FITTED NON-POLYNOMIAL CUBIC SPLINE METHOD FOR SOLVING SINGULARLY PERTURBED ROBIN TYPE BOUNDARY VALUE PROBLEMS WITH DISCONTINUOUS SOURCE TERM

A Research Paper Submitted to the Department of Mathematics, Jimma University in Partial Fulfillment for the Requirements of the Degree of Masters of Science in Mathematics (Numerical Analysis)

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Declaration

I, undersigned declare that, this thesis entitled “Fitted non-polynomial cubic spline method for solving singularly perturbed robin type boundary value problem with discontinuous source term” is my own original work and it has not been submitted for the award of any academic degree or the like in any other institution or university, and that all the sources I have used or quoted have been indicated and acknowledged.

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Abstract

In this thesis, fitted non polynomial cubic spline method for solving singularly perturbed robin type boundary value problems with discontinuous source term is considered. The stability and parameter uniform convergence of the proposed method are proved. To validate the applicability of the scheme, two model problems are considered for numerical experimentation and solved for different values of the perturbation parameter, ε and mesh size, *h*. The numerical results are tabulated in terms of maximum absolute errors and rate of convergence and it is observed that the present method is more accurate and ε-uniformly convergent for $h \geq \varepsilon$ where the classical numerical methods fails to give good result and it also improves the results of the methods existing in the literature.

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CHAPTER ONE

INTRODUCTION

1.1. Background of the Study

Numerical analysis is the area of mathematics and computer science that creates, analyzes, and implements algorithms for solving numerically the problems of discretized mathematics. Such problems originate generally from real-world applications of algebra, geometry and calculus, and they involve variables which vary continuously; these problems occur throughout the natural sciences, social sciences, engineering, medicine, and business **(** Rajasekar an, S. (1992).

A differential equation is said to be singularly perturbed differential equation, if the highest derivative term is multiplied by small parameter. It is well known that singularly perturbed problem often have very thin boundary and internal layers where the solution varies rapidly change, whereas away from the layer, solution behaves regularly and varies slowly, so the Miller numerical treatment of singularly perturbed problems faces major difficulties (Miller, 1974, Riordan, 2003). Due to the variation in the width of the layer with respect to small perturbation parameters several difficulties are experienced in solving the singularly perturbed problems using the standard numerical methods with uniform mesh (Kadalbajoo, 2005).

Singular perturbation problems (SPPs) model convection–diffusion process in applied mathematics that arise in diverse areas, including linearized Navier–Stokes equation at high Reynolds number and the drift-diffusion equation of semiconductor device modeling, heat and mass transfer at high Pe´clet number etc (Roos, et al, 1996, Doolan et al, 1980).

The novel aspect of the problem under consideration is that we take a source term in the differential equation which has a jump discontinuity at one or more points in the interior of the domain. This gives rise to an interior layer in the exact solution of the problem, in addition to the boundary layer at the outflow boundary point. Our goal is to construct an ε uniform numerical method for solving this problem, that is a numerical method which generates ε uniformly convergent numerical approximations to the solution and its derivatives. Note that problems with discontinuous data are treated theoretically, in the case of the solution of the convection-diffusion with Dirichlet case problem (Farrell et al, 2004a, Farrell et al, 2004b). Authers such as (Chandru and Prabha, 2014), (Chandru et al, 2014), (Roos and Zarin, 2010), (Farrell et.al, 1998) were discussed a self-adjoint Dirichlet type problem with discontinuous source term. Shanthi et al, (2006), Shanthi and Ramanujam (2002), Shanthi and Ramanujam (2004), have examined two parameter singularly perturbed BVPs for second order ODEs with discontinuous source term.

Boundary value problem of the type (1)–(2) model confinement of a plasma column by reaction pressure and geophysical fluid dynamics (Chin and Krasny, 1983).

Ansari (2003) was discussed the nature of problem in the Dirichlet case

$\alpha_1 = 1, \beta_1 = 0, \alpha_2 = 1, \beta_2 = 0$ and in the Neumann case $\alpha_1 = 0, \beta_1 = 1, \alpha_2 = 0, \beta_2 = 1.$

For detailed study one may refer (Ansari, 2003) and (Farrell, 2000). Various methods are available in literature to obtain numerical solution to singularly perturbed differential equation (1) subject to Robin boundary conditions when f is smooth on Ω (Ansari, 2003), (Natesan and Ramanujam, 2000), (Natesan and Bawa, 2007) and (Natesan, and Ramanujam, 1999). Some recent works have been done in similar type of problem with smooth data as follows. The numerical integration method for general singularly perturbed boundary value problem with mixed boundary condition is presented in Andargie and Reddy (2008). Prasad and Reddy (2011) have shown the advantages of Differential Quadrature Method (DQM) for finding the numerical solution (Mohapatra et al, 2011). Das et al. (2013) have discussed on system of reaction diffusion differential equations for Robin or mixed type boundary value problems by a cubic spline approximation. From this investigation the author considered a non-self adjoint Robin type problem with discontinuous source term, and obtained a parameter uniform convergent solution for equation (1)–(2).

Recently, Shandru and shanthi (2015) and Abagero *et al*.,(2021), presented fitted mesh and nonstandard finite difference method to solve singularly perturbed robin type boundary value problems with discontinuous source term. As far as the researcher's knowledge is concerned, numerical solution of singularly perturbed robin type boundary value problem with discontinuous source terms via fitted non-polynomial spline method is first being considered. Additionally, still there is a room to increase the accuracy because of the treatment of singular perturbation problem is not trivial distributions and the solution depend on perturbation parameter ε and mesh size h (Doolen *et.al* 1980). Due to this numerical treatment of singularly perturbed boundary value problems are needs improvement.

Therefore, it is important to develop more accurate and convergent numerical method for solving singularly perturbed boundary value problems. Thus, the purpose of this study is to developed stable, convergent and more accurate numerical method for solving singularly perturbed boundary value problems by using fitted non polynomial cubic spline method.

1.2. Objectives of the Study

1.2.1. General Objective

The general objective of this study is to develop a fitted non-polynomial cubic spline method for solving singularly perturbed robin type boundary value problem with discontinuous source terms.

1.2.2. Specific Objectives

The specific objectives of the present study are:

1. To formulate fitted non-polynomial cubic spline method for solving singularly perturbed robin type boundary value problem with discontinuous source terms.
2. To analyze the convergence of the scheme.
3. To investigate the accuracy of the proposed method.

1.3. Significance of the Study

The outcomes of this study may have the following importance:

- ✓ Provide some background information for other researchers who work on this area.
- ✓ To introduce the application of numerical methods in different field of studies.

1.4. Delimitation of the Study

Singularly perturbed problems are perhaps arises in variety of mathematical and physical problems. However, this study is delimited to solve singularly perturbed robin type with discontinuous source terms of the form:

$$Ly(x) \equiv \varepsilon y''(x) + p(x)y'(x) - q(x)y(x) = r(x), \quad x \in (\Omega^- \cup \Omega^+) \qquad (1)$$

subject to boundary conditions,

$$L_1 y(0) = \alpha_1 y(0) - \beta_1 \varepsilon y'(0) = s, \quad L_2 y(1) = \alpha_2 y(1) + \beta_2 y'(1) = t, \qquad (2)$$

where $\alpha_1, \beta_1 \geq 0,\ \ \alpha_1 + \beta_1 > 0,\ \beta_2 \geq 0,\ \alpha_1 > 0,$ and $\varepsilon > 0$ is small parameter $p(x)$, $q(x)$ are smooth functions $\bar{\Omega}$ such that $p(x) \geq \alpha > 0, q(x) \geq \beta \geq 0.$

It is convenient to introduce the notation $\Omega = (0,1)$, $\Omega^- = (0,d)$, $\Omega^+ = (d,1)$, $d \in \Omega$ and to denote the jump at d in any function.

CHAPTER TWO

REVIEW OF RELATED LITERATURE

2.1. Singularly Perturbed Problems

Science and technology develops many practical problems, such as the mathematical boundary layer theory or approximation of solution of various problems described by differential equations involving small parameters have become increasingly complex and therefore require the use of asymptotic methods. The term 'singular perturbations' was first used by Friedrichs *et al*. (1946) in a paper presented at a seminar on non-linear vibrations at New York University. Singularly perturbed problems arise frequently in applications including geophysical fluid dynamics, oceanic and atmospheric circulation, chemical reactions, civil engineering, optimal control, etc. The classification of singularly perturbed higher order problems depend on how the order of the original equation is affected if one sets $\varepsilon = 0,$ where ε is a small positive parameter multiplying the highest derivative occurring in the differential equation. If the order is reduced by one, we say that the problem is of convection-diffusion type and of reaction-diffusion type if the order is reduced by two. It is well known that the solution of singularly perturbed boundary value problems is described by slowly and rapidly varying parts. So there are thin transition layers where the solution can jump suddenly, while away from the layers the solution varies slowly and behaves regularly (Akram and Afia, 2013). Many scholars have studied the analytical and numerical solutions of these problems.(Abrahamsson *et al*. 1974) solved singularly perturbed ordinary differential equations using difference approximations. Numerical treatment of singularly perturbed boundary value problems for higher-order non-linear ordinary differential equations has a great role in fluid dynamics. The development of numerical methods for solving singularly perturbed problems started with methods aimed at solving ordinary differential equations, an account of which can be found in the first monograph on this subject by Doolan et al. (1980). Ilicash and Schultz (2004) introduced three finite-difference techniques for second-order singularly perturbed linear boundary value problems using convergent tension spline and on uniform tension spline methods. Valaramathi and Ramanujam

(2002) solved singularly perturbed two-point boundary value problems for third-order ordinary differential equations.

2.2. Spline Based Method

The approximation theory is one of the main topics of numerical analysis. It is a foundation for numeral algorithms in the different fields of applied mathematics. Polynomials are the most easily handled in practice, since they can be represented by restricted information, evaluated in limited number of basic operations and easily integrated or differentiated. Spline is a piecewise polynomial function defined in a region, such that there exists a decomposition of the region into sub-regions in each of which the function is a polynomial of some degree d. Also the function, as a rule, is continuous in the region, together with its derivatives of order up to d-1.

Numerical methods with spline functions in getting the approximate solution of the differential equations lead to a matrices which are solvable easily with algorithms having low cost of computation. Non-polynomial spline method has turned out to be an effective tool for solving ordinary and partial differential equations. Most of non-polynomial spline functions are consists of a polynomial and trigonometric parts as well as exponential parts. In many papers various techniques using quadratic, cubic, quartic, quintic, sextic, septic and higher degree non polynomial splines have been discussed for the numerical solution of linear and nonlinear boundary value problem.

Hence, in the recent times, many researchers have been trying to develop spline based methods. For example, Taha and Khlefha, (2016) concerned with the approximated solution of linear two-points boundary value problem using non-polynomial spline method.

2.3. Numerical versus Analytical Methods

A numerical solution means making guesses at the solution and testing whether the problem is solved well enough to stop. An analytical solution involves framing the problem in a well-understood form and calculating the exact solution. The best is when we can find out the exact solution using calculus, trigonometry and other techniques. The techniques used for calculating the exact solution are known as analytic methods because we used the analysis to figure it out.

Numerical methods are commonly used for solving mathematical problems that are formulated in science and engineering where it is difficult or even impossible to obtain exact solutions. Numerical solution is discrete. Numerical methods, on the other hand, can give an approximate solution to solve equation.

Numerical methods to solve singular perturbation problems have been widely used in many fields of fluid dynamics, reaction-diffusion processes, particle physics, and combustion processes. These types of problems are represented by differential equations including ε which is assumed to be a small parameter and solutions of the problems have non-uniform behavior when the parameter $\varepsilon \to 0$. Analytic solution is exact solution to a problem that can be calculated symbolically. Numerical methods give an approximate solution to solve equations. It is important to realize that a numerical solution is always numeric but analytical methods usually a result in terms of mathematical functions that can be evaluated for specific instances. However, numerical results can be plotted to show some of the behavior of the solution. A variety of numerical methods to solve singularly perturbed boundary value problem are available for ordinary differential equations.

2.4. Finite Difference Method (FDM)

The finite difference methods are a class of numerical techniques for solving differential equations by approximating derivatives with finite differences. Both the spatial domain and time interval (if applicable) are discretized, or broken into a finite number of steps, and the value of the solution at these discrete points is approximated by solving algebraic equations containing finite differences and values from nearby points.

FDM is used to solve ordinary differential equations that have conditions imposed on the boundary rather than at the initial point. These problems are called boundary-value problems. Bo Strand (1994).

Finite difference methods convert ordinary differential equations (ODE) or partial differential equations (PDE),in to nonlinear a system or system linear equations that can be solved by matrix algebra techniques.

CHAPTER THREE

METHODOLOGY

3.1. Study Area and period

This study was conducted at Jimma University under the department of Mathematics from Feb. 2023 to June 2023. Conceptually, the study focus on Fitted non-polynomial cubic spline Method for solving singularly perturbed robin type convection-diffusion problem with source term.

3.2. Study Design

The study employed mixed design (i.e., documentary review and numerical experimentation design).

3.3. Source of Information

The relevant sources of information for this study are books, published articles and related studies from internet.

3.4. Mathematical Procedure

In order to achieve the stated objectives, the study followed the following procedures:

1. Describing the problem;
2. Analyzing the properties of the continuous solution;
3. Discretizing the solution domain;
4. Developing fitted non-polynomial cubic spline scheme for the problem;
5. Establishing the convergence analysis of the developed scheme;
6. Developing an algorithm and writing code for the presented scheme;
7. Validating the scheme using numerical examples
8. Comparing of the result with the existing literature.

CHAPTER FOUR

DESCRIPTION OF THE METHODS, RESULTS AND DISCUSSION

4.1. Properties of continuous solution

The differential operator for the above problem is given by

$$L_\varepsilon \equiv \varepsilon \frac{d^2}{dx^2} + p\frac{d}{dx} - q,$$

and it satisfies the following minimum principle for boundary value problems (BVPs). The following lemmas (Doolan *et al.*, 1980) are necessary for the existence and uniqueness of the solution and for the problem to be well-posed.

Lemma 1: Suppose that the function $y \in C^1(\bar{\Omega}) \cap C^2(\Omega^- \cup \Omega^+)$, satisfies $L_1 y(0) \geq 0, L_2 y(1) \geq 0$ and $Ly(x) \leq 0, \forall x \in \Omega^- \cup \Omega^+$ and $[y'](d) \leq 0$, then $y(x) \geq 0, \forall x \in \bar{\Omega}$.

Proof: For the proof refer (Shandru and shanthi, 2015)

Lemma 2: (Stability result) Consider the boundary value problem (1)-(2) subject to the condition $p(x) \geq \alpha > 0, q(x) \geq \beta \geq 0$ If $y \in C^1(\bar{\Omega}) \cap C^2(\Omega^- \cup \Omega^+)$, then

$\|y\|_{\bar{\Omega}} \leq C \max\{|L_1 y(0)|, |L_2 y(1)|, |Ly|_{\Omega^- \cup \Omega^+}\}.$

Proof: For the proof refer (Shandru and shanthi, 2015)

Lemma 3: For each integer k, satisfying $0 \leq k \leq 4$, the solution of y of (1)-(2) satisfy the bounds

$$\left\|y^{(k)}\right\|_{\bar{\Omega}\setminus\{d\}} \leq C\varepsilon^{-k}.$$

Proof: For the proof refer (Shandru and shanthi, 2015)

Lemma 4: Let y_ε be the solution of (P_ε). Then, for $k = 0,1,2,3,$

$$\left|y_\varepsilon^{(k)}(x)\right| \leq C\left\{1 + \varepsilon^{-k}\left(\exp\left(\frac{-\alpha x}{\varepsilon}\right)\right)\right\}, \text{ for all } x \in \Omega^-.$$

Proof: For the proof refer (Shandru and shanthi, 2015)

4.2. Formulation of the numerical scheme

The linear differential equation in equation (1) cannot, in general, be solved analytically because of the dependence of $p(x)$ and $q(x)$ on the spatial coordinate. We divide in to interval $\left(\Omega^- \cup \Omega^+\right)$ in to N equal parts with constant mesh length h. Let $0 = x_1, x_2, ..., x_N = 1$ be mesh points. Then we have $x_i = ih,\ i = 1, 2, ..., N.$

If we consider, the interval $[0,1]$, the discretized form of Eq. (1) becomes

$$\varepsilon y''(x_i) + p(x_i)y'(x_i) - q(x_i)y(x_i) = r(x_i) \qquad x \in (\Omega^- \cup \Omega^+) \qquad (3)$$

For each segment $[x_{i,}x_{i+1}], i = 1, 2, ..., N-1$ the non-polynomial cubic spline $S_\Delta(x)$ has the following form:

$$S_\Delta(x) = a_i + b_i(x - x_i) + c_i(e^{w(x-x_i)} - e^{-w(x-x_i)}) + d_i(e^{w(x-x_i)} + e^{-w(x-x_i)}), \qquad (4)$$

where, a_i, b_i, c_i and d_i are unknown coefficient, and $w \neq 0$ arbitrary parameter which was used to increase the accuracy of the method.

To determine the unknown coefficients in Eq. (4) in terms of y_i, y_{i+1}, M_i and M_{i+1} first we define:

$$\begin{cases} S_\Delta(x_i) = y_i, & S''_\Delta(x_i) = M_i, \\ S_\Delta(x_{i+1}) = y_{i+1}, & S''_\Delta(x_{i+1}) = M_{i+1}. \end{cases} \qquad (5)$$

Differentiating Eq.(4), successively, we get ,

$$S'_\Delta(x) = b_i + c_i w(e^{w(x-x_i)} + e^{-w(x-x_i)}) + d_i w(e^{w(x-x_i)} - e^{-w(x-x_i)}), \qquad (6)$$

$$S''_\Delta(x) = c_i w^2(e^{w(x-x_i)} - e^{-w(x-x_i)}) + d_i w^2(e^{w(x-x_i)} + e^{-w(x-x_i)}), \qquad (7)$$

Substituting M_i in Eq. (5) into Eq. (7), we have

$$S''_\Delta(x_i) = M_i = c_i w^2(e^{w(x_i-x_i)} - e^{-w(x_i-x_i)}) + d_i w^2(e^{w(x_i-x_i)} + e^{-w(x_i-x_i)})$$

$$\Rightarrow M_i = 2d_i w^2$$

$$\Rightarrow d_i = \frac{M_i}{2w^2} \qquad (8)$$

Substituting y_i in Eq. (5) in to Eq.(4),we have

$$S_\Delta(x_i) = y_i = a_i + b_i(x_i - x_i) + c_i(e^{w(x_i-x_i)} - e^{-w(x_i-x_i)}) + d_i(e^{w(x_i-x_i)} + e^{-w(x_i-x_i)})$$

$$\Rightarrow y_i = a_i + 2d_i$$

$$\Rightarrow a_i = y_i - 2d_i \tag{9}$$

Substituting M_{i+1} in Eq. (5) into Eq. (7) , we have:

$$S''_\Delta(x_{i+1}) = M_{i+1} = c_i w^2 (e^{w(x_{i+1}-x_i)} - e^{-w(x_{i+1}-x_i)}) + d_i w^2 (e^{w(x_{i+1}-x_i)} + e^{-w(x_{i+1}-x_i)})$$

$$\Rightarrow M_{i+1} = c_i w^2 (e^{wh} - e^{-wh}) + d_i w^2 (e^{wh} + e^{-wh})$$

$$\Rightarrow c_i w^2 (e^{wh} - e^{-wh}) = M_{i+1} - d_i w^2 (e^{wh} + e^{-wh})$$

$$\Rightarrow c_i = \frac{M_{i+1}}{w^2 (e^{wh} - e^{-wh})} - d_i \frac{(e^{wh} + e^{-wh})}{(e^{wh} - e^{-wh})}. \tag{10}$$

Substituting Eq.(8) into Eq.(10), we get

$$c_i = \frac{M_{i+1}}{w^2 (e^{wh} - e^{-wh})} - \frac{M_i (e^{wh} + e^{-wh})}{2w^2 (e^{wh} - e^{-wh})}. \tag{11}$$

Substituting y_{i+1} in Eq. (5) into Eq. (4) we have:

$$S_\Delta(x_{i+1}) = y_{i+1} = a_i + b_i (x_{i+1} + x_i) + c_i (e^{w(x_{i+1}-x_i)} - e^{-w(x_{i+1}-x_i)}) + d_i (e^{w(x_{i+1}-x_i)} + e^{-w(x_{i+1}-x_i)})$$

$$\Rightarrow y_{i+1} = a_i + b_i h + c_i (e^{wh} - e^{-wh}) + d_i (e^{wh} + e^{-wh})$$

$$\Rightarrow b_i h = y_{i+1} - y_i + \frac{M_i}{w^2} - c_i (e^{wh} - e^{-wh}) - d_i (e^{wh} + e^{-wh})$$

$$\Rightarrow b_i = \frac{y_{i+1} - y_i}{h} + \frac{M_i}{hw^2} - c_i \frac{(e^{wh} - e^{-wh})}{h} - d_i \frac{(e^{wh} + e^{-wh})}{h} \tag{12}$$

Substituting Eq.(8) and Eq.(11) into Eq.(12), we have:

$$\Rightarrow b_i = \frac{y_{i+1} - y_i}{h} + \frac{M_i}{hw^2} - \frac{M_{i+1}}{hw^2} + \frac{M_i (e^{wh} + e^{-wh})}{2hw^2} - \frac{M_i (e^{wh} + e^{-wh})}{2hw^2},$$

$$\Rightarrow b_i = \frac{y_{i+1} - y_i}{h} + \frac{M_i - M_{i+1}}{hw^2}. \tag{13}$$

Therefore the coefficients in Eq. (4) are determined as

$$\begin{cases} a_i = y_i - \dfrac{M_i}{w^2}, & c_i = \dfrac{M_{i+1}}{w^2 (e^\theta - e^{-\theta})} - \dfrac{M_i (e^\theta + e^{-\theta})}{2w^2 (e^\theta - e^{-\theta})}, \\ b_i = \dfrac{y_{i+1} - y_i}{h} + \dfrac{M_i - M_{i+1}}{w\theta}, & d_i = \dfrac{M_i}{2w^2}, \end{cases} \tag{14}$$

where $\theta = wh$.

Using the continuity condition of the first derivative at $x_i, S'_{\Delta-1}(x_i) = S'_\Delta(x_i)$, we have

$$b_{i-1} + wc_{i-1}(e^\theta + e^{-\theta}) + wd_{i-1}(e^\theta - e^{-\theta}) = b_i + 2wc_i. \tag{15}$$

Reducing indices of Eq.(14) by one and substituting into Eq.(15), we obtain

$$\frac{y_i - y_{i-1}}{h} + \frac{M_{i-1} - M_i}{w\theta} + w\left(\frac{2M_i - (e^{\theta} + e^{-\theta})M_{i-1}}{2w^2(e^{\theta} + e^{-\theta})}\right)(e^{\theta} + e^{-\theta}) + \left(\frac{M_{i-1}}{2w^2}\right)w(e^{\theta} - e^{-\theta})$$

$$= \frac{y_{i+1} - y_i}{h} + \frac{M_i - M_{i+1}}{w\theta} + 2w\left(\frac{2M_{i+1} - (e^{\theta} + e^{-\theta})M_i}{2w^2(e^{\theta} + e^{-\theta})}\right)$$

$$\Rightarrow \frac{y_{i-1} - 2y_i + y_{i+1}}{h} = \alpha M_{i-1} + 2\beta M_i + \alpha M_{i+1}, \tag{16}$$

where,

$$\alpha = \frac{1}{\theta^2}\left(1 - \frac{2\theta}{(e^{\theta} - e^{-\theta})}\right) \text{ and } \beta = \frac{1}{\theta^2}\left(\frac{\theta(e^{\theta} + e^{-\theta})}{(e^{\theta} - e^{-\theta})} - 1\right).$$

For $h \to 0, \theta \to 0,$ since $\theta = wh,$ as $\theta \to 0$ by using L-Hopital's rule, we obtain

$$\lim_{\theta \to 0} \alpha = \frac{1}{6} \text{ and } \beta = \frac{1}{3}.$$

Using $S''_{\Delta}(x_i) = y'' = M_i$ in to Eq.(3), we get

$$\begin{cases} \varepsilon M_i = r_i - p_i y'_i - q_i y_i \\ \varepsilon M_{i-1} = r_{i-1} - p_{i-1} y'_{i-1} - q_{i-1} y_{i-1} \\ \varepsilon M_{i+1} = r_{i+1} - p_{i+1} y'_{i+1} - q_{i+1} y_{i+1} \end{cases} \tag{17}$$

Using Taylor's series expansions of $y_{i-1}, y_{i+1}, y'_{i-1}, y'_{i+1}$ and simplifying, we have

$$\begin{cases} y'_i = \dfrac{y_{i+1} - y_{i-1}}{2h} + T_{1,} \\ y'_{i-1} = \dfrac{-y_{i+1} + 4y_i - 3y_{i-1}}{2h} + T_2 \\ y'_{i+1} = \dfrac{3y_{i+1} - 4y_i + y_{i-1}}{2h} + T_2, \end{cases} \tag{18}$$

where

$$T_1 = -\frac{h^2}{6} y'''(\xi) \text{ and } T_2 = \frac{h^2}{12} y'''(\xi), \; for \xi \in (x_{i-1}, x_i).$$

Using Eq. (18) in to Eq. (17), we get

$$
\begin{cases}
M_i = \dfrac{1}{\varepsilon}\left\{r_i - p_i\left(\dfrac{y_{i+1}-y_{i-1}}{2h}+T_1\right)-q_i y_i\right\} \\
M_{i-1} = \dfrac{1}{\varepsilon}\left\{r_{i-1} - p_{i-1}\left(\dfrac{-y_{i+1}+4y_i-3y_{i-1}}{2h}+T_2\right)-q_{i-1} y_{i-1}\right\} \\
M_{i+1} = \dfrac{1}{\varepsilon}\left\{r_{i+1} - p_{i+1}\left(\dfrac{3y_{i+1}-4y_i-y_{i-1}}{2h}+T_2\right)-q_{i+1} y_{i+1}\right\}
\end{cases}
\tag{19}
$$

Substituting Eq. (19) into Eq. (16) and rearranging, we get

$$
\frac{\varepsilon}{h^2}\left(y_{i-1}-2y_i+y_{i+1}\right)+\frac{\alpha p_{i-1}}{2h}\left(-y_{i+1}+4y_i-3y_{i-1}\right)+\frac{2\beta p_i}{2h}\left(y_{i+1}-y_{i-1}\right)+\frac{\alpha p_{i+1}}{2h}\left(3y_{i+1}-4y_i-y_{i-1}\right)
$$
$$
=\alpha(R_{i-1}-q_{i-1}y_{i-1}+R_{i+1}-q_{i+1}y_{i+1})+2\beta(R_i-q_i y_i)+T, \tag{20}
$$

where $T=(4\beta p_i-\alpha p_{i-1}-\alpha p_{i+1})\dfrac{h^2}{12}y'''(\xi)$ is the local truncation error.

From the theory of singular perturbations described in O'Malley, (1991) and the Taylor's series expansion of *p(x)* about the point '0' in the asymptotic solution of the problem in Eq.(3), we have

$$
y(x_i)\approx y_0(x_i)+(\phi_0-y_0(0))e^{-p(0)\frac{ih}{\varepsilon}}
$$

and letting $\rho=\dfrac{h}{\varepsilon}$, we get

$$
\lim_{h\to 0} y(ih)\approx y_0(0)+(\phi_0-y_0(0))e^{-p(0)i\rho}
$$

Introducing fitting factor $\sigma(\rho)$ in to Eq. (20), we get

$$
\begin{aligned}
&\frac{\sigma(\rho)\varepsilon}{h^2}(y_{i-1}-2y_i+y_{i+1})+\frac{\alpha p_{i-1}}{2h}(-y_{i+1}+4y_i-3y_{i-1})+\frac{2\beta p_i}{2h}(y_{i+1}-y_{i-1}) \\
&+\frac{\alpha p_{i+1}}{2h}(3y_{i+1}-4y_i+y_{i-1})=\alpha(R_{i-1}-q_{i-1}y_{i-1}+R_{i+1}-q_{i+1}y_{i+1})+2\beta(R_i-q_i y_i)+T
\end{aligned}
\tag{21}
$$

Multiplying Eq. (21) by and taking a limit as $h\to 0$, we get :

$$
\begin{aligned}
&\frac{\sigma}{\rho}\lim_{h\to 0}(y_{i-1}-2y_i+y_{i+1})+\frac{\alpha p(0)}{2}\lim_{h\to 0}(-y_{i+1}+4y_i-3y_{i-1})+\beta p(0)\lim_{h\to 0}(y_{i+1}-y_{i-1}) \\
&+\frac{\alpha p(0)}{2}(3y_{i+1}-4y_i+y_{i-1})=0
\end{aligned}
\tag{22}
$$

Thus, we consider the left boundary layers.

For *p(x)* > 0 (left –end boundary layer), we have

$$\begin{cases} \lim_{h\to 0}(y_{i-1}-2y_i+y_{i+1})=(\phi_0-y_0(0))e^{-p(0)i\rho}(e^{p(0)\rho}+e^{-p(0)\rho}-2) \\ \lim_{h\to 0}(-y_{i+1}-4y_i-3y_{i-1})=(\phi_0-y_0(0))e^{-p(0)i\rho}(-3e^{p(0)\rho}-e^{-p(0)\rho}+4) \\ \lim_{h\to 0}(y_{i+1}-y_{i-1})=(\phi_0-y_0(0))e^{-p(0)i\rho}(e^{p(0)\rho}+3e^{-p(0)\rho}-4) \\ \lim_{h\to 0}(3y_{i+1}-3y_i+y_{i-1})=(\phi_0-y_0(0))e^{-p(0)i\rho}(e^{-p(0)\rho}-e^{p(0)\rho}) \end{cases} \tag{23}$$

Substituting Eq. (22) in to (23) and simplifying, we get

$$\sigma_0=\rho p(0)(\alpha+\beta)\coth\left(\frac{p(0)\rho}{2}\right).$$

In general,we take a variable fitting parameter as

$$\sigma_i=\rho_i p(x_i)(\alpha+\beta)\coth\left(\frac{p(x_i)\rho_i}{2}\right) \tag{24}$$

$$where, \rho=\frac{h}{\varepsilon}.$$

Thus,(21) can be written as

$$\begin{aligned} &\left\{\frac{\varepsilon\delta_i}{h^2}-\frac{3\alpha p_{i-1}}{2h}+\alpha q_{i-1}-\frac{\beta p_i}{h}+\frac{\alpha p_{i+1}}{2h}\right\}y_{i-1}-\left\{\frac{2\varepsilon\delta_i}{h^2}-\frac{2\alpha p_{i-1}}{h}-2\beta q_i+\frac{2\alpha p_{i+1}}{h}\right\}y_i \\ &+\left\{\frac{\varepsilon\delta_i}{h^2}-\frac{\alpha p_{i-1}}{2h}+\alpha q_{i+1}-\frac{\beta p_i}{h}+\frac{3\alpha p_{i+1}}{2h}\right\}y_{i+1}=\alpha(R_{i-1}+R_{i+1})+2\beta R_i \end{aligned} \tag{25}$$

Further, (21) can be written as three term recurrence relation of the form

$$L^N\equiv E_i y_{i-1}-F_i y_i+G_i y_{i+1}=H_i, i=1,2,...,N-1, \tag{26}$$

where

$$\begin{cases} E_i=\frac{\varepsilon\delta_i}{h^2}-\frac{3\alpha p_{i-1}}{2h}+\alpha q_{i-1}-\frac{\beta p_i}{h}+\frac{\alpha p_{i+1}}{2h} \\ F_i=\frac{2\varepsilon\delta_i}{h^2}-\frac{2\alpha p_{i-1}}{h}-2\beta q_i+\frac{2\alpha p_{i+1}}{h} \\ G_i=\frac{\varepsilon\delta_i}{h^2}-\frac{\alpha p_{i-1}}{2h}+\alpha q_{i+1}-\frac{\beta p_i}{h}+\frac{3\alpha p_{i+1}}{2h} \\ H_i=\alpha(R_{i-1}+R_{i+1})+2\beta R_i \end{cases}$$

The tri-diagonal system in Eq. (26) can be easily solved by the method of Discrete Invariant Imbedding Algorithm.

4.3. Truncation Error

Let expand the term $y \pm 1$ and $M \pm 1$ from (16), using Taylors series as

$$\begin{cases} y_{i+1} = y_i + hy_i' + \frac{h^2}{2!}y_i'' + \frac{h^3}{3!}y_i''' + \frac{h^4}{4!}y_i^{(4)} + \frac{h^5}{5!}y_i^{(5)} + \frac{h^6}{6!}y_i^{(6)} + O(h^7), \\ y_{i-1} = y_i - hy_i' + \frac{h^2}{2!}y_i'' - \frac{h^3}{3!}y_i''' + \frac{h^4}{4!}y_i^{(4)} - \frac{h^5}{5!}y_i^{(5)} + \frac{h^6}{6!}y_i^{(6)} + O(h^7), \\ M_{i+1} = y_{i+1}'' = y_i'' + hy_i''' + \frac{h^2}{2!}y_i^{(4)} + \frac{h^3}{3!}y_i^{(5)} + \frac{h^4}{4!}y_i^{(6)} + O(h^7), \\ M_{i-1} = y_{i-1}'' = y_i'' - hy_i''' + \frac{h^2}{2!}y_i^{(4)} - \frac{h^3}{3!}y_i^{(5)} + \frac{h^4}{4!}y_i^{(6)} + O(h^7), \end{cases} \tag{27}$$

The local truncation error $T_i(h)$ obtain from Eq. (16) as

$$T_i(h) = \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} - \alpha(M_{i-1} + M_{i+1}) - 2\beta M_i \tag{28}$$

Substituting the series of $y_{i\pm1}$ and $M_{i\pm1}$ from (26) into (27) and collecting like terms gives

$$T_i(h) = (1 - 2(\alpha + \beta))y_i'' + h^2(\frac{1}{12} - \alpha)y_i^{(4)} + O(h^4) \tag{29}$$

But from the values of $\alpha = \frac{1}{6}$ and $\beta = \frac{1}{3}$, Eq. (29) becomes

$$T_i(h) = h^2(-\frac{1}{12})y_i^{(4)} + O(h^4),$$

which implies

$$\|T_i(h)\| \le Ch^2, \tag{30}$$

where $C = \frac{1}{12}\left|y_i^{(4)}\right|$.

This establishes that the developed method is second order accurate or its order of convergence is O(h^2).

To treat the boundary condition we used forward finite difference formula for $i = 0$ and backward difference formula for $i = N$ respectively for the first derivative term.

That is, for $i = 0$, from Eq. (2), we have

$$\alpha_1 y(0) - \beta_1 \varepsilon y'(0) = p \Rightarrow \alpha_1 y_0 - \beta_1 \varepsilon y_0' = p$$

$$\Rightarrow \alpha_1 y_0 - \beta_1 \varepsilon \left[\frac{y_1 - y_0}{h} \right] = p$$

$$\Rightarrow \left(\alpha_1 + \frac{\beta_1 \varepsilon}{h} \right) y_0 - \frac{\beta_1 \varepsilon}{h} y_1 = p \tag{31}$$

Similarly, for $i = N$, from Eq. (2), we have

$$\alpha_2 y(N) + \beta_2 y'(N) = q \Rightarrow \alpha_2 y_N + \beta_2 y_N' = q$$

$$\Rightarrow \alpha_2 y_N + \beta_2 \left[\frac{y_N - y_{N-1}}{h} \right] = q$$

$$\Rightarrow -\frac{\beta_2}{h} y_{N-1} + \left(\alpha_2 + \frac{\beta_2}{h} \right) y_N = q \tag{32}$$

Thefore, the problem in Eq. (1) with given boundary condition in Eq. (2), can be solved using the scheme Eqs. (26), (31) and (32) which forms $N \times N$ system of algebraic equations.

4.4. Convergence Analysis

Local truncation error refers to the differences between the original differential equation and its finite difference approximation at a mesh points. Finite difference scheme is called consistent if the limit of truncation error $(T_i(h))$ is equal to zero as the mesh size h goes to zero. Hence, the proposed method in (26) with local truncation error in Eq. (30) satisfy the definition of consistency as

$$\lim_{h \to 0} T_i(h) = \lim_{h \to 0} Ch^2 = 0 \tag{33}$$

Thus, the proposed scheme is consistent.

4.5. Stability Analysis

Consider the developed scheme in (26)

$$E_i y_{i-1} - F_i y_i + G_i y_{i+1} = H_i, \tag{34}$$

where, the coefficients E_i, F_i and G_i are as in Eq. (26).

If we multiply both sides of (26) by h^2 and consider the value of E_i, F_i and G_i for sufficiently small *h*, we get

$$E_i = G_i = \varepsilon\delta_i, F_i = 2\varepsilon\delta_i, \tag{35}$$

Considering Eq. (35) in to Eq. (26) the one which is multiplied by h^2 the developed scheme can be written in matrix form

$$AY=B, \tag{36}$$

where the matrixes

$$A=\begin{bmatrix} -2\varepsilon\sigma_i & \varepsilon\sigma_i & 0 & \cdots & 0 \\ \varepsilon\sigma_i & -2\varepsilon\sigma_i & \varepsilon\sigma_i & \cdots & 0 \\ 0 & - & - & & 0 \\ \vdots & & & & \varepsilon\sigma_i \\ 0 & - & - & \varepsilon\sigma_i & -2\varepsilon\sigma_i \end{bmatrix}, Y=\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{N-2} \\ y_{N-1} \end{bmatrix} \text{ and } B=\begin{bmatrix} h^2H_1 - E_1y_0 \\ h^2H_2 \\ \vdots \\ h^2H_{N-2} \\ h^2H_{N-1} - G_{N-1}y_N \end{bmatrix}$$

Matrix *A* is a tri-diagonal matrix with size (*N*-1)*(N-1). Matrix A is irreducible if its co-diagonal s contains non-zero elements only. The co-diagonals contains E_i and G_i. It is clearly seen that, for sufficiently small h and $E_i \neq 0$ and $F_i \neq 0$, for $i = 1,2,3,...,N-1$. Hence, A is irreducible. Again we can see that all $|E_i|$, $|F_i|$, $|G_i| > 0$, for $i = 1,2,3,...,N-1$ and in each row of A, the modulus of diagonal element is greater than or equal to the sum of modulus of the two co-diagonal elements i.e., $|F_i| \geq |E_i| + |G_i|$.This implies that *A* is diagonally dominant .Under this condition the Thomas Algorithm is stable for sufficiently small *h*.

The eigenvalues of a tri-diagonal matrix A are given by

$$\lambda_s = -2\varepsilon\sigma_i + 2\left\{\sqrt{(\varepsilon\sigma_i)(\varepsilon\sigma_i)}\right\}\cos\frac{s\pi}{N}, s = 1(1)N-1. \tag{37}$$

Hence, the eigenvalue of matrix A in Eq.(35)are

$$\lambda_s = -2\varepsilon\sigma_i + 2\left\{\sqrt{\left(\varepsilon\sigma_i\right)^2}\right\}\cos\frac{s\pi}{N} = -2\varepsilon\sigma_i(1-\cos\frac{s\pi}{N}), s = 1(1)N-1. \tag{38}$$

But from trigonometric identity, we have $1-\cos\frac{s\pi}{N} = 2\sin^2\frac{s\pi}{N}$.

Thus, the eigenvalues of A becomes

$$\lambda_s = -2\varepsilon\sigma_i(2\sin^2\frac{s\pi}{N}) = -4\varepsilon\sigma_i \sin^2\frac{s\pi}{2N} \le -4\varepsilon\sigma_i \quad (39)$$

A finite difference method for the boundary value problems is stable if A is non-singular and $\|A^{-1}\| \le C,$ for $0 < h < h_{0,}$ where, C and $h_{0,}$ are two constants that are independent of h.

Since A is real and symmetric it follows that A^{-1} is also real and symmetric so that, its eigenvalue are real and given by $\frac{1}{\lambda_s}$.

Hence, as Siraj et al.(2019), the stability condition of the method will be satisfied when; $\|A^{-1}\| = \left|\frac{1}{\lambda_s}\right| = \left|\frac{-1}{4\varepsilon\sigma_i}\right| \le \frac{1}{4\varepsilon\sigma_i} \le C,$ where C is in dependentof h.

Thus, the developed scheme in Eq.(26) is stable. A consistent and stable finite difference method is convergent by Smith (1985). Hence, as we have shown above, the proposed method is satisfying both the criteria of consistency and stability which are equivalent to convergence of the method.

4.6 Numerical Example and Results

To validate the established theoretical results, we perform numerical experiments using the model problem of the form in Eqs. (1)-(2).

Having $y_j \equiv y_j^h$ (the approx. solution obtained via the present method) for different values of h and ε, since the exact solution is not available; the maximum errors (denoted by E_ε^h) are evaluated using the formula given by the double mesh principle

$$E_\varepsilon^h := \max_{0\le j\le N}\left|y_j^h - y_{2j}^{2h}\right|.$$

Further, we will tabulate the errors

$$E^h = \max_{0<\varepsilon\le 1} E_\varepsilon^h.$$

The numerical rates of convergence are computed using the formula

$$r_\varepsilon^h := \log_2^{\left(\frac{E_\varepsilon^h}{E_\varepsilon^{\frac{h}{2}}}\right)}$$

and the numerical rate of "ε-uniform convergence" is computed using

$$R^h = \log_2^{\left(\frac{E^h}{E^{\frac{h}{2}}}\right)}.$$

Example 4.1

$$\varepsilon y''(x) + y'(x) = f(x), \quad x \in \Omega^- \cup \Omega^+,$$
$$y(0) - \varepsilon y'(0) = 1, \; y(1) + y'(1) = -1,$$

where
$$f(x) = \begin{cases} 0.7, & \text{for } 0 \le x \le 0.5, \\ -0.6, & \text{for } 0.5 < x \le 1. \end{cases}$$

Table 4.1: Maximum absolute errors and order of convergence for Example 4.1 at different mesh points N

ε	$N = 32$	$N = 64$	$N = 128$	$N = 256$	$N = 512$
10^{-4}	2.0313e-02	1.0156 e-02	5.0781e-03	2.5391e-03	1.2695e-03
10^{-8}	2.0313e-02	1.0156 e-02	5.0781e-03	2.5391e-03	1.2695e-03
10^{-12}	2.0313e-02	1.0156 e-02	5.0781e-03	2.5391e-03	1.2695e-03
10^{-16}	2.0313e-02	1.0156 e-02	5.0781e-03	2.5391e-03	1.2695e-03
10^{-20}	2.0313e-02	1.0156 e-02	5.0781e-03	2.5391e-03	1.2695e-03

Table 4.2: Comparison of Maximum absolute errors and order of convergence for Example 4.1 at different of mesh points N

$\varepsilon \downarrow N \rightarrow$	64	128	256	512	1024
Present Method					
2^{0}	1.2486e-03	6.2308e-04	3.1124e-04	1.5554e-04	7.7752e-05
2^{-2}	3.9279e-03	1.9488e-03	9.7064e-04	4.8438e-04	2.4195e-04
2^{-4}	5.3915e-03	2.6166e-03	1.2885e-03	6.3930e-04	3.1841e-04
2^{-6}	6.3219e-03	2.8548e-03	1.3488e-03	6.5460e-04	3.2234e-04
2^{-8}	8.9456e-03	3.7124e-03	1.5805e-03	7.1370e-04	3.3719e-04
2^{-10}	1.0153e-02	4.9868e-03	2.2364e-03	9.2810e-04	3.9512e-04
2^{-12}	1.0156e-02	5.0781e-03	2.5382e-03	1.2467e-03	5.5910e-04
E_N	**1.0156e-02**	**5.0781e-03**	**2.5382e-03**	**1.2467e-03**	**5.5910e-04**
R_N	**1.0000**	**1.0007**	**1.0257**	**1.0257**	
Chandru and Shanthi, (2015)					
2^{0}	1.1329e−02	5.5712e−03	2.6949e−03	1.2574e−03	5.3885e−04
2^{-2}	8.7403e−03	4.4761e−03	2.2085e−03	1.0406e−03	4.4812e−04
2^{-4}	2.0405e−02	1.0005e−02	4.8332e−03	2.2538e−03	9.6554e−04

2^{-6}	2.0072e−02	1.1136e−02	6.0306e−03	3.1191e−03	1.4580e−03
2^{-8}	2.3563e−02	1.2830e−02	6.7717e−03	3.4147e−03	1.5689e−03
2^{-10}	2.5198e−02	1.3801e−02	7.3129e−03	3.6779e−03	1.6744e−03
2^{-12}	2.5658e−02	1.4128e−02	7.5359e−03	3.8225e−03	1.7536e−03
E_N	**2.5658e−02**	**1.4128e−02**	**7.5359e−03**	**3.8225e−03**	**1.7536e−03**
R_N	**0.86085**	**0.90671**	**0.97927**	**1.1242**	

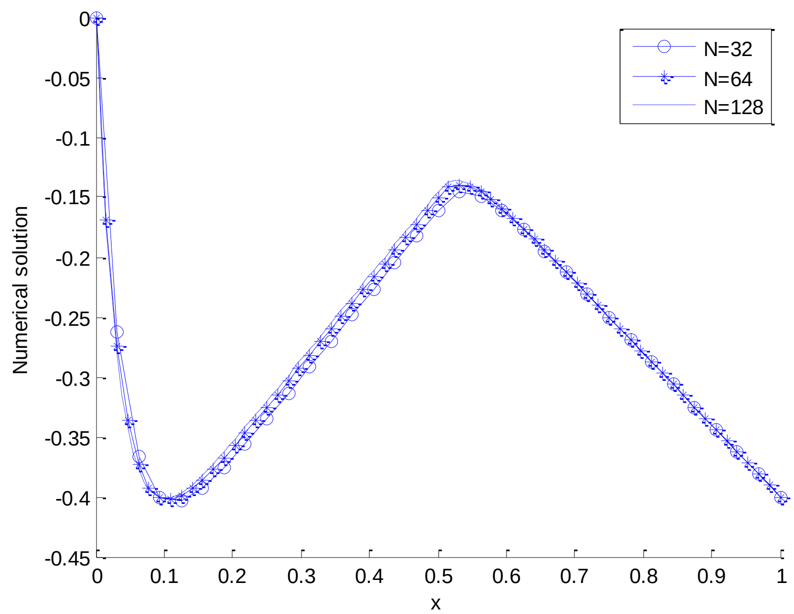

Figure 4.1: Solution plot of N = 32, 64, 128 and $\varepsilon=2^{-5}$ for the Example 4.1

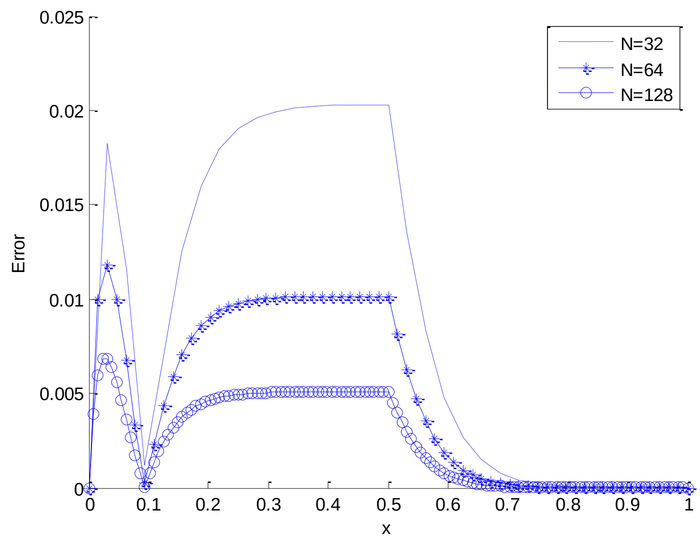

Figure 4.2: Point wise absolute error plot of N = 32, 64, 128 and $\varepsilon=2^{-5}$ for the Example 4.1.

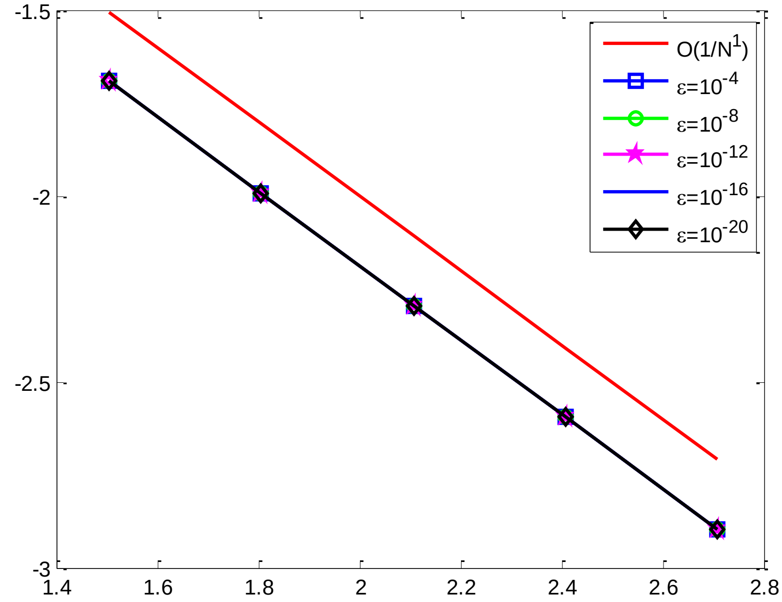

Figure 4.3: ε-uniform convergence with NSFDM in Log-Log scale for Example 4.2.

Example 4.2

$$\varepsilon y''(x)+\frac{1}{1+x}y'(x)=f(x),\quad x\in\Omega^{-}\cup\Omega^{+},$$
$$y(0)-\varepsilon y'(0)=1,\ y(1)+y'(1)=1,$$

where
$$f(x)=\begin{cases}1+x, & \text{for } 0\le x\le 0.5,\\ 4.0, & \text{for } 0.5<x\le 1.\end{cases}$$

Table 4.3: Maximum absolute errors and order of convergence for Example 4.2 at different mesh points N

$\varepsilon\downarrow N\rightarrow$	$N=32$	$N=64$	$N=128$	$N=256$	$N=512$
10^{-4}	8.3431e-02	4.1854e-02	2.0962e-02	1.0489e-02	5.2329e-03
10^{-8}	8.3431e-02	4.1854e-02	2.0962e-02	1.0489e-02	5.2329e-03
10^{-12}	8.3431e-02	4.1854e-02	2.0962e-02	1.0489e-02	5.2329e-03
10^{-16}	8.3431e-02	4.1854e-02	2.0962e-02	1.0489e-02	5.2329e-03
10^{-20}	8.3431e-02	4.1854e-02	2.0962e-02	1.0489e-02	5.2329e-03

Table 4.4: Comparison of Maximum absolute errors and order of convergence for Example 4.2 at number of mesh points N

$\varepsilon \downarrow N \rightarrow$	64	128	256	512	1024
Present Method					
2^{0}	2.4822e-03	1.2368e-03	6.1735e-04	3.0841e-04	1.5414e-04
2^{-2}	9.2976e-03	4.5948e-03	2.2838e-03	1.1385e-03	5.6841e-04
2^{-4}	1.6317e-02	7.9207e-03	3.9029e-03	1.9371e-03	9.6494e-04
2^{-6}	1.9827e-02	8.5708e-03	3.9917e-03	1.9270e-03	9.4663e-04
2^{-8}	3.2254e-02	1.2786e-02	4.9464e-03	2.1290e-03	9.8941e-04
2^{-10}	4.1526e-02	1.9138e-02	8.0742e-03	3.2014e-03	1.2359e-03
2^{-12}	4.1854e-02	2.0960e-02	1.0408e-02	4.7906e-03	2.0192e-03
E_N	**4.1854e-02**	**2.0960e-02**	**1.0408e-02**	**4.7906e-03**	**2.0192e-03**
R_N	**0.9977**	**1.0099**	**1.1194**	**1.2464**	
Chandru and Shanthi, (2015)					
2^{0}	6.6163e−02	3.2658e−02	1.5827e−02	7.3916e−03	3.1691e−03
2^{-2}	2.1351e−01	1.0537e−01	5.1051e−02	2.3838e−02	1.0219e−02
2^{-4}	4.7760e−01	2.4152e−01	1.1853e−01	5.5717e−02	2.3969e−02
2^{-6}	9.0232e−01	5.4579e−01	3.0795e−01	1.6300e−01	7.7457e−02
2^{-8}	9.4993e−01	5.7117e−01	3.2238e−01	1.7024e−01	8.0147e−02
2^{-10}	9.6347e−01	5.7862e−01	3.2680e−01	1.7253e−01	8.1223e−02
2^{-12}	9.6698e−01	5.8056e−01	3.2795e−01	1.7313e−01	8.1501e−02
E_N	**9.6698e−01**	**5.8056e−01**	**3.2795e−01**	**1.7313e−01**	**8.1501e−02**
R_N	**0.73604**	**0.82399**	**0.92161**	**1.08700**	

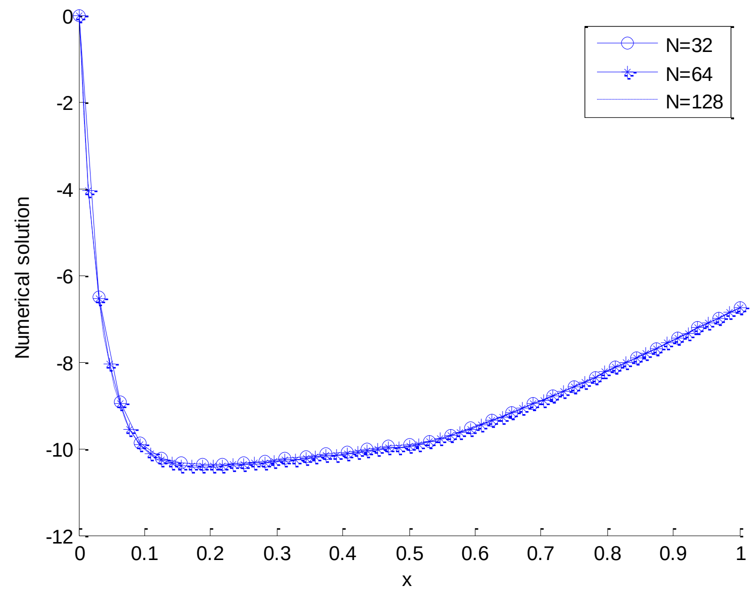

Figure 4.4: Solution plot of N = 32, 64, 128 and $\varepsilon=2^{-5}$ for the Example 4.2.

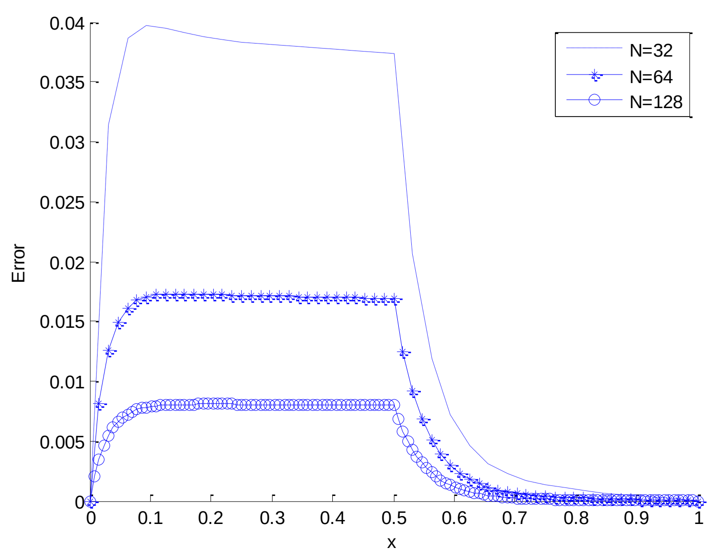

Figure: 4.5 Point wise absolute error plot of N = 32, 64, 128 and $\varepsilon=2^{-5}$ for the Example 4.2.

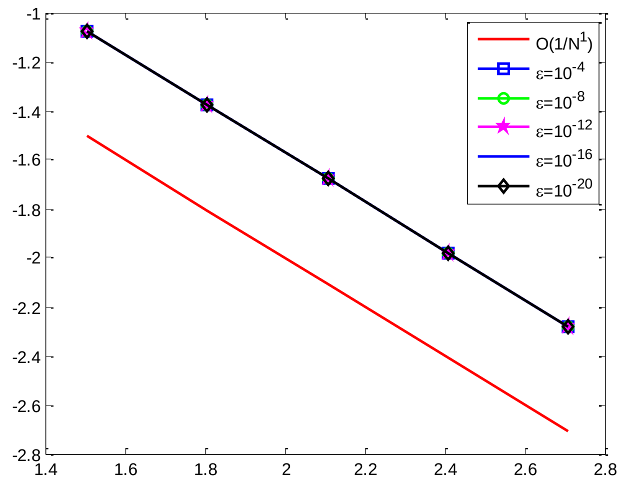

Figure 4.6: ε-uniform convergence with NSFDM in Log-Log scale for Example 4.2.

4.4. Discussion

The numerical results are tabulated in terms of maximum absolute errors, numerical rate of convergence and uniform errors (see Tables 4.1-4.4) and compared with the results of the previously developed numerical methods existing in the literature (Table 4.2 and 4.4). Further, the ε-uniform convergence of the method is shown by the log-log plot of the ε-uniform error (Figure 4.2 and 4.4) and the numerical solution for various values of N and ε are given (see Figure 4.2-4.4). Unlike other fitted finite difference methods constructed in standard ways, the method that we presented in this thesis is relatively simple to construct.

CHAPTER FIVE

CONCLUSION AND SCOPE FOR FUTURE WORK

5.1 Conclusion

This study introduces uniformly convergent numerical method based on fitted non-polynomial cubic spline method for solving singularly perturbed second-order ODEs of Robin type BVPs with discontinuous source term. Due to discontinuity in the source term there is an interior layer occurring. To fit the interior and boundary layer a suitable fitted operator method on uniform mesh is constructed. The behavior of the continuous solution of the problem is studied and shown that it satisfies the continuous stability estimate and the derivatives of the solution are also bounded. The numerical scheme is developed on uniform mesh. The Robin type BVPs is treated using numerical finite difference techniques; and the results are compared accordingly. The stability of the developed scheme is established and its uniform convergence is proved. To validate the applicability of the method, two model problems are considered for numerical experimentation for different values of the perturbation parameter and mesh points.

5.2. Scope of the Future Work

In this thesis, fitted cubic spline method for solving singularly perturbed second-order ODEs of Robin type BVPs with discontinuous source term is introduced. Hence, the scheme proposed in this thesis can also be extended to higher order fitted finite difference method for solving singularly perturbed robin type boundary value problems with discontinuous source term.

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