ESSENTIAL NORM OF GENERALIZED INTEGRATION OPERATOR ON FOCK SPACE

A THESIS SUBMITTED TO THE DEPARTMENT OF MATHEMATICS IN PARTIAL FULFILLMENT FOR THE REQUIREMENTS OF THE DEGREE OF MASTERS OF SCIENCE IN MATHEMATICS

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Declaration

I, Misganu Mekonnen, with student ID number S30374/10, declare that this thesis entitled "Essential norm of generalized integration operator on Fock space " is my own original work and it has not been submitted to any institution or University elsewhere for the award of any academic degree, and sources of information that I have been used or quoted are indicated and acknowledged.
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Contents

Abstract

In the past two decades, different properties of the integral operator have been studied in several functional spaces. In particular, on Fock spaces boundedness and compactness of the operator were studied by (Constantin, 2012 and Mengestie, 2013). Recently, after the generalized integration operator is introduced, boundedness and compactness properties of $T_g^{n,m}$ have been studied by (Hafiz, 2022).However the essential norm of $T_g^{n,m}$ on Fock spaces was not studied yet. So, the purpose of this thesis is to study the essential norm of $T_g^{n,m}$ on Fock spaces. Our result generalizes the result of (Mengestie,2018) from Volterra type integral to the generalized integral operator.

Chapter 1

Introduction

1.1 Background of the study

For a given spaces $H(U)$ of holomorphic functions f on a domain $U \subseteq \mathbb{C}$. The Volterra type integral operator T on $H(U)$ induced by a holomorphic function symbol g is defined as,

$$T_g f(z) = \int_o^z f(w)g'(w)dw$$

Volterra type integral operator was first introduced by (Pommerenke, 1977) for which he studied the boundedness property of the operator on the Hilbert space of Hardy spaces H^2. The study was continued by (Aleman and Siskakis, 1995) on the whole Hardy spaces H^p. Later, (Aleman and Siskakis, 1997) gave the analogous characterization on the Bergman space A^p. But, those studies are considered on spaces of analytic functions defined over a disk. Similarly, the study of the operator on Fock space was introduced by (Constantin, 2012) and (Mengestie, 2013). They characterized the bounded and compact properties of the operator in terms of the inducing symbol g.
(Li and Stevic, 2008 and 2009) raised an idea to extend the Volterra-type integral operator T_g by considering its product with composition operator $C_\phi f = f(\phi)$ and they studied their operator theoretic properties in terms of the inducing pair of symbols on some spaces of analytic functions on the unit disk. They eventually considered the following operator induced by analytic functions g and ϕ .

$$T_{g,\phi} f(z) = \int_o^z f(\phi(w))g'(w)dw$$

Since a particular choice of $\phi(z) = z$ reduce $T_{g,\phi}$ to the Volterra-type integral operator T_g,the operator $T_{g,\phi}$ is called the generalized Volterra-type integral operator. Boundedness and compactness of this operator have been studied on different spaces and the characterization of these properties on Fock space have been forwarded by (Mengestie, 2013 and 2014) and later improved by (Mengestie and Worku, 2018).
(Chalmoukis, 2020) introduced a new generalization of Volterra type integral operators called generalized integration operator. For an entire function symbol g and non-negative

integers n,m with $0 \leq m < n$, the generalized integration operator is defined as

$$T_g^{n,m} f = I^n[f^m g^{n-m}]$$

Where I^n is the n^{th} iterate of the integration operator $If(z) = \int_o^z f(w)dw$. (Chalmoukis, 2020) studied the operator on the Hardy spaces and very recently other researchers considered the operator on other spaces (see for example, Du and Qu, 2021 and Qian and Zhu, 2021). In particular, for the values of $n = 1$ and $m = 0$, it gives the Volterra type integral operator T_g. Recently, (Hafiz, 2022) studied the boundedness and compactness of the generalized integration operator on a Fock spaces. We next recall that definition of boundedness, compactness and essential norm of linear operator.

Definition 1.1: Let X and Y can be any normed spaces. A linear operator $T : X \to Y$ is bounded, if there exists a positive real number $c > 0$ such that

$$\|Tx\| \leq c\|x\|, x \in X.$$

Moreover, if $\|Tx_n\|$ goes to zero whenever x_n goes to zero weakly in X, then we say that T is compact.

Definition 1.2: Let X and Y are two Banach spaces and T be a bounded linear operator from X to Y. Then the essential norm $\|T\|_e$ of T is defined as the distance from T to the compact operators.

That is,

$$\|T\|_e = inf\|T - K\|$$

where K is compact operator from X to Y.

From this and since the set of all compact operators is a closed subset of the set of bounded operators, it follows that an operator from X to Y is compact if and only if its essential norm is zero. Computing essential norms of operators has been an interesting research problem in operator related function theory, and different estimates have been made for instance for the composition operator and Volterra-type integral operator on various functional spaces. (Mengestie, 2018) studied essential norm of T_g when they act between generalized weighted Fock spaces and his result gives asymptotic estimate of essential norm of T_g in terms of a function involving the inducing function g and the weight function ϕ in the space. Following this line of research, we estimate the essential norm of generalized integration operator $T_g^{n,m}$ acting between Fock spaces.

Notation: The notation $T(z) \preceq S(z)$ (or equivalently, $S(z) \succeq T(z)$) means that there is a constant c such that $T(z) \leq cS(z)$ for all $z \in c$ and $T(z) \sim S(z)$ means both $T(z) \preceq S(z)$ and $S(z) \preceq T(z)$.

1.1.1 Fock Space

Definition 1.1.1: The Fock space $F_p, 0 < p \leq \infty$, is a space of analytic functions f on $\mathbb{C}$ for which,

$$\|f\|_p = \begin{cases} \left(\frac{p}{2\pi} \int_{\mathbb{C}} |f(z)|^p e^{\frac{-p|z|^2}{2}} dA(Z) \right)^{\frac{1}{p}}, if p < \infty, \\ sup_{z \in \mathbb{C}} |f(z)| e^{\frac{-|z|^2}{2}}, if p = \infty \end{cases}$$

is finite.The space F_p for $1 \leq p \leq \infty$ is a Banach space and for each $f \in F_p$, we have a pointwise estimate given by $|f(z)| \leq e^{\frac{-|z|^2}{2}} \|f\|_p$. As an example polynomials are in all Fock spaces. Fock spaces play important role in quantum physics, harmonic analysis on the Heisenberg group and partial differential equations. In particular, the normalized reproducing kernels in the Fock space are exactly the so-called coherent states in quantum physics.

For any fixed $w \in \mathbb{C}$ the mapping $f \to f(w)$ is a bounded linear functional F_2 and by Riesz representation theorem in functional analysis, there exists a unique function K_w in F_2 such that $f(w) = \langle f, K_w \rangle$ for all $f \in F_2$. This unique function is called reproducing kernel function and F_2 is called a reproducing kernel Hilbert space. Thus, the space F_2 is a reproducing kernel Hilbert space with kernel given by,

$$K_w(z) = e^{z\bar{w}},$$

and the normalized kernel function is given by

$$k_w(z) = \frac{K_w(z)}{\|K_w(z)\|_2} = e^{\bar{z}w - \frac{|z|^2}{2}}.$$

To show how the explicit expression for kernel function given above obtained, we recall that for any orthonormal bases $\{e_n\}$ and $f \in F_2$ we have

$$f(z)=\sum_{k=0}^{\infty}\langle f,e_n\rangle e_n(z).$$

This implies that

$$\begin{aligned}K_w(z) &= \sum_{k=0}^{\infty}\langle K_w,e_n\rangle e_n(z)\\ &= \sum_{k=0}^{\infty}\langle\overline{e_n,k_w}\rangle e_n(z)\\ &= \sum_{k=0}^{\infty}\langle\overline{e_n(w)}\rangle e_n(z).\end{aligned}$$

Since $\{e_n(z)=\sqrt{\frac{a^n}{n!}z^n}\}$ is an orthonormal bases for F_2, substituting this in the above equation gives,

$$K_w(z)=e^{z\bar{w}}.$$

Example: Let $f(z)$=z^n. Then f belongs to F_p, $0<p\leq\infty$, since f is entire and its norm is estimated as follows.
First for $p<\infty$,

$$\|z^n\|_p^p=\frac{p}{2\pi}\int_0^{\infty}|z^n|^p e^{\frac{-p|z|^2}{2}}dA(z).$$

By the sterlings formula:

$$\begin{aligned}\|z^n\|_p^p &= p\int_0^{\infty}r^{np}e^{\frac{-pr^2}{2}}rdr\\ &= (\frac{1}{p})^{\frac{np}{2}}\Gamma(\frac{np}{2}+1)\\ &\sim (\frac{n}{\alpha e})^{\frac{np}{2}}\sqrt{n},\end{aligned}$$

where Γ is a gamma function and $\Gamma(\frac{np}{2}+1)\sim\sqrt{n}$.
Thus

$$\|z\|_p^p\sim(\frac{n}{\alpha e})^{\frac{n}{2}}n^{\frac{1}{2p}}.$$

Similarly, for $p=\infty$, we have

$$\|z^n\|_{\infty}=(\frac{n}{\alpha e})^{\frac{n}{2}}.$$

In (Hu 2013 and Ueki 2016), Fock spaces have been characterized in terms of the Little wood-Paley type estimate formula, which characterize the space in terms of n^{th} derivative,

$n \geq 1$, and stated as follows.

Lemma 1.1.1: For any function $f \in F_p$, $0 < p \leq \infty$, we have

$$\|f\|_p = \begin{cases} \left(\sum_{i=0}^{n-1} |f^{(i)}(0)| + \int_{\mathbb{C}} \frac{|f^{(n)}(z)|^p}{(1+|z|)^n} e^{\frac{-p|z|^2}{2}} dA(z) \right)^{\frac{1}{p}}, p < \infty \\ sup_{z \in \mathbb{C}} \frac{|f^{(n)}(z)|}{(1+|z|)^n} e^{\frac{-|z|^2}{2}}, p = \infty \end{cases}$$

From Corollary 2.8 of (Zhu, 2012) we have the following Pointwise estimate, which is useful to prove our main results.

Lemma 1.1.2: Let $0 < p \leq \infty$ and $w \in \mathbb{C}$ be a fixed point. Then the map

$$U_w f(z) = f(z-w) e^{z\bar{w} - \frac{|w|^2}{2}}$$

is a surjective isometry on F_p.

proof:

$$\begin{aligned} \|U_w f\|_p^p &= \frac{p}{2\pi} \int_{\mathbb{C}} |f(z-w)|^p |e^{z\bar{w}}|^p e^{\frac{-p|z|^2}{2}} dA(z) \\ &= \frac{p}{2\pi} e^{\frac{-p|w|^2}{2}} \int_{\mathbb{C}} |f(z-w)|^p |e^{z\bar{w}}|^p e^{\frac{-p|z|^2}{2}} dA(z). \end{aligned}$$

Making change of variables $z - w = s$, we have

$$\begin{aligned} \|U_w f\|_p^p &= \frac{p}{2\pi} e^{\frac{-p|w|^2}{2}} \int_{\mathbb{C}} |f(s)|^p |e^{s+w\bar{w}}|^p e^{\frac{-p|s+w|^2}{2}} dA(z) \\ &= \frac{p}{2\pi} e^{\frac{-p|w|^2}{2}} \int_{\mathbb{C}} |f(s)|^p |e^{s\bar{w}}|^p e^{\frac{-p|s+w|^2}{2}} dA(z) \\ &= \frac{p}{2\pi} \int_{\mathbb{C}} |f(s)|^p |e^{\frac{-p|s+w|^2}{2}} dA(z) \\ &= \|f\|_p^p \end{aligned}$$

Lemma 1.1.3: For any $r > 0, p > 0$ and $f \in F_p$,we have

$$|f(z)|^p e^{\frac{-p|z|^2}{2}} \leq \|f\|_p^p,$$

for all $z \in \mathbb{C}$

Proof: If $p = \infty$,then the inequality follows from the definition ,i.e.,

$$|f(z)|e^{\frac{-|z|^2}{2}} \leq \|f(z)\|_\infty \Rightarrow |f(z)| \leq e^{\frac{|z|^2}{2}}\|f\|_\infty.$$

Assume $0 < p < \infty$,then for each $f \in F_p$,then the function $|f(z)|^p$ is subharmonic and Hence

$$|f(0)|^p = \frac{1}{2\pi}\int_0^\infty |f(re^{(it)})|^p dt.$$

Multplying both sides $2re^{\frac{-pr^2}{2}}$ and integration on$[0,\infty)$ gives

$$2\int_0^\infty |f(0)|^p re^{\frac{-pr^2}{2}}dr \leq \frac{1}{\pi}\int_0^\infty re^{\frac{-pr^2}{2}}(\int_0^{2\pi})|f(re^{(it)})|^p dt)dr.$$

The left hand side becomes $\frac{2}{p}|f(0)|^p$ and applying rectangular representation

$$\begin{aligned} \frac{2}{p}|f(0)|^p &\leq \frac{1}{\pi}\int_0^\infty \int_0^{2\pi} |f(re^{(it)}|^p dtre^{\frac{-pr^2}{2}}dr \\ &= \frac{1}{\pi}\int_{\mathbb{C}} |f(z)|^p e^{\frac{-p|z|^2}{2}} dA(z). \end{aligned}$$

From which we have

$$\begin{aligned} |f(0)|^p &\leq \frac{p}{2\pi}\int_{\mathbb{C}} |f(z)|^p e^{\frac{-p|z|^2}{2}} dA(z) \\ &= \|f\|_p^p. \end{aligned}$$

From the above inequality and the above Lemma 1.1.2 we have

$$|U_{-z}f(0)|^p = \|U_{-z}f\|_p^p = \|f\|_p^p,$$

which completes the proof as

$$|U_{-z}f(0)|^p = |f(z)|^p e^{\frac{-p|z|^2}{2}}.$$

Lemma 1.1.4: For any $r > 0, p > 0$ and $f \in F_p$,where n is non-negative integer;then we have

$$|f^{(n)}(z)| = c(1+|z|)^n e^{\frac{|z|^2}{2}}\|f\|_p.$$

From (Zhu,2012) we have also the following inclusion property of the spaces.

Theorem 1.1.5: For $0 < p \leq q \leq \infty$,then $F_p \subseteq F_q$.

proof: For any entire function f,if $q < \infty$,then applying the pointwise estimate in Lemma 1.1.2,we obtain

$$\begin{aligned}
\|f\|_q^q &= \frac{q}{2\pi}\int_{\mathbb{C}} |f(z)e^{\frac{-|z|^2}{2}}|^q dA(z) \\
&= \frac{q}{2\pi}\int_{\mathbb{C}} |f(z)|^p |f(z)|^{q-p} e^{\frac{-q|z|^2}{2}} dA(z) \\
&\leq \frac{q}{2\pi}\|f\|_q^{q-p}\int_{\mathbb{C}} |f(z)|^p e^{\frac{-q|z|^2}{2}} dA(z) \\
&= \frac{q}{p}\|f\|_p^q
\end{aligned}$$

□

Therefore,

$$\|f\|_q \leq \left(\frac{q}{p}\right)^{\frac{1}{p}} \|f\|_p^q$$

for all $f \in F_p$ and Hence $F_p \subseteq F_q$.

On the other hand,if $q = \infty$,then using Lemma1.1.2,we have

$$\begin{aligned}
\|f\| &= sup_{z\in\mathbb{C}}|f(z)|e^{\frac{-|z|^2}{2}} \\
&\leq \left(c\frac{2\pi}{p}\right)^{\frac{1}{p}} \|f\|_p, \forall f \in F_p.
\end{aligned}$$

Therfore, $F_p \subseteq F_\infty$.

1.2 Statements of the problem

In several functional spaces, different properties of the integral operator have been studied in the past two decades. Volterra type integral operator was introduced by (Pommerenke, 1977) for which he studied the boundedness property of the operator on the Hilbert space of Hardy spaces H^2. In particular, on Fock spaces boundedness and compactness of the operator were studied by (Constantin, 2012 and Mengestie, 2013). Recently after the generalized integration operator was introduced, boundedness and compactness properties of

$T_g^{n,m}$ have been studied by (Hafiz, 2022). However the essential norm of $T_g^{n,m}$ on Fock spaces was not studied yet. So, the purpose of this thesis is to study the essential norm of $T_g^{n,m}$ on Fock spaces. Our result generalizes the result of (Mengestie, 2018) from Volterra type integral operator to the generalized integral operator.

1.3 Objectives

1.3.1 General objective

The general objective of this research is to study the essential norm of the generalized integration operators on Fock spaces.

1.3.2 Specific objectives

The specific objectives of this study are:

- To estimate lower and upper for essential norms of the generalized integration operators on Fock spaces.
- To formulate essential norms of Volterra type integral operators in particular.

1.4 Significance of the study

The result of this thesis may have the following importance.

- It may be used as a base for any researcher who is interested to study other properties of generalized integration operators on Fock spaces.
- It may help the graduate students to acquire research skills and scientific procedures.

1.5 Delimitation of the Study

This study focused only on estimating the essential norm of generalized integration operators acting between Fock spaces.

Chapter 2

Review of Related Literatures

(Pommerenke, 1977) introduced and studied the boundedness property of Volterra-type integral operator T_g on the Hardy space H^2. The study was continued by (Aleman and Siskakis, 1995) on the whole Hardy spaces H^p Then (Aleman and Siskakis, 1997) characterized bounded and compact properties of the operator on the Hardy and Bergman space A^p. Following that, several researchers are motivated to study different properties of the Volterra-type integral operator in different spaces. (Constantin, 2012) studied bounded, compact and other properties of T_g on the Fock spaces F_p. Then the study was continued by (Mengestie, 2013) on the growth type Fock space F_∞

Theorem 2.1: (Constantin, 2012 and Mengestie, 2013) Let $0 < p \le q \le \infty$.

Then $T_g : F_p \to F_q$ is

a) bounded if and only if $g(z) = az^2 + bz + c$; $a, b, c \in \mathbb{C}$;

b) compact if and only if $g(z) = az + b$; $a, b, \in \mathbb{C}$;

For the case when the operator maps from larger space to the smaller, there is a stronger condition in which boundedness and compactness are equivalent.

Theorem 2.2: (Constantin, 2012 and Mengestie, 2013) Let $0 < q < p \le \infty$. Then the following are equivalent

a) $T_g : F_p \to F_q$ is bounded;

b) $T_g : F_p \to F_q$ is compact;

c) $q > \begin{cases} \frac{2p}{p+2}, if p < \infty \\ 2, if p = \infty \end{cases}$ and $g(z) = az + b$ for some $a, b \in \mathbb{C}$,

(Mengestie, 2014) studied the extended operator, namely the generalized Volterra type integral operator on Fock spaces F_p in 2014. Recently (Mengestie and Worku; 2018) studied also boundedness and compactness of generalized Volterra type integral operator $T_{g,\phi}$ with the simpler characterization on Fock spaces F_p, which we state it by the following theorem.

Theorem 2.3: (Mengestie and Worku; 2018) Let $0 < p, q \le \infty$ and (g, ϕ) be non-constant entire functions. Then

1.If $p \le q$,then $T_{g,\phi}$ is bounded (respectively,compact) if and only if the function

$\frac{|g'(z)|e^{|\phi(z)|^2-|z|^2}}{(1+|z|)}$ is bounded (respectively,$lim_{z\to\infty}\frac{|g'(z)|e^{|\phi(z)|^2-|z|^2}}{(1+|z|)}=0$)

2,If $q < p$,then $T_{g,\phi}$ is bounded or compact if and only if

$$\int_{\mathbb{C}}\left(\frac{|g'(z)|e^{|\phi(z)|^2-|z|^2}}{(1+|z|)}\right)^{\frac{pq}{p-q}}dA(z)<\infty, for p<\infty$$

$$\int_{\mathbb{C}}\left(\frac{|g'(z)|e^{|\phi(z)|^2-|z|^2}}{(1+|z|)}\right)^{q}dA(z)<\infty, for p=\infty$$

In (Mengestie, 2018), the author studied the essential norm of Volterra type integral operators on Fock spaces which we state as follows:

Theorem 2.4: (Mengestie, 2018): Let $1\leq p\leq q\leq\infty$ and $T_g : F_p \to F_q$ is a bounded operator. That is $g(z) = az^2+bz+c$. Then

$$\|T_g\|_e \cong |a|.$$

Recently, (Hafiz, 2022), studied boundedness and compactness properties of generalized integration operators $T_g^{n,m}$ on Fock spaces F_p.

Theorem 2.5: (Hafiz, 2022) Let $0 < p$, $q \leq \infty$ and $g \in H(\mathbb{C})$,m,n nonnegative integers with $0\leq m<n$ and $T_g^{n,m}$ maps from F_p into F_p.

1. If $p \leq q$, then $T_g^{n,m}$ is bounded (respectively, compact) if and only if the function $\frac{|g^{n-m}(z)||z|^m}{(1+|z|)^n}$ is bounded (respectively, $lim_{z\to\infty}\frac{|g^{n-m}(z)||z|^m}{(1+|z|)^n}=0$)

2. If $q < p$, then $T_g^{n,m}$ is bounded or compact if and only if

$$\int_{\mathbb{C}}\left(\frac{|g^{n-m}(z)||z|^m}{(1+|z|)^n}\right)^{\frac{pq}{p-q}}dA(z)<\infty, for p<\infty$$

$$\int_{\mathbb{C}}\left(\frac{|g^{n-m}(z)||z|^m}{(1+|z|)^n}\right)^{q}dA(z)<\infty, for p=\infty$$

The purpose of this thesis is to continue the study of $T_g^{n,m}$ and estimate the essential norm of the operators.

Chapter 3

Methodology

3.1 Study area and period

The study was conducted in Jimma University department of mathematics under the functional analysis stream from January 2022 G.C to December 2022 G.C.

3.2 Study Design

In this research work, we followed analytical method of design.

3.3 Source of Information

The relevant sources of information for this study were journals, books, published articles, and related studies on the Internet.

3.4 Mathematical Procedure of the Study

The mathematical procedure that we followed for this research work are the following:

$\Rightarrow$ Establishing theorems.

$\Rightarrow$ Proving upper and lower estimates of essential norm for the operators.

$\Rightarrow$ Characterizing compactness of the generalized integral operator in terms of essential norm.

Chapter 4

Main Result and Discussion

4.1 Essential norm

We begin the section with the following lemma from (Tien and Khoi, 2019) which is useful in proving our main result.

Theorem 4.1: Let $0 < p, q \leq \infty$ and $\phi \in H(\mathbb{C})$. Then

1. If $0 < p \leq q \leq \infty$, then composition operator, $C_\phi f(z) = f(\phi(z))$,
$C_\phi : F_p \to F_q$ is bounded (respectively, compact)
if and only if $\phi(z) = az + b$, $a, b \in \mathbb{C}$ with $|a| \leq 1$ (respectively, $\phi(z) = az + b$, $a, b \in \mathbb{C}$ with $|a| < 1$).

2. If $0 < p \leq q \leq \infty$, then $C_\phi : F_p \to F_q$ is bounded or compact
if and only if $\phi(z) = az + b$, $a, b \in \mathbb{C}$ with $|a| < 1$.

Our next result expresses the essential norm of $T_g^{n,m}$ in terms of the inducing symbol function g and it generalizes some essential norm estimates of T_g on Fock spaces.

Theorem 4.2: Let $1 \leq p \leq q \leq \infty$ and $T_g^{n,m} : F_p \to F_q$ is a bounded operator where n and m are non negative integer and $n > m$

$$||T_g^{n,m}||_e = lim_{|z| \to \infty} sup \frac{|g^{n-m}(z)||z|^m}{(1+|z|)^n}.$$

Proof: To prove this theorem we consider two cases:

1. First, we determine the lower estimate. Let Q: $F_p \to F_q$ be a compact operator. Then since k_w converges weakly to zero on F_p as $|w| \to \infty$, we have

$$\begin{aligned} ||T_g^{n.m} - Q|| &\geq lim_{|w| \to \infty} Sup ||T_g^{n,m} k_w - Q k_w||_q \\ &\geq lim_{|w| \to \infty} Sup (||T_g^{n,m} k_w||_q - ||Q k_w||_q) \\ &= lim_{|w| \to \infty} Sup ||T_g^{n,m} k_w||_q - lim_{|w| \to \infty} Sup ||Q k_w||_q \\ &= lim_{|w| \to \infty} Sup ||T_g^{n,m} k_w||_q \end{aligned}$$

Then applying monotonicity of the Fock spaces in Theorem 1.1.5, Littlewood-Paley type estimate in Lemma 1.1.1 , and setting $|z| = |w|$, we get

$$\begin{aligned} lim_{|w|\to\infty} sup \|T_g^{n.m} k_w\|_q &\gtrsim lim_{|w|\to\infty} sup \|T_g^{n.m} k_w\|_q \\ &\geq lim_{|w|\to\infty} sup \frac{|K_w^m(z)||g^{n-m}(z)|}{(1+|z|)^n} e^{\frac{-|z|^2}{2}} \\ &= lim_{|z|\to\infty} sup \frac{|z|^m |g^{n-m}(z)|}{(1+|z|)^n} \end{aligned}$$

This completes the proof of the lower estimates.

2. Next, we determine the upper estimate. We consider a sequence $\phi_k(z) = \frac{k}{k+1}(z)$ for each $k \in N$. Since $\frac{k}{k+1} < 1$, by Theorem 4.1 c_{ϕ_k} is compact and if $T_g^{n,m}$ is bounded, then $T_g^{n,m}(c_{\phi_k}) : F_p \to F_q$ is compact.

Using this,

$$\begin{aligned} \|T_g^{n,m}\|_e &\leq \|T_g^{n,m} - T_g^{n,m}(c_{\phi_k})\| \\ &= sup_{\|f\|\leq 1} \|T_g^{n,m} - T_g^{n,m}(c_{\phi_k}) f\|_q \end{aligned}$$

If $q = \infty$, then using Littlewood paley estimate in Lemma 1.1.1

$$\begin{aligned} sup_{\|f\|\leq 1} \|T_g^{n,m} - T_g^{n,m}(c_{\phi_k}) f\|_q &\cong sup_{\|f\|\leq 1} sup_{z\in\mathbb{C}} \frac{|g^{n-m}(z)||f^m(z) - f^m(\phi_k(z))|}{(1+|z|)^n} e^{\frac{-|z|^2}{2}} \\ &\leq sup_{\|f\|\leq 1} sup_{|z|>r} \frac{|g^{n-m}(z)||f^m(z) - f^m(\phi_k(z))|}{(1+|z|)^n} e^{\frac{-|z|^2}{2}} \\ &+ sup_{\|f\|\leq 1} sup_{|z|\leq r} \frac{|g^{n-m}(z)||f^m(z) - f^m(\phi_k(z))|}{(1+|z|)^n} e^{\frac{-|z|^2}{2}}. \end{aligned}$$

Then, the first summand above by Lemma 1.1.4 is bounded by

$$sup_{\|f\|\leq 1} sup_{|z|>r} \frac{|g^{n-m}(z)||f^m(z) - f^m(\phi_k(z))|}{(1+|z|)^n} e^{\frac{-|z|^2}{2}}$$

$$
\begin{aligned}
&\leq sup_{|z|>r}\frac{|g^{n-m}(z)|}{(1+|z|)^n}sup_{\|f\|\leq 1}|f^m(z)-f^m(\phi_k(z))|e^{\frac{-|z|^2}{2}}\\
&\leq sup_{|z|>r}\frac{|g^{n-m}(z)||z|^m}{(1+|z|)^n}\|f\|_p\\
&\leq sup_{|z|>r}\frac{|g^{n-m}(z)||z|^m}{(1+|z|)^n}.
\end{aligned}
$$

For the second summand, we need the following estimate. Integrating the function f^{m+1} along the radial segment $[\frac{K}{k+1}(z),z]$, we find

$$|f^{(m+1)}(z)-f^m(\phi_k(z))|\leq |z||f^{(m+1)}(z^*)|,$$

for some z^* in the segment $[\frac{K}{k+1}(z),z]$.
Using Cauchy estimate for the $f^{(m+1)}$, we also have

$$|f^{(m+1)}(z^*)|\leq \frac{1}{r}max_{|z|>r}|f^m(z)|$$

Thus

$$|f^m(z)-f^m(\phi_k(z))|\leq \frac{|z|}{r(k+1)}max_{|z|=2r}|f^m(z)|.$$

Using the estimate in Lemma 1.1.4, we have

$$max_{|z|=2r}|f^m(z)|\leq r^m e^{2r^2}\|f\|_p.$$

Combining the above estimates, we have

$$sup_{\|f\|\leq 1}sup_{|z|\leq r}\frac{|g^{n-m}(z)||f^m(z)-f^m(\phi_k(z))|}{(1+|z|)^n}e^{\frac{-|z|^2}{2}}$$

$$
\begin{aligned}
&\leq \left(sup_{z\in\mathbb{C}}\frac{|g^{n-m}(z)||z|^m}{(1+|z|)^n}e^{\frac{-|z|}{2}}\right)^2\left(\frac{|z|}{r(k+1)}e^{2r^2}\|f\|_p\right)\\
&\leq \frac{e^{2r^2}}{r(k+1)}\to 0
\end{aligned}
$$

as $k \to \infty$.
Therefore, from the above two summand estimates, we have

$$||T_g^{n,m}||_e = lim_{|z|\to\infty} sup \frac{|g^{n-m}(z)||z|^m}{(1+|z|)^n}.$$

For $q < \infty$, using Littlewood paley estimate,

$$||T_g^{n,m}||_e \leq sup_{\|f\|_p\leq 1}\|T_g^{n,m} - T_g^{n,m}(c_{\phi_k})f\|_q$$

$$\begin{aligned}
&\cong \quad sup_{\|f\|_p\leq 1}\left(\int_{\mathbb{C}} \frac{|g^{n-m}(z)|^q|f^m(z)-f^m(\phi_k(z))|^q}{(1+|z|)^{nq}} e^{\frac{|z|^2}{2}} dA(z)\right)^{\frac{1}{q}} \\
&\leq \quad sup_{\|f\|_p\leq 1}\left(\int_{|z|\leq R} \frac{|g^{n-m}(z)|^q|f^m(z)-f^m(\phi_k(z))|^q}{(1+|z|)^{nq}} e^{\frac{|z|^2}{2}} dA(z)\right)^{\frac{1}{q}} \\
&+ \quad sup_{\|f\|_p\leq 1}\left(\int_{|z|>R} \frac{|g^{n-m}(z)|^q|f^m(z)-f^m(\phi_k(z))|^q}{(1+|z|)^{nq}} e^{\frac{|z|^2}{2}} dA(z)\right)^{\frac{1}{q}}
\end{aligned}$$

We now estimates the above two summands separately

$$sup_{\|f\|_p\leq 1}\left(\int_{|z|>R} \frac{|g^{n-m}(z)|^q|f^m(z)-f^m(\phi_k(z))|^q}{(1+|z|)^{nq}} e^{\frac{|z|^2}{2}} dA(z)\right)^{\frac{1}{q}}$$

$$\begin{aligned}
&\leq \quad \left(sup_{|z|>r}\frac{|g^{n-m}(z)|}{(1+|z|)^n}\right) sup_{\|f\|\leq 1}\|f\|_q + sup_{\|f\|\leq 1}\|C_{\phi_k} f\|_q \\
&\leq \quad sup_{|z|>r}\frac{|g^{n-m}(z)|}{(1+|z|)^n} sup_{\|f\|_p\leq 1}\|f\|_p
\end{aligned}$$

Similarly for the first summand it is bounded by

$$\left(sup_{z\in\mathbb{C}}\frac{|g^{n-m}(z)|}{(1+|z|)^n} e^{\frac{|z|^2}{2}}\right)^2 \left(sup_{\|f\|_p\leq 1}\left(\int_{|z|\leq R} \frac{|f^m(z)-f^m(\phi_k(z))|^q}{(1+|z|)^{nq}} e^{\frac{|z|^2}{2}} dA(z)\right)^{\frac{1}{q}}\right)$$

$$\leq \quad sup_{\|f\|_p \leq 1} sup_{|z| \leq R} |f^m(z) - f^m(\phi_k(z)|$$

Following similar procedures as in the case of $q = \infty$, we have

$$sup_{\|f\|_p \leq 1} sup_{|z| \leq R} |f^m(z) - f^m(\phi_k(z)| \quad \leq \quad \frac{e^{2r^2}}{r(k+1)} \to 0$$

as $k \to \infty$.
Therefore, from the above estimates we have

$$\|T_g^{n,m}\|_e \leq lim_{|z| \to \infty} sup \frac{|g^{n-m}(z)||z|^m}{(1+|z|)^n}.$$

Next, we simplify the above Theorem into more simplified result.
Theorem 4.3: Let $1 \leq p \leq q \leq \infty$ and $T_g^{n,m} : F_p \to F_q$ be a bounded operator. Then g has the form where $k = n - m + 1$.

$$\|T_g^{n,m}\|_e = |a_k|.$$

Proof: First observe from Theorem 2.5 that $T_g^{n,m} : F_p \to F_q$ is a bounded if and only if the function $\frac{|g^{n-m}(z)||z|^m}{(1+|z|)^n}$ is uniformly bounded, which by Liouvilles Theorem gives,

$$g(z) = a_k z^k + ... + a_0,$$

where $k = n - m + 1$.
Using this and Theorem 4.2 we obtain

$$lim_{|z| \to \infty} sup \frac{|g^{n-m}(z)||z|^m}{(1+|z|)^n} = |a_k|.$$

In particular for n=1 and m=0 our result simplified to the result in Theorem 2.4.

Chapter 5

Conclusion and future scope

5.1 Conclusion

This thesis includes a number of results, which characterize generalized integration operators acting between Fock spaces. Our results in Chapter 4, which is about the essential norm is new and may be applied to study other properties defined whenever the operator is bounded. In addition, our results generalizes some of the results that have been obtained for the Volterra-type integral operators. In particular, it generalizes the results of (Mengestie, 2018) from Volterra type integral operator to the generalized integration operators on Fock space , which is stated in Theorem 2.4.

5.2 Future Scope

In this thesis ,to estimate lower and upper for the essential norm of generalized integration operator on Fock spaces. Hence, the process proposed to estimate lower and upper for the essential norm of generalized of the different properties of integration operator acting between Fock spaces.

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