An Iterative Algorithm for Split Equality Fixed Point Problem of Pseudo-pseudo Contractive Mappings in Hilbert Space.

A RESEARCH SUBMITTED TO THE DEPARTMENT OF MATHEMATICS IN PARTIAL FULFILLMENT FOR THE REQUIREMENTS OF THE DEGREE OF MASTERS OF SCIENCE IN MATHEMATICS

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Declaration

I,Teyib A/Oli A/Chebsa,with student ID number MR0196/14-0,the undersigned declare that,this thesis paper entitled that An Iterative Algorithm for Split Equality Fixed Point Problem of Peudo-Pseudo Contractive Mappings in Hilbert Spaces is my own original work and it has not been submitted to any institution or university elsewhere for the award of any academic degree or like. Where other sources of information have been used or quoted,they have been indicated and acknowledged.
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Acronym

Throughout this research, we denote the following.

- $\mathbb{N}$ is stands for the set of positive integers.
- $\mathbb{R}$ is stands forthe set of real numbers.
- $\mathbb{R}^+$ is stands for the set of non-negative real numbers.
- $\|.\|$ is the norm symbol.
- $\langle .,. \rangle$ is the inner symbol.
- SEP is split equality problem.
- VIP is variational inequality problem.
- $\nabla\ h$ is gradient of h and h is a function.
- ∇g is gradient of g and g is a function.
- SMVIP is split monotone variational inequality problem.
- SEFP is split equality feasibility problem.
- SEVIP is split equality variational inclusion problem.
- SEFPP is split equality fixed point problem.

Notation in this thesis

- H denotes Hilbert Space.
- H_1, H_2, H_3 denotes Real Hilbert Spaces.
- C and D denotes nonempty closed convex subset of real Hilbert Space .
- $F(T)$ denotes the set of all fixed point of a map T.
- $F(S)$ denotes the set of all fixed point of a map S.

Abstract

In this thesis we introduced an iterative algorithm for approximating a solution of split equality fixed point problems (SEFPP) of peudo-pseudo contractive mappings in Hilbert spaces and proved the strong convergence of a sequence generated by proposed algorithm to a solution of the problem in Hilbert spaces provided that the mappings are uniformly continuous. Finally,we applied our main results to solve convex minimization problem. Our results extended and generalized some related results in the literature.

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Chapter 1

Introduction

1.1 Background of the study

The theory of nonlinear functional analysis is quite rich and a fascinating area of research, in particular, the fixed point theory. The theory of fixed point can be classified into theories for the existence of fixed points and approximation of fixed points. It has proven to be at the heart of technological development and numerous applications ranging from engineering, computer science, mathematical sciences and social sciences. In particular, fixed point techniques have been applied in diversified fields, such as science, economics, and engineering. Consequently, many authors concentrated on providing iterative algorithms for approximation of fixed points of mappings when they exist or assuming existence (see, e.g., Mann (1953), Berinde (2007), Browder (1968), Khan (2008) and Krasnoselskii (1955)).
The well known method for approximating a fixed point of contraction mapping is the Picard iterations. However, this iteration method may not always converge to a fixed point of T, when T is non-expansive mapping.

So, for approximating fixed points of the classes of mappings are more general than the class of contraction mappings. Many iterative schemes, such as Mann iteration, Halpern Iteration, Ishikawa iteration, are introduced by different authors (see, e.g., Mann (1953), Halpern (1964), and Ishikawa (1974)).

Many authors have also constructed iterative algorithms called hybrid Mann and hybrid Ishikawa algorithms to obtain strong convergence of the sequence proposed by their method of converging a fixed point of Lipchitz pseudocontractive mappings (see, e.g., Liu et al. (2011), Marino et al. (2009)).

In 2020, Zegeye and Wega (2020), introduced a new class of mapping which is more general than the class of pseudocotractive mappings called pseudo-pseudocontractive and established an iterative algorithm which converges strongly to a fixed point of pseudo-pseudocontractive mapping provided that the mapping T is uniformly continuous which is sequentially weakly continuous.

We also remark that many authors have studied different algorithms for approximating solutions of split equality fixed point problem (SEFPP) for nonlinear mappings in the settings of Hilbert spaces (see, e.g., Moudafi (2014), Zhao (2014), Chang et al. (2015), Boikanyo. O.A (2019), Wega (2022)). Split equality fixed point problem was first introduced by Moudafi (2014) in 2014.The split equality fixed point problem can be mathematically formulated as the problem of finding x and y with the property:

$$x \in F(T), y \in F(S) \text{ such that } \bar{A}x = \bar{B}y. \tag{1.1}$$

where $\bar{A} : H_1 \to H_2$ and $\bar{B} : H_2 \to H_3$ are bounded linear mappings and F(T) and F(S) denote the sets of fixed points of non-expansive mappings T and S, respectively.

In 2015, Zhao (2015) introduced an iterative process for the class of quasi-nonexpansive mappings and proved that under certain assumptions, the algorithm aforementioned converges weakly to a solution of the split equality fixed point problem without prior knowledge of norms. In 2015, Chang et al. (2015) introduced an iterative process for the class of quasi-pseudo contractive mappings and proved that under certain assumptions, the algorithm aforementioned converges weakly to a solution of the split equality fixed point problem.
Motivated and Inspired by the above works the purpose of this thesis is to introduce a new iterative algorithm for approximating a solution of split equality fixed point problem of pseudo-pseudo contractive mappings in Hilbert Spaces.

Moreover,we give an application to the convex minimization problems.Our results extend and generalize many results in the literature.

Now, we recall some definitions that we used in the sequel.
Definition 1: A mapping $T : H \to H$ is said to be :

i) $L-$Lischitz mapping if and only if there exists $L > 0$ such that

$$||Tx - Ty|| \leq L||x - y||, forall x, y \in H. \tag{1.2}$$

ii) nonexpansive if and only if

$$||Tx - Ty|| \leq ||x - y||, forall x, y \in H. \tag{1.3}$$

iii) pseudocontractive mapping if and only if

$$\langle Tx - Ty, x - y \rangle \leq ||x - y||^2, forall x, y \in H. \tag{1.4}$$

iv) Directed if $Fix(T) \neq \emptyset$ and $\langle Tx - x^*, Tx - x \rangle \leq 0$,forall $x \in H$ and $x^* \in Fix(T)$

v) Demicontractive if $Fix(T) \neq \emptyset$ and there exists $k \in [0, 1)$ such that

$$||Tx - x^*|| \leq ||x - x^*||^2 + k||Tx - x||^2, forall x \in H \text{ and } x^* \in Fix(T). \tag{1.5}$$

vi) Quasi-pseudo contractive if $Fix(T) \neq \emptyset$ and

$$||Tx - x^*|| \leq ||x - x^*||^2 + ||Tx - x||^2, forall x \in H \text{ and } x^* \in Fix(T). \tag{1.6}$$

vii) pseudo-pseudocontractive mapping if and only if.

$$\langle x - Tx, y - x \rangle \geq 0 \Rightarrow \langle y - Ty, y - x \rangle \geq 0, forall$$

x,y$\in H$. (1.7)

viii) A point $x^* \in H$ is said to be a fixed point of T if $Tx^* = x^*$.

1.2 Statements of the Problem

In past years, some iterative methods have been proposed to solve the split equality fixed point problems in Hilbert Spaces for different classes of mappings.

Recently, Boikanyo, (2019) introduced an iterative method for the class of quasi-pseudo-contractive mappings that converge strongly to some solution of the split equality fixed point problem.

Very recently,Wega, (2022) established an iterative algorithm for approximating a solution of SEFPP of pseudocontractive mappings and proved strong convergence of the sequence generated by the proposed scheme to a solution of the problem in the settings of Hilbert spaces. However, a scheme which converges strongly to a solution of split equality fixed point problem of pseudo-pseudocontractive mappings is not yet studied in real Hilbert spaces.
Motivated and inspired by the researcher works of Boikanyo,(2019) and Wega,(2022),now in this thesis the researcher established a new iterative algorithm for approximating a solution of split equality fixed point problem of pseudo-pseudo contractive mappings in Hilbert Spaces.

1.3 Objectives of the Study

1.3.1 General Objective

The general objective of this thesis was to study an iterative algorithm for split equality fixed point problem of pseudo-pseudocontractive mappings in Hilbert Spaces.

1.3.2 Specific Objectives

The specific objectives of this thesis is to:

- investigate an iterative algorithm for approximating a solution of split equality fixed point problem of pseudo-pseudocontractive mappings in Hilbert spaces.
- prove the sequence generated by the proposed algorithm is bounded in Hilbert Spaces.
- prove the sequence generated by the proposed algorithm converges strongly to a solution of split equality fixed point problem.
- apply our main result to find a solution of split equality minimum point problem for a convex function in Hilbert Spaces.

1.4 Significance of the Study

The study may have the following importance:

- It may give basic research skills to the researcher.
- It may be applied to solve some rea world problem (optimation problem).
- It may provide some background information for other researchers who want to conduct a research on related topics.

1.5 Delimitation of the Study

This study was delimited to study iterative algorithm for split equality fixed point problem of Pseudo-pseudocontractive mappings in Hilbert spaces.

Chapter 2

Review of Related Literatures

Let C and D be nonempty closed convex subsets of real Banach spaces H_1 and H_2 respectively. The split feasibility problem (SFP) was introduced by Censor and Elfving (1994) and is formulated as to finding

$$z \in C \text{ such that } \bar{A}x \in D, \tag{2.1}$$

where $\bar{A} : H_1 \to H_2$ is a bounded linear operator. Such models were successfully developed for resolution enhancement, instance in radiation therapy treatment planning, sensor networks, and etc (see, e.g., Chedume CE. and Ido K.(2016),Zegeye and Wega(2020),Bortifeld et al.(2006) and references therein). Motivated by the this problem, several authors have been studied and investigated several split type problems (see, e.g.,Zegeye H.(2008),Su Y and Xu.HK.(2012),Zhao J.(2014)). In 2012, Censor et al (2010) established the following split variational inequality problem (SVIP) which can mathematically be formulated as the problem of finding:

$$z \in C \text{ such that } \langle A_1 z, x - z\rangle \forall x \in C, \tag{2.2}$$

and such that

$$y = \bar{A}z \in D \text{ solves } \langle A_2 z, x - z\rangle, \forall x \in D. \tag{2.3}$$

Where $A_1 : H_1 \to H_1$ and $A_2 : H_2 \to H_2$ are two mappings and $\bar{A} : H_1 :\to H_2$ is bounded linear mappings.

In 2011, Moudafi (2011) introduced the split monotone variational inclusion problem (SMVIP) given by

$$z \in H \text{ such that } 0 \in T_1 z + G_1 z \tag{2.4}$$

and such that

$$y = \bar{A}z \in H \text{ solves } o \in T_2 y + G_2 y. \tag{2.5}$$

Where $\bar{A} : E_1 :\to H_2$ is bounded linear mappings and
$G_1 : H_1 \to 2^{H_1*}$ and $G_2 : H_2 \to 2^{H_2*}$

are multi-valued maximal monotone mappings and $T_1 : H_1 \to H_1$ and $T_2 : H_2 \to H_2$ are two single valued and -inverse strongly monotone mappings and he proved that the sequence generated by his algorithm converges weakly to a solution of problem (2.6) in Hilbert spaces. Note that by taking $G_1 = N_C$ and $G_2 = N_D$ normal cones of the closed convex sets $C \subseteq H_1$ and $D \subseteq H_2$in problem (2.6) we recover the split variational inequality problem (2.5) introduced by Censor et al (2010).
In 2013, Moudafi (2013) introduced the following split equality problem which can be formulated as finding:

$$x \in C, y \in D \text{ such that } \bar{A}x = \bar{B}y. \tag{2.6}$$

Where $C \subseteq H_1$ and $D \subseteq H_2$ are nonempty, closed and convex sets and $\bar{A} : H_1 \to H_2$ and $\bar{B} : H_2 \to H_3$ are bounded linear mappings. Obviously, $\bar{B} = I$ and $H_2 = H_3$, then problem (2.7)reduces to problem (2.5).
Moreover, in 2014, Moudafi (2014) established the following split equality fixed point problem (SEFPP) which can be formulated as finding:

$$x \in F(T), y \in F(S) \text{ such that } \bar{A}x = \bar{B}y, \tag{2.7}$$

where $\bar{A} : H_1 \to H_3$ and $\bar{B} : H_2 \to H_3$ are bounded linear mappings and $T : H_1 \to H_2$ and $S : H_2 \to H_2$ are two nonlinear operators such that $F(T) \neq \emptyset$ and $F(S) \neq \emptyset$ this problem recently studied by several authors (see, e.g., Boikanyo (2019), Wega (2022)).
Recently, Boikanyo (2019) introduced an iterative method for the class of quasi-pseudo-contractive mappings that always converge strongly to some solution of the split equality fixed point problem.

More recently, Wega (2022) established an iterative scheme for approximating a solution of SEFPP of pseudocontractive mappings and proved strong convergence of the sequence generated by the proposed scheme to a solution of the problem in the settings of Hilbert spaces.

2.1 Preliminaries

In this section we recall some known results which are used in our subsequent analysis.

For $x \in H$, the projection mapping $P_c : H \to C$ is defined by

$$||P_c x - x|| = inf_{y \in c} ||x - y||, \tag{2.8}$$

and hence, P_c satisfies: $||P_c x - P_c y||^2 \leq \langle P_c x - P_c y, x - y \rangle$, for all $x, y \in H$.

Definition 2.1.1 *A mapping $T : C :\to H$ is called sequentially weakly continuous if for each sequence $\{x_n\}$, we have $\{x_n\}$ converges weakly to p implies $\{Tx_n\}$ converges to Tp.*

Lemma 2.1.1 *For all $x, y \in H$, the following inequalities hold.*

i) $2\langle x, y \rangle = ||x||^2 + ||y||^2 - ||x - y||^2$.

ii) $||x + y||^2 \leq ||x||^2 + 2\langle y, x + y \rangle$.

Lemma 2.1.2 *(Albert,1996). let $x \in H$, then*

$$P_c x \in C \textit{ if and only if } \langle y - P_c x, x - P_c x \rangle \leq 0, \textit{ for every } y \in C. \tag{2.9}$$

This result implies that for all $x \in H$

$$||P_C x - z||^2 \leq ||x - z||^2 - ||x - P_c x||^2 z \in C. \tag{2.10}$$

Lemma 2.1.3 *(Mainge, 2008). Let $\{a_k\}$ be a sequence of real numbers that does not decrease at infinity, in the sense that there exists a subsequence $\{a_{k_j}\}$ of $\{a_k\}$ such that $a_{k_j} < a_{k_j+1}$ for all $j \geq 0$. Define an integer sequence $\{m_k\}_{k \geq k_0}$ as*

$$m_k = \max\{k_0 \leq l \leq k : a_l < a_{l+1}\}.$$

Then, $m_k \to \infty$ as $k \to \infty$ and for all $k \geq k_0$

$$\max\{a_{m_k}, a_k\} \leq a_{m_k+1}.$$

Lemma 2.1.4 *(Xu, 2002).Let* $\{\alpha_n\}$ *be a sequence of nonnegative real numbers satisfying the following relation:*

$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n\gamma_n$ *for* $n \geq n_0$ *where* $\{\alpha_n\} \subseteq (0,1)$ *and* $\{\gamma_n\} \subseteq R$, *satisfies*

$$\sum_{n=1}^{\infty} \alpha_n = \infty, \text{ and } \limsup_{n\to\infty} \gamma_n \leq 0. \text{ Then } \lim_{n\to\infty} a_n = 0.$$

Lemma 2.1.5 *(Osilike and Igbokwe, 2000).Let* H *be a real Hilbert space, for all* $x_i \in H$ *and* $\alpha_i \in [0,1]$ *for* $i = 1,2,3,...n$, *such that* $\alpha_1 + \alpha_2 + \alpha_3 + ... + \alpha_n = 1$, *the following holds:*
$\|\alpha_0 x_0 + \alpha_1 x_1 + ... + \alpha_n x_n\|^2 = \sum_{i=0}^{n} \alpha_i \|x_i\|^2 - \sum_{0\leq i,j\leq n} \alpha_i\alpha_j \|x_i - x_j\|^2.$

Lemma 2.1.6 *(He,2006).Let* C*, be a nonempty, closed and convex subset of* H. *Let* $r(x)$*, be a real valued function on* H *and defined* $K := \{x \in C : r(x) \leq 0\}$. *If* K*, is nonempty and* r *is L-Lipshitz continuous with* $L > 0$, *then*
$||P_K x - x|| \geq \frac{1}{L}\max\{r(x), 0\}$, *for* $x \in C$.

Chapter 3

Research Design and Methodology

In this section,the study area and period, mathematical study design, source of information and mathematical procedures will be presented.

3.1 Study Area and Period

The study was conducted at Jimma University under the department of mathematics from September, 2022 G.C. to June, 2023 G.C.

3.2 Study Design

In order to achieve the objectives of this study employed analytical methods of designing.

3.3 Sources of Information

The relevant sources of information for this study were published articles, different mathematics books which are related to our research topic.

3.4 Mathematical Procedures

In this study we followed the procedures stated below:

- Establishing an iterative algorithm and constructing theorem for approximating a solution of split equality fixed point problem of pseudo-pseudocontractive mappings in Hilbert spaces.

- Proving boundedness of the sequence generated by the proposed algorithm in Hilbert Spaces.

- Proving strong convergence of the sequence generated by proposed method to a split equality fixed point problem of pseudo-pseudocontractive mappings.

- Applying our main result to solve minimization problems.

Chapter 4

Main Result and Discussion

In this section, we shall make use of the following assumptions:

Assumption 1:

A_1: Let $T : H_1 \to H_1$ and $S : H_2 \to H_2$ be uniformly continuous Pseudo-pseudo contractive mappings on bounded subset of H_1 and H_2, respectively.

A_2 : Let $\Omega := \{(x,y) \in H_1 \times H_2 : x \in F(T), y \in F(S)$ and $Ax = By\} \neq 0$ where $A : H_1 \to H_3$ and $B : H_2 \to H_3$ are bounded linear mappings with adjoints A^* and B^*, respectively. where H_3 is another real Hilbert space.

A_3 :Let $\iota \in (0,1)$, $\mu > 0$ and $\delta \in [\delta_-, \bar{\delta}] \subset (0, \frac{1}{\mu})$.

A_4 : Let $\{\alpha_n\} \subset (0, \varepsilon)$ for some constant real number $\varepsilon > 0$ be real sequence such that $\lim_{n\to\infty} \alpha_n = 0$, and $\sum_{n=1}^{\infty} \alpha_n = \infty$.

A_5 : Let $g_1 : H_1 \times H_1 \to H_1 \times H_1$ and $g_2 : H_2 \times H_2 \to H_2 \times H_2$ be contraction mappings with constants $\alpha_1, \alpha_2 \in (0, \frac{1}{\sqrt{2}})$, respectively and we denote $\alpha = max\{\alpha_1, \alpha_2\}$.

A_6 : Let the sequence $\{\gamma_n\}$ satisfies

$0 \leq \gamma_n \leq \frac{||(Ax_n - Bt_n)||^2}{||A^*(Ax_n - Bt_n)||^2 + ||B^*(Bt_n - Ax_n)||^2}$,for $n \in \gamma$

otherwise,γ_n=$\gamma > 0$, such that the indexes $\gamma = \{n \in N : Ax_n - Bt_n \neq 0\}$.

4.1 Algorithm:1

For arbitrary $(x_0, t_0) \in H_1 \times H_2$, define an iterative algorithm by

Step 1. Compute

$$\begin{cases} z_n = (1-\delta)x_n + \delta T x_n \text{ and } d(x_n) = x_n - z_n, \\ u_n = (1-\delta)t_n + \delta S t_n \text{ and } d'(t_n) = t_n - u_n. \end{cases} \tag{4.1}$$

Step 2. Compute

$$\begin{cases} y_n = x_n - \Upsilon_n d(x_n), \\ v_n = t_n - \Upsilon'_n d'(t_n), \end{cases} \tag{4.2}$$

where, $\Upsilon_n = \iota^{j_n}$ such that j_n is the smallest nonnegative integer j satisfying

$$\langle \iota^j(d(x_n)) + T(x_n - \iota^j d(x_n)) - Tx_n, d(x_n)\rangle \leq \mu\|d(x_n)\|^2,$$

and $\Upsilon_n' = \iota^{j_n'}$ such that j_n' is the smallest nonnegative integer j' satisfying

$$\langle \iota^{j'} d'(t_n) + S(t_n - \iota^{j'} d'(t_n)) - St_n, d'(t_n)\rangle \leq \mu\|d'(t_n)\|^2.$$

Step 3. Compute

$$\begin{cases} a_n = P_C(x_n - \gamma_n A^*(Ax_n - Bt_n)), \\ b_n = P_D(t_n - \gamma_n B^*(Bt_n - Ax_n)), \\ w_n = \theta_n a_n + (1-\theta_n)p_n \\ r_n = \theta_n b_n + (1-\theta_n)q_n, \end{cases} \tag{4.3}$$

where $C_n = \{x \in H : \langle y_n - Ty_n, x - y_n\rangle \leq 0\}$,
$D_n = \{x \in H : \langle v_n - Sv_n, x - v_n\rangle \leq 0\}$ and $\{\theta_n\} \subset [\rho, 1)$ for $\rho > 0$ such that $p_n = P_{C_n}x_n$ and $q_n = P_{D_n}x_n$.

Step 4. Compute

$$\begin{cases} x_{n+1} = \alpha_n g_1(x_n) + (1-\alpha_n)w_n, \\ t_{n+1} = \alpha_n g_2(t_n) + (1-\alpha_n)r_n. \end{cases} \tag{4.4}$$

Step 5. Set $n := n+1$ and go to **Step 1**.

Lemma 4.1.1 *Suppose that the assumption $A_1 - A_2$ hold, and $\{x_n\}, \{y_n\}, \{z_n\}$, $\{u_n\}, \{v_n\}$ are sequences, generated by Algorithm 1.Then the search rules in step two are well defined.*

Proof: Since $\iota \in (0,1)$, T and S are uniformly continuous mappings on H, We have

$$\langle \iota^j(d(x_n)) + T(x_n - \iota^j d(x_n)) - Tx_n, d(x_n)\rangle \to 0 \text{ as } j \to \infty,$$

and

$$\langle \iota^{j'}(d'(x_n)) + S(x_n - \iota^{j'} d'(x_n)) - Sx_n, d'(x_n)\rangle \to 0 \text{ as } j' \to \infty.$$

Moreover, since $\|d(x_n)\| > 0$ and $\|d'(x_n)\| > 0$ there exist a non-negative integers j_n and j'_n , satisfying the inequalities in Step 2. □

Lemma 4.1.2 *Suppose that the assumption $A_1 - A_3$ hold. If $\{x_n\}, \{y_n\}, \{z_n\}, \{u_n\}, \{v_n\}$ are sequences generated by Algorithm 1 then,*

$$\langle x_n - Tx_n, d(x_n)\rangle = \frac{1}{\delta}\|d(x_n)\|^2.$$

and

$$\langle t_n - St_n, d'(t_n)\rangle = \frac{1}{\delta}\|d'(t_n)\|^2. \tag{4.5}$$

Proof: From equations (4.1), we have, $z_n = (1-\delta)x_n + \delta Tx_n$ which gives us, $z_n - x_n = \delta(Tx_n - x_n)$ and hence

$$\frac{z_n - x_n}{\delta} = Tx_n - x_n \tag{4.6}$$

Thus, from the fact that $d(x_n) = x_n - z_n$, we get

$$\begin{aligned} \langle x_n - Tx_n, d(x_n)\rangle &= \langle \frac{x_n - z_n}{\delta}, x_n - x_n\rangle \\ &= \frac{1}{\delta}\langle x_n - z_n, x_n - z_n\rangle \\ &= \frac{1}{\delta}\|x_n - z_n\|^2 \end{aligned} \tag{4.7}$$

Similarly, we get

$$\langle t_n - St_n, d'(t_n)\rangle = \frac{1}{\delta}\|t_n - u_n\|^2. \tag{4.8}$$

□

Lemma 4.1.3 *Suppose the assumptions $A_1 - A_3$ holds. Let $(p,q) \in \Omega$, let $h_n(x_n) = \langle y_n - Ty_n, x_n - y_n\rangle$, and let $g_n(t_n) = \langle y = v_n - Sv_n, t_n - v_n\rangle$. Then, $h_n(p) \leq 0$, $g_n(q) \leq 0$, $h_n(x_n) \geq \Upsilon_n(\frac{1}{\delta} - \mu)\|d(x_n)\|^2$, and $g_n(t_n) \geq \Upsilon'_n(\frac{1}{\delta} - \mu)\|d'(t_n)\|^2$. In particular, if $d(x_n) \neq 0$ and $d'(t_n) \neq 0$, then $h(x_n) > 0$ and $g(t_n) > 0$.*

Proof: For the fact that $(p,q) \in \Omega$, we have

$$\langle p - Tp, y_n - p\rangle \geq 0. \tag{4.9}$$

This inequality and the fact that T is pseudo-pseudocontractive mapping, we obtain

$$h_n(p) = \langle y_n - Ty_n, y_n - p\rangle \geq 0,$$

which gives us,

$$h_n(p) = \langle y_n - Ty_n, p - y_n\rangle \leq 0.$$

In addition, from Step 2, of Algorithm 1, we have,

$$\begin{array}{rcl} h_n(x_n) & = & \langle y_n - Ty_n, x_n - y_n\rangle \\ & = & \langle y_n - Ty_n, x_n - (x_n - \Upsilon_n d(x_n)\rangle \\ & = & \Upsilon_n \langle y_n - Ty_n, d(x_n)\rangle. \end{array}$$

Furthermore, from the inequalities in Step 2, we have,

$$\langle x_n - y_n + Ty_n - Tx_n, d(x_n)\rangle \leq \mu \|d(x_n)\|^2,$$

which implies

$$\langle y_n - Ty_n, d(x_n)\rangle \geq \langle x_n - Tx_n, d(x_n)\rangle - \mu \|d(x_n)\|^2. \tag{4.10}$$

From Lemma 4.0.2 and inequality above, we obtain

$$\langle y_n - Ty_n, d(x_n)\rangle \geq (\frac{1}{\delta} - \mu)\|d(x_n)\|^2 \tag{4.11}$$

By combining (4.10) and (4.11), we obtain,

$$h_n(x_n) \geq \Upsilon_n(\frac{1}{\delta} - \mu)\|d(x_n)\|^2.$$

Similarly, we obtain,

$$g_n(t_n) \geq \Upsilon'_n(\frac{1}{\delta} - \mu)\|d'(t_n)\|^2.$$

□

Theorem 4.1.4 *Suppose the assumptions* $A_1 - A_4$ *hold. Then, the sequence* $\{(x_n, t_n)\}$*, generated by the Algorithm* 1 *is bounded in Hilbert space,* $H_1 \times H_2$*.*

Proof: Let $(p,q) \in \Omega$ from Lemma 2.1.5 and and (2.10), we get

$$\begin{aligned}
\|w_n - p\|^2 &= \|\theta_n a_n + (1-\theta_n)p_n - p\|^2 \\
&= \|\theta_n(a_n - p) + (1-\theta_n)(P_{C_n}x_n - p)\|^2 \\
&\leq \theta_n\|a_n - p\|^2 + (1-\theta_n)\|P_{C_n}x_n - p\|^2 \\
&\leq \theta_n\|a_n - p\|^2 + (1-\theta_n)[\|x_n - p\|^2 - ||x_n - P_{C_n}x_n||^2]. \quad (4.12)
\end{aligned}$$

Similarly, we obtain

$$\|r_n - q\|^2 \leq \theta_n\|b_n - q\|^2 + (1-\theta_n)[\|t_n - q\|^2 - ||t_n - P_{D_n}t_n||^2]. \quad (4.13)$$

Thus, by adding inequalities (4.12) and (4.13), we get

$$\begin{aligned}
\|w_n - p\|^2 + \|r_n - q\|^2 &\leq \theta_n[\|a_n - p\|^2 + \|b_n - q\|^2] \\
&\quad +(1-\theta_n)[\|x_n - p\|^2 - ||x_n - P_{C_n}x_n||^2] \\
&\quad +(1-\theta_n)[\|t_n - q\|^2 - ||t_n - P_{D_n}t_n||^2]. \quad (4.14)
\end{aligned}$$

In addition from (4.3) and (2.10), we obtain

$$\begin{aligned}
\|a_n - p\|^2 &= ||P_C(x_n - \gamma_n A^*(Ax_n - Bt_n) - p||^2 \\
&\leq ||x_n - \gamma_n A^*(Ax_n - Bt_n) - p||^2 - ||a_n - (x_n - \gamma_n A^*(Ax_n - Bt_n)||^2 \\
&\leq ||x_n - p||^2 + \gamma_n^2||A^*(Ax_n - Bt_n)||^2 - \gamma_n||Ax_n - Bt_n||^2 \\
&\quad -||x_n - a_n - \gamma_n A^*(Ax_n - Bt_n)||^2. \quad (4.15)
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
\|b_n - q\|^2 &\leq ||t_n - q||^2 + \gamma_n^2||B^*(Bt_n - Ax_n)||^2 - \gamma_n||Ax_n - Bt_n||^2 \\
&\quad -||t_n - b_n - \gamma_n B^*(Bt_n - Ax_n)||^2. \quad (4.16)
\end{aligned}$$

By adding inequalities (4.15) and (4.16), we get

$$\begin{aligned}\|a_n-p\|^2+\|b_n-q\|^2 \leq\ & ||x_n-p||^2+||t_n-q||^2 \\ & +\gamma_n^2[||A^*(Ax_n-Bt_n)||^2+||B^*(Bt_n-Ax_n)||^2] \\ & -2\gamma_n||Ax_n-Bt_n||^2-||x_n-a_n-\gamma_nA^*(Ax_n-Bt_n)||^2 \\ & -||t_n-b_n-\gamma_nB^*(Bt_n-Ax_n)||^2. \qquad (4.17)\end{aligned}$$

Moreover, from (4.17) and (A6), we obtain

$$\begin{aligned}\|a_n-p\|^2+\|b_n-q\|^2 \leq\ & ||x_n-p||^2+||t_n-q||^2-\gamma_n||Ax_n-Bt_n||^2 \\ & -||x_n-a_n-\gamma_nA^*(Ax_n-Bt_n)||^2 \\ & -||t_n-b_n-\gamma_nB^*(Bt_n-Ax_n)||^2. \qquad (4.18)\end{aligned}$$

Now, by substituting (4.18) in (4.14), we get

$$\begin{aligned}\|w_n-p\|^2+\|r_n-q\|^2 \leq\ & \|x_n-p\|^2+\|t_n-q\|^2 \\ & -(1-\theta_n)[||P_{C_{1,n}}x_n-x_n||^2+||P_{D_{1,n}}t_n-t_n||^2] \\ & -\theta_n\gamma_n||Ax_n-Bt_n||^2-\theta_n||x_n-a_n-\gamma_nA^*(Ax_n-Bt_n)||^2 \\ & -\theta_n||t_n-b_n-\gamma_nB^*(Bt_n-Ax_n)||^2. \qquad (4.19)\end{aligned}$$

From (4.19), Lemma 2.1.5 and (A5), we get

$$\begin{aligned}
||x_{n+1}-p||^2+||t_{n+1}-q||^2 &= ||\alpha_n g_1(x_n)+(1-\alpha_n)w_n-p||^2 \\
&\quad +||\alpha_n g_2(t_n)+(1-\alpha_n)r_n-q||^2 \\
&\leq \alpha_n||g_1(x_n)-p||^2+(1-\alpha_n)||w_n-p||^2 \\
&\quad +\alpha_n||g_2(t_n)-q||^2+(1-\alpha_n)|r_n-p||^2 \\
&\leq \alpha_n(||g_1(x_n)-g_1(p)||+||g_1(p)-p||)^2+(1-\alpha_n)||x_n-p||^2 \\
&\quad +\alpha_n(||g_2(t_n)-g_2(q)||+||g_2(q)-q||)^2+(1-\alpha_n)||t_n-q||^2 \\
&\leq \alpha_n[\alpha^2||x_n-p||^2+||g_1(x_n)-p||^2]+(1-\alpha_n)||x_n-p||^2 \\
&\quad +\alpha_n[\alpha^2||t_n-q||^2+||g_2(t_n)-q||^2]+(1-\alpha_n)||t_n-q||^2 \\
&\leq 2\alpha_n(\alpha^2||x_n-p||^2+||g_1(p)-p||^2)+(1-\alpha_n)||x_n-p||^2 \\
&\quad +2\alpha_n(\alpha^2||t_n-q||^2+||g_2(q)-q||^2) \\
&\quad +(1-\alpha_n)||t_n-q||^2. \qquad (4.20)
\end{aligned}$$

By setting $R_n(p,q)=||x_n-p||^2+||t_n-q||^2$, from inequality (4.20), we get

$$\begin{aligned}
R_{n+1}(p,q) &\leq (1-\alpha_n(1-2\alpha^2))R_n(p,q) \\
&\quad +2\alpha_n(||g_1(p)-p||^2+||g_2(q)-q||^2) \\
&\leq \max\left\{R_n(p,q),\frac{2}{1-2\alpha^2}(||g_1(p)-p||^2+||g_2(q)-q||^2)\right\},
\end{aligned}$$

and hence by induction

$$R_n(p,q)\leq \max\left\{R_0(p,q),\frac{2(||g_1(p)-p||^2+||g_2(q)-q||^2)}{1-2\alpha^2}\right\},$$

which implies that $\{x_n\}$, $\{t_n\}$ and hence $\{y_n\}$, $\{v_n\}$, $\{Tx_n\}$ and $\{Sv_n\}$ are bounded.
□

Theorem 4.1.5 *Suppose the assumption $(A1)-(A6)$ hold. Then, the sequence $\{(x_n,t_n)\}$, generated by Algorithm 1 converges strongly to $(p,q)=P_\Omega(g_1(p),g_2(q))$.*

Proof: Now, let $(p,q)=P_\Omega(g_1(p),g_2(q))$. From equation (2.8), we have,

$$\|p-p_n\|^2\leq\|p-x_n\|^2-\|x_n-p_n\|^2.$$

Similarly, we get

$$\|q-q_n\|^2 \leq \|q-t_n\|^2 - \|t_n-q_n\|^2. \tag{4.21}$$

Since T is bounded on bounded subset of H. then their exists $L>0$, such that

$$\|Ty_n - y_n\| \leq L,$$

for all $n \geq 0$. Thus,

$$\begin{aligned}|h_n(z)-h_n(w)| &= |\langle y_n - Ty_n, z-y_n\rangle - \langle y_n - Ty_n, w-y_n\rangle| \\ &= |\langle y_n - Ty_n, z-w\rangle| \\ &\leq \|y_n - Ty_n\|\|z-w\| \\ &\leq L\|z-w\|,\end{aligned}$$

which gives us that h_n is L- Lipschitz continuous on H. Thus, from Lemma 2.1.6 and Lemma 4.1.3, we obtain

$$\|x_n-p_n\|^2 \geq \frac{h_n x_n}{2L^2} \geq \Upsilon_n^2(\frac{1}{\delta}-\mu)^2\|d(x_n)\|^4. \tag{4.22}$$

Thus, from (4.21) and (4.22), we get

$$\|p-p_n\|^2 \leq \|p-x_n\|^2 - \Upsilon_n^2(\frac{1}{\delta}-\mu)^2\|d(x_n)\|^4. \tag{4.23}$$

Similarly, we get

$$\|q-q_n\|^2 \leq \|q-t_n\|^2 - \Upsilon_n'^2(\frac{1}{\delta}-\mu)^2\|d'(t_n)\|^4. \tag{4.24}$$

Now, from Lemma 2.1.5, ((4.23)), (4.19) and (4.15), we get

$$\begin{aligned}\|w_n-p\|^2+\|r_n-q\|^2 \;&=\; \|\theta_n a_n+(1-\theta_n)p_n-p\|^2+\|\theta_n b_n+(1-\theta_n)q_n-q\|^2 \\ &\leq\; \theta_n\|a_n-p\|^2+(1-\theta_n)\|p_n-p\|^2+\theta_n\|b_n-q\|^2+(1-\theta_n)\|q_n-q\|^2 \\ &\leq\; \theta_n\|x_n-p\|^2+(1-\theta_n)\|x_n-p\|^2+\theta_n\|t_n-q\|^2+(1-\theta_n)\|t_n-q\|^2 \\ &\quad -\Big(\Upsilon_n^2(\frac{1}{\delta}-\mu)^2\|d(x_n)\|^4+\Upsilon_n'^2(\frac{1}{\delta}-\mu)^2\|d'(t_n)\|^4\Big) \\ &\leq\; \|x_n-p\|^2+\|t_n-q\|^2 \\ &\quad -\Big(\Upsilon_n^2(\frac{1}{\delta}-\mu)^2\|d(x_n)\|^4+\Upsilon_n'^2(\frac{1}{\delta}-\mu)^2\|d'(t_n)\|^4\Big). \end{aligned} \tag{4.25}$$

By Lemma 2.1.1, Lemma 2.1.5 and (4.25), we obtain

$$\begin{aligned}
R_{n+1}(p,q) &= \|\alpha_n g_1(x_n)+(1-\alpha_n)w(n)-p\|^2+\|\alpha_n g_2(t_n)+(1-\alpha_n)r(n)-q\|^2 \\
&\leq \|\alpha_n(g_1(x_n)-g_1(p))+(1-\alpha_n)(w_n-p)+\alpha_n(g_1(p)-p)\|^2 \\
&\quad +\|\alpha_n(g_2(t_n)-g_2(q))+(1-\alpha_n)(r_n-q)+\alpha_n(g_2(q)-q)\|^2 \\
&\quad +2\alpha_n[\langle g_1(p)-p,x_{n+1}-p\rangle+\langle g_2(q)-q,t_{n+1}-q\rangle] \\
&\leq \alpha\alpha_n\|x_n-p\|^2+(1-\alpha_n)\|w_n-p\|^2+\alpha\alpha_n\|t_n-q\|^2+(1-\alpha_n)\|r_n-q\|^2 \\
&\quad +2\alpha_n\|g_1(x_n)-p\|\|x_{n+1}-x_n\|+2\alpha_n\|g_2(t_n)-q\|\|t_{n+1}-t_n\| \\
&\quad +2\alpha_n[\langle g_1(p)-p,x_n-p\rangle+\langle g_2(q)-q,t_n-q\rangle] \\
&\leq \big(1-(1-\alpha)\alpha_n\big)R_n(p,q) \\
&\quad +2\alpha_n\|g_1(x_n)-p\|\|x_{n+1}-x_n\|+2\alpha_n\|g_2(t_n)-q\|\|t_{n+1}-t_n\| \\
&\quad +2\alpha_n[\langle g_1(p)-p,x_n-p\rangle+\langle g_2(q)-q,t_n-q\rangle] \\
&\quad -(1-\alpha_n)-\Big(\Upsilon_n^2(\frac{1}{\delta}-\mu)^2\|d_1(x_n)\|^4+\Upsilon_n'^2(\frac{1}{\delta}-\mu)^2\|d'(t_n)\|^4\Big),
\end{aligned}$$

which gives us

$$\begin{aligned}
&(1-\alpha_n)-\Big(\Upsilon_n^2(\frac{1}{\delta}-\mu)^2\|d_1(x_n)\|^4+\Upsilon_n'^2(\frac{1}{\delta}-\mu)^2\|d'(t_n)\|^4\Big) \\
&\leq R_n(p,q)-R_{n+1}(p,q) \\
&+2\alpha_n\|g_1(x_n)-p\|\|x_{n+1}-x_n\|+2\alpha_n\|g_2(t_n)-q\|\|t_{n+1}-t_n\| \\
&+2\alpha_n[\langle g_1(p)-p,x_n-p\rangle+\langle g_2(q)-q,t_n-q\rangle]. \qquad (4.26)
\end{aligned}$$

In addition, from (4.19), we get

$$\begin{aligned}
R_{n+1}(p,q) &\leq R_n(p,q)+2\alpha_n[\langle g_1(p)-p,x_{n+1}-p\rangle+\langle g_2(q)-q,t_{n+1}-q\rangle] \\
&\quad -(1-\theta_n)[||P_{C_n}x_n-x_n||^2+||P_{D_n}t_n-t_n||^2] \\
&\quad -\theta_n\gamma_n||Ax_n-Bt_n||^2-\theta_n||x_n-a_n-\gamma_nA^*(Ax_n-Bt_n)||^2 \\
&\quad -\theta_n||t_n-b_n-\gamma_nB^*(Bt_n-Ax_n)||^2. \qquad (4.27) \\
& \qquad (4.28)
\end{aligned}$$

Next,we show that the sequence $\{R_n(p,q)\}$ converges strongly to zero. For this we consider two cases as follows:

Case 1: Assume that there exist $n_0 \in N$, such that the sequence of real numbers

$\{R_n(p,q)\}$ is decreasing for all $n \geq n_0$. Thus, the sequence $\{R_n(p,q)\}$ convergent and hence from (4.27) and the fact that $\alpha_n \to 0$, we obtain

$$\lim_{n\to\infty} ||P_{C_n}x_n - x_n||^2 = \lim_{n\to\infty} ||P_{D_n}t_n - t_n||^2 = \lim_{n\to\infty} ||Ax_n - Bt_n|| = 0,$$

and

$$\lim_{n\to\infty} ||x_n - a_n - \gamma_n A^*(Ax_n - Bt_n)|| = \lim_{n\to\infty} ||t_n - b_n - \gamma_n B^*(Bt_n - Ax_n)|| = 0,$$

which implies that

$$\lim_{n\to\infty} ||x_n - a_n|| \leq \lim_{n\to\infty} ||x_n - a_n - \gamma_n A^*(Ax_n - Bt_n)|| + \lim_{n\to\infty} ||\gamma_n A^*(Ax_n - Bt_n)|| = 0, \tag{4.29}$$

and

$$\lim_{n\to\infty} ||t_n - b_n|| \leq \lim_{n\to\infty} ||t_n - b_n - \gamma_n B^*(Bt_n - Ax_n)|| + \lim_{n\to\infty} ||\gamma_n B^*(Bt_n - Ax_n)|| = 0, \tag{4.30}$$

In addition, from (4.26), we have

$$\lim_{n\to\infty} \Upsilon_n^2 \|d(x_n)\|^4 = \lim_{n\to\infty} \Upsilon_n'^2 \|d(t_n)\|^4 = 0.$$

Then, from this we get

$$\lim_{n\to\infty} \Upsilon_n \|d(x_n)\|^2 = \lim_{n\to\infty} \Upsilon_n' \|d(t_n)\|^2 = 0. \tag{4.31}$$

Since the sequence $\{(x_n,t_n)\}$ is bounded, there exists a subsequence $\{(x_{n_k},(x_{n_k})\}$, of $\{(x_n,t_n)\}$ which converges weakly to $(\overline{p},\overline{q}) \in H_1 \times H_2$ and

$$\begin{aligned} &\limsup_{n\to\infty}[\langle g_1(p) - p, x_n - p\rangle + \langle g_2(q) - p, t_n - q\rangle] \\ &= \lim_{k\to\infty}[\langle g(p) - p, x_{n_k} - p\rangle + g_2(q) - q, t_{n_k} - q\rangle]. \end{aligned} \tag{4.32}$$

Now, we prove that

$$\lim_{k\to\infty} \|x_{n_k} - z_{n_k}\| = \lim_{k\to\infty} \|t_{n_k} - u_{n_k}\| = 0 \tag{4.33}$$

First consider the case, when $\liminf_{k\to\infty} \Upsilon_{n_k} > 0$. In this case there is $\Upsilon > 0$, such that $\Upsilon_{n_k} > \Upsilon > 0$, for all $k \in N$. Thus, we have

$$\|x_{n_k} - z_{n_k}\|^2 = \frac{1}{\Upsilon_{n_k}} \Upsilon_{n_k} \|x_{n_k} - z_{n_k}\|^2 \leq \frac{1}{\Upsilon} \Upsilon_{n_k} \|x_{n_k} - z_{n_k}\|^2.$$

From this inequality and ((4.31)), we obtain

$$\lim_{k\to\infty} \|x_{n_k} - z_{n_k}\|^2 = 0$$

and hence

$$\lim_{k\to\infty} \|x_{n_k} - z_{n_k}\|$$

Second consider, when $\lim inf_{k\to\infty} \Upsilon_{n_k} = 0$. In this case

$$\lim_{k\to\infty} \Upsilon_{n_k} = 0 \text{ and } \lim_{k\to\infty} \|x_{n_k} - z_{n_k}\|^2 = c > 0 \tag{4.34}$$

Consider, $y'_{n_k} = \frac{1}{\iota}\Upsilon_{n_k} z_{n_k} + (1 - \frac{1}{\iota}\Upsilon_{n_k})x_{n_k}$
Thus, from (4.34), we have

$$\lim_{k\to\infty} \|y'_{n_k} - z_{n_k}\| = \lim_{k\to\infty} \frac{1}{\iota} \Upsilon_{n_k} \|.x_{n_k} - z_{n_k}\| = 0 \tag{4.35}$$

From inequality in Step 2 and definition of y'_{n_k}, we obtain

$$\begin{aligned} \mu\|x_{n_k} - z_{n_k}\|^2 &< \langle x_{n_k} - y'_{n_k} + Ty'_{n_k} - Tx_{n_k}, x_{n_k} - z_{n_k}\rangle \\ &\leq \langle x_{n_k} - y'_{n_k}, x_{n_k} - z_{n_k}\rangle + \langle Ty'_{n_k} - Tx_{n_k}, x_{n_k} - z_{n_k}\rangle \\ &\leq \|x_{n_k} - y'_{n_k}\|\|x_{n_k} - z_{n_k}\| + \| + Ty'_{n_k} - Tx_{n_k}\|\|x_{n_k} - z_{n_k}\| \end{aligned} \tag{4.36}$$

From (4.35), (4.36) and the fact that T is uniformly continuous, we get $\lim_{n\to\infty} \|x_{n_k} - z_{n_k}\| = 0$, which contradict (4.34). Thus, from this fact the equation (4.33) holds. Furthermore, from Step 1 of the Algorithm 1, we have

$$z_{n_k} = (1-\delta)x_{n_k} + \delta T x_{n_k},$$

which gives as

$$\|z_{n_k} - x_{n_k}\| = \delta\|x_{n_k} - Tx_{n_k}\|. \tag{4.37}$$

Hence, from equation (4.33), we obtain,

$$\lim_{k\to\infty} \|x_{n_k} - Tx_{n_k}\| = 0. \tag{4.38}$$

Similarly, we get

$$\lim_{k\to\infty} \|t_{n_k} - St_{n_k}\| = 0. \tag{4.39}$$

Moreover, since $\{(x_{n_k}, t_{n_k})\}$, which converges weakly to $(\overline{p}, \overline{q})$, then $x_{n_k} \rightharpoonup \overline{p}$ and $t_{n_k} \rightharpoonup \overline{q}$. Thus, from (4.34), we get $\overline{p} \in F(T)$ and $\overline{q} \in F(S)$.
Next we show that $A\overline{p} = B\overline{q}$. But, observe that from Lemma 2.1.1 (ii) we get

$$\begin{aligned} ||A\overline{p} - B\overline{q}||^2 &= ||A\overline{p} - Ax_{n_k} + Bt_{n_k} - B\overline{q} + Ax_{n_k} - Bt_{n_k}||^2 \\ &\leq ||Ax_{n_k} - Bt_{n_k}||^2 + 2\langle A\overline{p} - B\overline{q}, A\overline{p} - Ax_{n_k} + Bt_{n_k} - B\overline{q}\rangle, \\ &\to 0 \text{ as } k \to \infty, \end{aligned} \tag{4.40}$$

and this implies $A\overline{p} = B\overline{q}$. That is $(\overline{p}, \overline{q}) \in \Omega$. From the definition of x_{n+1} and t_{n+1}, we have $\|x_{n+1} - w_n\| = \alpha_n \|g_1(x_n) - w_n\| \to 0$, as $n \to \infty$, and $\|t_{n+1} - r_n\| = \alpha_n \|g_2(t_n) - r_n\| \to 0$, as $n \to \infty$, since $\alpha_n \to \infty$, as $n \to \infty$. From (4.26) and (4.29), we get

$$\begin{aligned} \|x_{n+1} - x_n\| &\leq \|x_{n+1} - w_n\| + \|w_n - x_n\| \\ &\leq \|x_{n+1} - w_n\| + \|\theta_n a_n + (1-\theta_n)p_n - x_n\| \\ &\leq \|x_{n+1} - w_n\| + \theta_n \|a_n - x_n\| \qquad (4.41) \\ &\quad + (1-\theta_n)\|p_n - x_n\| \to 0 \text{ as } n \to 0. \qquad (4.42) \end{aligned}$$

Moreover,

$$\|x_n - w_n\| \leq \theta_n \|a_n - x_n\| + (1-\theta_n)\|p_n - x_n\| \to 0, \text{ as } n \to 0. \tag{4.43}$$

Thus, from (4.41) and (4.43), we obtain

$$\begin{aligned} \|x_{n+1} - x_n\| &= \|x_{n+1} - w_n + w_n - x_n\| \\ &\leq \|x_{n+1} - w_n\| + \|w_n - x_n\| \to 0, \text{ as } n \to 0. \end{aligned} \tag{4.44}$$

Similarly we can show that

$$\|t_{n+1}-t_n\| \quad \to 0, \text{ as } n\to 0. \tag{4.45}$$

From (4.32) and Lemma 2.1.2, we have

$$\begin{aligned}
\limsup_{n\to\infty}[\langle g_1(p)-p,x_n-p\rangle+\langle g_2(q)-q,t_n-q\rangle] &\leq \lim_{k\to\infty}[\langle g_1(p)-q,x_{n_k}-p\rangle \\
&\quad +\langle g_2(q)-q,t_{n_k}-q\rangle] \\
&= \langle g_1(p)-p,\overline{p}-p\rangle+\langle g_2(q)-q,\overline{q}-q\rangle \\
&\leq 0. \qquad (4.46)
\end{aligned}$$

Now, we show that the sequence $\{R_n(p,q)\}$ converges strongly to 0. Indeed, from Lemma 2.1.1 and 4.25, we obtain

$$\begin{aligned}
R_{n+1}(p,q) &\leq (1-(1-\alpha)2\alpha_n)R_n(p,q) \\
&\quad +\alpha_n(1-\alpha)\frac{2}{1-\alpha}[\langle g_1(p)-p,x_{n+1}-p\rangle \\
&\quad +\langle g_2(q)-q,t_{n+1}-q\rangle]. \qquad (4.47)
\end{aligned}$$

Finally, from (4.47), (4.44), (4.46) and Lemma 2.1.4, we get $R_n(p,q)\to 0$, as $n\to\infty$ and hence $x_n\to p$ and $t_n\to q$ as $n\to\infty$.

Case 2: Suppose that there exists a subsequence $\{R_{n_j}(p,q)\}$ of $\{R_n(p,q)\}$ such that

$$R_{n_j}(p,q) < R_{n_j+1}(p,q), \text{ for } j\geq 0. \tag{4.48}$$

Thus by Lemma 2.1.3 there exists a non-decreasing sequence $\{m_k\}$, of the set of positive integer of numbers such that $m_k\to\infty$, as $k\to\infty$,
$\|x_{m_k}-p\|^2\leq\|x_{m_k+1}-p\|^2$ and

$$\max\{R_{m_k}(p,q),R_k(p,q)\}\leq R_{m_k+1}(p,q) \text{ for all } k\geq 1. \tag{4.49}$$

Following the method of Case 1, we obtain

$$\lim_{k\to\infty}||P_{C_{m_k}}x_{m_k}-x_{m_k}||^2=\lim_{k\to\infty}||P_{C_{m_k}}x_{m_k}-x_{m_k}||^2=0,$$

$$\lim_{k\to\infty}||P_{D_{m_k}}t_{m_k}-t_{m_k}||=\lim_{k\to\infty}||P_{D_{m_k}}x_{m_k}-t_{m_k}||=0,$$

$$\lim_{k\to\infty} ||Ax_{m_k} - Bt_{m_k}|| = 0,$$

and

$$\lim_{k\to\infty} ||x_{m_k} - a_{m_k} - \gamma_{m_k} A^*(Ax_{m_k} - Bt_{m_k})|| = \lim_{k\to\infty} ||t_{m_k} - b_{m_k} - \gamma_{m_k} B^*(Bt_{m_k} - Ax_{m_k})|| = 0,$$

In addition, by following the method of Case 1, from the inequality (4.32), for $i = 1,2$, we obtain

$$\lim_{k\to\infty} \|x_{m_k} - p_{i,m_k}\| = \lim_{k\to\infty} \|t_{n_k} - q_{i,m_k}\| = 0.$$

In addition, for $i = 1,2$

$$\lim_{k\to\infty} \|x_{m_k} - z_{i,m_k}\| = 0 = \lim_{k\to\infty} \|t_{m_k} - u_{i,m_k}\| = 0,$$

$$\lim_{k\to\infty} \|x_{m_k} - x_{m_k+1}\| = 0 = \lim_{k\to\infty} \|t_{m_k} - t_{m_k+1}\| = 0,$$

and

$$\limsup_{k\to\infty} [\langle g_1(p) - p, x_{m_k+1} - p\rangle + \langle g_2(q) - q, t_{m_k+1} - q\rangle] \le 0. \tag{4.50}$$

Now, from (4.47), we get

$$\begin{aligned} R_{m_k+1}(p,q) &\le (1-(1-\alpha)\alpha_{m_k})R_{n_k}(p,q) \\ &\quad +\alpha_{m_k}(1-\alpha)\frac{2}{1-\alpha}[\langle g_1(p)-p, x_{m_k+1}-p\rangle \\ &\quad +\langle g_2(q)-q, t_{m_k+1}-q\rangle] \\ &\le (1-(1-\alpha)2\alpha_n)R_{m_k+1}(p,q) \\ &\quad +\alpha_{m_k}(1-\alpha)\frac{2}{1-\alpha}[\langle g_1(p)-p, x_{m_k+1}-p\rangle \\ &\quad +\langle g_2(q)-q, t_{m_k+1}-q\rangle] \end{aligned} \tag{4.51}$$

which implies that

$$\begin{aligned} \alpha_{m_k}(1-\alpha)R_{m_k+1}(p,q) &\le \alpha_{m_k}(1-\alpha)\frac{2}{1-\alpha}[\langle g_1(p)-p, x_{m_k+1}-p\rangle \\ &\quad +\langle g_2(q)-q, t_{m_k+1}-q\rangle]. \end{aligned} \tag{4.52}$$

Thus, from (4.48) and (4.52), we have

$$R_k(p,q) \leq R_{m_k+1}(p,q) \leq \frac{2}{1-\alpha}[\langle g_1(p)-p, x_{m_k+1}-p\rangle + \langle g_2(q)-q, t_{m_k+1}-q\rangle]. \quad (4.53)$$

Hence using (4.50), we get

$$\limsup_{k\to\infty} R_k(p,q) \leq \limsup_{k\to\infty} \frac{2}{1-\alpha}[\langle g_1(p)-p, x_{m_k+1}-p\rangle + \langle g_2(q)-q, t_{m_k+1}-q\rangle] \leq 0, \quad (4.54)$$

which implies

$$\limsup_{k\to\infty} R_k(p,q) = 0, \quad (4.55)$$

and hence $x_k \to p$ and $t_k \to q$ as, $k \to \infty$. □

Corollary 4.1.6 *Suppose the assumption* $(A2)-(A6)$ *hold. Let* $T : H_1 \to H_1$ *and* $S : H_2 \to H_2$ *be Uniformly Pseudo contractive mapping .Then , the sequence* (x_n,t_n)*, generated by Algorithm 1 converges strongly to* $(p,q) = P_\Omega(g_1(p), g_2(q))$.

In Algorithm 1 if we assume $g_1 = m_1$ and $g_2 = m_2$. we obtain the following Algorithm:

4.2 Algorithm:2

For arbitrary $(x_0,t_0) \in H_1 \times H_2$, define an iterative algorithm by

Step 1. Compute

$$\begin{cases} z_n = (1-\delta)x_n + \delta T x_n \text{ and } d(x_n) = x_n - z_n, \\ u_n = (1-\delta)t_n + \delta S t_n \text{ and } d'(t_n) = t_n - u_n. \end{cases} \quad (4.56)$$

Step 2. Compute

$$\begin{cases} y_n = x_n - \Upsilon_n d(x_n), \\ v_n = t_n - \Upsilon'_n d'(t_n), \end{cases} \tag{4.57}$$

where, $\Upsilon_n = \iota^{j_n}$ such that j_n is the smallest nonnegative integer j satisfying

$$\langle \iota^j(d(x_n)) + T(x_n - \iota^j d(x_n)) - Tx_n, d(x_n)\rangle \le \mu\|d(x_n)\|^2,$$

and $\Upsilon'_n = \iota^{j'_n}$ such that j'_n is the smallest nonnegative integer j' satisfying

$$\langle \iota^{j'} d'(t_n) + S(t_n - \iota^{j'} d'(t_n)) - St_n, d'(t_n)\rangle \le \mu\|d'(t_n)\|^2.$$

Step 3. Compute

$$\begin{cases} a_n = P_C(x_n - \gamma_n A^*(Ax_n - Bt_n)), \\ b_n = P_D(t_n - \gamma_n B^*(Bt_n - Ax_n)), \\ w_n = \theta_n a_n + (1-\theta_n)p_n \\ r_n = \theta_n b_n + (1-\theta_n)q_n, \end{cases} \tag{4.58}$$

where $C_n = \{x \in H : \langle y_n - Ty_n, x - y_n\rangle \le 0\}$,
$D_n = \{x \in H : \langle v_n - Sv_n, x - v_n\rangle \le 0\}$ and $\{\theta_n\} \subset [\rho, 1)$ for $\rho > 0$ such that $p_n = P_{C_n}x_n$ and $q_n = P_{D_n}x_n$.

Step 4. Compute

$$\begin{cases} x_{n+1} = \alpha_n m_1 + (1-\alpha_n)w_n, \\ t_{n+1} = \alpha_n m_2 + (1-\alpha_n)r_n. \end{cases} \tag{4.59}$$

Step 5. Set $n := n+1$ and go to **Step 1**.

Corollary 4.2.1 Suppose the assumption $(A_1) - (A_6)$ hold. Then the sequence (x_n, t_n), generated by Algorithm 2 converges strongly to $(p, q) = P_\Omega(m_1, m_2))$.

Corollary 4.2.2 Suppose the assumption $(A2) - (A6)$ hold. Let $T : H_1 \to H_1$ and $S : H_2 \to H_2$ be Pseudo contractive mapping.Then , the sequence (x_n, t_n), generated by Algorithm 2 converges strongly to $(p, q) = P_\Omega(m_1, m_2))$.

4.3 Application to Convex Minimization Problem

In this section, we apply our main result Theorem 4.4.1 to approximate a solution of split equality minimum point problem (SEMPP) for a convex function in Hilbert spaces.

Let $g : H_1 \to \mathbb{R}$ and $h : H_2 \to \mathbb{R}$ be a convex smooth function. We consider the problem of approximating $p^* \in H_1$ and $q^* \in H_2$ such that

$$g(p^*) = \min_{x \in H_1}\{g(x)\}, h(q^*) = \min_{x \in H_2}\{h(x)\} \text{ and } Ap^* = Bq^*. \tag{4.60}$$

This problem is equivalent, by Fermat's rule, to the problem of finding $p^* \in H_1$ and $q^* \in H_2$ such that

$$0 = \nabla g(p^*),\ 0 = \nabla\ h(q^*) \text{ and } Ap^* = Bq^*, \tag{4.61}$$

where ∇g and $\nabla\ h$ are gradients of g and h, respectively. We note that $\nabla\ h$ and ∇g are monotone maps (see, e.g.,Baillon and Haddad, (1977) and Rockafellar, (1970)). One way of solving problem (4.61) is finding a solution of SEGFPP pseudo-contractive maps of $S = I_2 - \nabla g$ and $T = I_1 - \nabla\ h$ as a fixed point of S and a fixed point of T are zero of ∇g and $\nabla\ h$, respectively. Thus, Algorithm 1 reduces to Algorithm 3 given below.

4.4 Algorithm:3

For arbitrary $(x_0, t_0) \in H_1 \times H_2$, define an iterative algorithm by

Step 1. Compute

$$\begin{cases} z_n = x_n - \delta \nabla g x_n \text{ and } d(x_n) = x_n - z_n, \\ u_n = t_n - \delta \nabla h t_n \text{ and } d'(t_n) = t_n - u_n. \end{cases} \tag{4.62}$$

Step 2. Compute

$$\begin{cases} y_n = x_n - \Upsilon_n d(x_n), \\ v_n = t_n - \Upsilon'_n d'(t_n), \end{cases} \tag{4.63}$$

where, $\Upsilon_n = \iota^{j_n}$ such that j_n is the smallest nonnegative integer j satisfying

$$\langle \nabla g(x_n - \iota^j d(x_n)) - \nabla g x_n, d(x_n) \rangle \leq \mu \|d(x_n)\|^2,$$

and $\Upsilon_n' = \iota^{j_n'}$ such that j_n' is the smallest nonnegative integer j' satisfying

$$\langle \nabla h(t_n - \iota^{j'} d'(t_n)) - \nabla h t_n, d'(t_n) \rangle \leq \mu \|d'(t_n)\|^2.$$

Step 3. Compute

$$\begin{cases} a_n = P_C(x_n - \gamma_n A^*(Ax_n - Bt_n), \\ b_n = P_C(t_n - \gamma_n B^*(Bt_n - Ax_n), \\ w_n = \theta_n a_n + (1-\theta_n) p_n \\ r_n = \theta_n b_n + (1-\theta_n) q_n, \end{cases} \tag{4.64}$$

where $C_n = \{x \in H : \langle \nabla g y_n, x - y_n \rangle \leq 0\}$,
$D_n = \{x \in H : \langle \nabla h v_n, x - v_n \rangle \leq 0\}$ and $\{\theta_n\} \subset [\rho, 1)$ for $\rho > 0$ such that $p_n = P_{C_n} x_n$ and $q_n = P_{D_n} x_n$.

Step 4. Compute

$$\begin{cases} x_{n+1} = \alpha_n g_1(x_n) + (1-\alpha_n) w_n, \\ t_{n+1} = \alpha_n g_2(t_n) + (1-\alpha_n) r_n. \end{cases} \tag{4.65}$$

where $g_1 : H_1 \to H_1$ and $g_2 : H_2 \to H_2$, are contraction mappings with constant coefficient α_1 and α_2, respectively.

Step 5. Set $n := n+1$ and go to **Step 1**.

The method of proof of Theorem 4.1.5 provides the following theorem for approximating a solution of SEMPP for convex functions in Hilbert spaces.

Theorem 4.4.1 Suppose assumption $(A1)$ and $(A3)-(A6)$ hold. Let $g : H_1 \to \mathbb{R}$ be a convex smooth function with $I_1 - \nabla g$ is uniformly continuous and $h : H_2 \to \mathbb{R}$ be a convex smooth function with $I_2 - \nabla\, h$ is uniformly continuous such that $\Omega = \{(p^*, q^*) \in H_1 \times H_2 : g(P^*) = \min_{x \in H_1}\{g(x)\},\ h(q^*) = \min_{x \in H_2}\{h(x)\}$ and $Ap^* =$

$Bq^*\}\neq \emptyset$. Then, the sequence generated by Algorithm 3 converges strongly to an element $(p^*,q^*) = P_\Omega(g_1(p^*),g_2(q^*))$.

Proof: Consider $S = I_2 - \nabla g$ and $T = I_2 - \nabla\ h$. Then, we get that S and T are uniformly continuous pseudocontractive mappings and fixed points of S and T are zeros of ∇g and $\nabla\ h$, respectively, and hence minimum points of g and h, respectively. Thus, Theorem 4.1.5 provides the conclusion of Theorem 4.4.1. □

If in Algorithm 3, we assume $H_1 = H_2 = H_3$ and $A = I = B$, then SEMPP reduced to a common minimum point problem for convex functions and the method of proof of Theorem 4.1.5 provides the following Corollary for approximating a solution of common minimum point problem for convex functions in Hilbert spaces.

Corollary 4.4.2 Suppose assumption(A_1) and $(A_3)-(A_6)$ hold with $H_1 = H_2 = H_3$ and $A = I = B$. Let $g : H_1 \to \mathbb{R}$ be a convex smooth function with $I - \nabla g$ is uniformly continuous and $h : H_2 \to \mathbb{R}$ be a convex smooth function with $I - \nabla\ h$ is uniformly continuous such that $\Omega = \{(p^*,q^*) \in H_1 \times H_2 : g(p^*) = \min_{x\in H_1}\{g(x)\},\ h(q^*) = \min_{x\in E}\{h(x)\}$ and $Ap^* = Bq^*\} \neq \emptyset$.
Then, the sequence generated by Algorithm 3 converges strongly to an element $(p^*,q^*) = P_\Omega(g_1(p^*),g_2(q^*))$.

Chapter 5

Conclusion and Future scope

5.1 Conclusion

In this thesis, we established an iterative algorithm for approximating a solution of split equality fixed point problem for pseudo-pseudocontractive mappings in real Hilbert space.
In addition, we also proved boundedness and a strong convergence of a sequence generated by the proposed algorithm to a solution of split equality fixed point problem for pseudo-pseudocontractive mappings in real Hilbert spaces. Our result generalizes some related results in the literature. In particular, Theorem 4.1.5 generalize the results of Moudafi(2014), Chang et al.(2015),Chidume et al.(2015) and Zhao(2014). Also, we applied our main result to solve optimization problem.

5.2 Future Scope

In this thesis we obtained an iterative algorithm for approximating a solution of split equality fixed point problem for pseudo-pseudocontractive mappings in real Hilbert space. However, extending this result to a Banach spaces which are more general than Hilbert spaces is an open problem. So, any interested researchers can use this opportunity to conduct their research work in this area.

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