - \alpha u(t) = Gu(t), t \in (0,1), \tag{4.26}$$

$$u(0) = u(1) = 0, u''(0) = \int_0^1 [u-w]^*(s)\varphi_1(s)ds, u''(1) = \int_0^1 [u-w]^*(s)\varphi_2(s)ds. \tag{4.27}$$

Notice that w(t) satisfies (4.26)-(4.27),it is easy to see that $\overline{u} \in C^4(0,1) \cap C^2[0,1]$ is a solution of (4.28)-(4.29) if and only if $\overline{u} - w \in C^4[0,1]$ is a solution of the following BVP:

$$u^{(4)}(t) + \beta u''(t) - \alpha u(t) = Gu(t), t \in (0,1), \tag{4.28}$$

$$\begin{aligned} u(0) = u(1) = 0, u''(0) &= \int_0^1 u^*(s)\varphi_1(s)ds, \\ u''(1) &= \int_0^1 u^*(s)\varphi_2(s)ds. \end{aligned} \tag{4.29}$$

Thus, if and only if $\overline{u}(t) \geq w(t), t \in [0,1]$, then $\overline{u} - w$ is a solution of BVP(1.2)-(1.3).
Now,by Lemma 4.1.3,$\overline{u} \in C^4(0,1) \cap C^2[0,1]$ is a solution of (4.26)-(4.27) and (4.28)-(4.29) if $\overline{u} \in X$ is a Leggett-william fixed point of the operator KG+B.So,we only desire focusing our attention on the existence of the Legget William fixed point of KG+B.
For the remainder of this section,we give the definition of positive solution.
By a positive solution of boundary value problem(BVP)(1.2)-(1.3),we mean a function $u \in C^4(0,1) \cap C^2[0,1]$ such that $u(t) \geq 0$, $t \in [0,1], u(t) > 0, t \in (0,1)$, and u satisfies (1.2)-(1.3) . □

Define $B = (C[0,1], \|.\|)$ where $\|u\| = \max_{0 \leq t \leq 1} |u(t)|$. Then B is a Banach space.Define the cone $K \subset B$ by $K = \{u \in B : u(t) \geq 0, t \in [0,1]\}$; and the operator $T : B \to B$ by

$$Tu(t) = \int_0^1 G(t,s)e(s)f(u(s))ds$$

Note that fixed points of T are solutions of (1.2),(1.3).

In order to use Theorem 2.1.1.

$$\max_{0\le t\le 1}\int_0^1 G(t,s)ds = \frac{1}{\alpha(\alpha-1)\Gamma(\alpha)} \tag{4.30}$$

we must show that $T : K \to K$ is completely continuous.

Lemma 4.1.5 *Let (H1)-(H3) hold. The operator $T : K \to K$ is completely continuous.*

Proof: Since $G(t,s) \ge 0$,then $Tu(t) \ge 0 \forall u \in K$.Hence if $u \in K$ then $Tu \in K$.
Fix $a > 0$ and let $M = \{u \in \mathbf{B} : \|u\| < a\}$.Let $L = max_{0\le s\le a}(f(s))$. Then for $u \in M$,

$$\begin{aligned} Tu(t) = \int_0^1 G(t,s)e(s)f(u(s))ds \\ &\le \|e\|_\infty L\int_0^1 G(t,s)ds \\ &\le \frac{L\|e\|_\infty}{\alpha(\alpha-1)\Gamma(\alpha)} \end{aligned}$$

Hence,

$$\|Tu\| \le \frac{L\|e\|_\infty}{\alpha(\alpha-1)\Gamma(\alpha)}$$

and so $T(M)$ is uniformly bounded.
Define $\gamma = (\frac{(\alpha-1)\Gamma(\alpha)}{\|a\|_\infty})^{\frac{1}{(\alpha-1)}}$ and let $t_1, t_2 \in [0,1]$ be such that t_1, t_2 and $t_2 - t_1 < \gamma$.Then , for all $u \in M$

$$\begin{aligned} |Tu(t_2) - Tu(t_1)| &\le \int_0^1 |G(t_2,s) - G(t_1,s)|e(s)f(u(s))ds \\ &\le \frac{\|e\|_\infty L}{(\alpha-1)\Gamma(\alpha)}(t_2^{\alpha-1} - t_1^{\alpha-1}) \\ &\le \frac{\|e\|_\infty L}{(\alpha-1)\Gamma(\alpha)}(t_2 - t_1)^{\alpha-1} \\ &\le \varepsilon \end{aligned} \tag{4.31}$$

Thus T is equicontinuous on M. An application of Arzela-Ascoli Theorem shows that T is completely continuous and the proof is complete. □

Theorem 4.1.6 *Let $\beta \in (0,1)$, M=$\|e\|_\infty$,$0 < A \leq \frac{\alpha(\alpha-1)\Gamma(\alpha)}{M}$ and $B \geq (\beta m \int_\beta^1 sG(s,s)ds)^{-1}$. Let a,b and c be such that $0 < a < b < c$.*
Assume that the following hypothesis are satisfied

(i) $f(u) < Aa$,$\forall (t,u) \in [0,1]x[0,a]$,
(ii) $f(u) > Bb$,$\forall (t,u) \in [\beta,1]x[b,c]$,
(iii)$f(u) \leq Ac$,$\forall (t,u) \in [0,1]x[0,c]$.

Then the boundary value problem (1.2)-(1.3) *has at least three positive solutions $u_1,u_2,u_3 \in K$ satisfying $\|u_1\| < a$, $b < \alpha(u_2)$, $a < \|u_3\|$ with $\alpha(u_3) < b$.*

Proof: Define a nonnegative functional on B by $\alpha(u) = min_{\beta \leq t \leq 1}|u(t)|$ we show that the conditions of theorem 2.1.1 are satisfied.
Let $u \in K_c$. Then $\|u\| \leq c$ and by (iii) and (4.30) ,we have

$$\|Tu\| = \max_{0 \leq t \leq 1} \int_0^1 G(t,s)e(s)f(u(s))ds$$
$$< \quad \frac{MA}{\alpha(\alpha-1)\Gamma(\alpha)}dsc$$
$$\leq c$$

Hence $T : K_c \to K_c$ and by Lemma 4.5,T is completely continuous.
Using an analogous argument,it follows from conditions(i) that if $u \in K_a$ then , $\|Tu\| < a$.Conditions(B2) of theorem 2.1.1 holds.
Let d be a fixed constant such that $b < d \leq c$
Finally,if $u \in K(\alpha,b,c)$ with $\|Tu\| \leq c$.Then $\alpha(d)$=$d > b$ and $\|d\| = d$.
As such ,$K(\alpha,b,d) \neq \phi$.Let $u \in K(\alpha,b,d)$ then $\|u\| \leq d \leq c$ and $min_{\beta \leq t \leq 1}u(s) \geq b$.By assumption (ii),and (iii),

$$\alpha(Tu) = min_{\beta \leq t \leq 1} \int_0^1 G(t,s)e(s)f(u(s))ds$$
$$> m \textstyle\int_0^1 sG(s,s)dsBb$$
$$> b$$

That is for all $u \in K(\alpha,b,d)$.$\alpha(Tu) > b$. Condition (B1) of theorem 2.1.1 holds.
Finally,if $u \in K(\alpha,b,c)$ with $\|Tu| > d$ then $\|u\| \leq c$ and $min_{\beta \leq t \leq 1}u(s) \geq b$ and from assumption (2) we can show $\alpha(Tu) > b$.

Condition(B3) of Theorem 2.1.1 holds.
As a consequence of Theorem 2.1.1,T has at least three fixed point u_1, u_2, u_3 such that $\|u_1\| < a$,$b < \alpha(u_2)$,$a < \|u_3\|$ with $\alpha(u_3) < b$.
These fixed points are solutions of (1.2) ,(1.3) and the proof is complete. □

Chapter 5

Conclusion and future scope

5.1 Conclusion

Based on the obtained result the following conclusion can be derived:
In this study,we have considered fourth order integral boundary value problems with two parameters and used the properties of Green's function to construct corresponding homogeneous equation.
After these we formulated equivalent integral equation for boundary value problem (1.2)-(1.3) in the given interval and determined the existence of positive solution of the integral equation by Applying Leggett-William fixed point theorem.
We established the existence of positive solutions for fourth order integral boundary value problems with two parameters by applying Leggett -William fixed point theorem.Finally,we established the existence of at least three positive solutions for fourth order integral boundary value problem with two parameters.

5.2 Future Scope

This study focused on existence of multiple positive solution for fourth order integral boundary value problems with two parameters.Any interested researchers may conduct the research on existence of multiple positive solutions by taking different coefficient and considering orders greater than four.

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