JIMMA UNIVERSITY COLLEGE OF NATURAL SCIENCES

DEPARTMENT OF MATHEMATICS

NONSTANDARD FINITE DIFFERENCE METHOD FOR SOLVING SECOND ORDER SINGULARLY PERTURBED PROBLEM HAVING LARGE DELAY

A Thesis Proposal Submitted to the Department of Mathematics Jimma University in Partial Fulfillment of the Requirements for the Degree of Master of Science in Mathematics.

(Numerical Analysis)

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Declaration

I,undersigned, declare that "Nonstandared finite difference method for solving second order singularly perturbed problem having large delay" is original and it has not been submitted to any institution elsewhere for the award of any academic degree or like and that all the sources I have used or quoted have been indicated and acknowledged as complete references.

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Date.........................

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Date:..........................

Dedication

To my beloved brother Yetsedaw Emagne. I never forget you ever.

Acknowledgment

First of all, I am indebted to my almighty God who gave me long life and helped me to pass through different up and down to reach this time. Next, my special heartfelt thanks go to my advisor Dr. Habtamu Garoma and co-advisor Mr. Worku Tilahun for their constructive and critical comments throughout the preparation of this research work. Finally, my thanks reach to my lovely wife and my darling families for this ideological and financial support.

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Abstract

In this thesis, nonstandard fitted finite difference method has been presented for the numerical solution of a second order singularly perturbed problems having large delay. The behavior of the continuous solution of the problem is studied and shown that it satisfies the continuous stability estimate and the derivatives are also bounded. The numerical scheme is developed on a uniform mesh using non standard finite difference method. To validate the applicability of the method, one model problem is considered for numerical experimentation for different values perturbation parameter and mesh points. The method is shown to be ε-uniformly convergent with order of convergence $O(h)$. The proposed method gives more accurate and ε-uniform numerical result.

Acronyms

* SPDE - Singularly Perturbed Differential Equation.

* SPPs - Singularly Perturbed Problems

* SPDDE -Singularly Perturbed Delay Differential Equation.

* NSFDM -Nonstandard Finite Difference Method

* DDE -Delay Differential Equation

Chapter 1

Introduction

1.1 Background of the study

Numerical analysis plays a significant role and helps us to find an approximate solution for problems which are difficult to solve analytically. In the field of computational mathematics, numerical method is widely used to solve equations arising in the field of physics, engineering and other sciences. The design and computation of the numerical algorithm is one of the mathematical challenges that researchers are facing, but scientists in the field of computational mathematics are trying to develop numerical methods by using computers for further application (Burden and Faires, 2011).

An equation involving the derivative of one/more dependent variable(s) with respect to one/more independent variable(s) is said to be a differential equation (DE). There are two broad categories of such differential equations, i.e PDE and ODE. If the derivative of the dependent variable is with respect to a single variable, it is called ordinary differential equation (ODE). But if the derivative is with respect two/more independent variables, then the differential equation is called partial differential equation (PDE).

A differential equation in which the highest order derivative term multiplied by a small positive parameter ε , where $0 < \varepsilon \ll 1$, is called singularly perturbed differential equation (SPDE) and the parameter is called a perturbation parameter. The classification of singularly perturbed problem depends on how the order of the original equation is affected if one sets $\varepsilon = 0$. If the order is reduced by one, we say that the problem is of convection-diffusion type and of reaction-diffusion type if the order is reduced by two.

Any system involving a feedback control will almost involve time delays. These arise because a finite time is required to sense information and then react to it. If we restrict the class of delay differential equation to a class in which the highest derivative is multiplied by a small positive parameter and involving at least one delay term, then it is said to be singularly perturbed delay differential equation. We call delay differential equations retarded type if the delay argument does not occur in the highest order derivative term; otherwise it is known as neutral delay differential equations. As ε tends to $(\varepsilon \to 0)$, the solution of problems exhibits interesting behaviors (rapid changes). The region where these rapid changes occur is called inner region or boundary layer and the region in which the solution changes regularly is called outer region. The behavior of the solutions of these types of differential equations depends on the magnitude of the parameters. In this problem typically there are thin transition layers where the solution varies rapidly or jumps abruptly, while away from the layers the solution behaves regularly and varies slowly.

In the recent years, there has been a growing interest in the numerical treatment of such differential equations. This is due to the versatility of such type of differential equations in the mathematical modeling of various physical and biological phenomena such as, population ecology, control theory, viscous elasticity, and materials with thermal memory, Elsgolt's and Norkin (1973). Hence in the recent times, many researchers have been trying to develop numerical methods for solving these problems. For example, Andargie and Reddy (2013) presented parameter fitted scheme to solve singularly perturbed delay differential equations Chakravarthy et al., (2015) presented fitted numerical scheme to solve singular perturbed delay differential equation, Mickens (2000) presented a non-standard finite difference schemes to solve singular perturbed delay differential equation, Pratima and Sharma (2011) presented numerical approximation for a class of singularly perturbed delay differential equations with boundary and interior layer(s). But, still the accuracy of such numerical methods needs attention, because the treatment of the singular perturbation is not trivial. Due to this, numerical treatment of singular perturbation needs improvement. Thus, this study will presents nonstandard finite difference method for solving second order singularly perturbed problem having large delay.

1.2 Statement of the problem

The numerical treatment of singularly perturbed problems yield major computational difficulties and the usual numerical methods fail to produce accurate results for all independent values of x when ε is very small related to the mesh size h (i.e. $\varepsilon \ll h$) for the solution singularly perturbation two point boundary value problems Khan and Khandelwal (2013). That is there are thin transition layers, where the solution varies rapidly. The field of delay differential equation (DDE) attracted mathematicians and engineers due to the following reasons. Firstly, we have to find an appropriate approximation of the solution at the delayed arguments. Secondly, the algorithm has to take care of the jump in the discontinuity due to the delay parameter and thirdly, its solution behavior is very interesting with boundary layers, interior layers and oscillations. However, the computation of its solution has been a great challenge and has been of great importance due to the versatility of such equations in the mathematical modeling of processes in various application fields, where they provide the best simulation of observed phenomena and hence the numerical approximation of such equations has been growing more and more. The increasing desire for the numerical solutions to such mathematical problems, which are more difficult or impossible to solve analytically, has become the present day scientific research area. Gülsu and Öztürk (2011) present an approximate solution of the singularly perturbed delay convection-diffusion equation. Chebyshev et al., (2015) present numerical treatment of singularly perturbed delay convection-diffusion equation by employing modified upwind finite difference scheme, but they mainly focuses only on the constant coefficient. Kadalbajoo and Ramesh (2007) states that, the accuracy of the problem increased by increasing the resolution of the grid which might be impractical in some cases like higher dimensions. It is well known that Taylor's series expansion methods for solving a class of second order singularly perturbed delay differential equations with boundary and interior layer(s) are fail to give accurate results when the delay is large. Recently, Rai and Sharma (2020) considered singularly perturbed delay differential equations using fitted mesh method. But, still there is a room to increase the accuracy. Besides, as far as the researchers' knowledge is concerned the problem under consideration via nonstandard finite difference method is not yet considered. Hence, the aim of this thesis is to formulate uniformly convergent non-standard finite difference methods to solve singularly perturbed problem having large delay. Owing to this, the present study attempt to answer the following questions:

- How does we construct the nonstandard finite difference method for solving second order singularly perturbed problem having large delay?
- To what extent the proposed method is convergent?
- To what extent the proposed method approximate the solution?

1.3 Objectives

1.3.1 General objective

The general objective of this study is to develop nonstandard finite difference method for solving second order singularly perturbed problem having large delay.

1.3.2 Specific objectives

The specific objectives of the study are:

* To formulate the nonstandard finite difference method for solving second order singularly perturbed problem having large delay.

* To establish the convergence of the present scheme.

* To investigate the accuracy of the proposed method.

1.4 Significance of the study

The results obtained in this study may:

* Help the graduate students to acquire research skills and scientific procedures.

* To introduce the application of numerical methods in different field of studies.

* Serve as a reference material for scholars who works on this area.

1.5 Delimitation of the study

- This study is delimited to nonstandard finite difference method for solving second order singularly perturbed problem having large delay of the form:

$$\varepsilon y''(x) + a(x)y'(x) + b(x)y(x) + c(x)y(x-1) = f(x), \quad 0 < x < 2, \tag{1.1}$$

$$y(x) = \phi(x), x \in [-1, 0], \tag{1.2}$$

$$y(2) = l, \tag{1.3}$$

where $\phi(x)$ is sufficiently smooth on $[-1, 0]$. For all $x \in \Omega$, it is assumed that the sufficient smooth functions $a(x), b(x)$, $c(x)$ and $f(x)$ satisfy at $a(x) > a > 0, b(x) > b \geq 0, c(x) \leq c < 0$, and $a + b + c > 0$ and l is a constant.

Chapter 2

Review of Related Literature

2.1 Singularly perturbed problems

Science and technology develops many practical problems, such as the mathematical boundary layer theory or approximation of solution of various problems described by differential equations involving small parameters have become increasingly complex and therefore require the use of asymptotic methods. The term "singular perturbations" was first used by Friedrichs and Wasow (1946) in a paper presented at a seminar on non-linear vibrations at New York University. Singularly perturbed problems arise frequently in applications including geophysical fluid dynamics, oceanic and atmospheric circulation, chemical reactions, civil engineering, optimal control, etc.

It is well known that the solution of singularly perturbed boundary value problems is described by slowly and rapidly varying parts. So there are thin transition layers where the solution can jump suddenly, while away from the layers the solution varies slowly and behaves regularly Akram and Afia (2013). Many scholars have studied the analytical and numerical solutions of these problems. Abrahamsson et al.,(1974) solved singularly perturbed ordinary differential equations using difference approximations. Numerical treatment of singularly perturbed boundary value problems for higher-order non linear ordinary differential equations has a great role in fluid dynamics. The development of numerical methods for solving singularly perturbed problems started with methods aimed at solving ordinary differential equations, an account of which can be found in the first monograph on this subject by Doolan et al (1980)

2.2 Singularly Perturbed Delay Differential Equation

Singularly perturbed delay differential equation is an equation in which evolution of system at a certain time depends on the rate at an earlier time. The delay in process arises due to requirement of definite time to sense the instruction and react to it. The delay differential equation in which the highest derivative is multiplied by perturbation parameter is known as perturbed delay differential equation. The delay differential equation can be classified as retarded delay differential equation and neutral differential equation. A delay differential equation is said to be of retarded delay differential equation if the delay argument does not occur in the highest order derivative term, otherwise it is known as neutral delay differential equations. If we restrict it to a class in which the highest derivative term is multiplied by a small parameter, then we obtain singularly perturbed delay differential equation of the retarded type. Frequently, delay differential equations have been reduced to differential equations with coefficients that depend on the delay by means Taylor's series expansions of the terms that involve delay and the resulting differential equation have been solved either analytically when the coefficients of these equations are constant or numerically, when they are not. The theory and numerical solution of singularly perturbed delay differential equation are still at the initial stage. In the past, only every few people had worked in the area of numerical methods on singularly perturbed delay differential equations (SPDDEs). Lange and Miura (1994) gave an asymptotic approximation to solve singularly perturbed second order delay differential equations. Duressa and Reddy (2013) presented a numerical method that does not depend on the asymptotic expansion and matching of the coefficients for solving a class of singularly perturbed delay differential equations with negative shift in the differentiated term. Andargie and Reddy (2013) provided a parameter fitted scheme and effect of small shifts on the boundary layer solution of the problem to solve singularly perturbed delay differential equations in the differentiated term of second order with left or right boundary. Accordingly, when the delay parameter is smaller than the perturbation parameter, the layer behavior is maintained. When the delay argument is sufficiently small, to tackle the delay term Kadalbajoo and Sharma (2004) used Taylor's series expansion and presented an asymptotic as well as numerical approach to solve such type boundary value problem.

But the existing methods in the literature fail in the case when the delay argument is bigger one because in this case, the use of Taylor's series expansion for the term containing delay may lead to a bad approximation. The numerical treatment of singularly perturbed problems preserves some major computational difficulties and in recent years a large number of special purpose methods have been proposed to provide accurate numerical solutions. This type of problem has been intensively studied analytically and it is known that its solution generally has boundary layers where the solution varies rapidly. The outer solution corresponds to the reduced problem, i.e., that obtained by setting the small perturbation parameter to zero.

2.3 Recent Development

Lange and Miura (1994) gave an asymptotic approximation to solve singularly perturbed second order delay differential equations. Chakravarthy et al., (2015) deals with the singularly perturbed boundary value problem for the second order delay differential equation. Similar boundary value problems are associated with expected first-exit times of the membrane potential in models of neurons. Chakravarthy et al., (2017) deals with singularly perturbed boundary value problem for a linear second order delay differential equation. Kumar and Rao (2020) presented a stabilized central difference method for the boundary value problem of singularly perturbed differential equations with a large negative shift. The central difference approximations for the derivatives are modified by re-approximating the error terms, leading to a stabilizing effect. The method is found to be second order convergent.

As introduced in the literature, most researchers have been tried to find approximate solution for singularly perturbed differential equations with a large delay, but mainly focuses on constant coefficients, and some others those who have done for variable co-efficients did not get more accurate solutions. Owing this, this study presents a more accurate and convergent numerical method for singularly perturbed differential equations with a large delay, by using nonstandard finite difference method.

Chapter 3

Methodology

3.1 Study Site and Period

The study is conducted at Jimma University, College of Natural Science, Department of Mathematics from August 2021 to January 2021.

3.2 Study Design

The study is applied both the documentation review and numerical experimentation or mixed design

3.3 Source of Information

The relevant source of information for this study were books, published articles on reputable journals.

3.4 Mathematical Procedure

In order to achieve the stated objectives, the study followed the following mathematical procedure:

1. Defining (or describing) the problem.
2. Discretizing the solution domain /interval.
3. Constructing nonstandard finite difference scheme for the governing problem and obtain system of linear equation.
4. Writing an algorithm for the developed schemes.
5. Establishing the stability and convergence analysis of the formulated schemes.
6. Solve the obtained system using Guasian elimination method.
7. Validating the schemes using numerical experiments.
8. Discussing and providing conclusions.

Chapter 4

Description of The Method, Result and Discussion

4.1 Description of the method

From Eq. (1.1) and Eq. (1.2), we have singularly perturbed problem having large delay of the form:

$$\varepsilon y''(x) + a(x)y'(x) + b(x)y(x) + c(x)y(x-1) = f(x), \quad 0 < x < 2, \tag{4.1}$$

subject to the interval and boundary conditions,

$$\begin{cases} y(x) = \phi(x), & -1 \leq x \leq 0, \\ y(2) = l. \end{cases} \tag{4.2}$$

As we observed from Eq. (4.1) and Eq. (4.2), the values of $y(x-1)$ is known for the domain $\Omega_1 = (0, 1]$ and unknown for the domain $\Omega_2 = (1, 2)$ due to the large delay at $x = 1$. So, it impossible to treat the problem throughout the domain $(\bar{\Omega})$. Thus, we have to treat the problem at Ω_1 and Ω_2 separately.

So, Eqs. (4.1)-(4.2) is equivalent to

$$Ly(x) = R(x), \tag{4.3}$$

where

$$Ly(x) = \begin{cases} L_1y(x) = \varepsilon y''(x) + a(x)y'(x) + b(x)y(x), x \in \Omega_1, \\ L_2y(x) = \varepsilon y''(x) + a(x)y'(x) + b(x)y(x) + c(x)y(x-1), x \in \Omega_2, \end{cases} \tag{4.4}$$

$$R(x) = \begin{cases} f(x) - c(x)\phi(x-1), x \in \Omega_1, \\ f(x), x \in \Omega_2, \end{cases} \tag{4.5}$$

with boundary conditions

$$\begin{cases} y(x) = \phi(x), x \in [-1, 0], \\ y(1^-) = y(1^+), y'(1^-) = y'(1^+), \\ y(2) = l. \end{cases} \tag{4.6}$$

4.2 Properties of Continuous Solution

Lemma 4.2.1 *(Minimum Principle) Let $\psi(x)$ be any function in X such that $\psi(0) \geq 0, \psi(2) \geq 0, L_1\psi(x) \geq 0, \forall x \in \Omega_1, L_2\psi(x) \geq 0, \forall x \in \Omega_2$ and $[\psi'](1) \leq 0$ then $\psi(x) \leq 0, \forall x \in \bar{\Omega}$.*

Proof: Define a test function

$$s(x) = \begin{cases} \frac{1}{8} + \frac{x}{2}, x \in [0, 1], \\ \frac{3}{8} + \frac{x}{4}, x \in [1, 2]. \end{cases}$$

Note that $s(x) > 0, \forall x \in \bar{\Omega}, Ls(x) > 0, \forall x \in \Omega_1 \cup \Omega_2, s(0) > 0, s(2) > 0$ and $[s'](1) < 0$. Let $\mu = \max\{\frac{-\psi(x)}{s(x)} : x \in \bar{\Omega}\}$. Then, there exists $x_0 \in \bar{\Omega}$ such that $\psi(x_0) + \mu s(x_0) = 0$ and $\psi(x) + \mu s(x) \geq 0, \forall x \in \bar{\Omega}$.Therefore, the function $(\psi + \mu s)$ attains its minimum at

$x = x_0$. Suppose the lemma does not hold true, then $\mu > 0$.

Case (i): $x_0 = 0$

$$0 < (\psi + \mu s)(0) = \psi(0) + \mu s(0) = 0,$$

it is a contradiction.

Case (ii): $x_0 \in \Omega_1$

$$0 < L(\psi + \mu s)(x_0) = \varepsilon(\psi + \mu s)''(x_0) + a(x_0)(\psi + \mu s)'(x_0) + b(x_0)(\psi + \mu s)(x_0) \geq 0,$$

it is a contradiction.

Case (iii): $x_0 = 1$

$$0 \leq [(\psi + \mu s)'](1) = [\psi'](1) + \mu[s'](1) < 0,$$

it is a contradiction.

Case (iv): $x_0 \in \Omega_2$

$$\begin{aligned} 0 < L(\psi + \mu s)(x_0) = \varepsilon(\psi + \mu s)''(x_0) + a(x_0)(\psi + \mu s)'(x_0) + b(x_0)(\psi + \mu s)(x_0) \\ + c(x_0)(\psi + \mu s)(x_0 - 1) \geq 0, \end{aligned}$$

it is a contradiction.

Case (iv): $x_0 = 2$

$$0 < (\psi + \mu s)(2) = (\psi + \mu s)(2) \leq 0,$$

it is a contradiction. Hence, the proof of the Lemma.

Lemma 4.2.2 *(Stability Result) The solution $y(x)$ of Eqs.* (1.1)-(1.2)*, satisfies the bound*

$$|y(x)| \leq C \max\{\big|y(0)\big|, \big|y(2)\big|, \sup_{x \in \Omega*} \big|Ly(x)\big|\}, \quad x \in \overline{\Omega}.$$

Proof: This Lemma can be proved by using Lemma 4.2.1 and the barrier functions

$\theta^{\pm}(x) = CMs(x) \pm y(x), \quad x \in \overline{\Omega}, \quad$ where $M = \max\left\{ \left|y(0)\right|, \left|y(2)\right|, \sup_{x\in\Omega*} \left|Ly(x)\right| \right\}$ and $s(x)$ is the test function as in Lemma 4.2.1.

Lemma 4.2.3 *Let $y(x)$ be the solution of Eqs.* (1.1)-(1.2)*. Then, we have the following bounds*

$$|y^{(k)}(x)|_{\Omega^*} \le C\varepsilon^{-k}, \quad for \quad k = 1, 2, 3.$$

Proof: To bound $y'(x)$ on the interval Ω_1, we consider

$$L_1 y(x) = \varepsilon y''(x) + a(x)y'(x) + b(x)y(x) = R(x).$$

Integrating the above equation on both sides, we have

$$\varepsilon\big(y'(x) - y'(0)\big) = [a(x)y(x) - a(0)y(0)] + \int_0^x a'(t)y(t)dt - \int_0^x b(t)y(t)dt + \int_0^x [f(t) - c(t)\phi(t-1)]dt,$$

Therefore,

$$\varepsilon y'(0) = \varepsilon y'(x) - [a(x)y(x) - a(0)y(0)] + \int_0^x a'(t)y(t)dt - \int_0^x b(t)y(t)dt + \int_0^x [f(t) - c(t)\phi(t-1)]dt,$$

Then by the Mean value theorem, there exits $z \in (0, \varepsilon)$ such that

$$|\varepsilon y'(z)| \le C(|y(x)|, |R(x)|, |\phi(x)|_{[-1,0]}) \quad and \quad |\varepsilon y'(0)| \le C(|y(x)| + |R(x)| + |\phi(x)|).$$

Hence,

$$|\varepsilon y'(x)| \le C \max(|y(x)|, |R(x)|, |\phi(x)|).$$

By a similar argument we can bound $y'(x)$ on Ω_2, as $|\varepsilon y'(x)| \le C$. From Eqs. (4.4) and (4.5), we have

$$|y^{(k)}(x)|_{\Omega^*} \le C\varepsilon^{-k}, \quad k = 2, 3.$$

Hence, the proof.

Lemma 4.2.4 *Let y_ε be the solution of (P_ε). Then, for $k = 0, 1, 2, 3,$*

$$| y_\varepsilon^{(k)}(x) | \leq C(1 + \varepsilon^{-k} \exp(\frac{-a}{\varepsilon} x)), \forall x \in [0, l].$$

Proof: For the proof refer Bansal and Sharma (2017).

4.3 Formulation of the method

The theoretical basis of non-standard discrete numerical method is based on the development of exact finite difference method. Mickens (2005) presented techniques and rules for developing non-standard finite difference methods for different problem types. In Mickens's rules, to develop a discrete scheme, denominator function for the discrete derivatives must be expressed in terms of more complicated functions of step sizes than those used in the standard procedures. These complicated functions constitute a general property of the schemes, which is useful while designing reliable schemes for such problems.
For the problem of the form in Eqs. (1.1)–(1.2), in order to construct exact finite difference scheme, we follow the procedures used in Bansal and Sharma (2017). Let us consider the following singularly perturbed differential equation of the form

$$\varepsilon y''(x) + a(x)y'(x) + b(x)y(x) = f(x). \tag{4.7}$$

The constant coefficient homogeneous problems corresponding to Eq. (4.7)

$$\varepsilon y''(x) + ay'(x) + by(x) = 0, \tag{4.8}$$

$$\varepsilon y''(x) + ay'(x) = 0, \tag{4.9}$$

where $a(x) \geq a$ and $b(x) \geq b$. Two linear independent solutions of Eq. (4.8) are $\exp(\lambda_1 x)$ and $\exp(\lambda_2 x)$, where

$$\lambda_{1,2} = \frac{-a \pm \sqrt{a^2 - 4\varepsilon b}}{2\varepsilon}. \tag{4.10}$$

We discretize the domain $[0, 1]$ using uniform mesh length $\Delta x = h$ such that, $\Omega^N = \{x_i = x_0 + ih, 1, 2, ..., N, x_0 = 0, x_N = 1, h = \frac{1}{N}\}$, where N denotes the number of mesh points. We denote the approximate solution to $y(x)$ at grid point x_i by Y_i. Now our main objective is to calculate a difference equation which has the same general solution as the differential equation Eq. (4.8) at the grid point x_i given by $Y_i = A_1 \exp(\lambda_1 x_i) + A_2 \exp(\lambda_2 x_i)$. Using the theory of difference equations and the procedures used in Bansal and Sharma (2017), we have

$$\det \begin{bmatrix} Y_{i-1} & \exp(\lambda_1 x_{i-1}) & \exp(\lambda_2 x_{i-1}) \\ Y_i & \exp(\lambda_1 x_i)) & \exp(\lambda_2 x_i) \\ Y_{i+1} & \exp(\lambda_1 x_{i+1}) & \exp(\lambda_2 x_{i+1}) \end{bmatrix} = 0. \tag{4.11}$$

Simplifying Eq. (4.11), we obtain

$$-\exp(-\frac{ah}{2\varepsilon})Y_{i-1} + 2\cosh(\frac{h\sqrt{a^2 - 4\varepsilon b}}{2\varepsilon})Y_i - \exp(\frac{ah}{2\varepsilon})Y_{i+1} = 0, \tag{4.12}$$

which is an exact difference scheme for Eq. (4.8).

Since $\varepsilon \to 0$, we use an approximation $\frac{h\sqrt{a^2-4\varepsilon b}}{2\varepsilon} \approx \frac{ah}{2\varepsilon}$ in Eq. (4.12). Hence, multplying both side of Eq. (4.12) by $\exp(\frac{ah}{2\varepsilon})$ and after doing the arithmetic manipulation and rearrangement on Eq. (4.12), for the constant coefficient problem Eq. (4.9), we get

$$\varepsilon \frac{Y_{i-1} - 2Y_i + Y_{i+1}}{\frac{h\varepsilon}{a}(\exp(\frac{ah}{\varepsilon}) - 1)} + a\frac{Y_{i+1} - Y_i}{h} = 0. \tag{4.13}$$

The denominator function becomes $\Psi^2 = \dfrac{h\varepsilon}{a}\left(\exp\left(\dfrac{ha}{\varepsilon}\right) - 1\right)$. Adopting this denominator function for the variable coefficient problem, we write it as

$$\Psi_i^2 = \frac{h\varepsilon}{a_i}\left(\exp\left(\frac{ha_i}{\varepsilon}\right) - 1\right), \tag{4.14}$$

where Ψ_i^2 is the function of ε, a_i and h.

Assume that $\bar{\Omega}^{2N}$ denote partition of [0,2] in to $2N$ subintervals such that $0 = x_0 < x_1 < ... < x_N = 1$ and $1 < x_{N+1} < x_{N+2} < ... < x_{2N} = 2$ with $x_i = ih$, $h = \frac{2}{2N} = \frac{1}{N}$, $i = 0, 1, 2, ..., 2N$.

Case 1: Consider Eq. (4.1) on the domain $\Omega_1 = (0, 1)$ which is given by

$$\begin{cases} \varepsilon y''(x) + a(x)y'(x) + b(x)y(x) = f(x) - c(x)\phi(x-1), \\ y_0 = y(0) = \phi(0). \end{cases} \tag{4.15}$$

Undertaking the notation $Y_i = y(x_i)$ and using the nonstandard finite difference methodology of Mickens(1991), for right layer in the domain Ω_1 the scheme to solve Eq. 4.15 is given by

$$\varepsilon\left(\frac{Y_{i+1} - 2Y_i + Y_{i-1}}{\psi_i^2}\right) + a_i\left(\frac{Y_{i+1} - Y_i}{h}\right) + b_iY_i + \tau_1 = f_i - c_i\phi(x_i - 1), \tag{4.16}$$

where

$$\Psi_i^2 = \frac{hc_\varepsilon}{a_i}\left(\exp\left(\frac{hp_i}{c_\varepsilon}\right) - 1\right) = h^2 + O\left(\frac{h^3}{\varepsilon}\right),$$

the local truncation term $\tau_1 = h\frac{a_i}{2}y_i'' + O(h^2) = O(h)$.

Eq. (4.16) can be written as three term recurrence relation as

$$E_iY_{i-1} + F_iY_i + G_iY_{i+1} = H_i, i = 1, 2, ..., N, \tag{4.17}$$

where $E_i = \frac{\varepsilon}{\Psi_i^2}$, $F_i = \frac{-2\varepsilon}{\psi_i^2} - \frac{a_i}{h} + b_i$, $G_i = \frac{\varepsilon}{\Psi_i^2} + \frac{a_i}{h}$ and $H_i = f_i$.

Case 2: Consider Eq. (4.1) on the domain $\Omega_2 = (1, 2)$, for right layer in the domain Ω_2

using the nonstandard finite difference method which is given by

$$\varepsilon\left(\frac{Y_{i+1}-2Y_{i+1}+Y_i}{\psi_i^2}\right)+a_i\left(\frac{Y_{i+1}-Y_i}{h}\right)+b_iY_i+c_iY(x_i-1)+\tau_1=f_i,$$

Similarly, this equation can be written as

$$c_iY_j+E_iY_{i-1}+F_iY_i+G_iY_{i+1}=H_i, i=N+1,N+2,...,2N-1, \tag{4.18}$$

where $Y_j=y(x_i-1), j=1,2,...,N,\ E_i=\frac{\varepsilon}{\psi_i^2}-\frac{p_i}{h},\ F_i=\frac{2\varepsilon}{\psi_i^2}+\frac{a_i}{h}+b_i,\ G_i=\frac{\varepsilon}{\psi_i^2}$ and $H_i=f_i$.

Therefore, on the whole domain $\overline{\Omega}=[0,2]$, the basic schemes to solve Eq. (1.1)-(1.2) are the schemes given in Eq. (4.17) and Eq. (4.18).together with the local truncation error of τ_1.

4.4 Uniform Convergence Analysis

In this section, we need to show the discrete scheme in Eq. (4.17), satisfy the discrete minimum principle, uniform stability estimates, and uniform convergence.

Lemma 4.4.1 *(Discrete Minimum Principle) Let Y_i be any mesh function that satisfies $Y_0\geq 0$, $Y_N\geq 0$ and $L_\varepsilon^NY_i\leq 0$, $i=1,2,3,...,N-1$,then $Y_i\geq 0$, for $i=0,1,2,...,N$.*

Proof: The proof is by contradiction. Let j be such that $Y_j=\min_i Y_i$ and suppose that $Y_j\leq 0$. Clearly, $j\notin\{0,N\}$. $Y_{j+1}-Y_j\geq 0$ and $Y_j-Y_{j-1}\leq 0$.
Therefore,

$$\begin{aligned}
L_\varepsilon^NY_j=&\varepsilon\left(\frac{Y_{j+1}-2Y_j+Y_{j-1}}{\Psi_i^2}\right)+a_j\left(\frac{Y_{j+1}-Y_j}{h}\right),\\
=&\frac{\varepsilon}{\Psi_i^2}(Y_{j+1}-2Y_j+Y_{j-1})+\frac{a_j}{h}(Y_{j+1}-Y_j),\\
=&\frac{\varepsilon}{\Psi_i^2}((Y_{j+1}-Y_j)-(Y_j-Y_{j-1}))+\frac{a_j}{h}(Y_{j+1}-Y_j),\\
\geq&\ \ 0,
\end{aligned}$$

where the strict inequality holds if $Y_{j+1} - Y_j > 0$. This is a contradiction and therefore $Y_j \geq 0$. Since j is arbitrary, we have $Y_i \geq 0, \ \ \text{for} \ \ i = 0, 1, 2, ..., N$.
From the discrete minimum principle we obtain an $\varepsilon-$ uniform stability property for the operator L_ε^N.

We provide above the discrete operator L_ε^N satisfy the minimum principle. Next we analyze the uniform convergence analysis.
Using Taylor series expansion, the bound for $y(x_{i-1})$ and $y(x_{i+1})$ at x_i as

$$\begin{cases} y(x_{i-1}) = y(x_i) - hy'(x_i) + \frac{h^2}{2!}y''(x_i) - \frac{h^3}{3!}y^{(3)}(x_i) + \frac{h^4}{4!}y^{(4)}(x_i) + O(h^5), \\ y(x_{i+1}) = y(x_i) + hy'(x_i) + \frac{h^2}{2!}y''(x_i) + \frac{h^3}{3!}y^{(3)}(x_i) + \frac{h^4}{4!}y^{(4)}(x_i) + O(h^5). \end{cases}$$

We obtain the bound for

$$\begin{cases} |D^+D^-y(x_i)| \leq C|y''(x_i)|, \\ |y''(x_i) - D^+D^-y(x_i)| \leq Ch^2|y^{(4)}(x_i)|. \end{cases} \tag{4.19}$$

Similarly, for the first derivative term

$$|y'(x_i) - D^+y(x_i)| \leq Ch|y''(x_i)|, \tag{4.20}$$

where $|y^{(k)}(x_i)| = \sup_{x_i \in (x_0, x_N)} |y^{(k)}(x_i)|, \quad k = 2, 3, 4.$

Theorem 4.1 *Let the coefficients functions $a(x)$ and the source function $f(x)$ in Eqs. (1.1)-(1.2) of the domain Ω be sufficiently smooth, so that $y(x) \in C^4[0,1]$. Then, the discrete solution Y_i satisfies*

$$|L^N(y_i - Y_i)| \leq Ch\left(1 + \sup_{x \in (0,1)} \left(\frac{\exp(\frac{-ax_i}{\varepsilon})}{\varepsilon^3}\right)\right).$$

Proof: We consider the truncation error discretization as

$$
\begin{aligned}
|L^N(y_i - Y_i)| =& |L^N y_i - L^N Y_i|,\\
\le& C|\varepsilon y_i'' + a_i y_i' - \{\varepsilon \frac{D^+D^-h^2}{\Psi_i^2} y_i + a_i D^+ y_i\}|,\\
\le& C|\varepsilon (y_i'' - \frac{D^+D^-h^2}{\Psi_i^2} y_i) + a_i(y_i' - D^+y_i)|,\\
\le& C\varepsilon|y_i'' - D^+D^-y_i| + C\varepsilon|(\frac{h^2}{\Psi_i^2} - 1)D^+D^-y_i| + Ch|y_i''|,\\
\le& C\varepsilon h^2|y_i^{(4)}| + Ch|y_i''| + Ch|y_i''|,\\
\le& C\varepsilon h^2|y_i^{(4)}| + Ch|y_i''|.
\end{aligned}
$$

We used the estimate $\varepsilon|\frac{h^2}{\Psi_i^2} - 1| \le Ch$ which can be derived from Eq. (4.14). Indeed, define $\rho = \dfrac{a_i h}{\varepsilon}, \rho \in (0,\infty)$ .Then,

$$
\varepsilon|\frac{h^2}{\Psi_i^2} - 1| = \varepsilon|\frac{h^2}{\frac{h\varepsilon}{a_i}\left(\exp\left(\frac{ha_i}{\varepsilon}\right) - 1\right)} - 1| = a_i h|\frac{1}{\exp(\rho) - 1} - \frac{1}{\rho}| =: a_i h Q(\rho).
$$

By simplifying and writing explicitly we obtain

$$
Q(\rho) = \frac{\exp(\rho) - \rho - 1}{\rho(\exp(\rho) - 1)},
$$

and we obtain the limit is bounded as

$$
\lim_{\rho \longrightarrow 0} Q(\rho) = \frac{1}{2}, \quad \lim_{\rho \longrightarrow \infty} Q(\rho) = 0.
$$

Hence, for all $\rho \in (0,\infty)$ we have $Q(\rho) \le C$. So, the error estimate in the discretization is bounded as

$$
|L^N(y_i - Y_i)| \le C\varepsilon h^2|y_i^{(4)}| + Ch|y_i''|. \tag{4.21}
$$

From Eq. (4.21) and boundedness of derivatives of solution in Lemma 4.2.3 , we obtain

$$
\begin{aligned}
|L^N(y(x_i) - Y_i)| &\leq C\varepsilon h^2 \left| \left(1 + \varepsilon^{-4} \exp\left(\frac{-ax_i}{\varepsilon}\right)\right) \right|, \\
&+ Ch \left| \left(1 + \varepsilon^{-2} \exp\left(\frac{-ax_i}{\varepsilon}\right)\right) \right|, \\
&\leq Ch^2 \left| \left(\varepsilon + \varepsilon^{-3} \exp\left(\frac{-ax_i}{\varepsilon}\right)\right) \right|, \\
&+ Ch \left| \left(1 + \varepsilon^{-2} \exp\left(\frac{-ax_i}{\varepsilon}\right)\right) \right|, \\
&\leq Ch \left(1 + \sup_{x \in (0,1)} \left(\frac{\exp(\frac{-ax_i}{\varepsilon})}{\varepsilon^3}\right)\right),
\end{aligned}
$$

since $\varepsilon^{-3} > \varepsilon^{-2}$. Most of the time during analysis, one encounters with exponential terms involving divided by the power function in ε which are always the main cause of worry. For their careful consideration while proving the ε -uniform convergence, we prove as follows.

Lemma 4.4.2 *For a fixed mesh and for $\varepsilon \to 0$, it holds*

$$
\lim_{\varepsilon \to 0} \max_{1 \leq i \leq N-1} \left(\frac{\exp(\frac{-ax_i}{\varepsilon})}{\varepsilon^m}\right) = 0, \quad m = 1, 2, 3, ...
$$

$$
\lim_{\varepsilon \to 0} \max_{1 \leq i \leq N-1} \left(\frac{\exp(\frac{-a(1-x_i)}{\varepsilon})}{\varepsilon^m}\right) = 0, \quad m = 1, 2, 3, ...
$$

where $x_i = ih, h = \frac{1}{N}, i = 1, 2, ..., N-1$.

Proof: Consider the partition $[0, 1] := \{0 = x_0 < x_1 < .... < x_{N-1} < x_N = 1\}$ for the interior grid points, we have

$$
\max_{1 \leq i \leq N-1} \frac{\exp\left(\frac{-ax_i}{\varepsilon}\right)}{\varepsilon^m} \leq \frac{\exp\left(\frac{-ax_1}{\varepsilon}\right)}{\varepsilon^m} = \frac{\exp\left(\frac{-ah}{\varepsilon}\right)}{\varepsilon^m},
$$

$$
\max_{1 \leq i \leq N-1} \frac{\exp\left(\frac{-a(1-x_i)}{\varepsilon}\right)}{\varepsilon^m} \leq \frac{\exp\left(\frac{-a(1-x_{N-1})}{\varepsilon}\right)}{\varepsilon^m} = \frac{\exp\left(\frac{-ah}{\varepsilon}\right)}{\varepsilon^m},
$$

as $x_1 = 1 - x_{N-1} = h$.

Then, by the application of L'Hospital's rule m times gives

$$\lim_{\varepsilon \longrightarrow 0} \frac{\exp\left(\frac{-ah}{\varepsilon}\right)}{\varepsilon^m} = \lim_{r=\frac{1}{\varepsilon} \longrightarrow \infty} \frac{r^m}{\exp(ahr)} = \lim_{r=\frac{1}{\varepsilon} \longrightarrow \infty} \frac{m!}{(ah)^m \exp(ahr)} = 0.$$

Hence, the proof is completed.

Theorem 4.2 *Under the hypothesis of boundness of discrete solution (i.e., it satisfies the discrete minimum principle), Lemma 4.4.2 and Theorem 4.1, the discrete solution satisfy the following bound.*

$$\sup_{0 \le \varepsilon \le 1} \max_i |y_i - Y_i| \le CN^{-1}. \tag{4.22}$$

Proof: Results from boundness of solution, Lemma 4.4.2 and Theorem 4.1 gives the required estimates. Hence the proof.

4.5 Numerical Examples and Results

In this section, one example is given to illustrate the numerical method discussed above. The considered problem contain large delay parameter on the reaction term and small delay parameter on the convection term. The solution of the problem exhibits interior layer due to the delay parameter and strong left boundary layer due to the small perturbation parameter ε (see Fig 4.1). Fig 4.2 shows, as the number of mesh point increases (as the mesh size decreases), the absolute error deceases which shows the convergence of the scheme and Fig 4.3 and Table 4.1 shows, the ε-uniform convergence of our scheme for $h \ge \varepsilon$ where the classical numerical method fails.

The exact solutions of the test problems are not known. Therefore, we use the double mesh principle to estimate the error and compute the experiment rate of convergence to the computed solution. For this we put

$$E_{\varepsilon}^{N} = \max_{0\le i\le 2N} |Y_i^N - Y_i^{2N}|, \tag{4.23}$$

where Y_i^N and Y_i^{2N} are the i^{th} components of the numerical solutions on meshes of N and $2N$ respectively. We compute the uniform error and the rate of convergence as

$$E^N = \max_{\varepsilon} E_{\varepsilon}^{N}, \text{and} R^N = \log_2\left(\frac{E^N}{E^{2N}}\right). \tag{4.24}$$

The numerical results are presented for the values of the perturbation parameter $\varepsilon \in \{\ 10^{-4}, 10^{-8}, ..., 10^{-20}\}$.

Example 4.5.1 *Consider the model singularly perturbed boundary value problem*

$$\varepsilon y''(x) + 4(x+1)y'(x) - 3y(x) + y(x-1) = x+1 \quad x \in (0,1)\cup(1,2),$$

subject to the boundary conditions

$$y(x) = 1, \quad x \in [-1,0], \quad y(2) = 1$$

Table 4.1: Maximum absolute errors for Example 4.5.1 at number of mesh points N

ε	N=16	N=32	N=64	N=128	N=256
2^{-2}	2.8610e-03	1.3216e-03	6.3113e-04	3.0778e-04	1.5189e-04
2^{-4}	4.1609e-03	1.7580e-03	7.5901e-04	3.4577e-04	1.6413e-04
2^{-6}	4.9381e-03	2.4245e-03	1.0777e-03	4.4913e-04	1.9263e-04
2^{-8}	4.9619e-03	2.5372e-03	1.2765e-03	6.1620e-04	2.7180e-04
2^{-10}	4.9619e-03	2.5374e-03	1.2829e-03	6.4497e-04	3.2178e-04
2^{-12}	4.9619e-03	2.5374e-03	1.2829e-03	6.4497e-04	3.2178e-04
2^{-14}	4.9619e-03	2.5374e-03	1.2829e-03	6.4497e-04	3.2178e-04
....					
2^{-28}	4.9619e-03	2.5374e-03	1.2829e-03	6.4497e-04	3.2178e-04
E^N	4.9619e-03	2.5374e-03	1.2829e-03	6.4497e-04	3.2178e-04
R^N	0.9828	0.9915	0.9957	0.9979	

Table 4.2: Comparision of maximum absolute errors and rate of convergence for Example 4.5.1 at number of mesh points.

ε	N=16	N=32	N=64	N=128	N=256
Present method					
E^N	4.9619e-03	2.5374e-03	1.2829e-03	6.4497e-04	3.2178e-04
R^N	0.9828	0.9915	0.9957	0.9979	
Rai and Sharma, (2020)					
E^N	1.15431e-02	6.43596e-03	3.42406e-03	1.75301e-03	8.83454e-04
R^N	0.84	0.91	0.97	0.99	

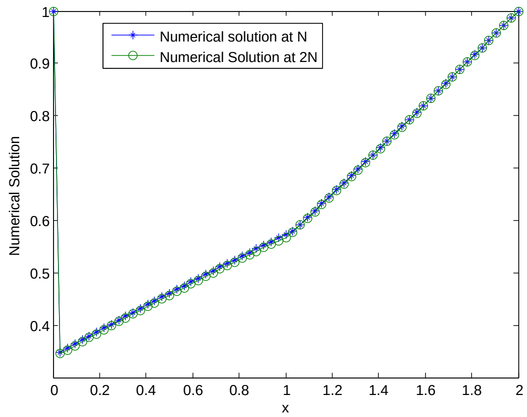

Figure 4.1: Numerical Solution for Example 4.5.1 at $\varepsilon = 10^{-8}$ and $N = 64$.

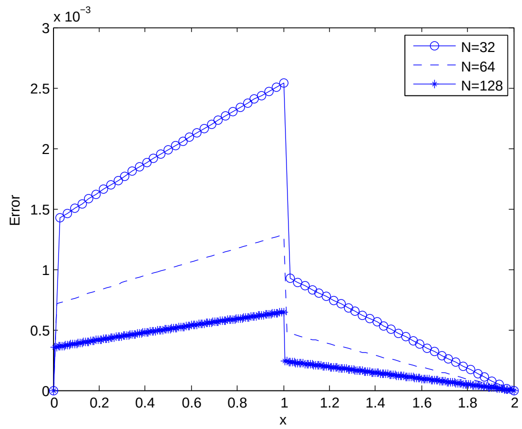

Figure 4.2: Point wise error for our Example 4.5.1 at diffrent mesh points.

4.6 Discussion and Conclusion

This study introduces non-standard finite difference numerical method for solving singularly perturbed differential equations having large delay. The behavior of the continuous solution of the problem is studied and shown that it satisfies the continuous stability

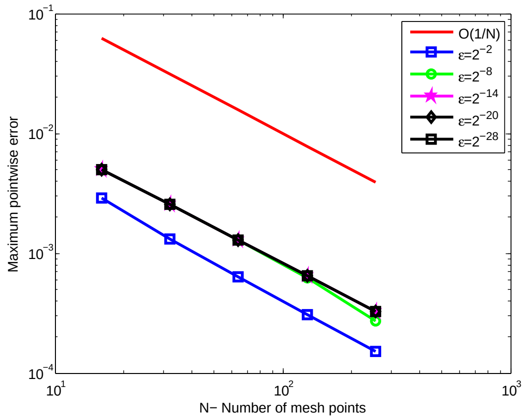

Figure 4.3: ε-uniform convergence of the method using log-log plot for Example 4.5.1.

estimate and the derivatives of the solution are also bounded. The numerical scheme is developed on uniform mesh using non-standard finite difference method in the given differential equation. The stability of the developed numerical method is established and its uniform convergence is proved. To validate the applicability of the method, one model problem is considered for numerical experimentation for different values of the perturbation parameter and mesh points. The numerical results are tabulated in terms of maximum absolute errors, numerical rate of convergence and uniform errors (see Table 4.1). Further, behavior of the numerical solution (Fig 4.1), point-wise absolute errors (Fig 4.2) and the ε-uniform convergence of the method is shown by the log-log plot (Fig 4.3). The method is shown to be ε-uniformly convergent with order of convergence $O(h)$. The proposed method gives more accurate, stable and ε-uniform numerical result.

Reference

Abrahamsson, L. R., Keller, H. B., and Kreiss, H. O. (1974). Difference approximations for singular perturbations of systems of ordinary differential equations. Numerische Mathematik, 22(5), 367-391.

Akram, G. and Afia, N. (2013a). (2013b). Solution of a fourth order singularly perturbed boundary value problem using quintic spline. Middle-east journal of specific research,15(2):, 302-311.

Andargie, A., and Reddy, Y. N. (2013). Parameter fitted scheme for singularly perturbed delay differential equations. International Journal of Applied Science and Engineering, 11(4), 361-373.

Bansal, K., and Sharma, K. K. (2017). Parameter uniform numerical scheme for time dependent singularly perturbed convection-diffusion-reaction problems with general shift arguments. Numerical Algorithms, 75(1), 113-145.

Burden, R. L. and Fairs, J.D.(2011). Numerical analysis, 9th edition, Young stown univeristy.

Chakravarthy P. P, Kumar S.D., Rao R. N. and Ghate D.P. . (2015). A fitted numerical scheme for second order singularly perturbed delay differntial equations via cubic spline in comparession,. Advances in Difference Equations, 1-14.

Chebyshev-gauss grid,and phaneendra, (2015). Numerical treatment of singularly perturbed delay differential equations, IOSR Journal of Mathematics (IOSRJM),1(2), 01-07.

Doolan, E.P.,Miller, J.J.H., Schilders, W.H.A. (1980). Uniform Numerical Methods for problems with initial and boundary layers, Boole Press,Dublin.

Elsgolt's L. E. and Norkin S. B. (1973). Introduction to the Theory and Applications of Differntial quations with Deviating Arguments, Academic Press New york,.

Friedrichs, K.O., and Wasow, W. . (1946). Singular Perturbations of Non-linear Oscillations,. Duke.Mathematical Journal,13, 367-381.

Gemechis File and Reddy Y.N. (2013). Computational method for solving singularly perturbed delay differential equations with negative shift,. International Journal of Applied Science and Engineering,11 (1), 101-113.

Gülsu M. and Öztürk Y. (2011). Approximate solution of the singular-perturbation problem on chebyshev-gauss grid, American Journal of Computational Mathematics, 1, 209-218.

Kadalbajoo M. K. and Sharma K.K. (2004). Numerical analysis of singularly perturbed delay differntial equations with layer behavior, Appl.Math.Comput.157, 11-28.

Kadalbajoo. K. and Ramesh V.P. (2007). Numerical methods on Shishkin mesh for singularly perturbed delay diffential equations with a grid adaptation strategy,. Applied Mathematics and computation,188, 1816-1831..

Khan,A and Khandelwal, P. (2013). Khan,A and Khandelwal, P. (2013). Non-polynomial sextic spline solution of singularly perturbed boundery value plems,rob . International Journal of computer Mathematics.

Kumar and Rao. (2020). stabilized central difference method for the boundary valueproblem of singularly perturbed differential equations with a large negative shift.

Lange C. G. and Miura R. M. (1994). Singular perturbation analysis of boundary-value problems for differential-difference equations very small shifts with layer behavior,. SIAMJ.Appl.Math.54, 249-272.

Mickens, R. E. (2005). Advances in the applications of nonstandard finite difference schemes. World Scientific.

Mickens . R.E. (2000). Applications of Nonstandard Finite Difference Schemes,. World Scintific,Singapore.

Pratima R. and SharmaK. K. . (2011). Numerical analysis of singularly perturbed delay differential turning point problem, . Applied Mathematics and Computation,218, 3483-3498.

Rai, P., & Sharma, K. K. (2020). Numerical approximation for a class of singularly perturbed delay differential equations with boundary and interior layer (s). Numerical

Algorithms, 85(1), 305-328.