CONTINUITY AND COMPACTNESS OF GENERALIZED VOLTERRA TYPE INTEGRAL OPERATORS ON FOCK-SOBOLEV SPACES

A THESIS SUBMITTED TO THE DEPARTMENT OF MATHEMATICS IN PARTIAL FULFILLMENT FOR THE REQUIREMENTS OF THE DEGREE OF MASTERS OF SCIENCE IN MATHEMATICS

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Declaration

I, Mohammed Abdella, with student ID number RM0689/15, declare that this thesis entitled ”CONTINUITY AND COMPACTNESS OF GENERALIZED VOLTERRA TYPE INTEGRAL OPERATORS ON FOCK-SOBOLEV SPACES” is my own original work and it has not been submitted to any institution or University elsewhere for the award of any academic degree and sources of information that I have been used or quoted are indicated and acknowledged.

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Abstract

The study of continuity and compactness properties of Volterra type integral operator on spaces of holomorphic functions over different domains is a wide history. Specifically, on Fock spaces with domain the whole complex plane, it was initiated by (Constantin, 2012) and continued by (Mengestie, 2013). In (Mengestie, 2013) and (Mengestie and Worku, 2018) the authors have considered these properties of the generalized Volterra type integral operators acting between Fock spaces with Gaussian weight $\frac{|z|^2}{2}$. Recently, (Mengestie and Takele, 2023) characterized bounded and compact generalized Volterra type integral operators on generalized Fock spaces with weight functions growing faster than the Gaussian weight. Their result shows that the operator experiences richer bounded and compact structure in this spaces than Fock spaces with Gaussian weight. In this thesis, we studied these properties of generalized Volterra type integral on Fock-Sobolev spaces with weight functions growing slower than Gaussian weight. We first characterize bounded and compact properties in terms Brezin type integral transform and then give a simplified characterization. Our result shows that, the operator has similar structure when compared with Fock spaces with Gaussian weights. The results obtained in this thesis unify and extend a number of results in the area. In particular, it generalizes the results of (Mengestie, 2016), (Mengestie, 2017) and (Mengestie and Worku, 2018).

Contents

Chapter 1

Introduction

1.1 Background of the study

The study of integral operators is an important area of study, as these operators are widely used in various fields of mathematics and science. One such type of integral operator is the Volterra type integral operator, which arises in many areas, such as in control theory and mathematical modeling.

Definition 1.1.1. For a given space $\mathcal{H}(U)$ of analytic functions on U, the Volterra-type integral operator on $\mathcal{H}(U)$, induced by a analytic symbol function g, is defined as

$$V_g f(z) = \int_0^z f(w) g'(w) dw.$$

The operator is first introduced by (Pommerenke, 1977) and then studied by other authors with the aim to investigate the connection between their behaviors and the function-theoretic properties of the symbol g.

A natural question that arises is whether these operators are continuous or compact in the setting of different spaces, as this has significant implications for their properties and behavior. In this context, the study of continuity and compactness of Volterra type integral operators has been the subject of much research in modern analysis.

Definition 1.1.2. Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces and $T : \mathcal{X} \to \mathcal{Y}$ a linear operator.

i) If there is a constant $c > 0$ such that $\|Tx\| \leq c\|x\|$, $x \in \mathcal{X}$, then we say that T is a bounded (or equivalently continuous) linear operator .

ii) If $\|Tx_n\| \to 0$ whenever $x_n \to 0$ weakly in X, then we say that T is compact.

(Pommerenke, 1977) investigated continuity of Volterra type integral operator on the Hardy space H^p, for $p = 2$, and almost after two decades it was extended to whole H^p space, for $0 < p < \infty$, by (Aleman and Siskakis, 1995), further studying compactness property also. Later (Aleman and Siskakis, 1997), gave the analogous characterization on the Bergman space. (Constantin, 2012) and (Mengestie, 2013) considered the problem of continuity and compactness, among with other properties, over a space defined over the whole complex plane $\mathbb{C}$, namely classical Fock spaces $\mathcal{F}_p$.

Definition 1.1.3. Let $0 < p \leq \infty$. Then the classical Fock space, $\mathcal{F}_p$, is space of entire functions f for which

$$\|f\|_p = \begin{cases} \left(\frac{p}{2\pi}\int_{\mathbb{C}} |f(z)|^p e^{-p\frac{|z|^2}{2}} dA(z)\right)^{\frac{1}{p}} < \infty, & 0 < p < \infty \\ \sup_{z\in\mathbb{C}} |f(z)| e^{-\frac{|z|^2}{2}} < \infty, & p = \infty \end{cases}$$

where dA denotes the Lebesgue area measure.

The idea to extend the Volterra type integral operator V_g into a more general operator called generalized Volterra type integral operator, $V_{(g,\psi)}$, was first raised by (Li and Stevic, 2008), as this newly introduced operator has an application in the study of space of linear isometry of analytic functions.

Definition 1.1.4. For a given a space $\mathcal{H}(U)$ of analytic functions on U, the generalized Volterra type integral operator, $V_{(g,\psi)}$, on $\mathcal{H}(U)$, induced by a pair analytic symbol functions (g, ψ), is defined as

$$V_{(g,\psi)} f(z) = \int_0^z f(\psi(w)) g'(w) dw.$$

We note that, if $\psi(z) = z$ then $V_{(g,\psi)}$ reduces to the Volterra type integral operator V_g.

(Li and Stevic, 2008) studied the operator theoretic properties of $V_{(g,\psi)}$ in terms of the inducing pair of symbols (g, ψ) on some spaces of analytic functions on the unit disk. (Mengestie, 2014, and Mengestie and Worku, 2018) studied continuity and compactness of the operator on the classical Fock spaces. Recently, (Mengestie and Takele, 2023) considered the problem over generalized Fock spaces $\mathcal{F}^p_\varphi$ with weight function φ satisfying the following smoothness conditions;

Let $\varphi : [0, \infty) \to [0, \infty)$ be a twice continuously differentiable function, which can be extended to the whole complex plane by setting $\varphi(z) = \varphi(|z|)$, $z \in \mathbb{C}$. We assume that $\Delta\varphi > 0$ and set

$$\tau(z) \simeq \begin{cases} (\Delta\varphi(z))^{-1/2}, & |z| \geq 1 \\ 1, & 0 \leq |z| < 1 \end{cases}$$

where $\tau(z)$ is a radial positive differentiable function satisfying the admissibility conditions

$$\lim_{r\to\infty}\tau(r)=0 \quad \text{and} \quad \lim_{r\to\infty}\tau'(r)=0,$$

and there exists a constant $C>0$ such that $\tau(r)r^C$ increases for large r or

$$\lim_{r\to\infty}\tau'(r)\log\frac{1}{\tau(r)}=0.$$

The notation $U(z)\simeq V(z)$ means both $U(z)\lesssim V(z)$ and $V(z)\lesssim U(z)$, where $U(z)\lesssim V(z)$ (or equivalently $V(z)\gtrsim U(z)$) means that there is a constant C such that $U(z)\le CV(z)$ holds for all z in the set of a question. An example of such kind weight φ function which satisfy the above smoothness and admissibility conditions are power functions $\varphi_\alpha(r)=r^\alpha,\ \alpha>2$ and the exponential type functions such as $\varphi_\beta(r)=e^{\beta r},\ \ \beta>0$.

Definition 1.1.5. Let $0<p\le\infty$ and φ be a weight function satisfying the above conditions. Then, the generalized Fock space $\mathcal{F}^p_\varphi$ is defined as a space consisting of all entire functions f for which

$$\|f\|_{\varphi,p}=\begin{cases}\left(\int_{\mathbb{C}}|f(z)|^pe^{-p\varphi(z)}dA(z)\right)^{\frac{1}{p}}<\infty, & 0<p<\infty\\ \sup_{z\in\mathbb{C}}|f(z)|e^{-p\varphi(z)}<\infty, & p=\infty\end{cases}$$

where dA denotes the usual Lebesgue area measure on $\mathbb{C}$.

The weight function in the generalized Fock space grows faster than the classical weight function in the classical Fock space. The study in (Mengestie and Takele, 2023) aims at investigating the effects of faster growthness of weight function to the properties of the operator. In particular, continuity and compactness of Volterra type integral operator V_g, on generalized Fock space $\mathcal{F}^p_\varphi$, was investigated in (Constantin and Peleaz, 2015, and Mengestie and Ueki, 2019). Results in the above papers show that the operators $V_{(g,\psi)}$ and V_g experience richer property on generalized Fock space than the classical counterpart. A next question to raise is what will happen to these properties if the weight function grows slower than the Gaussian weight function, specifically, on the Fock Sobolev spaces with weight function growing much slower than Gaussian weight in the classical Fock spaces. A result in (Mengestie, 2017) shows that this is not the case for the Volterra type integral operators, that is, it experiences similar continuity and compactness properties when compared with the classical Fock spaces.

Definition 1.1.6. Let $0<p\le\infty$ and m be a nonnegative integer. Then, Fock Sobolev

space, $\mathcal{F}_{(m,p)}$, is a space of all entire functions f for which

$$\|f\|_{(m,p)} = \begin{cases} \left(\int_{\mathbb{C}} |f(z)|^p e^{-p(\frac{1}{2}|z|^2 - m\log(1+|z|))} dA(z) \right)^{\frac{1}{p}} < \infty, & 0 < p < \infty \\ \sup_{z\in\mathbb{C}} |f(z)| e^{-p(\frac{1}{2}|z|^2 - m\log(1+|z|))} < \infty, & p = \infty \end{cases}$$

where dA denotes the usual Lebesgue area measure on $\mathbb{C}$.

In particular, the choice of $m = 0$ reduces the space to the classical Fock space. The space is named after the Russian mathematicians Vladimir Fock and Sergei Sobolev, as an extensions of the classical Sobolev spaces to the setting of holomorphic functions on the complex plane. These spaces play a crucial role in the study of quantum systems with an infinite number of degrees of freedom, such as quantum harmonic oscillators and quantum fields. The central idea is to generalize the concept of Sobolev spaces to functions that are not only smooth but also holomorphic, making them well-suited for addressing problems in the complex domain.

The aim of this thesis is to characterize continuity and compactness of generalized Volterra type integral operator, $V_{(g,\psi)}$, on Fock Sobolev spaces, and investigate whether the operator show similar structure or not, compared with when the operator plays between classical Fock spaces.

1.2 Statement of the problem

Different properties including continuity and compactness of generalized Volterra type integral operators are widely studied acting between different spaces of analytic functions, see for example the materials in (Li and Stevi, 2008 and 2009, Mengestie, 2014, and Mengestie and Worku, 2018). In particular, in the past few years there is a high interest to study properties of the operator on different Fock type spaces. In (Mengestie, 2014, and Mengestie and Worku, 2018), continuity and compactness of the operator have been characterized on classical Fock spaces with Gaussian weight function $\frac{|z|^2}{2}$. Recently, (Mengestie and Takele, 2023) studied these properties on generalized Fock spaces with weight function growing faster than Gaussian weight, showing that the operator experience richer continuity and compactness properties than on the classical case. A natural question to ask is what will happen to those properties if the weight function grows slower than Gaussian weight. So, the purpose of this thesis is to characterize continuity and compactness of generalized Volterra type integral operators on Fock Sobolev spaces, which is a typical example of slower growing weight, and analyze these properties of the operators acting between different Fock type spaces.

1.3 Objectives of the study

1.3.1 General objectives

The general objective of this study is to describe continuity and compactness properties of generalized Volterra type integral operators, $V_{(g,\psi)}$, in terms of function theory of inducing symbol functions g and ψ, acting between Fock Sobolev spaces $\mathcal{F}_{(m,p)}$ and $\mathcal{F}_{(m,q)}$, for $0 < p, q \leq \infty$.

1.3.2 Specific objectives

The specific objective of this study are:

- To find a necessary and sufficient condition for continuity of generalized Volterra type integral operators on Fock Sobolev spaces.
- To find a necessary and sufficient condition for generalized Volterra type integral operators to be compact on Fock Sobolev spaces.
- To find a condition on which both properties continuity and compactness of the operator are equivalent.
- To compare and analyze properties that the operator shows when it acts between different Fock type spaces and Fock Sobolev spaces.

1.4 Significance of the study

The result of this study may have the following importance:

- It may be used as a base for any researcher who is interested to study other properties of generalized Volterra type integral operators on Fock Sobolev spaces.
- It may be applied in the study of linear isometries of spaces of analytic functions and other area's of Physics.
- It may help graduate students to acquire research skills and scientific procedures.

1.5 Delimitation of the study

This study was delimited to establishing continuity and compactness properties of generalized Volterra type integral operators on Fock Sobolev spaces.

Chapter 2

Review of Related Literature

As noted in the introduction, (Pommerenke, 1977) introduced the Volterra type integral operator which was later generalized to the generalized Volterra type integral operator by (Li and Stevic, 2008). In the context of Fock spaces, the generalized Volterra type integral operator was studied by (Mengestie, 2013) and later by (Mengestie and Worku, 2018), which is stated by the following theorem.

Theorem 2.0.1 (Mengestie and Worku, 2018).
Let $0 < p, q \leq \infty$, (g, ψ) be pairs of nonconstant entire functions and

$$M_{(g,\psi)}(z) := \frac{|g'(z)|}{1+|z|} e^{\frac{1}{2}(|\psi(z)|^2 - |z|^2)}.$$

i) If $0 < p \leq q \leq \infty$, then $V_{(g,\psi)} : \mathcal{F}_p \to \mathcal{F}_q$ is continuous (respectively, compact) if and only if the function $M_{(g,\psi)}$ is bounded over $\mathbb{C}$ (respectively, $\lim_{|z|\to\infty} M_{(g,\psi)}(z) = 0$).

ii) If $0 < q < p \leq \infty$, then $V_{(g,\psi)} : \mathcal{F}_p \to \mathcal{F}_q$ is continuous or compact if and only if $\int_{\mathbb{C}} M_{(g,\psi)}^{\frac{pq}{p-q}}(z)$ is finite.

In particular, for the case $\psi(z) = z$, that is, for the Volterra type integral operators V_g, we have the following simplified form of symbol function g for which the operator is continuous or compact.

Theorem 2.0.2 (Constantin, 2012 and Mengestie, 2013).

(i) If $0 < p \leq q \leq \infty$, then $V_g : \mathcal{F}_p \to \mathcal{F}_q$ is continuous (respectively, compact) if and only if $g(z) = az^2 + bz + c$, $a, b, c \in \mathbb{C}$ (respectively, $g(z) = az + b$, $a, b \in \mathbb{C}$).

(ii) If $0 < q < p \leq \infty$, then $V_g : \mathcal{F}_p \to \mathcal{F}_q$ is continuous or compact if and only if $g(z) = az + b$, $a, b \in \mathbb{C}$.

Recently, in 2023, the generalized Volterra type integral was studied by (Mengestie and Takele, 2023) acting between generalized Fock spaces. We will state it by the following theorem. Define a function $M_{(g,\psi,\varphi)}$ to be

$$M_{(g,\psi,\varphi)}(z) := \frac{|g'(z)|}{1+\varphi'(z)} e^{\varphi(\psi(z))-\varphi(z)}.$$

Theorem 2.0.3. *Let $0 < p \le q \le \infty$ and (g,ψ) be a pair of nonconstant entire functions. Then*

(i) $V_{(g,\psi)} : \mathcal{F}^p_\varphi \to \mathcal{F}^q_\varphi$ is continuous if and only if

$$\begin{cases} \sup_{z\in\mathbb{C}}(\Delta\varphi(z))^{\frac{q-p}{pq}} M_{(g,\psi,\varphi)}(z) < \infty, & p \le q < \infty \\ \sup_{z\in\mathbb{C}}(\Delta\varphi(z))^{\frac{1}{p}} M_{(g,\psi,\varphi)}(z) < \infty, & p < q = \infty \\ \sup_{z\in\mathbb{C}} M_{(g,\psi,\varphi)}(z) < \infty, & p = q = \infty. \end{cases}$$

(ii) $V_{(g,\psi)} : \mathcal{F}^p_\varphi \to \mathcal{F}^q_\varphi$ is compact if and only if

$$\begin{cases} \lim_{|z|\to\infty}(\Delta\varphi(z))^{\frac{q-p}{pq}} M_{(g,\psi,\varphi)}(z) = 0, & p \le q < \infty \\ \lim_{|z|\to\infty}(\Delta\varphi(z))^{\frac{1}{p}} M_{(g,\psi,\varphi)}(z) = 0, & p < q = \infty \\ \lim_{|z|\to\infty} M_{(g,\psi,\varphi)}(z) = 0, & p = q = \infty. \end{cases}$$

For the case where the operator maps from $\mathcal{F}^p_\varphi$ to $\mathcal{F}^q_\varphi$ with $0 < q < p \le \infty$, we have the following.

Theorem 2.0.4. *Let $0 < q < p \le \infty$ and (g,ψ) a pair of nonconstant entire functions. Then the following are equivalent.*

(i) $V_{(g,\psi)} : \mathcal{F}^p_\varphi \to \mathcal{F}^q_\Phi$ is compact.

(ii) $V_{(g,\psi)} : \mathcal{F}^p_\varphi \to \mathcal{F}^q_\Phi$ is continuous.

(iii) The function $M_{(g,\psi,\varphi)} \in L^r(\mathbb{C}, dm)$, where $r = \begin{cases} \frac{pq}{p-q}, & p < \infty \\ q, & p = \infty. \end{cases}$.

The purpose of these thesis was to find analogous characterization of boundedness and compactness for generalized Volterra type integral operator on Fock-Sobelov spaces.

Chapter 3

Methodology of the study

3.1 Study area and Period

The study was conducted in Jimma University department of mathematics under the functional analysis stream from September, 2023 G.C. to June, 2024 G.C.

3.2 Study design

In this research work we employed analytical method of design.

3.3 Source of information

The relevant sources of information for this study were books and published articles.

3.4 Mathematical Procedure of the study

The mathematical procedure we followed for this research work are the following:

- Providing a sufficient and necessary condition for continuity and compactness of the generalized Volterra type integral operators on Fock Sobolev spaces.
- Characterizing continuity and compactness of Volterra type integral operator on the spaces.
- Establishing and proving theorems.

Chapter 4

Main results

4.1 Preliminaries

In this section, we give some definitions and theorems which are used for this work.

Let $0 < p \leq \infty$ and $0 < q < \infty$. Then we call a nonnegative measure μ on $\mathbb{C}$;

(i) a (p, q) Fock-Carleson measure for Fock-Sobolev spaces if

$$\int_{\mathbb{C}} |f(z)|^q e^{-\frac{q}{2}|z|^2} d\mu(z) \lesssim \|f\|^q_{(m,p)}$$

for all $f \in \mathcal{F}_{(m,p)}$.

(ii) a vanishing (p, q) Fock-Carleson measure for Fock-Sobolev spaces if

$$\lim_{j \to \infty} \int_{\mathbb{C}} |f_j(z)|^q e^{-\frac{q}{2}|z|^2} d\mu(z) = 0$$

whenever f_j is a uniformly bounded sequence in $\mathcal{F}_{(m,p)}$ that converges to zero on a compact subsets of $\mathbb{C}$.

In other words, μ is a (p, q) Fock-Carleson measure for Fock-Sobolev space (respectively, a vanishing (p, q) Fock-Carleson measure for Fock-Sobolev space) if and only if the embedding map $I_\mu : \mathcal{F}_{(m,p)} \to L^q(\lambda_q)$ is bounded (respectively, compact), where $d\lambda_q(z) = e^{-\frac{q}{2}|z|^2} d\mu(z)$. These type of measures for Fock-Sobolev spaces have been completely characterized by (Mengestie, 2015), see Theorem 2.1–2.4, which we stated by the following two theorems for further use. For this, we define (t, s)-Berezin type integral transform, $t, s > 0$, of μ by

$$\tilde{\mu}_{(t,s)}(w) = \int_{\mathbb{C}} \frac{e^{-\frac{t}{2}|z-w|^2}}{(1 + |z|)^s} d\mu(z).$$

Shortly, we write L^p for the space $L^p(\mathbb{C}, dA)$.

Theorem 4.1.1 (Mengestie, 2015). *Let $0 < p \leq q < \infty$ and $\mu \geq 0$. Then*

(i) μ is a (p, q) Fock-Carleson measure for Fock-Sobolev space if and only if $\tilde{\mu}_{(t,mq)} \in L^{\infty}$ for some (or any) $t > 0$. Moreover, $\|\mu\|^q \simeq \|\tilde{\mu}_{(t,mq)}\|_{L^{\infty}}$.

(ii) μ is a vanishing (p, q) Fock-Carleson measure for Fock-Sobolev space if and only if $\tilde{\mu}_{(t,mq)} \to 0$ as $|z| \to \infty$ for some (or any) $t > 0$.

Theorem 4.1.2 (Mengestie, 2015). *Let $0 < q < p \leq \infty$ and $\mu \geq 0$. Then the following are equivalent;*

(i) μ is a (p, q) Fock-Carleson measure for Fock-Sobolev space.

(ii) μ is a vanishing (p, q) Fock-Carleson measure for Fock-Sobolev space.

(iii) $\tilde{\mu}_{(t,mq)} \in \begin{cases} L^{\frac{p}{p-q}}, & p < \infty \\ L^1, & p = \infty. \end{cases}$ *for some (or any) $t > 0$.*

Moreover, $\|\mu\|^q \simeq \begin{cases} \|\tilde{\mu}_{(t,mq)}\|_{L^{\frac{p}{p-q}}}, & p < \infty \\ \|\tilde{\mu}_{(t,mq)}\|_{L^1}, & p = \infty. \end{cases}$.

In (Mengestie, 2015 and 2016), the spaces $\mathcal{F}_{(m,p)}$ have been characterized in terms of Littlewood–Paley type derivative formula and the characterization for the case $m = 0$ was given by (Constantin, 2012) for finite p and (Mengestie, 2013) for p is infinite.

Lemma 4.1.3 (Mengestie, 2015 and 2016). *For $f \in \mathcal{F}_{(m,p)}$,*

$$\|f\|_{(m,p)} \simeq \begin{cases} \left(|f(0)|^p + \int_{\mathbb{C}} \frac{|f'(z)|^p (1+|z|)^{mp+p}}{(1+|z|+||z|^2+|z|-m|)^p} e^{-p\frac{1}{2}|z|^2} dA(z)\right)^{\frac{1}{p}}, & p < \infty \\ |f(0)| + \sup_{z\in\mathbb{C}} \frac{|f'(z)|(1+|z|)^{m+1}}{1+|z|+||z|^2+|z|-m|} e^{-\frac{1}{2}|z|^2}, & p = \infty \end{cases}$$

From Lemma 3 of (Cho and Zhu, 2012), we have the following useful pointwise estimate in our consideration,

$$|f(z)| \leq \frac{\|f\|_{(m,p)}}{(1+|z|)^m} e^{\frac{1}{2}|z|^2} \tag{4.1.1}$$

for $0 < p \leq \infty$.

4.2 Main results

In this section we stated and prove our main results. We begin with the following useful lemma which we may use without mentioning. The lemma can proved following standard arguments.

Lemma 4.2.1. *Let* $0 < p, q \leq \infty$ *and* (g, ψ) *be a pair of entire functions. Then* $V_{(g,\psi)} : \mathcal{F}_{(m,p)} \to \mathcal{F}_{(m,q)}$ *is compact if and only if* $\|V_{(g,\psi)} f_k\|_{(m,q)} \to 0$ *as* $k \to \infty$ *for each bounded sequence of functions* $(f_k)_{k\in\mathbb{N}}$ *in* $\mathcal{F}_{(m,p)}$ *converging to zero uniformly on compact subsets of* $\mathbb{C}$ *as* $k \to \infty$.

For simplicity, we define

$$\mathcal{B}^{\infty}_{(m,\psi)}(|g|)(w) := \frac{|g'(w)|(1+|w|)^{m+1}}{(1+|\psi(w)|)^m(1+|w|+||w|^2+|w|-m|)} e^{\frac{1}{2}(|\psi(w)|^2-|w|^2}$$

and

$$\mathcal{B}_{(m,\psi)}(|g|^p)(w) := \int_{\mathbb{C}} \frac{|k_w(\psi(z))|^p |g'(z)|^p (1+|z|)^{mp+p} e^{-\frac{p}{2}|z|^2}}{(1+|\psi(z)|)^{mp}(1+|z|+||z|^2+|z|-m|)^p} dA(z),$$

where $k_w(z) = e^{z\bar{w}} - \frac{|w|^2}{2}$. Now, we may state our first main result.

Theorem 4.2.2. *Let* $0 < p \leq q \leq \infty$ *and* (g, ψ) *be a pair of entire functions on* $\mathbb{C}$. *Then*

(i) for $q < \infty$, $V_{(g,\psi)} : \mathcal{F}_{(m,p)} \to \mathcal{F}_{(m,q)}$ *is;*

  (a) bounded (or continuous) if and only if $\mathcal{B}_{(m,\psi)}(|g|^q) \in L^\infty$. *Moreover,*

$$\|V_{(g,\psi)}\| \simeq \|\mathcal{B}_{(m,\psi)}(|g|^q)\|^{\frac{1}{q}}_{L^\infty}.$$

  (b) compact if and only if $\mathcal{B}_{(m,\psi)}(|g|^q)(z)$ *goes to zero as* $|z| \to \infty$.

(ii) for $q = \infty$, $V_{(g,\psi)} : \mathcal{F}_{(m,p)} \to \mathcal{F}_{(m,\infty)}$ *is;*

  (a) bounded (or continuous) if and only if $\mathcal{B}^{\infty}_{(m,\psi)}(|g|) \in L^\infty$. *Moreover,*

$$\|V_{(g,\psi)}\| \simeq \|\mathcal{B}^{\infty}_{(m,\psi)}(|g|)\|_{L^\infty}.$$

  (b) compact if and only if $\mathcal{B}^{\infty}_{(m,\psi)}(|g|)(z)$ *goes to zero as* $|\psi(z)| \to \infty$.

Proof. (i) An application of Littlewood-Paley type estimate (Lemma 4.1.3), we obtain

$$\begin{aligned}\|V_{(g,\psi)} f\|^q_{(m,q)} &\simeq \int_{\mathbb{C}} \frac{|f(\psi(z))|^q |g'(z)|^q (1+|z|)^{mq+q}}{(1+|z|+||z|^2+|z|-m|)^q} e^{-\frac{q}{2}|z|^2} dA(z) \\ &= \int_{\mathbb{C}} |f(z)|^q d\beta_{(m,q)}(z)\end{aligned}$$

where $\beta_{(m,q)}$ is a pull back measure on $\mathbb{C}$ defined by

$$\beta_{(m,q)}(E) = \int_{\psi^{-1}(E)} \frac{|g'(z)|^q(1+|z|)^{mq+q}}{(1+|z|+||z|^2+|z|-m|)^q} e^{-\frac{q}{2}|z|^2} dA(z)$$

for every Borel subset E of $\mathbb{C}$. Let $d\lambda_{(m,q)}(z) = e^{\frac{q}{2}|z|^2} d\beta_{(m,q)}(z)$. Then

$$\|V_{(g,\psi)}f\|^q_{(m,q)} \simeq \int_{\mathbb{C}} |f(z)|^q e^{-\frac{q}{2}|z|^2} d\lambda_{(m,q)}(z). \tag{4.2.1}$$

(a) From the above estimate we conclude that $V_{(g,\psi)} : \mathcal{F}_{(m,p)} \to \mathcal{F}_{(m,q)}$ is bounded if and only if $\lambda_{(m,q)}$ is a (p,q) Fock-Carleson measure for Fock-Sobolev space. By Theorem 4.1.1 this holds if and only if $\tilde{\lambda}_{(m,q)}$ belongs to L^∞. To arrive at the conclusion of the theorem it is enough to show $\tilde{\lambda}_{(m,q)} = \mathcal{B}_{(m,\psi)}(|g|^q)$. Substituting back $d\lambda_{(m,q)}$ and $\beta_{(m,q)}$ we get

$$\begin{aligned}
\tilde{\lambda}_{(m,q)}(w) &= \int_{\mathbb{C}} \frac{e^{-\frac{q}{2}|z-w|^2}}{(1+|z|)^{mq}} d\lambda_{(m,q)}(z) \\
&= \int_{\mathbb{C}} \frac{e^{\frac{q}{2}|z|^2} e^{-\frac{q}{2}|z-w|^2}}{(1+|z|)^{mq}} d\beta_{(m,q)}(z) \\
&= \int_{\mathbb{C}} \frac{|k_w(\psi(z))|^q |g'(z)|^q (1+|z|)^{mq+q} e^{-\frac{q}{2}|z|^2}}{(1+|\psi(z)|)^{mq}(1+|z|+||z|^2+|z|-m|)^q} dA(z) \\
&= \mathcal{B}_{(m,\psi)}(|g|^q)(w).
\end{aligned} \tag{4.2.2}$$

From which the conclusion follows. Moreover, we have the norm estimate

$$\|V_{(g,\psi)}\| \simeq \|\tilde{\lambda}_{(m,q)}\|^{\frac{1}{q}}_{L^\infty} = \|\mathcal{B}_{(m,\psi)}(|g|^q)\|^{\frac{1}{q}}_{L^\infty}.$$

(b) Similarly, the operator $V_{(g,\psi)} : \mathcal{F}_{(m,p)} \to \mathcal{F}_{(m,q)}$ is compact if and only if $\lambda_{(m,q)}$ is a vanishing (p,q) Fock-Carleson measure for Fock-Sobolev space and by Theorem 4.1.1 this holds if and only if $\tilde{\lambda}_{(m,q)}(z)$ goes to zero as $|z| \to \infty$. From estimate 4.2.2, we conclude that $V_{(g,\psi)}$ is compact if and only if $\mathcal{B}_{(m,\psi)}(|g|^q)(z)$ goes to zero as $|z| \to \infty$.

(ii) (a) Suppose $\mathcal{B}^\infty_{(m,\psi)}(|g|)$ belongs to L^∞. Using Lemma 4.1.3 and the pointwise estimate in 4.1.1,

$$\begin{aligned}
\|V_{(g,\psi)}f\|_{(m,\infty)} &\simeq \sup_{z\in\mathbb{C}} \frac{|f(\psi(z))||g'(z)|(1+|z|)^{m+1}}{1+|z|+||z|^2+|z|-m|} e^{-\frac{1}{2}|z|^2}\\
&\leq \sup_{z\in\mathbb{C}} \frac{\|f\|_{(m,p)}|g'(z)|(1+|z|)^{m+1}}{(1+|\psi(z)|)(1+|z|+||z|^2+|z|-m|)} e^{\frac{1}{2}(|\psi(z)|^2-|z|^2)}\\
&= \|f\|_{(m,p)} \sup_{z\in\mathbb{C}} \mathcal{B}^{\infty}_{(m,\psi)}(|g|)(z).
\end{aligned}$$

This implies that

$$\|V_{(g,\psi)}\| \lesssim \sup_{z\in\mathbb{C}} \mathcal{B}^{\infty}_{(m,\psi)}(|g|)(z) = \|\mathcal{B}^{\infty}_{(m,\psi)}(|g|)\|_{L^\infty} < \infty \tag{4.2.3}$$

and hence $V_{(g,\psi)}$ is bounded. To prove the other direction we apply $V_{(g,\psi)}$ to the sequence of test functions $\eta_{(m,w)}(z) = (1+|w|)^{-m}k_w(z)$ belonging to all $\mathcal{F}_{(m,p)}$ with $\|\eta_{(m,w)}\|_{(m,p)} \lesssim 1$, see Lemma 20 of (Cho and Zhu, 2012).

$$\begin{aligned}
\infty > \|V_{(g,\psi)}\| &\geq \|V_{(g,\psi)}\eta_{(m,w)}\|_{(m,\infty)}\\
&\simeq \sup_{z\in\mathbb{C}} \frac{|\eta_{(m,w)}(\psi(z))||g'(z)|(1+|z|)^{m+1}}{1+|z|+||z|^2+|z|-m|} e^{-\frac{1}{2}|z|^2}\\
&\geq \frac{|g'(z)|(1+|z|)^{m+1}}{(1+|w|)^m(1+|z|+||z|^2+|z|-m|)} e^{\frac{1}{2}(|\psi(z)|^2-|\psi(z)-w|^2-|z|^2)}
\end{aligned}$$

for all w and z in $\mathbb{C}$. Setting $w=\psi(z)$ in the above estimate gives

$$\infty > \|V_{(g,\psi)}\| \gtrsim \mathcal{B}^{\infty}_{(m,\psi)}(|g|)(z) \tag{4.2.4}$$

and therefore $\mathcal{B}^{\infty}_{(m,\psi)}(|g|)$ belongs to L^∞. Moreover, from the estimates in 4.2.3 and 4.2.4 we have norm estimate

$$\|V_{(g,\psi)}\| \simeq \|\mathcal{B}^{\infty}_{(m,\psi)}(|g|)\|_{L^\infty}.$$

(b) First, suppose $V_{(g,\psi)}$ is compact. Observe that the sequence $\eta_{(m,w)} \to 0$ as $|w| \to \infty$ uniformly on a compact subsets of $\mathbb{C}$ and hence $\limsup_{|w|\to\infty} \|V_{(g,\psi)}\eta_{(m,w)}\|_{(m,\infty)} = 0$. Now, using Lemma 4.1.3 and then putting $w = \psi(z)$

$$\begin{aligned}
0 &= \limsup_{|w|\to\infty} \|V_{(g,\psi)}\eta_{(m,w)}\|_{(m,\infty)} \\
&\simeq \limsup_{|w|\to\infty} \left(\sup_{z\in\mathbb{C}} \frac{|\eta_{(m,w)}(\psi(z))||g'(z)|(1+|z|)^{m+1}}{1+|z|+||z|^2+|z|-m|} e^{-\frac{1}{2}|z|^2} \right) \\
&\gtrsim \limsup_{|\psi(z)|\to\infty} \left(\frac{|g'(z)|(1+|z|)^{m+1}}{(1+|\psi(z)|)^m(1+|z|+||z|^2+|z|-m|)} e^{\frac{1}{2}(|\psi(z)|^2-|z|^2)} \right) \\
&= \limsup_{|\psi(z)|\to\infty} \mathcal{B}^{\infty}_{(m,\psi)}(|g|)(z).
\end{aligned}$$

Therefore, $\lim_{|\psi(z)|\to\infty} \mathcal{B}^{\infty}_{(m,\psi)}(|g|)(z) = 0$. On the other hand, if $\lim_{|\psi(z)|\to\infty} \mathcal{B}^{\infty}_{(m,\psi)}(|g|)(z) = 0$, then we have the following;

(1) By (i) above, $V_{(g,\psi)}$ is bounded and applying $V_{(g,\psi)}$ to the constant function $f(z) = 1$, we have

$$\infty > \|V_{(g,\psi)}\| \gtrsim \|V_{(g,\psi)}f\|_{(m,\infty)} \simeq \sup_{z\in\mathbb{C}} \frac{|g'(z)|(1+|z|)^{m+1}}{1+|z|+||z|^2+|z|-m|} e^{-\frac{1}{2}|z|^2} \simeq \|g\|_{(m,\infty)}$$

which implies that $g \in \mathcal{F}_{(m,\infty)}$.

(2) For each $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $\mathcal{B}^{\infty}_{(m,\psi)}(|g|)(z) < \epsilon$ for all $|\psi(z)| > N$.

Now, let f_j be a sequence of bounded functions in $\mathcal{F}_{(m,p)}$ converging to zero uniformly on a compact subsets of $\mathbb{C}$ as $j \to \infty$. Then applying Lemma 4.1.3 and the estimate in 4.1.1 together with the above two points,

$$\begin{aligned}
\|V_{(g,\psi)}f_j\|_{(m,\infty)} &\simeq \sup_{z\in\mathbb{C}} \frac{|f_j(\psi(z))||g'(z)|(1+|z|)^{m+1}}{1+|z|+||z|^2+|z|-m|} e^{-\frac{1}{2}|z|^2} \\
&\leq \sup_{z:|\psi(z)|\leq N} \frac{|f_j(\psi(z))||g'(z)|(1+|z|)^{m+1}}{1+|z|+||z|^2+|z|-m|} e^{-\frac{1}{2}|z|^2} \\
&\quad + \sup_{z:|\psi(z)|>N} \frac{|f_j(\psi(z))||g'(z)|(1+|z|)^{m+1}}{1+|z|+||z|^2+|z|-m|} e^{-\frac{1}{2}|z|^2} \\
&\lesssim \|g\|_{(m,\infty)} \sup_{z:|\psi(z)|\leq N} |f_j(\psi(z))| \\
&\quad + \|f_j\|_{(m,p)} \sup_{z:|\psi(z)|>N} \frac{|g'(z)|(1+|z|)^{m+1}}{(1+|\psi(z)|)^m(1+|z|+||z|^2+|z|-m|)} e^{\frac{1}{2}(|\psi(z)|^2-|z|^2)} \\
&\lesssim \sup_{z:|\psi(z)|\leq N} |f_j(\psi(z))| + \sup_{z:|\psi(z)|>N} \mathcal{B}^{\infty}_{(m,\psi)}(|g|)(z) \\
&< \sup_{z:|\psi(z)|\leq N} |f_j(\psi(z))| + \epsilon.
\end{aligned}$$

Letting $\epsilon \to 0$ and then $j \to \infty$, we get $\lim_{j\to\infty} \|V_{(g,\psi)}f_j\|_{(m,\infty)} = 0$ and therefore the operator $V_{(g,\psi)}$ is compact. □

For the case $0 < q < p \leq \infty$, we have the following strong condition on which boundedness (or continuity) and compactness are equivalent.

Theorem 4.2.3. *Let* $0 < q < p \leq \infty$ *and* (g, ψ) *be a pair of entire functions on* $\mathbb{C}$. *Then*

(i) for $p < \infty$, $V_{(g,\psi)} : \mathcal{F}_{(m,p)} \to \mathcal{F}_{(m,q)}$ *is bounded (or continuous) if and only if it is compact if and only if* $\mathcal{B}_{(m,\psi)}(|g|^q) \in L^{\frac{p}{p-q}}$. *Moreover,*

$$\|V_{(g,\psi)}\| \simeq \|\mathcal{B}_{(m,\psi)}(|g|^q)\|^{\frac{1}{q}}_{L^{\frac{p}{p-q}}}.$$

(ii) for $p = \infty$, $V_{(g,\psi)} : \mathcal{F}_{(m,\infty)} \to \mathcal{F}_{(m,q)}$ *is bounded (or continuous) if and only if it is compact if and only if* $\mathcal{B}_{(m,\psi)}(|g|^q) \in L^1$. *Moreover,*

$$\|V_{(g,\psi)}\| \simeq \|\mathcal{B}_{(m,\psi)}(|g|^q)\|^{\frac{1}{q}}_{L^1}.$$

Proof. (a) From the estimate in 4.2.1 observe that $V_{(g,\psi)}$ is bounded (respectively, compact) if and only if $\lambda_{(m,q)}$ is a (respectively, vanishing) (p, q) Fock-Carleson measure for Fock-Sobolev space. By Theorem 4.1.2, the two are equivalent and this holds if and only if $\tilde{\lambda}_{(m,q)}$ belongs to $L^{\frac{p}{p-q}}$. But, from the estimate in 4.2.2 we conclude that $V_{(g,\psi)}$ is bounded (or compact) if and only if $\mathcal{B}_{(m,\psi)}(|g|^q)$ belongs to $L^{\frac{p}{p-q}}$. Moreover, we have the norm estimate

$$\|V_{(g,\psi)}\| \simeq \|\tilde{\lambda}_{(m,q)}\|^{\frac{1}{q}}_{L^{\frac{p}{p-q}}} \simeq \|\mathcal{B}_{(m,\psi)}(|g|^q)\|^{\frac{1}{q}}_{L^{\frac{p}{p-q}}}.$$

(b) Similar procedure as above and an application of Theorem 4.1.2 shows that $V_{(g,\psi)}$ is bounded (or compact) if and only if $\mathcal{B}_{(m,\psi)}(|g|^q)$ belongs to L^1. Moreover,

$$\|V_{(g,\psi)}\| \simeq \|\mathcal{B}_{(m,\psi)}(|g|^q)\|^{\frac{1}{q}}_{L^1}.$$

□

Lemma 4.2.4. *Let* $0 < p < \infty$ *and* (g, ψ) *be a pair of nonconstant entire functions. Then if* $\mathcal{B}^{\infty}_{(m,\psi)}(|g|)$ *or* $\mathcal{B}_{(m,\psi)}(|g|^p)$ *belongs to* L^∞, *then* $\psi(z) = a_m z + b_m$ *for some* $a_m, b_m \in \mathbb{C}$ *with* $|a_m| \leq 1$.

Proof. First, suppose $\mathcal{B}^{\infty}_{(m,\psi)}(|g|)$ belongs to L^∞. Then,

$$|g'(z)| \lesssim \frac{(1 + |\psi(z)|)^m (1 + |z| + ||z|^2 + |z| - m|)}{(1 + |z|)^{(m+1)} e^{\frac{1}{2}(|\psi(z)|^2 - |z|^2)}}.$$

Since g is nonconstant, we must have $|\psi(z)|^2 - |z|^2 \leq 0$. Therefore, $|\psi(z)| \leq |z|$ and by Liouville's theorem ψ has the form $\psi(z) = a_m z + b_m$ with $|a_m| \leq 1$. On the other hand, if $\mathcal{B}_{(m,\psi)}(|g|^p)$ belongs to L^∞, then

$$\begin{aligned}\mathcal{B}_{(m,\psi)}(|g|^p)(w) &= \int_{\mathbb{C}} \frac{|k_w(\psi(z))|^p|g'(z)|^p(1+|z|)^{mp+p}e^{-\frac{p}{2}|z|^2}}{(1+|\psi(z)|)^{mp}(1+|z|+||z|^2+|z|-m|)^p}dA(z)\\ &\geq \int_{D(\zeta,1)} \frac{|k_w(\psi(z))|^p|g'(z)|^p(1+|z|)^{mp+p}e^{-\frac{p}{2}|z|^2}}{(1+|\psi(z)|)^{mp}(1+|z|+||z|^2+|z|-m|)^p}dA(z)\end{aligned}$$

for all $w, \zeta \in \mathbb{C}$, where $D(\zeta, 1)$ is a disc of center ζ and radius 1. For $z \in D(\zeta, 1)$, we have the estimate $1 + |z| \simeq 1 + |\zeta|$, $1 + |\psi(z)| \simeq 1 + |\psi(\zeta)|$ and

$$1 + |z| + ||z|^2 + |z| - m| \simeq 1 + |\zeta| + ||\zeta|^2 + |\zeta| - m|.$$

Using this and subharmonicity of $|k_w(\psi)g'|$, we further estimate the above integral as

$$\begin{aligned}&\int_{D(\zeta,1)} \frac{|k_w(\psi(z))|^p|g'(z)|^p(1+|z|)^{mp+p}e^{-\frac{p}{2}|z|^2}}{(1+|\psi(z)|)^{mp}(1+|z|+||z|^2+|z|-m|)^p}dA(z)\\ &\simeq \left(\frac{(1+|\zeta|)^{mp+p}}{(1+|\psi(\zeta)|)^{mp}(1+|\zeta|+||\zeta|^2+|\zeta|-m|)^p}\right)\int_{D(\zeta,1)} |k_w(\psi(z))|^p|g'(z)|^p e^{-\frac{p}{2}|z|^2}dA(z)\\ &\qquad\gtrsim \frac{(1+|\zeta|)^{mp+p}|k_w(\psi(\zeta))|^p|g'(\zeta)|^p e^{-\frac{p}{2}|\zeta|^2}}{(1+|\psi(\zeta)|)^{mp}(1+|\zeta|+||\zeta|^2+|\zeta|-m|)^p}.\end{aligned}$$

Setting $w = \psi(\zeta)$, we obtain

$$\mathcal{B}_{(m,\psi)}(|g|^p)(\psi(\zeta)) \gtrsim \left(\mathcal{B}^\infty_{(m,\psi)}(|g|)\right)^p(\zeta) \tag{4.2.5}$$

and the conclusion follows from boundedness of $\mathcal{B}^\infty_{(m,\psi)}(|g|)$. □

Now, we may state our simplified characterization of Theorem 4.2.2 part (i).

Theorem 4.2.5. *Let $0 < p \leq q < \infty$ and (g, ψ) be a pair of nonconstant entire functions. Then $V_{(g,\psi)} : \mathcal{F}_{(m,p)} \to \mathcal{F}_{(m,q)}$ is*

(a) bounded if and only if $\mathcal{B}^\infty_{(m,\psi)}(|g|)$ belongs to L^∞.

(b) compact if and only if $\mathcal{B}^\infty_{(m,\psi)}(|g|)(z)$ goes to zero as $|z| \to \infty$.

Proof. (a) Suppose $V_{(g,\psi)}$ is bounded. Then, by Theorem 4.2.2, $\mathcal{B}_{(m,\psi)}(|g|^q)$ is uniformly bounded over $\mathbb{C}$ and from the estimate in 4.2.5, $\mathcal{B}^\infty_{(m,\psi)}(|g|)$ is also bounded. On the other hand, if $\mathcal{B}^\infty_{(m,\psi)}(|g|)$ is uniformly bounded over $\mathbb{C}$, then by Lemma 4.2.4, $\psi(z) = a_m z + b_m$

with $0 < |a_m| \leq 1$, and hence

$$\begin{aligned}
\mathcal{B}_{(m,\psi)}(|g|^q)(w) &= \int_{\mathbb{C}} \frac{|k_w(\psi(z))|^q |g'(z)|^q (1+|z|)^{mq+q} e^{-\frac{q}{2}|z|^2}}{(1+|\psi(z)|)^{mq}(1+|z|+||z|^2+|z|-m|)^q} dA(z) \\
&\leq \left(\sup_{z\in\mathbb{C}} \left(\mathcal{B}^{\infty}_{(m,\psi)}(|g|)\right)^q(z) \right) \int_{\mathbb{C}} |k_w(\psi(z))|^q e^{-\frac{q}{2}|\psi(z)|^2} dA(z) \\
&= \left(\sup_{z\in\mathbb{C}} \left(\mathcal{B}^{\infty}_{(m,\psi)}(|g|)\right)^q(z) \right) \frac{1}{|a_m|^2} \int_{\mathbb{C}} |k_w(z)|^q e^{-\frac{q}{2}|z|^2} dA(z) \\
&= \frac{1}{|a_m|^2} \sup_{z\in\mathbb{C}} \left(\mathcal{B}^{\infty}_{(m,\psi)}(|g|)\right)^q(z).
\end{aligned}$$

From Theorem 4.2.2 we conclude that the operator $V_{(g,\psi)}$ is bounded.

(b) If the operator $V_{(g,\psi)}$ is compact, then the conclusion easily follows from Theorem 4.2.2 and the estimate in 4.2.5. We will prove the other side. Assume $\lim_{|z|\to\infty} \mathcal{B}^{\infty}_{(m,\psi)}(|g|)(z) = 0$ and therefore $\mathcal{B}^{\infty}_{(m,\psi)}(|g|)$ is bounded over $\mathbb{C}$. By Lemma 4.2.4, $\psi(z) = a_m z + b_m$ with $0 < |a_m| \leq 1$, using this we get

$$\begin{aligned}
\mathcal{B}_{(m,\psi)}(|g|^q)(w) &= \int_{\mathbb{C}} \frac{|k_w(\psi(z))|^q |g'(z)|^q (1+|z|)^{mq+q} e^{-\frac{q}{2}|z|^2}}{(1+|\psi(z)|)^{mq}(1+|z|+||z|^2+|z|-m|)^q} dA(z) \\
= \int_{|z|\leq|w|} & \frac{\left(\mathcal{B}^{\infty}_{(m,\psi)}(|g|)\right)^q(z) |k_w(\psi(z))|^q}{e^{\frac{q}{2}|\psi(z)|^2}} dA(z) + \int_{|z|>|w|} \frac{\left(\mathcal{B}^{\infty}_{(m,\psi)}(|g|)\right)^q(z) |k_w(\psi(z))|^q}{e^{\frac{q}{2}|\psi(z)|^2}} dA(z) \\
&\lesssim \sup_{z:|z|\leq|w|} |k_w(\psi(z))|^q \int_{|z|\leq|w|} \frac{\left(\mathcal{B}^{\infty}_{(m,\psi)}(|g|)\right)^q(z)}{e^{-\frac{q}{2}|\psi(z)|^2}} dA(z) \\
&\qquad + \left(\sup_{z:|z|>|w|} \left(\mathcal{B}^{\infty}_{(m,\psi)}(|g|)\right)^q(z) \right) \int_{|z|>|w|} \frac{|k_w(\psi(z))|^q}{e^{\frac{q}{2}|\psi(z)|^2}} dA(z) \\
&\leq \left(\sup_{z:|z|\leq|w|} |k_w(\psi(z))|^q \right) \left(\sup_{z\in\mathbb{C}} \left(\mathcal{B}^{\infty}_{(m,\psi)}(|g|)\right)^q(z) \right) \int_{|z|\leq|w|} e^{-\frac{q}{2}|\psi(z)|^2} dA(z) \\
&\qquad + \frac{1}{|a_m|^2} \sup_{z:|z|>|w|} \left(\mathcal{B}^{\infty}_{(m,\psi)}(|g|)\right)^q(z) \\
&\lesssim \sup_{z:|z|\leq|w|} |k_w(\psi(z))|^q + \sup_{z:|z|>|w|} \left(\mathcal{B}^{\infty}_{(m,\psi)}(|g|)\right)^q(z).
\end{aligned}$$

Since k_w uniformly converges to zero on a compact subsets of $\mathbb{C}$ as $|w| \to \infty$ and $\sup_{z:|z|>|w|} \left(\mathcal{B}^{\infty}_{(m,\psi)}(|g|)\right)^q(z)$ goes to zero as $|w| \to \infty$, we have

$$\lim_{|w|\to\infty} \mathcal{B}_{(m,\psi)}(|g|^q)(w) = 0.$$

The conclusion follows from Theorem 4.2.2. □

In particular, for $\psi(z) = z$, that is for the Volterra type integral operator V_g, we have the following corollary of the above theorem.

Corollary 4.2.6. *Let* $0 < p \leq q \leq \infty$ *and* g *is an entire function on* $\mathbb{C}$. *Then* $V_g : \mathcal{F}_{(m,p)} \to \mathcal{F}_{(m,q)}$ *is*

(i) bounded if and only if $g(z) = az^2 + bz + c$, *for some* $a, b, c \in \mathbb{C}$.

(ii) compact if and only if $g(z) = az + b$, *for some* $a, b \in \mathbb{C}$.

Theorem 4.2.7. *Let* $0 < q < p \leq \infty$ *and* (g, ψ) *be a pair of nonconstant entire functions. Then*

(i) for $p < \infty$, $V_{(g,\psi)} : \mathcal{F}_{(m,p)} \to \mathcal{F}_{(m,q)}$ *is bounded if and only if it is compact if and only if* $\mathcal{B}^{\infty}_{(m,\psi)}(|g|) \in L^{\frac{pq}{p-q}}$.

(ii) for $p = \infty$, $V_{(g,\psi)} : \mathcal{F}_{(m,\infty)} \to \mathcal{F}_{(m,q)}$ *is bounded if and only if it is compact if and only if* $\mathcal{B}^{\infty}_{(m,\psi)}(|g|) \in L^q$.

Proof. (i) By Theorem 4.2.3, it is enough to show that $\mathcal{B}_{(m,\psi)}(|g|^q) \in L^{\frac{p}{p-q}}$ if and only if $\mathcal{B}^{\infty}_{(m,\psi)}(|g|) \in L^{\frac{pq}{p-q}}$. First we suppose that $\mathcal{B}_{(m,\psi)}(|g|^q) \in L^{\frac{p}{p-q}}$ and proceed to show $\mathcal{B}^{\infty}_{(m,\psi)}(|g|) \in L^{\frac{pq}{p-q}}$. Then, using the estimate in 4.2.5 and Lemma 4.2.4, that is $\psi(z) = a_m z + b_m$ for some $a_m, b_m \in \mathbb{C}$ with $0 < |a_m| \leq 1$, we get

$$\begin{aligned}\int_{\mathbb{C}} \left(\mathcal{B}^{\infty}_{(m,\psi)}(|g|)\right)^{\frac{pq}{p-q}}(z)dA(z) &\lesssim \int_{\mathbb{C}} \mathcal{B}^{\frac{p}{p-q}}_{(m,\psi)}(|g|^q)(\psi(z))dA(z)\\ &= \frac{1}{|a_m|^2} \int_{\mathbb{C}} \mathcal{B}^{\frac{p}{p-q}}_{(m,\psi)}(|g|^q)(z)dA(z) < \infty.\end{aligned}$$

On the other hand, if $\mathcal{B}^{\infty}_{(m,\psi)}(|g|) \in L^{\frac{pq}{p-q}}$, then applying Hölder's inequality,

$$\begin{aligned}\mathcal{B}^{\frac{p}{p-q}}_{(m,\psi)}(|g|^q)(w) &= \left(\int_{\mathbb{C}} \frac{|k_w(\psi(z))|^q |g'(z)|^q (1+|z|)^{mq+q} e^{-\frac{q|z|^2}{2}}}{(1+|\psi(z)|)^{mq}(1+|z|+||z|^2+|z|-m|)^q} dA(z)\right)^{\frac{p}{p-q}}\\ &\leq \int_{\mathbb{C}} \frac{|k_w(\psi(z))|^q |g'(z)|^{\frac{pq}{p-q}} (1+|z|)^{(m+1)\frac{pq}{p-q}}}{(1+|\psi(z)|)^{\frac{mpq}{p-q}} (1+|z|+||z|^2+|z|-m|)^{\frac{pq}{p-q}}} e^{-\frac{pq|z|^2}{2(p-q)}} e^{\frac{q}{2}(\frac{p}{p-q}-1)|\psi(z)|^2} dA(z)\\ &\qquad \times \left(\int_{\mathbb{C}} |k_w(\psi(z))|^q e^{-\frac{q|\psi(z)|^2}{2}} dA(z)\right)^{\frac{q}{p-q}}\\ &\lesssim \int_{\mathbb{C}} \frac{|k_w(\psi(z))|^q |g'(z)|^{\frac{pq}{p-q}} (1+|z|)^{(m+1)\frac{pq}{p-q}}}{(1+|\psi(z)|)^{\frac{mpq}{p-q}} (1+|z|+||z|^2+|z|-m|)^{\frac{pq}{p-q}}} e^{-\frac{pq|z|^2}{2(p-q)}} e^{\frac{q}{2}(\frac{p}{p-q}-1)|\psi(z)|^2} dA(z).\end{aligned}$$

Making use of the above estimate, Fubini's Theorem and the fact that

$$\int_{\mathbb{C}} |K_w(z)|^q e^{-\frac{q}{2}|z|^2} dA(z) = e^{\frac{q}{2}|w|^2},$$

we get

$$
\begin{aligned}
&\int_{\mathbb{C}} \mathcal{B}_{(m,\psi)}^{\frac{p}{p-q}}(|g|^q)(w)dA(w) \lesssim \\
&\int_{\mathbb{C}}\int_{\mathbb{C}} \frac{|k_w(\psi(z))|^q|g'(z)|^{\frac{pq}{p-q}}(1+|z|)^{(m+1)\frac{pq}{p-q}}}{(1+|\psi(z)|)^{\frac{mpq}{p-q}}(1+|z|+||z|^2+|z|-m|)^{\frac{pq}{p-q}}}e^{-\frac{pq|z|^2}{2(p-q)}}e^{\frac{q}{2}(\frac{p}{p-q}-1)|\psi(z)|^2}dA(z)dA(w) \\
&\qquad = \int_{\mathbb{C}} \frac{|g'(z)|^{\frac{pq}{p-q}}(1+|z|)^{(m+1)\frac{pq}{p-q}}e^{-\frac{pq|z|^2}{2(p-q)}+\frac{q}{2}(\frac{p}{p-q}-1)|\psi(z)|^2}}{(1+|\psi(z)|)^{\frac{mpq}{p-q}}(1+|z|+||z|^2+|z|-m|)^{\frac{pq}{p-q}}}\int_{\mathbb{C}}\frac{|K_{\psi(z)}(w)|^q}{e^{\frac{q|w|^2}{2}}}dA(w)dA(z) \\
&\qquad\qquad = \int_{\mathbb{C}} \frac{|g'(z)|^{\frac{pq}{p-q}}(1+|z|)^{(m+1)\frac{pq}{p-q}}e^{-\frac{pq}{2(p-q)}(|\psi(z)|^2-|z|^2)}}{(1+|\psi(z)|)^{\frac{mpq}{p-q}}(1+|z|+||z|^2+|z|-m|)^{\frac{pq}{p-q}}}dA(z) \\
&\qquad\qquad\qquad = \int_{\mathbb{C}} \left(\mathcal{B}_{(m,\psi)}^{\infty}(|g|)\right)^{\frac{pq}{p-q}}(z)dA(z) < \infty.
\end{aligned}
$$

From which we conclude that $\mathcal{B}_{(m,\psi)}(|g|^q) \in L^{\frac{p}{p-q}}$.

(ii) By Theorem 4.2.3, we plan to show that, $\mathcal{B}_{(m,\psi)}(|g|^q) \in L^1$ implies $\mathcal{B}_{(m,\psi)}^{\infty}(|g|)$ belongs to L^q and $\mathcal{B}_{(m,\psi)}^{\infty}(|g|) \in L^q$ intern implies boundedness of the operator. But, if $\mathcal{B}_{(m,\psi)}(|g|^q) \in L^1$, then similar procedure as in the above proof shows that $\mathcal{B}_{(m,\psi)}^{\infty}(|g|)$ belongs to L^q. On the other hand, if $\mathcal{B}_{(m,\psi)}^{\infty}(|g|)$ belongs to L^q, then applying Lemma 4.1.3 and Lemma 4.2.4,

$$
\begin{aligned}
\|V_{(g,\psi)}f\|_{(m,q)} &\simeq \int_{\mathbb{C}} \frac{|f(\psi(z))|^q|g'(z)|^q(1+|z|)^{mq+q}}{(1+|z|+||z|^2+|z|-m|)^q}e^{-\frac{q}{2}|z|^2}dA(z) \\
&\le \sup_{z\in\mathbb{C}}\left(|f(\psi(z))|^q e^{-\frac{q}{2}|\psi(z)|^2}\right)\int_{\mathbb{C}}\frac{|g'(z)|^q(1+|z|)^{mq+q}}{(1+|z|+||z|^2+|z|-m|)^q}e^{\frac{q}{2}(|\psi(z)|^2-|z|^2)}dA(z) \\
&= \|f\|_{(m,\infty)}\int_{\mathbb{C}}\left(\mathcal{B}_{(m,\psi)}^{\infty}(|g|)\right)^q(z)dA(z),
\end{aligned}
$$

which implies that

$$
\|V_{(g,\psi)}\| \lesssim \int_{\mathbb{C}}\left(\mathcal{B}_{(m,\psi)}^{\infty}(|g|)\right)^q(z)dA(z) < \infty.
$$

Therefore, $V_{(g,\psi)}$ is bounded. □

Similarly, for $\psi(z) = z$, from the above theorem we obtain the following corollary.

Corollary 4.2.8. *Let $0 < q < p \le \infty$ and g be an entire function on $\mathbb{C}$. Then*

(i) for $p < \infty$, $V_g : \mathcal{F}_{(m,p)} \to \mathcal{F}_{(m,q)}$ is bounded if and only if it is compact if and only if $g(z) = az + b$ whenever $\frac{q}{2} > \frac{p-q}{p}$, and g is constant otherwise.

(ii) for $p = \infty$, $V_g : \mathcal{F}_{(m,\infty)} \to \mathcal{F}_{(m,q)}$ is bounded if and only if it is compact if and only if $g(z) = az + b$ whenever $q > 2$, and g is constant otherwise.

Comparing our results for $m \neq 0$, and Theorem 2.0.1, we conclude that the operator has similar bounded and compact structure as in the classical Fock spaces case. In fact we have the following proposition.

Proposition 4.2.9. *Let* $0 < p, q \leq \infty$, *m be a positive integer, and* (g, ψ) *be a pair of nonconstant entire functions. Then* $V_{(g,\psi)} : \mathcal{F}_{(m,\infty)} \to \mathcal{F}_{(m,q)}$ *is bounded (respectively, compact) if and only if* $V_{(g,\psi)} : \mathcal{F}_p \to \mathcal{F}_q$ *is bounded (respectively, compact).*

Proof. Suppose $V_{(g,\psi)} : \mathcal{F}_{(m,p)} \to \mathcal{F}_{(m,q)}$ is bounded. Then $\mathcal{B}^{\infty}_{(m,\psi)}(|g|)$ is bounded, and hence $1 + |\psi(z)| \leq 1 + |z|$. Using this and the inequality,

$$1 + |z| + ||z|^2 + |z| - m| \leq (1 + |z|)^2,$$

we get

$$M_{(g,\psi)}(z) \leq \mathcal{B}^{\infty}_{(m,\psi)}(|g|)(z).$$

From this, the above theorems and Theorem 2.0.1, we conclude that $V_{(g,\psi)} : \mathcal{F}_p \to \mathcal{F}_q$ is also bounded.

On the other hand, if $V_{(g,\psi)} : \mathcal{F}_p \to \mathcal{F}_q$ is also bounded, then $M_{(g,\psi)}$ is bounded, and by Lemma 4.2.4, $\psi(z) = az + b$ for some $a, b \in \mathbb{C}$. Thus, the function,

$$\frac{(1 + |z|)^{m+2}}{(1 + |\psi(z)|)^m (1 + |z| + ||z|^2 + |z| - m|)}$$

is bounded. From which, we obtain the following estimate,

$$\mathcal{B}^{\infty}_{(m,\psi)}(|g|)(z) \lesssim M_{(g,\psi)}(z).$$

This together with the above theorems, and Theorem 2.0.1 imply that $V_{(g,\psi)} : \mathcal{F}_{(m,p)} \to \mathcal{F}_{(m,q)}$ is bounded.

The compactness case follows from similar procedures as above. □

Chapter 5

Conclusion and future scope

5.1 Conclusion

This thesis includes a number of results, which characterizes bounded and compact properties of generalized Volterra type integral operators on Fock-Sobelov spaces. Our results are new and simple to apply when compared with Brezin type integral transform characterization. Moreover, our result extends and generalizes some previously obtained results for this class of operators. In particular, the results in Theorem 4.2.5 and 4.2.7 generalizes the results in (Mengestie, 2016) and (Mengestie, 2017) from Volterra type integral operators into generalized Volterra type integral operators, and extends the results in (Mengestie and Worku, 2018) from Fock spaces into Fock-Sobelov spaces.

5.2 Future scope

The study of different properties of generalized Volterra type integral operators acting between different spaces of analytic functions is an active area of research. So, any interested researchers can use this opportunity and conduct their research work in this area.

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