SOLVING TIME FRACTIONAL NEWELL-WHITEHEAD-SEGEL EQUATION VIA CONFORMABLE VARIATIONAL ITERATION METHOD

A THESIS SUBMITTED TO THE DEPARTMENT OF MATHEMATICS, COLLEGE OF NATURAL SCIENCES, JIMMA UNIVERSITY IN PARTIAL FULFILLMENT OF THE REQUIREMENTS OF THE DEGREE OF MASTERS OF SCIENCE IN MATHEMATICS

[DIFFERENTIAL EQUATION]

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DECLARATION

This is to certify that this thesis entitled ***"solving time fractional Newell-whitehead-Segel equation via conformable variational iteration method"*** submitted in partial fulfillment of the requirement for the award of the Degree of Masters of Science (M.Sc.) in Differential Equation to the school of Graduate studies, Jimma University, through the Department of Mathematics studies conducted by Mr. Gari Kenei Ligdi.

To best of my knowledge and beliefs, the substances included under this thesis work has not been submitted for the fulfillment of any award qualification in earlier time.

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LIST OF ACRONYMS AND ABBREVIATIONS

CFD - Conformable Fractional Derivative

CFRDTM - Conformable Fractional Reduced Differential Transform Method

CHAM - Conformable Homotopy Analysis Method

C-VIM - Conformable Variational Iteration Method

FPDEs - Fractional Partial Differential Equations

NWSE - Newell-Whitehead-Segel Equation

ODE - Ordinary Differential Equation

PDE - Partial Deferential Equation

SRPSM - Sumudu Residual Power Series Method

ABSTRACT

This study presents the application of the conformable variational iteration method to get approximate analytical solutions for the time fractional Newell-Whitehead-Segel equation. Time fractional order derivative Newell-Whitehead-Segel equation is the interaction of the effect of the diffusion term with the non-linear effect of the reaction term and it is one of the most important of amplitude equation which describes the appearance of the stripe pattern in two dimensional systems A powerful mathematical method known as the C-VIM combines the application of the newly defined fractional derivative introduced by Khalil et al. (2014) called conformable fractional derivative on a well-known variational iteration method. The conformable derivative is one of the admirable choices to handle nonlinear physical problems of different fields of interest. By applying the C-VIM, we derive an approximate analytical solution of time fractional order derivative Newell-Whitehead-Segel equation. Convergence analysis and numerical examples are presented to show the efficiency of the proposed method. Plotted graphs demonstrate the mightiness and accurateness of the proposed technique. The applications show that C-VIM gives solutions that coincide with the exact solutions, and it saves a lot of computational work in solving time fractional NWSEs.

Keywords: Conformable variational iteration method; Time fractional Newell– Whitehead–Segel equation; Lagrangian multiplier; Conformable fractional derivative.

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CHAPTER ONE

INTRODUCTION

1.1 Background of the Study

Fractional calculus is a field of applied mathematics, three centuries as old as the conventional calculus and deals with derivatives and integrals of arbitrary orders(Prakash & Verma, 2019). Nonlinear phenomena are always visible in the study of applied Mathematics, Physics, Chemistry and many related fields of engineering. In our daily life, we come across many real life models of Mathematics for numerical solution of nonlinear differential equations (Gupta et al., 2020), and references therein.

During the last decade, super improvements have been visualized in the field of fractional calculus, which is very popular amongst science and engineering community. In recent years, differential equation containing fractional order derivatives has been contributed in various fields of science and engineering such as diffusion equation, polarization, electro-magnetic waves, visco-elasticity, electrode electrolyte heat conduction, finance, control theory, biomedical engineering, biology(Prakash & Verma, 2019).

In order to achieve the goal of highly accurate solution, many authors illustrate various techniques such as Adomain Decomposition Method(Ray & Bera, 2005), Generalized Differential Transform Method(Odibat et al., 2008), Finite Element Method(Jiang & Ma, 2011),Fractional Differential Transform Method (Arikoglu & Ozkol, 2007),Homotopy Perturbation Method(Zhang et al., 2014;Prakash, 2016),Iterative Methods(Dhaigude & Nikam, 2012),Variation Iteration Method(Safari et al., 2009), Homotopy Analysis Method(Liao, 2004),Homotopy Perturbation Sumudu Transform Method(Rathore et al., 2012),Homotopy Analysis Transform Method(Kumar & Singh et al., 2018),Local Fractional Homotopy Perturbation Sumudu Transform Method and Local Fractional Reduced Differential Transform Method(Kumar & Tchier et al., 2018),Homotopy Analysis Sumudu Transform Method and others(Singh et al., 2018).

Acan et al. (2020) used C-VIM, CFRDM, and CHAM to solve a nonlinear fractional partial differential equation. They conclude that for $\alpha = 1$, the absolute error with C-VIM is approximately the same as with the exact solution, compared to the other methods. Similarly, they found that C-VIM achieved the best results among the three methods.

The advantages of using Conformable Virational Iteration Method is the free choice of initial approximation, free from rounding off errors as it does not involve discretization, and does not require large computer obtained memory or power. Therefore, we use this advantage to construct an initial values without unknown parameters(Shang & Han, 2010), and without perturbation, no use of Adomian polynomial or any restrictive assumptions.

The Newell-Whitehead-Segel equation model is the interaction of the effect of the diffusion term with the non-linear effect of the reaction term(Prakash & Verma, 2019),and it is one of the most important of amplitude equation which describes the appearance of the stripe pattern in two dimensional systems(Mahgoub, 2019).

Newell-Whitehead-Segel equations have broad pertinence in bio-engineering, biology, ecology, chemical and mechanical engineering. In general the non-linear time fractional Newell Whitehead-Sigel equation is of the form(Prakash et al., 2019):

$$\frac{\partial^{\alpha} U(x,t)}{\partial t^{\alpha}} = k\frac{\partial^{2} U(x,t)}{\partial x^{2}} + aU(x,t) - b(U(x,t))^{q}, \tag{1.1}$$

with initial condition $U(x, 0) = f(x)$. (1.2)

where $t > 0$, $0 < \alpha \leq 1$,a, b, $k > 0$ are real numbers ,and q is a positive integer. And α is a parameter, which describes the order of the time-fractional derivative. In equation (1.1), $\frac{\partial^{\alpha} U(x,t)}{\partial t^{\alpha}}$ represents the variation of U(x, t) with time at a fixed location, $\frac{\partial^{2} U(x,t)}{\partial x^{2}}$ represents the variation of $U(x,t)$, with spatial variable at a specific time, and term $aU(x,t) - b\big(U(x,t)\big)^{q}$, takes into account the effect of the source term.

Previously, Newell-Whitehead-Segel equations were solved by some methods like: homotopy perturbation method (Nourazar et al., 2011),He's variational iteration method (Prakash & Kumar,2016). Additionally, Kumar & Sharma(2016), and Prakash et al.(2019), solved fractional models of the Newell-Whitehead-Segel equation using the homotopy analysis Sumudu transform method and fractional variational iteration method, respectively.

1.2. Statement of the problem

NWSE is found in wide variety of engineering and scientific applications. It is one of the most important of amplitude equation which describes the appearance of the stripe pattern in two dimensional systems(Mahgoub, 2019).Equation(1.1) describe particle motion with memory in time. Therefore, the study of time-fractional Newell-Whitehead-Segel Eq. (1.1) is very important(Prakash et al., 2019).

Recently, the analytic and numerical Solution of two dimensional Newell-Whitehead-Segel model with time-fractional derivative was obtained by Sumudu residual power series method(Liu et al.,2023).But, solving two-dimensional NWSE given by equation (1.1) using C-VIM is not presumably presented in the existing literature. Motivated by the above discussions this study mainly focused on the following points related to Equation (1.1).

- Defining terminologies related to fractional order derivatives.
- Discussing the concept of Conformable fractional derivatives.
- Defining NWSE.
- Applying C-VIM for time fractional NWSE.
- The convergence of C-VIM.
- Verifying the proposed method using test examples.

1.3 Objectives of the study

1.3.1 General Objective

The general objective of this study is to investigate the solution of time fractional Newell-Whitehead-Segel equation by applying the Conformable Variational Iteration method.

1.3.2 Specific Objectives

The specific objectives of this study are:

- To present an overview of key terms and concepts in fractional order derivatives.
- To explain the concept of Conformable fractional derivatives.
- To define time fractional order derivative NWSE.
- To demonstrate the use of C-VIM in solving time fractional NWSE equation.
- To prove the convergence of the proposed method.
- To test the effectiveness of the proposed method through example cases.

1.4 Significance of the Study

This research was believed to have the following significances.

- It can help mathematics students to better understand fractional order derivatives.
- It provides alternative technique to find the approximate analytic solutions of two dimensional NWSEs by C-VIM.
- It expand our knowledge and to know in detail.
- It gives the researcher awareness and experience about writing effective project in the future.

1.5 Delimitation of the Study

This study is delimited to solve time fractional order derivative Newell-Whitehead-Segel Equation (1.1) using Conformable Variational Iteration method.

CHAPTER TWO
LITERATURE REVIEW

Fractional calculus is the generalization of integer-order calculus to arbitrary order integration and differentiation with non-integer order. Fractional calculus was introduced on 1695 when Leibniz wrote a letter to L'Hopital, questioning "what does it mean by $d^n f/dx^n$ when $n = \frac{1}{2}$?",where $d^n f/dx^n$ is the n^{th}derivative of the map f(x) (Jneid & Chaouk, 2020).

Several forms of fractional derivatives such as: Riemann-Liouville, Hadamard, Coimbra, Caputo, Riesz, Marchaud, Canavati and others were introduced(Oliveira et al., 2014).As this new fractional calculus came to light, many fractional partial differential equations unfolded in the process. Furthermore, FPDEs turned out to play a substantial role in modeling countless phenomena that arise in applied sciences as shown in(Jneid & Chaouk, 2020).

Fractional differential equations are located in the center of numerous research area in recent years because of their commonly appearance in different kinds of applied sciences. Therefore, it was dealt to solve fractional ordinary and partial differential equations by many researchers(Podlubny,1999; Beyer & Kemplfe, 1995; Mainardi,1996).

The advantage of using fractional models of differential equations in physical models is their non-local property. Fractional order derivative is non-local while integer order derivative is local in nature. Hence the fractional models are more realistic and fractional derivatives are often used to model problems in acoustics, fluid mechanics, diffusion, electromagnetism, signal processing, biology, finance and others(Prakash et al., 2019), and references therein.

The time-fractional NWSEs describe particle motion with memory in time while space-fractional derivative arises when variations are heavy-tailed and describes particle motion that accounts for variation in the flow field over the entire system. We are focusing more on motion with memory in time. Also, the fraction in time derivative suggests a modulation or weighting of system memory. Therefore, the study of time-fractional NWSE(1.1) is very important(Prakash et al., 2019).

In 1969, Newell, Whitehead and Segel derived this equation (Meiron & Newell, 1985), and ever since then it has been employed to model many nonlinear problems appearing in fluid mechanics. NWSE have huge applicability in ecology, biology, chemistry, engineering and so many other areas of science. In the previous years, many researchers have suggested different methods for solving this equation(Jneid & Chaouk, 2020).

Previously, the solution of the time-fractional NWSE has attracted the attention of many authors. Ali et al. (2019), solved the Caputo time fractional NWSE using the fractional power series methodology. Prakash et al. (2019), have used the fractional variation iteration method to obtain approximate analytical solutions for the fractional NWSEs in the sense of the Caputo derivative(Liaqat et al., 2022).

In the work of Prakash and Verma (2019), they used the Adomian Decomposition Method to solve time-fractional NWSEs in the sense of Caputo. Saadeh et al. (2019) obtained the approximate analytical solutions to the fractional-order NWSE by using the residual power series method. The Sumudu Decomposition approach was utilized by Ahmed and Elbadri (2020) to determine approximate and definite solutions to the Caputo time-fractional NWSE(Liaqat et al., 2022), and some references therein.

Recently, there is too much interest in new well-behaved simple fractional derivative introduced by Khalil et al.(2014), named conformable derivative depending just on the limit definition of the derivative. The conformable derivative is one of the admirable choices to handle nonlinear physical problems of different fields of interest(Zulfiqar & Ahmad, 2021).

In 2015, Abdeljawad provided fractional versions of the chain rule, exponential functions, and others. In the same year, Cenesiz and Kurt give the solutions of time and space fractional heat differential equations and approximate analytical solution of the time conformable fractional Burger’s equation via Homotopy Analysis Method. By this new defined conformable derivative, Acan et al. (2016), proposed the conformable fractional reduced differential transform method and gave some applications (Firat et al., 2017).

The solution of the time-fractional NWSEs in the sense of conformable derivative has attracted the attention of many authors. Jneid and Chaouk(2020),have used the CFRDTM to solve time fractional NWSEs. Acan et al. (2020), used C-VIM, CFRDM, and CHAM to solve a nonlinear fractional partial differential equation.

To solve NWSEs, Benattia and Belghaba (2021), employed the conformable Sumudu Transform and the Adomian Decomposition approach. In 2022, Liaqat et al. established a new procedure by using the Conformable Shehu Transform and Adomian Decomposition Method for solving fractional-order NWSEs in the sense of conformable fractional derivative for the first time(Liaqat et al., 2022).

The variation iteration method developed in 1999 by Chinese professor He. It was used to study the linear wave equation, nonlinear wave equation, and wave-like equation in bounded and unbounded domains. The method has been proved by many authors to be reliable and efficient for a wide variety of scientific applications(Wazwaz, 2007).

It was shown by many authors that this method is more powerful than existing techniques such as: the Adomian method, perturbation method, and others. The method gives rapidly convergent successive approximations of the exact solution if such a solution exists; otherwise a few approximations can be used for numerical purposes. The method was effectively used and has no specific requirements, such as linearization and small parameters for nonlinear operators(Wazwaz, 2007), and references therein.

The Conformable Virational Iteration Method is the variational iteration method constructed on the newly defined fractional derivative by Khalil et al. (2014), called conformable fractional derivative. It was presented by Acan et al. for the first time in 2020, and successfully applied to Dumped Burger equation. This method(C-VIM) finds the successive approximations of the exact solution of the problems (Chakraverty et al., 2019).

However, how to solve time fractional NWSE of (1.1) by applying C-VIM is not presumably discussed in Acan et al. (2020), and in other extant literature. It observes that C-VIM gives solutions that coincide with exact solution and it saves a lot of computation work. Therefore, this study has been aimed to develop a scheme to find approximate analytical solution of two dimensional NWSE (1.1) subject to initial condition (1.2) by C-VIM.

CHAPTER THREE

METHODOLOGY

3.1 Study Area and Period

This study was conducted at Jimma University College of Natural Sciences, Department of Mathematics, from September, 2023 to June, 2024.

3.2. Study Design

This study employed analytical design.

3.3. Sources of Information

The sources of information for the study were documentary reviews, such as journal articles, internet sources, and other scholar papers.

3.4. Mathematical Procedures

For the success of the objectives of the study, the following mathematical steps were performed.

1. Writing the problem as in equation (1.1).
2. Applying the definition of Conformable fractional derivative on equation (1.1) to get the conformable time fractional NWSE.
3. Constructing the correction functional according to the conformable variational iteration method.
4. Finding the successive iterated term using the correction functional as an iterative formula, $u_i(x,t)$ for i = 0, 1, 2, 3… n.
5. Obtaining the approximate analytic solutions for the governing problem by using the $\lim_{i\to\infty} u_i\,(x,t)$.
6. Implementing Mathematica Software to find numerical results, and for sketching graphs.

CHAPTER FOUR
PRELIMINARIES

4.1. Fractional Calculus

In this section, we will look at the basic theory of fractional calculus. Fractional calculus involves both fractional derivatives and fractional integrals. In this section, we define and discuss the properties related to fractional derivatives.

4.1.1 Fractional Derivatives

There are various definitions and theorems of fractional integrals and derivatives. In this section, we present and define the two commonly used fractional derivatives: Riemann-Liouville and Caputo's fractional derivative. We also define the newly introduced fractional derivative by (Khalil et al., 2014) called conformable fractional derivative.

Definition 4.1.1 Riemann-Liouville fractional derivative (Podlubny , 1999)**.**

Let $\alpha \in R$, $m - 1 \leq \alpha < m \in N$, the Riemann-Liouville time fractional partial derivative of order α for the function u(x, t) is defined as follows:

$$D_t^{\alpha} u(x,t) = \frac{\partial^m}{\partial t^m} \int_a^t \frac{(t-\tau)^{m-\alpha-1}}{\Gamma(m-\alpha)} u(x,\tau)\, d\tau, \mathrm{t} > 0.$$

Definition 4.1.2 Caputo fractional derivative(Podlubny , 1999)

Let $\alpha, t \in \mathbb{R}, m - 1 \leq \alpha < m \in \mathbb{N}, t > 0$,then the Caputo time fractional partial derivative for the function u(x, t) is given as:

$$\begin{cases} D_t^{\alpha} u(x,t) = \displaystyle\int_0^t \frac{(t-\tau)^{m-\alpha-1}}{\Gamma(m-\alpha)} \frac{\partial^m u(x,t)}{\partial t^m}\, d\tau, \\ D_t^{\alpha} u(x,t) = \dfrac{\partial^m u(x,t)}{\partial t^m}, \alpha = m \in \mathbb{N}. \end{cases}$$

Definition 4.1.3 Conformable Fractional Derivative (Khalil et al., 2014)

Given a function $f: [0, \infty) \longrightarrow \mathbb{R}$. Then, for all $t > 0$ and $\alpha \in (0,1]$ the "conformable fractional derivative" of f of order α is defined by:

$$T_{\alpha}(f)(t) = lim_{h \to 0} \frac{f(t + ht^{1-\alpha}) - f(t)}{h}.$$

Definition 4.1.4 $\boldsymbol{a}$**- Conformable fractional derivative** (Sarikaya et al., 2019)

Given a function f : [a, b] → R with $0 \leq a < b$. Then the "$a -$ conformable fractional derivative" of f of order α is defined by:

$$T_{\alpha}^{a} f(x,t) = lim_{h\to 0} \frac{f(x,t+ht^{-\alpha}\,(t-a))-f(x,t)}{h(1-at^{-\alpha})}, \text{ for all t} > a, t^{\alpha} \neq a,\ \alpha \in (0,1]. \quad (4.1)$$

Definition 4.1.5. Given a function U: [0,∞) → ℝ. Then the conformable time fractional partial derivative of order α for a function U(x, t) defined by (Khalil et al., 2014) is given as:

$$ {}_{t}T_{\alpha}\mathrm{U}(\mathrm{x},\mathrm{t}) = lim_{h\to 0} \frac{U(x,t+ht^{1-\alpha})-U(x,t)}{h}, \quad \text{for all t} > 0 \text{ and } \alpha \in (0, 1].$$

Definition 4.1.6(Thabet & Kendre, 2018)**.** Given a function U(x, t): $(0,\infty)$ → ℝ. Then the conformable space fractional partial derivative of order β for a function U(x, t) is defined as:

$$ {}_{x}T_{\beta}U(x,t) = lim_{\sigma\to 0} \frac{U(x+\sigma x^{1-\beta},t)-U(x,t)}{\sigma}, \quad \text{for all x} > 0,\ \beta \in (0, 1].$$

Note that for brevity, we used $\frac{\partial^{\alpha}U(x,t)}{\partial t^{\alpha}}$ as ${}_{t}T_{\alpha}\mathrm{U}(\mathrm{x},\mathrm{t})$ throughout this paper.

Theorem 4.1(Khalil et al., 2014) Let α ϵ (0, 1] and u(x, t), v(x, t) be $(\alpha\,, a)$ differentiable at (x, t) ϵ ℝ X (0,ℝ).Then,

i. ${}_{t}T_{\alpha}^{a}(au+bv) = a\,{}_{t}T_{\alpha}^{a}u + b\,{}_{t}T_{\alpha}^{a}v$.

ii. ${}_{t}T_{\alpha}^{a}(t^{p}) = \frac{pt^{p-1}(t-a)}{(t^{\alpha}-a)}$, for all p ∈ ℝ.

iii. ${}_{t}T_{\alpha}^{a}(\lambda) = 0$, for all constant function u(x, t) = λ.

iv. ${}_{t}T_{\alpha}^{a}(uv) = u({}_{t}T_{\alpha}^{a}v) + v({}_{t}T_{\alpha}^{a}u)$.

v. ${}_{t}T_{\alpha}^{a}\left(\frac{u}{v}\right) = \frac{v({}_{t}T_{\alpha}u)-u({}_{t}T_{\alpha}v)}{v^{2}}$.

vi. If u is differentiable with respect to t, then ${}_{t}\,T_{\alpha}^{a}u(x,t) = t^{1-\alpha}\frac{\partial u(x,t)}{\partial t}$.

vii. ${}_{t}\,T_{\alpha}^{a}\ \mathrm{u}\ (\mathrm{x},\mathrm{t}) = \frac{(\,t-a\,)}{(t^{\alpha-a})}\frac{\partial u(x,t)}{\partial t}$.

Proof*:* Parts i and iii follow directly from the definition. Now, we will prove ii, iv, v, vi, vii.
For a fixed and $t > a,\ t^{\alpha} \neq a$, we have,

$$ {}_{t}\,T_{\alpha}^{a}(t^{p}) = lim_{\varepsilon\to 0} \frac{(t+\varepsilon t^{-\alpha}(\,t-a))^{p}-t^{p}}{\varepsilon(1-at^{-\alpha})},$$

$$= \lim_{\varepsilon\to\infty} \frac{t^{p}+p\varepsilon t^{-\alpha}(\,t-a\,)t^{p-1}+O(\varepsilon^{2})-\,t^{p}}{\varepsilon(1-at^{-\alpha})},$$

$$= \frac{pt^{p-1}(t-a)}{(t^{\alpha}-a)}.$$ This completes the proof of ii.

Now, we will prove iv. For this purpose, since u, v are α-differentiable at a point $t > a$, and $t^{\alpha} \neq a$, we obtain:

$${}_{t}T_{\alpha}^{a}(uv) = lim_{\varepsilon\to 0}\frac{u(x,t+\varepsilon t^{-\alpha}(t-a))\,v(x,t+\varepsilon t^{-\alpha}(t-a))-u(x,t)v(x,t)}{\varepsilon(1-at^{-\alpha})},$$

by adding and subtracting $u(x,t+\varepsilon t^{-\alpha}(t-a))v(x,t)$, we get

$$= lim_{\varepsilon\to 0}\frac{u(x,t+\varepsilon t^{-\alpha}(t-a))\,v(x,t+\varepsilon t^{-\alpha}(t-a)) - u(x,t+\varepsilon t^{-\alpha}(t-a))v(x,t+\ u(x,t+\varepsilon t^{-\alpha}(t-a))v(x,t) - u(x,t)v(x,t)}{\varepsilon(1-at^{-\alpha})},$$

$$= lim_{\varepsilon\to 0}\frac{u(x,t+\varepsilon t^{-\alpha}(t-a))[\,v(x,t+\varepsilon t^{-\alpha}(t-a)) - v(x,t)] + v(x,t)[u(x,t+\varepsilon t^{-\alpha}(t-a))\text{-}\ u(x,t)]}{\varepsilon(1-at^{-\alpha})},$$

$$= lim_{\varepsilon\to 0}u(x,t+\varepsilon t^{-\alpha}(t-a))\frac{[\,v(x,t+\varepsilon t^{-\alpha}(t-a))-v(x,t)]}{\varepsilon(1-at^{-\alpha})} + lim_{\varepsilon\to 0}v(x,t)\frac{[u(x,t+\varepsilon t^{-\alpha}(t-a))\text{-}\ u(x,t)]}{(1-at^{-\alpha}},$$

$$= lim_{\varepsilon\to 0}u(x,t+\varepsilon t^{-\alpha}(t-a))lim_{\varepsilon\to 0}\frac{[\,v(x,t+\varepsilon t^{-\alpha}(t-a)) - v(x,t)]}{\varepsilon(1-at^{-\alpha})} + lim_{\varepsilon\to 0}v(x,t)lim_{\varepsilon\to 0}\frac{[u(x,t+\varepsilon t^{-\alpha}(t-a))\text{-}\ u(x,t)]}{1-at^{-\alpha}},$$

$$= u(x,t+0)lim_{\varepsilon\to 0}\frac{[\,v(x,t+\varepsilon t^{-\alpha}(t-a)) - v(x,t)]}{\varepsilon(1-at^{-\alpha})} + v(x,t)lim_{\varepsilon\to 0}\frac{[u(x,t+\varepsilon t^{-\alpha}(t-a))\text{-}\ u(x,t)]}{1-at^{-\alpha}},$$

$$= u(x,t)\frac{\partial v(x,t)}{\partial t} +\ v(x,t)\frac{\partial u(x,t),}{\partial t}$$

$= \mathrm{u(x,t)}\,{}_{t}T_{\alpha}v(x,t) + v(x,t)\,{}_{t}T_{\alpha}u(x,t)$.This completes the proof of (iv).

The proof of the v is similar to iv. Similarly, we can prove (vi) as follows.

By definition 4.1.5, we have

$${}_{t}T_{\alpha}^{a}\mathrm{u(x,t)} = lim_{h\to 0}\frac{u(x,t+ht^{1-\alpha})-u(x,t)}{h}.$$

Substituting $\lambda = ht^{1-\alpha}$ implies that ${}_{t}T_{\alpha}^{a}\mathrm{u(x,t)} = lim_{\lambda\to 0}\frac{u(x,t+\lambda)-u(x,t)}{\lambda t^{\alpha-1}}$,

$$= t^{1-\alpha}\ lim_{\lambda\to 0}\frac{u(x,t+\lambda)-u(x,t)}{\lambda},$$

$$= t^{1-\alpha}\frac{\partial}{\partial t}\mathrm{u(x,t)}.$$

To prove vii, let $\varepsilon = ht^{1-\alpha}(t-a)$ in Definition 4.1.5 and taking $h = \frac{\varepsilon t^{\alpha-1}}{t-a}$. Therefore, we get:

$${}_{t}T_{\alpha}^{a}\mathrm{u(x,t)} = lim_{h\to 0}\frac{u(x,t+ht^{1-\alpha})-u(x,t)}{h},$$

$$= lim_{h\to 0}\frac{(t-a)u(x,t+\varepsilon)-u(x,t)}{\varepsilon t^{\alpha}-a},$$

$$= \frac{(t-a)}{(t^{\alpha-a})}\ lim_{h\to 0}\frac{u(x,t+\varepsilon)-u(x,t)}{\varepsilon},$$

$= \frac{(t-a)}{(t^{\alpha-a})}\frac{\partial \mathrm{u(x,t)}}{\partial t}$. This completes the proof.

Table 1: conformable fractional derivative of certain functions(*(Khalil et al., 2014)*

Functions	Conformable fractional derivative
${}_t\,T_\alpha^a u(x,t)$	$t^{1-\alpha}\dfrac{\partial u(x,t)}{\partial t}$
$T_\alpha(1)$	0
$T_\alpha^a(t^n)$	$\dfrac{nt^{n-1}(t-a)}{t^\alpha - a}$
$T_\alpha(sin(at)$	$at^{1-\alpha}$cos(at)
$T_\alpha(cos(at)$	$-at^{1-\alpha}$(sin(at)
$T_\alpha(e^{at})$	$at^{1-\alpha}e^{at}$
$T_\alpha(\frac{1}{\alpha}t^\alpha)$	$\dfrac{t^{\alpha-1}(t-a)}{t^\alpha - a}$
$T_\alpha(e^{\frac{t^\alpha}{\alpha}})$	$\dfrac{t^{\alpha-1}(t-a)}{t^\alpha - a}e^{\frac{t^\alpha}{\alpha}}$
$T_\alpha(sin(\dfrac{1}{\alpha}t^\alpha)$	cos($\frac{1}{\alpha}t^\alpha$)
$T_\alpha(cos(\dfrac{1}{\alpha}t^\alpha)$	-sin($\frac{1}{\alpha}t^\alpha$)
${}_tT_\alpha^a\left(\dfrac{f}{g}\right)$	$\dfrac{g({}_tT_\alpha f) - f({}_tT_\alpha g)}{g^2}$
${}_tT_\alpha^a(fog)$	$f'(g)\,{}_tT_\alpha^a(g)$
${}_tT_\alpha^a(fg)$	$f({}_tT_\alpha^a g) + g({}_tT_\alpha^a f)$

4.2 Variational Iteration Method (VIM)

The variational iteration method was first proposed by Chinese professor He in 1999 to study the linear and nonlinear wave equation, and wave-like equation in bounded and unbounded domains. The method has been proven by many authors to be reliable, efficient, and powerful for a wide variety of scientific applications, both linear and nonlinear (Wazwaz, 2007).

In general VIM can be explained as follows. Consider the following general non-linear partial differential equation in the form:

$$T_\alpha U(x,t) + \mathrm{LU(x,t)} + \mathrm{NU(x,t)} = \mathrm{g(x,t)}, \qquad (4.2)$$

where L is the linear operator, N is the nonlinear operator and $g(x,t)$ is a known analytical function. The correction functional according to the variational iteration method is given by:

$$U_{n+1}(x,t) = U_n(x,t) + \int_0^t \lambda(\xi)\{LU_n(x,\xi) + NU_n(x,\xi) - g(x,\xi)\}\, d\xi, \qquad (4.3)$$

where λ is the Lagrange multiplier, which can be identified optimally via the variational theory. As in (Odibat, 2010), the Lagrange multiplier $\lambda(\xi)$ can be identified optimally by means of variation theory and give as follows:

$$\lambda(\xi) = \frac{(-1)^m}{(m-1)!}(\tau - t)^{m-1}, m \geq 1, \qquad (4.4)$$

Consequently, λ can be obtained as:

For $m = 1, the$ derivative is $(\frac{d}{dt})$ and $\lambda = -1$.

For $m = 2, \left(\frac{d^2}{dt^2}\right)$ and $\lambda = (\tau - t)$.

For $m = 3, \left(\frac{d^3}{dt^3}\right)$ and $\lambda = -\left(\frac{(\tau-t)^2}{2!}\right)$. and so on, and in our case the highest order derivative is $(\frac{d}{dt})$. Hence the Lagrange multiplier $\lambda = -1$.Then, using this value in equation (4.3) we get:

$$U_{n+1}(x,t) = U_n(x,t) - \int_0^t \lambda(\xi)\{LU_n(x,\xi) + NU_n(x,\xi) - g(x,\xi)\}\, d\xi. \qquad (4.5)$$

The subscript n denotes the n^{th} approximation, and $\widetilde{U}_n$ is considered as restricted variation, which means $\delta\widetilde{U}_n = 0$. The initial approximation U_0 can be chosen freely if it satisfies the given conditions. The solution of (4.2) is given by:

$$U(x,t) = \lim_{n\to\infty} U_n(x,t). \qquad (4.6)$$

4.3 The Conformable Variational Iteration Method (C-VIM)

The Conformable Virational Iteration Method is the variational iteration method constructed on the newly defined fractional derivative by Khalil et al. (2014) called conformable fractional derivative. It was presented by Acan et al. first the first time in 2020.

In order to apply the concept of C-VIM on non-linear partial differential equation expressed in eq.(4.2) above, we first write the non-linear FPDEs in the standard operator form as follows:

$$T_\alpha u(x,t) + Lu(x,t) + Nu(x,t) = g(x,t), \tag{4.7}$$

where T_α is conformable fractional derivative of order α with $0 < \alpha \leq 1$. To solve differential equation (4.7) via C-VIM we write the differential equation (4.7) in the form by the theorem (4.1) as follows:

$$t^{1-\alpha}\frac{\partial u(x,t)}{\partial t} + Lu(x,t) + Nu(x,t) = g(x,t). \tag{4.8}$$

As in classical VIM (section 4.2) above, the correction functional for equation (4.6) can be constructed as:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\xi)[\xi^{1-\alpha}\frac{\partial U_n(x,\xi)}{\partial \xi} + Lu_n(x,\xi) + N(\tilde{u}_n(x,\xi) - g(x,\xi)]d\xi. \tag{4.9}$$

where $\lambda, \tilde{u}_n$ are as explained in the past section.

Hence, we get the solution as:

$$u(x,t) = \lim_{n\to\infty} u_n(x,t).$$

CHAPTER FIVE
RESULTS AND DISCUSSION

5.1. Conformable variational iteration method procedures to find approximate analytic solution of time fractional Newell-Whitehead-Segel equation.

In this section, we consider time fractional NWSE (1.1) with initial condition (1.2) to apply the Conformable Variational Iteration method on the governing problem. Then we develop the final approximate solution depending on the Conformable Variational Iteration Method Procedures.

Consider the time fractional Newell-Whitehead-Segel equation of the form:

$$ {}_tT_\alpha U(x,t) = k\frac{\partial^2 U(x,t)}{\partial x^2} + aU(x,t) - b(U(x,t))^q, \qquad 0 < \alpha \le 1, \tag{5.1}$$

where $t > 0$, $0 < \alpha \le 1$, a, b, $k > 0$ are real numbers, and q is a positive integer, and T_α is conformable fractional derivative of order α mentioned earlier.

To solve differential equation (5.1) via C-VIM, we write the differential equation (5.1) in the standard form by the theorem 4.1,

$$\tau^{1-\alpha}\frac{\partial U(x,\tau)}{\partial \tau} - k\frac{\partial^2 U(x,t)}{\partial x^2} - aU(x,t) + b\big(U(x,t)\big)^q = 0. \tag{5.2}$$

According to the conformable variational iteration method, we can construct a correct functional of equation (5.2) as follows.

$$U_{n+1}(x,t) = U_n(\mathrm{x},\mathrm{t}) + \lambda(\tau)\int_0^t \left(\tau^{1-\alpha}\frac{\partial U_n(x,\tau)}{\partial \tau} - k\partial^2\frac{U_n(x,\tau)}{\partial x^2} - aU_n(x,\tau) + b(U_n(x,\tau))^q\right)d\tau, \tag{5.3}$$

where λ is a Lagrangian multiplier. As we discussed earlier we obtain $\lambda(\tau) = -1$ and hence, from equation (5.3), we get:

$$U_{n+1}(x,t) = U_n(\mathrm{x},\mathrm{t}) - \int_0^t \left(\tau^{1-\alpha}\frac{\partial U_n(x,\tau)}{\partial \tau} - k\partial^2\frac{U_n(x,\tau)}{\partial x^2} - aU_n(x,\tau) + b(U_n(x,\tau))^q\right)d\tau. \tag{5.4}$$

For an appropriate choice of a, b and q we can obtain successive approximations $U_n(x,t), n \ge 0$ leading to the solution:

$$U(x,t) = \lim_{n\to\infty} U_n(x,t). \tag{5.5}$$

5.2 Convergence analysis

In this section, we focus on the convergence of the proposed conformable variational iteration method when applied to equation (5.1). This can be analyzed in the same manner as the convergence of VIM (Prakash et al., 2019), but with a conformable fractional derivative. Here, we discuss the sufficient conditions that guarantee the existence and convergence of the solution. We also present sufficient conditions for the existence of a unique solution and the convergence of the proposed method, along with its error estimate.

Consider the time fractional Newell-Whitehead-Segel equation of the form:

$$\frac{\partial^\alpha U(x,t)}{\partial t^\alpha} = k\frac{\partial^2 U(x,t)}{\partial x^2} + aU(x,t) - b(U(x,t))^q, \quad 0 < \alpha \le 1 .$$

According to Conformable variational iteration method (C-VIM), the equation has the recurrence relation of the form:

$$U_{n+1}(x,t) = U_n(\mathrm{x,t}) - \int_0^t \left(\tau^{1-\alpha}\frac{\partial U_n(x,\tau)}{\partial \tau} - k\partial^2\frac{U_n(x,\tau)}{\partial x^2} - aU_n(x,\tau) + b(U_n(x,\tau))^q\right)d\tau,$$

and the solution is of the form:

$$U(x,t) = \lim_{n\to\infty} U_n(x,t).$$

Let we define the operator B as:

$$B = \int_0^t(-1)\left(\tau^{1-\alpha}\frac{\partial U_n(x,\tau)}{\partial \tau} - k\partial^2\frac{U_n(x,\tau)}{\partial x^2} - aU_n(x,\tau) + b(U_n(x,\tau))^q\right)d\tau, \qquad (5.6)$$

and we define the components U_k, $\mathrm{k} = 0,1,2,\ldots..$ as,

$$U(x,t) = \lim_{n\to\infty} U_n(x,t) = \sum_{k=0}^{\infty} U_k. \qquad (5.7)$$

Theorem 5.1(Odibat, 2010)**.** Let B, defined in (5.6) is an operator from a Banach space BS to BS. The series solution $U(x,t) = \lim_{n\to\infty} U_n(x,t) = \sum_{k=0}^{\infty} U_k$ as defined in (5.7), converges if 0 < p < 1 exists such that ‖B $[U_0 + U_1 + U_2 + \cdots + U_{k+1}]$ ‖ ≤ ‖ $P[U_0 + U_1 + U_2 + \cdots + U_k]$ ‖, (i.e. ‖ U_{k+1}‖ ≤ p‖U_k‖), $\forall_k \epsilon \mathrm{N} \cup \{0\}$.

This theorem is a special case of the Banach fixed point theorem, which was used in(Tatari & Dehghan, 2007) as a sufficient condition to study the convergence of the FVIM for some partial differential equations.

Theorem 5.2 (Odibat, 2010). If the series solution $U(x,t) = \sum_{k=0}^{\infty} U_k$ defined in (5.7) converges, then it is an exact solution of nonlinear problem (5.1).

Theorem 5.3 (Odibat, 2010). We assume that the series solution $\sum_{k=0}^{\infty} U_k$ defined in (5.7) is convergent to the solution U(x, t). If the truncated series $\sum_{k=0}^{j} U_{k,}$ is used as an approximation to the solution U(x, t) of problem (5.1), then the maximum error $E_j(x,t),$ is estimated as:

$$E_j(x,t) = \frac{1}{1-p} p^{j+1} \parallel U_0 \parallel.$$

If for every i ∈ N ∪ {0}, we define the parameters,

$$x_i = \begin{cases} \{\parallel U_i \parallel, & \parallel U_i \parallel \neq 0, \\ 0, & \parallel U_i \parallel = 0. \end{cases}$$

Then, the series solution $\sum_{k=0}^{\infty} U_k$ of problem (5.1) converges to an exact solution U (x, t), when:

$$0 \leq x_i < 1, \ \forall_i \in \mathrm{N} \cup \{0\}.$$

Moreover, as stated in theorem 5.3, the maximum absolute truncation error is estimated as:

$$\parallel U(x,t) - \sum_{k=0}^{\infty} U_k \parallel \leq \frac{1}{1-x} X^{j+1} \parallel U_0 \parallel,$$

where $X = max\{X_i, i = 0,1,2, \dots, j\}$.

Remark 1 If the first finite $X_i' s$, i $=$ $1, 2, .., $j, are not less than one and $X_i \leq 1$ for i > j, then, of course, the series solution $\sum_{k=0}^{\infty} U_k$ of problem (5.1) converges to an exact solution. In other words, the first finite terms do not affect the convergence of series solution. In this case, the convergence of C-VIM approach depends on x_i , for i > j.

5.3 Illustrative Examples

In this section, we consider two examples that demonstrate the performance and efficiency of the C-VIM for solving time fractional order derivative Newell-Whitehead-Segel equation.

Example 5.1 $D_t^{\alpha} U(x,t) = U_{xx} - 2U, 0 < \alpha \leq 1 \quad 0 < t \leq 1 \;, \quad 0 \leq x \leq 1 \,,$ (5.8)

with initial condition:

$$U(x,0) = U_0 = e^{x}. \tag{5.9}$$

When $\alpha = 1$, the exact solution of equation (5.8) is

$$U(x,t) = e^{x-t} (\text{Liu et al.}, 2023). \tag{5.10}$$

By C-VIM the correction functional of the equation (5.8) can be constructed as follows.

$$U_{n+1}(x,t) = U_n(x,t) - \lambda \int_0^t \left\{ \tau^{1-\alpha} \frac{\partial U_n(x,\tau)}{\partial \tau} - \frac{\partial^2 U_n(x,\tau)}{\partial x^2} + 2U_n(x,\tau) \right\} d\tau. \tag{5.11}$$

By using (5.9) in (5.11) we obtain,

$$U_1(x,t) = e^x - \int_0^t \{\tau^{1-\alpha}.0 - e^x + 2e^x\} (d\tau),$$

$$= e^x - e^x \int_0^t d\tau,$$

$$U_1(x,t) = e^x (1 - t). \tag{5.12}$$

$$U_2(x,t) = U_1(x,t) - \int_0^t \left\{ \frac{\partial^{\alpha} U_0}{\partial \tau^{\alpha}} - \frac{\partial^2 U_0}{\partial x^2} + 2U_0 \right\} (d\tau),$$

$$= e^x (1-t) - \int_0^t \left\{ \frac{\partial^{\alpha} e^x (1-t)}{\partial \tau^{\alpha}} - \frac{\partial^2 e^x (1-t)}{\partial x^2} + 2e^x (1-t) \right\} (d\tau),$$

$$= e^x (1-t) - \int_0^t \left\{ \tau^{1-\alpha} \frac{\partial}{\partial t} e^x (1-t) - e^x (1-t) + 2e^x (1-t \right\} (d\tau),$$

$$= e^x (1-t) - e^x \int_0^t \{1 - t - \tau^{1-\alpha}\} (d\tau),$$

$$= e^x (1-t) - e^x \left[t - t^2 - \frac{t^{2-\alpha}}{2-\alpha} \right],$$

$$U_2(x,t) = e^x [1 - 2t + t^2 - \frac{t^{2-\alpha}}{2-\alpha}] = \frac{1}{2} e^x \left(2 - 4t + 2t^2 - \frac{2t^{2-\alpha}}{-2+\alpha} \right). \tag{5.13}$$

$$U_3(x,t) = U_2(x,t) - \int_0^t \left\{ \tau^{1-\alpha} \frac{\partial U_2(x,t)}{\partial t} - \frac{\partial^2 U_2(x,t)}{\partial x^2} + 2U_2(x,t) \right\} (d\tau),$$

$$U_3(x,t) = \frac{1}{6} e^x \left(6 - 18t + 9t^2 - t^3 - \frac{18t^{2-\alpha}}{-2+\alpha} + \frac{6t^{3-\alpha}}{-2+\alpha} + \frac{6t^{3-2\alpha}}{-3+2\alpha} \right). \tag{5.14}$$

and so on.

Hence the approximate analytic solution is:

$U(x,t) = lim_{n \to 3} U_n(x,t) = \; U_3(x,t)$.The classical solution of the Newell-Whitehead-Segel equation is $U(x,t) = e^{x-t}$(Liu et al., 2023).

Example 2. Consider the non-linear time-fractional Newell-Whitehead-Segel equation:

$$T_\alpha u(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2} + u(x,t) - u^2(x,t), \qquad 0 < \; \alpha \leq 1, \tag{5.15}$$

subject to the initial condition:

$$u(x,0) = \frac{1}{\left(1+e^{\frac{x}{\sqrt{6}}}\right)^2}.$$

Based on the description of the C-VIM we get the following recurrence relation:

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left\{ \xi^{1-\alpha} \; \frac{\partial u_n(x,\xi)}{\partial \xi} - \frac{\partial^2 u_n(x,\xi)}{\partial x^2} - u_n(x,\xi) + \left(u_n(x,\xi)\right)^2 \right\} d\xi.$$

Then we get the following successive approximate solutions:

$$u_1(x,t) = u_0(x,t) - \int_0^t \left\{ \xi^{1-\alpha} \; \frac{\partial u_0(x,\xi)}{\partial \xi} - \frac{\partial^2 u_0(x,\xi)}{\partial x^2} - u_0(x,\xi) + \left(u_0(x,\xi)\right)^2 \right\} d\xi,$$

$$u_1(x,t) = \frac{1}{\left(1+e^{\frac{x}{\sqrt{6}}}\right)^2} + \frac{5e^{\frac{x}{\sqrt{6}}}t}{3\left(1+e^{\frac{x}{\sqrt{6}}}\right)^3}\;. \tag{5.16}$$

In a similar manner we obtain:

$$u_2(x,t) = \frac{1}{\left(1+e^{\frac{x}{\sqrt{6}}}\right)^2} +$$

$$\frac{5e^{\frac{x}{\sqrt{6}}}t}{3\left(1+e^{\frac{x}{\sqrt{6}}}\right)^3} \frac{5e^{\frac{x}{\sqrt{6}}}t^{1-\alpha}\left(36\left(1+e^{\frac{x}{\sqrt{6}}}\right)^3 t + t^\alpha\left(36-15t+6e^{\sqrt{\frac{3}{2}}x}(6+5t)+9e^{\sqrt{\frac{2}{3}}x}(12+5t)-4e^{\frac{x}{\sqrt{6}}}\left(-27+5t^2\right)\right)(-2+\alpha)\right)}{108\left(1+e^{\frac{x}{\sqrt{6}}}\right)^6(-2+\alpha)}, \tag{5.17}$$

u_3

$$
\begin{aligned}
&\frac{1}{\left(1+e^{\frac{x}{\sqrt{6}}}\right)^2}+\frac{5e^{\frac{x}{\sqrt{6}}}t}{3\left(1+e^{\frac{x}{\sqrt{6}}}\right)^3}+\frac{5e^{\frac{x}{\sqrt{6}}}t^{1-\alpha}}{108\left(1+e^{\frac{x}{\sqrt{6}}}\right)^6(-2+\alpha)}\left(36\left(1+e^{\frac{x}{\sqrt{6}}}\right)^3t+t^{\alpha}\left(36-15t+6e^{\sqrt{\frac{3}{2}}x}(6+5t)+9e^{\sqrt{\frac{2}{3}}x}(12+5t)-4e^{\frac{x}{\sqrt{6}}}(-27+5t^2)\right)(-2+\alpha)\right)-\frac{5e^{\frac{x}{\sqrt{6}}}t}{81648\left(1+e^{\frac{x}{\sqrt{6}}}\right)^{12}}\times\\
=&\left\{
\begin{array}{l}
504e^{3\sqrt{\frac{3}{2}}x}\left(-54-90t-25t^2-\frac{108t^{1-\alpha}}{-2+\alpha}-\frac{90t^{2-\alpha}}{-2+\alpha}+\frac{54t^{2-2\alpha}}{-3+2\alpha}\right)\\
+126\left(-216+180t-25t^2-\frac{432t^{1-\alpha}}{-2+\alpha}+\frac{180t^{2-\alpha}}{-2+\alpha}+\frac{216t^{2-2\alpha}}{-3+2\alpha}\right)\\
+378e^{4\sqrt{\frac{2}{3}}x}\left(-648-900t-175t^2-\frac{1296t^{1-\alpha}}{-2+\alpha}-\frac{900t^{2-\alpha}}{-2+\alpha}+\frac{648t^{2-2\alpha}}{-3+2\alpha}\right)\\
+252e^{\frac{7x}{\sqrt{6}}}\left(-3888-4320t-270t^2+205t^3+25t^4+\frac{180t^{4-2\alpha}}{(5-2\alpha)(-2+\alpha)^2}-\frac{7776t^{1-\alpha}}{-2+\alpha}-\frac{4320t^{2-\alpha}}{-2+\alpha}+\frac{3888t^{2-2\alpha}}{-3+2\alpha}-\frac{300t^{4-\alpha}}{10-7\alpha+\alpha^2}+\frac{180t^{3-\alpha}(-6+\alpha)}{8-6\alpha+\alpha^2}\right)\\
+756e^{\sqrt{6}x}\left(-3024-2520t+360t^2+290t^3+25t^4+\frac{360t^{4-2\alpha}}{(5-2\alpha)(-2+\alpha)^2}-\frac{6048t^{1-\alpha}}{-2+\alpha}-\frac{2520t^{2-\alpha}}{-2+\alpha}+\frac{3024t^{2-2\alpha}}{-3+2\alpha}-\frac{450t^{4-\alpha}}{10-7\alpha+\alpha^2}+\frac{360t^{3-\alpha}(-6+\alpha)}{8-6\alpha+\alpha^2}\right)\\
+63e^{\frac{x}{\sqrt{6}}}\left(-3888+2160t+720t^2-320t^3+25t^4+\frac{720t^{4-2\alpha}}{(5-2\alpha)(-2+\alpha)^2}-\frac{7776t^{1-\alpha}}{-2+\alpha}+\frac{2160t^{2-\alpha}}{-2+\alpha}+\frac{3888t^{2-2\alpha}}{-3+2\alpha}+\frac{600t^{4-\alpha}}{10-7\alpha+\alpha^2}+\frac{720t^{3-\alpha}(-6+\alpha)}{8-6\alpha+\alpha^2}\right)\\
+28e^{\sqrt{\frac{2}{3}}x}\left(-34992+9720t+12420t^2-2430t^3-720t^4+125t^5+\frac{9720t^{4-2\alpha}}{(5-2\alpha)(-2+\alpha)^2}-\frac{69984t^{1-\alpha}}{-2+\alpha}+\frac{9720t^{2-\alpha}}{-2+\alpha}+\frac{34992t^{2-2\alpha}}{-3+2\alpha}+\frac{1800t^{5-\alpha}}{12-8\alpha+\alpha^2}+\frac{4050t^{4-\alpha}}{10-7\alpha+\alpha^2}+\frac{9720t^{3-\alpha}(-6+\alpha)}{8-6\alpha+\alpha^2}\right)\\
+84e^{2\sqrt{\frac{2}{3}}x}\left(-40824-11340t+15525t^2+2400t^3-795t^4-125t^5+\frac{10800t^{4-2\alpha}}{(5-2\alpha)(-2+\alpha)^2}-\frac{81648t^{1-\alpha}}{-2+\alpha}-\frac{11340t^{2-\alpha}}{-2+\alpha}+\frac{40824t^{2-2\alpha}}{-3+2\alpha}+\frac{1800t^{5-\alpha}}{12-8\alpha+\alpha^2}-\frac{4500t^{4-\alpha}}{10-7\alpha+\alpha^2}+\frac{10800t^{3-\alpha}(-6+\alpha)}{8-6\alpha+\alpha^2}\right)\\
+7e^{\frac{5x}{\sqrt{6}}}\left(-489888-272160t+135000t^2+48600t^3-855t^4-1000t^5+\frac{97200t^{4-2\alpha}}{(5-2\alpha)(-2+\alpha)^2}-\frac{979776t^{1-\alpha}}{-2+\alpha}-\frac{272160t^{2-\alpha}}{-2+\alpha}+\frac{489888t^{2-2\alpha}}{-3+2\alpha}+\frac{7200t^{5-\alpha}}{12-8\alpha+\alpha^2}-\frac{81000t^{4-\alpha}}{10-7\alpha+\alpha^2}+\frac{97200t^{3-\alpha}(-6+\alpha)}{8-6\alpha+\alpha^2}\right),\\
+2e^{\sqrt{\frac{3}{2}}x}\left(-1143072+472500t^2-9450t^3-34965t^4+1000t^6+\frac{340200t^{4-2\alpha}}{(5-2\alpha)(-2+\alpha)^2}-\frac{2286144t^{1-\alpha}}{-2+\alpha}+\frac{1143072t^{2-2\alpha}}{-3+2\alpha}+\frac{75600t^{5-\alpha}}{12-8\alpha+\alpha^2}+\frac{340200t^{3-\alpha}(-6+\alpha)}{8-6\alpha+\alpha^2}\right).
\end{array}
\right.
\end{aligned}
$$

and so on.

Accordingly, the solution to the problem is:

$$u(x,t) = lim_{n\to 3} u_n(x,t) = \ u_3(x,t).$$

In particular when $\alpha = 1$, we get the solution in the form:

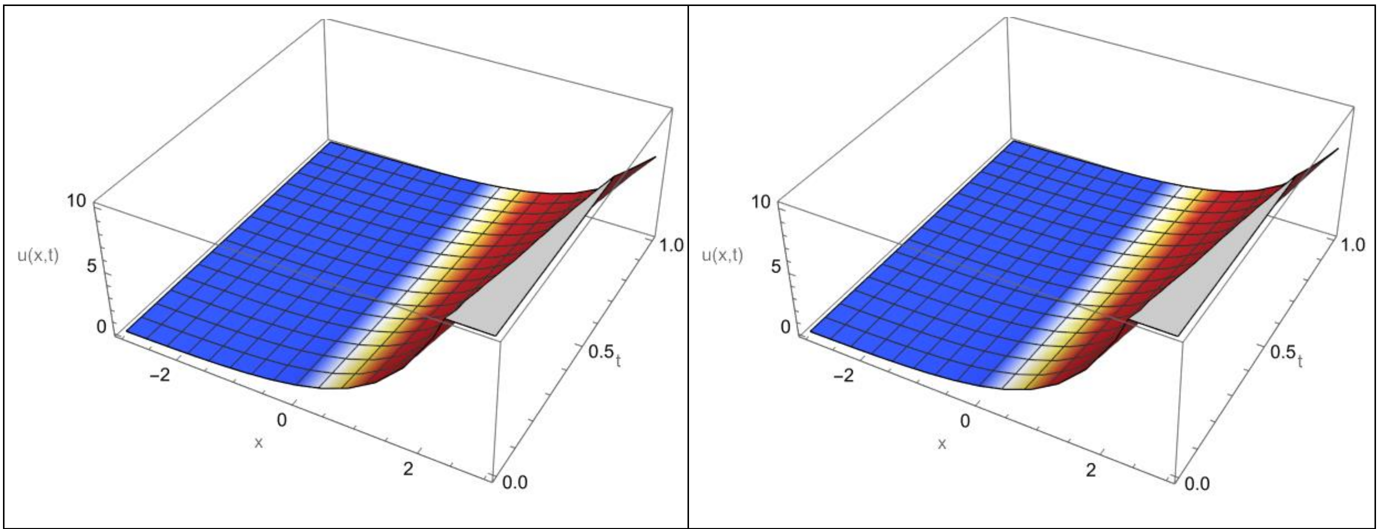
$$u(x,t) = \frac{1}{\left(1+e^{\frac{x}{\sqrt{6}}}\right)^2} + \frac{5e^{\frac{x}{\sqrt{6}}}t}{3\left(1+e^{\frac{x}{\sqrt{6}}}\right)^3} + \frac{25e^{\frac{x}{\sqrt{6}}}t^2\left(-3+9e^{\sqrt{\frac{2}{3}}x}+6e^{\sqrt{\frac{3}{2}}x}-4e^{\frac{x}{\sqrt{6}}}t\right)}{108\left(1+e^{\frac{x}{\sqrt{6}}}\right)^6}$$

$$+\frac{25e^{\frac{x}{\sqrt{6}}}t^3}{81648(1+e^{\frac{x}{\sqrt{6}}})^{12}}\begin{pmatrix} 630+13230e^{4\sqrt{\frac{2}{3}}x}+2520e^{3\sqrt{\frac{3}{2}}x} \\ -63e^{\frac{x}{\sqrt{6}}}(-48-34t+5t^2)-252e^{\frac{7x}{\sqrt{6}}}(-102+26t+5t^2)-378e^{\sqrt{6}x}(-48+71t+10t^2) \\ +14e^{\sqrt{\frac{2}{3}}x}(216+567t+144t^2-50t^3)+84e^{2\sqrt{\frac{2}{3}}x}(-225-255t+87t^2+25t^3) \\ +7e^{\frac{5x}{\sqrt{6}}}(-1080-5670t-117t^2+200t^3)+e^{\sqrt{\frac{3}{2}}x}(-7560+3780t+7938t^2-400t^4) \end{pmatrix},$$

which converges to the exact solution is $u(x,t) = \frac{1}{\left(1+e^{-\frac{5t}{6}+\frac{x}{\sqrt{6}}}\right)^2}$ (Liu et al., 2023).

5.4 Numerical Results and Discussion

In this section, we present and discuss the efficiency and capability of the proposed method in the form of tables and graphs based on the numerical results of Example 5.1 and 5.2.Numerical results were generated by using Mathematica software.



a	b

Figure 1: The physical nature of the solution of Example 1 for $\alpha = 1$ compared with the exact solution: a) Approximate b) Exact

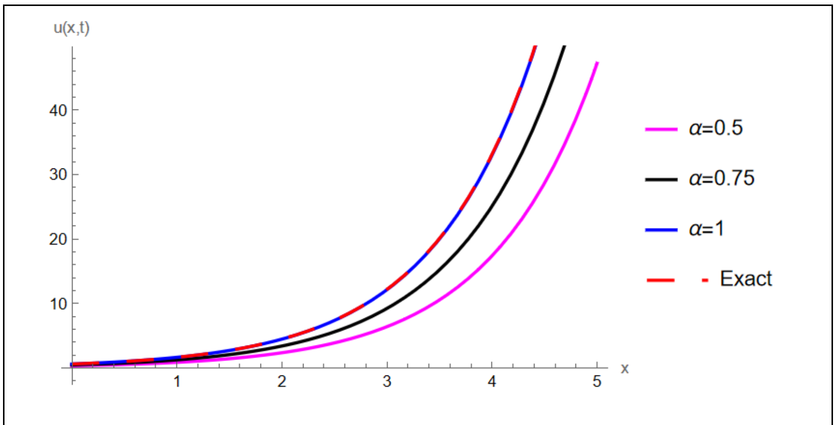


Figure 2: Comparisons of the physical nature the solutions of Example 1 by C-VIM for α∈ [0.5, 0.75,1] with the exact solution.

Figure 1(a) and (b) show the comparison of approximate solutions of Example 1 by C-VIM for $\alpha = 1$ with the exact solution. It can be observed from this figure that the approximate and the exact solutions are identical.

Figure 2 illustrates the comparisons of the approximate solutions of Example 1 by C-VIM for α∈ [0.5, 0.75, 1] with the exact solution. The results demonstrate that as α approaches one, the approximate solution become closer and closer to the exact solution.

Table 2 presents the numerical solutions of example 1 for different fractional order α derived from C-VIM when x=1. Furthermore, we compared the numerical values of our solution and absolute errors for Example1 for (α = 1) with the results obtained by Liu et al. (2023) as presented in Table 3. Our findings indicate that the C-VIM solution achieves the best approximation to the exact solution.

Table 2: Numerical values of Example 1 for different fractional order α derived from C-VIM and x=1.

t	$\alpha = 0.3$	$\alpha = 0.5$	$\alpha = 0.7$	$\alpha = 0.9$
0.2	1.513224664	1.646263262	1.839935709	2.096554971
0.4	0.8297559249	1.054247104	1.339540953	1.670355605
0.6	0.4733263419	0.727447018	1.028326186	1.349513411
0.8	0.3162679382	0.5548771724	0.8250103624	1.09218053
1.0	0.2531733076	0.4530469714	0.6708580154	0.8649078545

Table 3: Comparison of numerical values of Example 1 when $\alpha = 1$ with Liu et al. (2023).

x=0.5	t	Exact	Approximate Solution		Absolute Errors	
			Liu et al. (2023)	Present	Liu et al. (2023)	Present
0.5	0.05	1.568312185	1.56831	1.568311760	0.00000218	4.25×10^{-07}
	0.15	1.419067549	1.41903	1.419033789	0.00004	3.38×10^{-05}
	0.25	1.284025417	1.28377	1.283769948	0.00026	2.55×10^{-04}
	0.35	1.161834243	1.16087	1.160871516	0.00096	9.63×10^{-04}
	0.45	1.051271096	1.04869	1.048689773	0.00258	2.58×10^{-03}
1.0	0.05	2.585709659	2.58571	2.585708958	0.0000	7.01×10^{-07}
	0.15	2.339646852	2.33959	2.339591191	0.00006	5.57×10^{-05}
	0.25	2.117000017	2.11658	2.116578820	0.00042	4.21×10^{-04}
	0.35	1.915540829	1.91395	1.913953562	0.00159	1.59×10^{-03}
	0.45	1.733253018	1.729	1.728997136	0.00425	4.26×10^{-03}

Similarly, we present presents the numerical solutions of example 2 for different fractional order α derived from C-VIM when x=1 in table 4. As it can be observed from the table, the numerical solution of the problem converges to the exact solution as the values of the fractional order α approach to one.

Table 4: Numerical values of Example 2 for different fractional order α derived from C-VIM and x =1.

t	$\alpha = 0.3$	$\alpha = 0.5$	$\alpha = 0.7$	$\alpha = 0.9$
0.2	0.24373476	0.23480991	0.22160809	0.20342914
0.4	0.31764339	0.29996756	0.27666369	0.24731511
0.6	0.38445638	0.36034246	0.33005672	0.29313253
0.8	0.44470218	0.41667930	0.38217735	0.34054810
1.0	0.49831507	0.46880664	0.43258593	0.38881665

In Table 5 we present absolute errors of Example 2 obtained by C-VIM in comparison with SRPSM (Liu et al. 2023) when α = 1, at x =0.5 and 1 and for diverse values of t. As it can be observed from table our present work decreases the absolute error than SRPSM.

Table 5: Absolute errors of Example 2 obtained by C-VIM in comparison with SRPSM when α = 1, x = 0.5 and 1 and for diverse values of t.

x	t	Exact	Approximate Solution		Absolute Errors	
			SRPSM	C-VIM	SRPSM	C-VIM
0.5	0.05	0.21111702	0.210998	0.21111701	0.001	5.47×10^{-09}
	0.15	0.23062016	0.229679	0.23061973	0.001	4.33×10^{-07}
	0.25	0.25105340	0.248804	0.25105011	0.003	3.29×10^{-06}
	0.35	0.27235003	0.268668	0.27233749	0.004	1.25×10^{-05}
	0.45	0.29443139	0.289567	0.29439708	0.005	3.43×10^{-05}
1	0.05	0.16758157	0.167455	0.16758156	0.0001	5.25×10^{-09}
	0.15	0.18460565	0.183602	0.18460525	0.001	3.98×10^{-07}
	0.25	0.20266838	0.200255	0.20266549	0.002	2.89×10^{-06}
	0.35	0.22173509	0.217750	0.22172461	0.004	1.05×10^{-05}
	0.45	0.24175777	0.236813	0.24173054	0.004	2.72×10^{-05}

In Figure 3(a) and (b), the physical nature of the solution of Example 2 for $\alpha = 1$ is compared with the exact solution and Figure 4 shows the Comparisons of the physical nature the solutions of Example 2 by C-VIM for α∈ [0.5,0.75,1] with the exact solution. In both results the present method (C-VIM) approximate a solution rapidly agreed with the exact solution as the value of α approaches one.

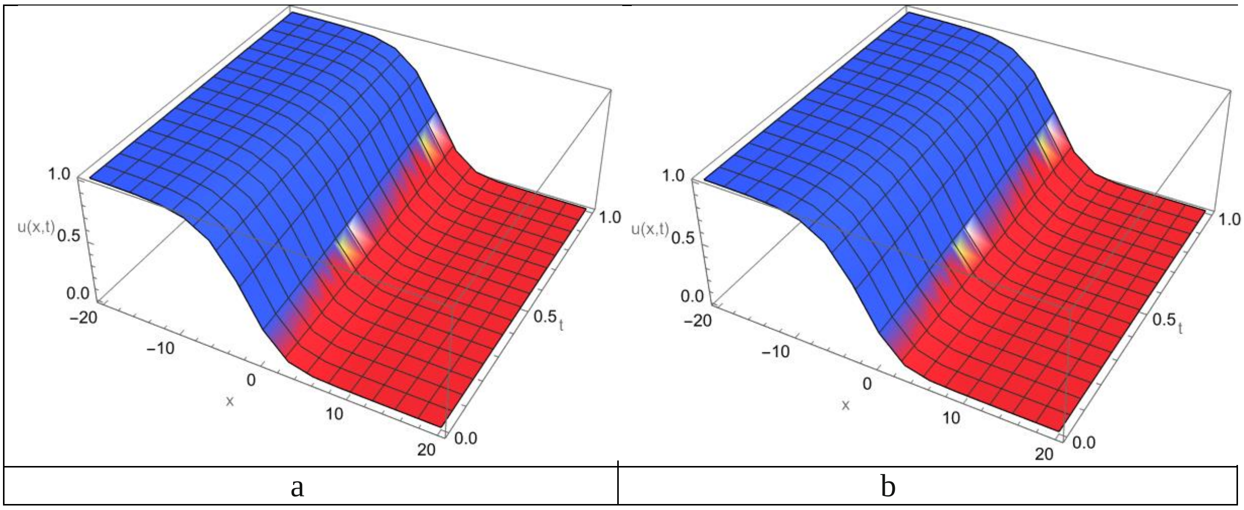


Figure 3: The physical nature of the solution of Example 2 for $\alpha = 1$ compared with the exact solution:

a) Approximate b) Exact

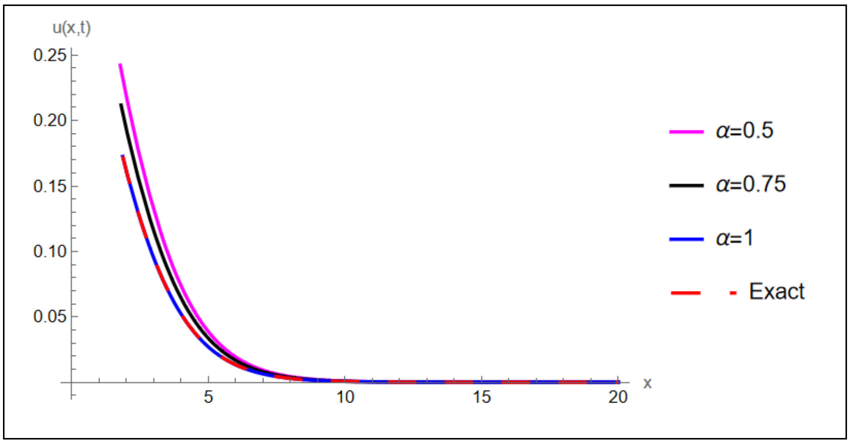


Figure 4: Comparisons of the physical nature the solutions of Example 2 by C-VIM for α ∈ [0.5, 0.75, and 1] with the exact solution.

In general, it can be observed that solution depends on the conformable time-fractional derivative. Furthermore, the maximum absolute errors of the approximate solutions of both problems in their domain of interest is small, assures that the method thought provides good approximate results. Of course, accuracy and efficiency can be enhanced by increasing the number of iterations.

CHAPTER SIX
CONCLUSION AND FUTURE SCOPE

6.1 Conclusion

This study investigates the conformable variational iteration method (C-VIM) for solving the time fractional Newell-Whitehead-Segel equation. The effectiveness of C-VIM is demonstrated through two examples.

Numerical values of the two examples for different fractional order α derived from C-VIM are presented in tables and figures. The solutions obtained by the proposed method are compared with the exact solutions for the two examples.

Numerical solutions obtained by C-VIM are compared with the work of Liu et al. (2023) and can be observed that they are in good agreement. 2D graphs of the solutions are plotted in Figures 2 and 4 to visualize the solution behavior of the proposed method. We used Mathematica software to obtain the approximate solutions and plotting the graphs.

The method has many advantages when compared with similar works in the existing literature, since; the approximate solutions are obtained without any discretization, perturbation, and restrictive conditions. But, saving us a lot of time and effort with minimal amount of computational work it gives solutions that agree perfectly with the exact solutions.

As a result, we conclude that C-VIM is a powerful, useful, effective and has high accuracy to solve time fractional NWSEs. Finally, we conclude that both the analytical and numerical results revealed that the solution of the time fractional NWSE can be successfully obtained by using the C-VIM. When dealing with fractional calculus, this method proven to be a practical and effective tool for solving such equations.

6.2 Future Work

In this study, the Conformable Variational Iteration Method (C-VIM) has been successfully applied to find approximate solution of NWSE (1.1) subject to an initial condition (1.2). Following the same procedure, as discussed in this research work, the Conformable Variational iteration Method (C-VIM) may be applied to solve second order linear and non-linear partial differential equations.

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