Positive Solutions of Fractional Order Differential Equations with Integral Boundary Conditions and a Parameter

A RESEARCH SUBMITTED TO JIMMA UNIVERSITY, COLLEGE OF NATURAL SCIENCES, DEPARTMENT OF MATHEMATICS IN PARTIAL FULFILLMENT FOR THE REQUIREMENTS OF THE DEGREE OF MASTER OF SCIENCE IN MATHEMATICS

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June, 2024

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Declaration

I, the undersigned declare that, this research paper entitled "**Positive Solutions of Fractional Order Differential Equations with Integral Boundary Conditions and a Parameter**" is my own original work and it has not been submitted for the award of any academic degree or the like in any other institution or university, and that all the sources I have used or quoted have been indicated and acknowledged.

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Acknowledgment

Firstly, I am indebted to my almighty God who gave me a long life, and strengthened and helped me in my life. Next, I would like to express my deepest gratitude to my advisors, Wesen Legesse (Ph.D.) and Solomon Bati (M.Sc.), for their support, constructive comments, and immediate response that helped me in preparing and writing this thesis.

Lists of Abbreviations

FDEs - Fractional Differential Equations
BVPs - Boundary Value Problems
BCs - Boundary Conditions
ICs - Initial Conditions
IBCs - Integral Boundary Conditions
Eq. - Equation
FBVPs – Fractional Boundary Value Problems

Abstract

In this study, we consider a fractional order differential equation with integral boundary conditions and a parameter. In the meantime, we investigate the existence and uniqueness of positive solutions. To accomplish the objectives, we determine the Green's function for the fractional boundary value problem and develop the operator equation by using Green's function as a kernel. By applying the Guo-Krasnoseliskii fixed-point theorem on a cone, we develop the existence results of at least one positive solution in a Banach space. To verify uniqueness of the solutions, we apply Banach contraction mapping principle. Finally, we provide particular example to illustrate the main result.

Contents

Chapter 1

Introduction

1.1 Background of the Study

The classical differential equations, which involve integer-order derivatives, have been widely used to describe a wide range of physical processes. However, they often fail to capture the memory and long-range interactions inherent in certain systems. This led to the development of fractional calculus, which incorporates fractional derivatives and fractional integration allowing for the representation of delayed and distributed effects(Mainardi et al., 2018).

Fractional calculus is the theory of integrals and derivatives of any arbitrary order. It is a generalization of the ordinary differentiation and integration to any arbitrary order(Oldham et al., 1974). The subject is as old as differential calculus and goes back to many great mathematicians such as Leibniz, Liouville, Grűnwald, Letnikov, Riemann, L'Hopital and Weyl when Leibniz and Newton invented differential calculus(Morel et al., 2013).

Fractional differential equation arises in various fields of science and engineering such as physics(Hilfer, 2000), aerodynamics(Mainardi et al., 2018), viscoelasticity (Mainardi et al., 2018), electromagnetism, control theory(Ahmad et al., 2016), chemistry, biology, economics (Cabada & Wang, 2012), etc. These applications and interest in this area of mathematics are growing fast across several disciplines, including applied mathematics(Podlubny, 1998).

A fractional differential equation refers to an equation in which the dependent variable's fractional derivative concerning the independent variable is involved (Salehi & Schiesser, 2018). Fractional derivatives are an excellent tool for describing inherited properties of different materials and processes as well as long-term memory. Compared with standard integer-order models, the primary benefit of fractional differential equations is this. There are numerous types of fractional derivative operators proposed in the literature, including Riemann–Louville, Caputo, Hadamard, Katugampola, Hilfer, etc(Yang, 2019).

Certain fractional derivatives have been found to include other kinds of fractional derivatives. For instance, Riemann–Liouville and Hadamard fractional derivatives are included in the generalized fractional derivative that Katugampola(Katugampola, 2011) introduced, whereas Riemann–Liouville and Caputo fractional derivatives are the primary focus of the Hilfer fractional derivative (Hilfer, 2000). This is another advantage of studying fractional calculus.

Boundary value problems (BVPs) arise in the study of various biological, physical, and chemical processes such as heat conduction, thermo elasticity, chemical engineering, and so on(Jajarmi & Baleanu, 2020). Due to fractional order models being more realistic and useful, fractional boundary value problems are crucial in the applied sciences and engineering fields. An intriguing and significant class of problems are boundary value problems with integral boundary conditions. As special cases, they include boundary value problems with two, three, multipoint, and nonlocal variables (Ahmad et al., 2016).

Integral boundary conditions (IBCs) are the type of boundary conditions that are commonly used in fractional order differential equation (FODE) problems. IBCs involve integrals of the solution over the entire domain of the problem. They have various applications in applied fields such as blood flow problems, chemical engineering, thermo-elasticity, underground water flow, population dynamics, and so forth(Cabada & Wang, 2012).

The study of positive solutions for Boundary Value Problems (BVPs) is paramount and an active area of research due to its significant applications in various fields, including population dynamics, chemical reactions, and diffusion processes (Mainardi, 2018). For example, positive solutions of a BVP arising from chemical reactor theory represent the temperature during the reaction(Morel et al., 2013).

Researchers have recently examined the existence and uniqueness of positive solutions for more general classes of FDE problems with IBCs using Guo-Kransnoseliskii's fixed-point theorems on a cone(H. Abbas et al., 2021; Hao et al., 2019; M. Li et al., 2020a). These recent advances on the study of positive solutions for fractional BVPs with IBCs conditions have led to a deeper understanding of these complex systems.

In 2009, by using the nonlinear alternative of the Leray-Schauder type and Guo-Krasnoselkii's fixed-point theorem, Bai et al(Bai et al.,2009) obtained the existence of a positive solution to the singular BVP:

$$
\begin{aligned}
{}^cD_{0+}^{\alpha}u(t) + f(t, u(t)) = 0, 0 < t < 1,\\
u(0) = u''(0) = u'(1) = 0,
\end{aligned}
$$

where $\alpha \in (2, 3]$ is a real number and $f(t, u)$ is singular at $t = 0$, ${}^cD_{0+}^{\alpha}$ denote Caputo fractional derivative of order α.

In 2012, Cabada (Cabada and Wang, 2012) studied the existence of a positive solution for the BVP with integral boundary conditions:

$$
\begin{aligned}
{}^cD_{0+}^{\alpha}u(t) + f(t, u(t)) = 0, 0 < t < 1,\\
u(0) = u''(0) = 0,\\
u(1) = \lambda \int_0^1 u(s)ds,
\end{aligned}
$$

where ${}^cD_{0+}^{\alpha}$ denote Caputo fractional derivative of order $\alpha, 2 < \alpha < 3$, $0 < \lambda < 2$, $f : (0, 1) \times (0, \infty) \to (0, \infty)$ is continuous.
The authors proved the existence of a positive solution of the equation by employing Guo-Krasnoselskii's fixed-point theorem.

In 2014, Wenquan Feng(Feng et al., 2014) studied Positive solutions to fractional boundary value problems with nonlinear boundary conditions:

$$
\begin{aligned}
D_{0+}^{\alpha}u(t) = f(t, u(t)), \quad 0 < t < 1,\\
u(0) = 0,\\
u(1) = H_1(\varphi(u)) + \int_E H_2(s, u(s))ds, 1 < \alpha < 2,
\end{aligned}
$$

where D_{0+}^{α} denotes Riemann-Lioville's fractional derivative of order α and φ having the form $\varphi(u) = \int_0^1 u(t)\,d\theta(t)$, the integral appearing in the boundary condition is taken in the Lebesgue-Stieltjes sense and θ is a function of bounded variation.

In 2019, X. Hao(Hao et al., 2019) studied positive Solutions of fractional differential equation with integral boundary conditions and a parameter:

$$
\begin{aligned}
-D_{0+}^{\eta-2}(u''(t)) + \lambda f(t, u(t)) = 0, t \in (0,1) \\
u''(0) = u'''(0) = \cdots = u^{(n-2)}(0) = 0, D_{0+}^{(k-2)}(u''(1)) = 0, \\
\alpha u(0) - \beta u'(0) = \int_0^1 u(s)dA(s), \\
\gamma u(1) + \delta u'(1) = \int_0^1 u(s)dB(s),
\end{aligned}
$$

where $D_{0+}^{\eta-2}$ and D_{0+}^{k-2} denote the standard Riemann-Liouville fractional derivative of order $\eta - 2$ and $k - 2$, respectively, with $n - 1 < \eta \leq n$, $\eta \geq 4$, and $2 \leq k \leq n - 2$. Additionally, $\alpha, \beta, \gamma, \delta > 0$, $f : [0,1] \times [0, +\infty) \to [0, +\infty)$ is continuous, $\lambda > 0$ is a parameter, and the integrals denote Riemann-Stieltjes integrals of u with respect to A and B, respectively, where $A(t)$ and $B(t)$ are nondecreasing on $[0,1]$.

In 2020, Min Li and others (Li et al., 2020b) applied the monotone iterative method and some inequalities associated with Green's function, obtained the existence of minimal and maximal positive solutions, and established two iterative sequences for approximating the solutions to the following BVP with integral boundary conditions:

$$
\begin{aligned}
{}^cD_{0+}^{q}u(t) + f(t, u(t)) = 0, t \in (0,1), \\
u''(0) = 0, \\
\alpha u(0) - \beta u'(0) = \int_0^1 h_1(s)u(s)ds, \\
\gamma u(1) + \delta({}^cD_{0+}^{\sigma}u)(t)(1) = \int_0^1 h_2(s)u(s)ds,
\end{aligned}
$$

where ${}^cD_{0+}^{q}$ and ${}^cD_{0+}^{\sigma}$ denote the standard Caputo fractional derivatives of order q and order σ, respectively.

In 2021, Hafida Abbas (Abbas et al., 2021) studied Positive solutions for fractional boundary value problems with integral boundary conditions and parameter dependence:

$$D^{\delta}u(t) + f(t, u(t)) = 0, 0 < t < 1,$$
$$u(0) = 0, \quad u(1) = \lambda \int_0^1 h(r)u(r)dr,$$

where D^{δ} denotes the Riemann-Liouville's fractional derivative of order δ, with $1 < \delta \leq 2$, h is a weighted function, and $\lambda > 0$ is a parameter.

However, the theory of boundary value problems for nonlinear fractional differential equations with integral boundary conditions is still in the initial stages and many aspects of this theory need to be explored.

1.2 Statement of the Problem

Fractional differential equations with IBCs are particularly challenging class of BVPs. The existence and uniqueness of positive solutions for such problems remain largely unexplored. To the best of our knowledge, the existence of a positive solution for the fractional differential equation (1.1) with boundary condition (1.2) below is not yet investigated. Therefore, Motivated by the paper by X.Hao (Hao et al., 2019) and others, the main objective of this study is to investigate the existence and uniqueness of positive solutions for the fractional order differential equation with integral boundary conditions and a parameter described in equations (1.1) and (1.2) below, and establish the existence theorem.

$$-D_{0+}^{\eta-2}\left(u''(t) + k^2 u(t)\right) + f\left(t, u(t)\right) = 0 \quad t \in (0,1), \tag{1.1}$$

$$\begin{aligned} u''(0) = -k^2u(0), \ldots, u^{(5)}(0) = -k^2u'''(0), u'''(1) = -k^2u'(1),& \\ \alpha u(0) - \beta u'(0) = \int_0^1 h_1(s)u(s)ds,& \\ \gamma u(1) + \delta u'(1) = \int_0^1 h_2(s)u(s)ds,& \end{aligned} \tag{1.2}$$

where $D_{0+}^{\eta-2}$ denotes the standard Riemann-Liouville fractional derivative of order $\eta - 2$, with $4 \leq \eta < 5$, $k \in (0, \pi/2)$ is a parameter, $\alpha, \beta, \gamma, \delta$ are positive constants such that $k^2 \leq \frac{\alpha\gamma}{\beta\delta}$, $\int_0^1 h_1(s)u(s)\,ds$ and $\int_0^1 h_2(s)u(s)\,ds$ denote

Riemann integrals of u with respect to s, $f : [0,1] \times [0,+\infty) \to [0,+\infty)$ is continuous, h_1 and h_2 are distinct given weighted functions.
Consequently, this study attempts to focus on the following problems:

1. Determine the existence and uniqueness of positive solutions for the given fractional boundary value problem.

2. Investigate conditions for the existence and uniqueness of a positive solution

3. Examine how the properties of the function $f(t, u(t))$ and the parameter k affect the existence and behavior of positive solutions.

1.3 Objectives of the study

1.3.1 General Objective

The general objective of this study is to investigate the existence and uniqueness of positive solutions for the fractional boundary value problem with integral boundary conditions described in equations (1.1) and (1.2).

1.3.2 Specific Objectives

1. To define fractional-order BVP.

2. To transform the given fractional BVP into an equivalent integral equation by construction of Green's function

3. To determine the fixed point of the integral equation.

4. To verify the existence of positive solutions by applying the Guo-Krasnoseliskii fixed-point theorem.

5. To verify the uniqueness of positive solutions by applying the Banach contraction mapping principle.

6. To develop an illustrative example.

1.4 Delimitation of the study

The study is delimited to positive solutions of fractional boundary value problem with integral boundary conditions described in equations (1.1) and (1.2).

1.5 Significance of the Study

It is believed that the result of this study will have the following importance:

- It will help the researcher to build a basic concept on how to conduct scientific research.
- It may help other researchers as a basis on related research topics.
- It may familiarize a researcher with scientific communication in applied mathematics.

Chapter 2

Literature Review

2.1 Overview of Fractional Calculus

The study of fractional calculus started at the end of the seventeenth century. Its basic theory was developed with the study of Lioville, Grunwald, Letnikov, and Riemann at the end of the nineteenth century(H. Abbas et al., 2021; S. Abbas et al., 2012). Fractional calculus is a generalization of classical integer-order calculus to non-integer-order derivatives and integrals (Oldham et al., 1974). Applications and interest in this area of mathematics are growing across several disciplines, including applied mathematics, engineering, and physics (Podlubny, 1998).

Fractional-order operators can be used to model memory and hereditary properties inherent in various materials and processes, leading to more accurate descriptions of complex dynamical systems (Baleanu, 2012). The theory of fractional calculus has been developed extensively, with a growing body of literature on fractional-order differential and integral equations, their properties, and numerical methods for their solution (Samko, 1993). The applications of fractional calculus continue to expand, with prominent examples in control theory, signal processing, and anomalous diffusion phenomena (Machado et al., 2011; Meerschaert et al. 2019).

2.2 Recent Studies on Fractional Boundary Value Problems

Recently, there has been much attention given to the study of the existence and multiplicity of solutions or positive solutions for boundary value problems of fractional differential equations. The researchers have utilized various techniques of nonlinear analysis to approach this problem, including fixed-point theorems, upper and lower solutions method fixed-point,index coincidence theory and Banach contraction mapping principle (H.Abbas et al., 2018; Feng et al., 2014; Hao et al., 2018).

Using fixed point theorem on a cone, there were some outstanding works accomplished between 2005 and 2010, regarding the existence of the solution for nonlinear BVPs. Positive solutions for boundary value problems of nonlinear fractional differential equations involving various boundary conditions were studied by authors like Zhanbing Bai and Yinghan Zhang (H. Abbas et al., 2018; Bai et al., 2014).

The advance in working with integral boundary conditions was started in 2012 by Alberto Cabada and Guotao Wang(Cabada & Wang, 2012). They studied positive solutions of nonlinear fractional differential equations with integral boundary conditions:

$$\begin{aligned} {}^cD^{\alpha}_{0+}u(t) + f(t, u(t)) = 0, 0 < t < 1, \\ u(0) = u''(0) = 0, \\ u(1) = \lambda \int_0^1 u(s)ds, \end{aligned}$$

where ${}^cD^{\alpha}_{0+}$ denote Caputo fractional derivative of order α, $2 < \alpha < 3$, $0 < \lambda < 2$, $f : (0, 1) \times (0, \infty) \to (0, \infty)$ is continuous. The existence of positive solutions for such BVPs was studied by Baleanu Mouffak et al(Baleanu et al., 2015).

In 2014, Wenquan Feng(Feng et al., 2014) studied Positive solutions to fractional boundary value problems with nonlinear boundary conditions:

$$\begin{aligned} D^{\alpha}_{0+}u(t) = f(t, u(t)), \quad 0 < t < 1, \\ u(0) = 0, \\ u(1) = H_1(\varphi(u)) + \int_E H_2(s, u(s))ds, 1 < \alpha < 2, \end{aligned}$$

where D^{α}_{0+} denotes Riemann-Lioville's fractional derivative of order α and φ having the form $\varphi(u) = \int_0^1 u(t)\, d\theta(t)$, the integral appearing in the boundary condition is taken in the Lebesgue-Stieltjes sense and θ is a function of bounded variation.

In 2015, Li et al (Li & Xu, 2015) studied boundary value problems of fractional order differential equations with integral boundary conditions and not instantaneous impulses.

In 2016, Mustafa et al (Mustafa et al., 2016) studied positive solutions of higher-order nonlinear multi-point fractional differential equations with integral boundary conditions:

$$
\begin{aligned}
-D_{0+}^{\eta-2}(u''(t)) + \lambda f(t, u(t)) = 0, t \in [0,1] \\
u''(0) = u'''(0) = \cdots = u^{(n-2)}(0) = 0, u'''(1) = 0, \\
\alpha u(0) - \beta u'(0) = \sum_{p=0}^{m-2} a_p \int_0^{\xi_p} u(s)ds, \\
\gamma u(1) + \delta u'(1) = \sum_{p=0}^{n-1} b_p \int_0^{\xi_p} u(s)ds,
\end{aligned}
$$

where $D_{0+}^{\eta-2}$ is the Riemann-Liouville fractional derivative of order $\eta - 2$, $m, n \geq 3$, $n - 1 < \eta \leq n$, where $n, m \in N$ and $\alpha, \beta, \gamma, \delta > 0$, $a_p, b_p \geq 0$ are given constants, and $0 < \xi_1 < \ldots < \xi_{m-2} < 1$, $f\colon [0,1] \times [0,\infty) \to [0,\infty)$ is continuous.

In 2018, Sun and Zhao(Zhao & sun, 2018c) studied Positive Solutions for Boundary Value Problems of Fractional Differential Equations with Integral Boundary Conditions:

$$
\begin{aligned}
D_{0+}^{\alpha} x(t) = f(t, x(t)), \quad t \in (0,1), \\
x(0) = x'(0), \\
x(1) = \lambda \int_0^1 x(t) dA(t),
\end{aligned}
$$

where D_{0+}^{α} is the Riemann-Liouville differential operator and $2 < \alpha < 3$, $A(t)$ is right continuous on $[0,1)$, left continuous at $t = 1$, and nondecreasing on $[0,1]$ with $A(0) = 0$, and $\int_0^1 x(t)\, \mathrm{d}A(t)$ denotes the Riemann-Stieltjes integral of x with respect to A.

In 2019, X. Hao (Hao et al., 2019) studied Positive Solutions of higher order fractional integral boundary value problems with a parameter:

$$
\begin{aligned}
-D_{0+}^{\eta-2}(u''(t)) + \lambda f(t, u(t)) = 0, t \in (0,1) \\
u''(0) = u'''(0) = \cdots = u^{(n-2)}(0) = 0, D_{0+}^{(k-2)}(u''(1)) = 0, \\
\alpha u(0) - \beta u'(0) = \int_0^1 u(s) dA(s), \\
\gamma u(1) + \delta u'(1) = \int_0^1 u(s) dB(s),
\end{aligned}
$$

where $D_{0+}^{\eta-2}$ and D_{0+}^{k-2} denote the standard Riemann-Liouville fractional derivative of order $\eta - 2$ and $k - 2$, respectively, with $n - 1 < \eta \leq n$, $\eta \geq 4$, and $2 \leq k \leq n - 2$. Additionally, $\alpha, \beta, \gamma, \delta > 0$, $f : [0,1] \times [0, +\infty) \to [0, +\infty)$ is continuous, $\lambda > 0$ is a parameter, and the integrals denote Riemann-Stieltjes integrals of u with respect to A and B, respectively, where $A(t)$ and $B(t)$ are nondecreasing on $[0, 1]$.

The work of Yuansheng et al in 2019, on the existence of positive solutions for a boundary value problem of fractional differential equation with a p-Laplacian operator and the study of new uniqueness results for the fractional differential equations with dependence on the first-order derivative by Zhenzhen Yue e(H. Abbas et al., 2018; Yue and Zou, 2019) are also recent advances on the study of fractional boundary value problem.

In 2020, Rezan Sevinik Adigüze (Sevinik Adigüzel et al., 2020) studied the solution of a boundary value problem associated with a fractional differential equation:

$$
\begin{aligned}
D^{\delta} w(x) + g(x, w(x)), \quad t \in (0,1) \\
w(0) = w'(0) = 0, \\
w(1) = \lambda \int_0^1 w(z) dz,
\end{aligned}
$$

where D^{δ} denotes the Riemann-Lioville's factional derivative of order δ, $1 < \delta \leq 2$, h is weighted function and λ is a parameter.

Researcher such as Min Li (M. Li et al., 2020a), Hanan A. (Wahash et al., 2020), and Amin Jajarmi (Jajarmi & Baleanu, 2020) studied Positive solutions for BVP of fractional differential equation with integral boundary conditions of Caputo type.

In 2021, Hafida (H. Abbas et al., 2021) studied Positive solutions for fractional boundary value problems with integral boundary conditions and parameter dependence:

$$D^{\delta}u(t) + f(t, u(t)) = 0, 0 < t < 1,$$
$$u(0) = 0, \quad u(1) = \lambda \int_0^1 h(r)u(r)dr,$$

where D^{δ} denotes the Riemann-Lioville's factional derivative of order δ ,$1 < \delta \leq 2$, h is weighted function and $\lambda > 0$ is a parameter.

The work of Mohammed S. Abdo et al FDE with Mittag-Leffler kernel and nonlinear integral conditions in 2021 and the study of Ning Wang (Wang & Zhou, 2023) in 2023 on FDEs with improper integral boundary conditions on the half-line are very recent and significant advancements of FDEs with IBCs.

Although the research on positive solutions for fractional order BVPs with IBCs is in advance, there are still several challenges in solving different classes of the problem. Establishing existence and uniqueness results becomes more difficult when non-linear terms and IBCs are included. Using standard fixed-point methods for fractional derivatives is difficult because of their memory and non-locality(H. Abbas et al., 2018; Ahmad et al., 2019, 2023). Another difficulty is to investigate the asymptotic behavior of solutions for large time variable values, the dependence of solutions on BVP parameters, and the existence and uniqueness of multiple positive solutions(Abdo et al., 2020).

To address these challenges, researchers have developed new mathematical techniques and tools, especially for fractional BVPs. These techniques incorporate the use of fractional Green's function(H. Abbas et al., 2021), fractional integral transform(Debnath et al., 2016), and fixed-point theorems(Ahmad et al., 2016). Furthermore, researchers have investigated numerical methods for approximating positive solutions of fractional BVPs(Ali, 2012; Jacobs et al., 2023; Jajarmi and Baleanu, 2020).

These advances have led to an increased knowledge of the qualitative and long-term behavior of solutions to fractional differential equations. However, further study is required to discover more general and reliable techniques for solving higher classes of fractional-order BVPs.

Chapter 3

Methodology

3.1 Study Area and Study period

This study was conducted at Jimma University, College of Natural Sciences, Department of Mathematics from October 2023 to June 2024.

3.2 Research Design

In this study, the documentary review design method has been employed.

3.3 Source of Information

The relevant sources of information that have been used in this study are:

- Different related mathematics books;
- Published articles or journals; and
- Internet sources.

3.4 Mathematical Procedures

To achieve the stated objectives, the following procedures has been followed.

1. Solving associated homogenous BVP for the FBVP.
2. Transforming the FBVP into an integral equation by constructing Green's function.
3. Determining the fixed point of the integral equation.
4. Proving the existence of positive solutions by applying the Guo-Krasnoseliskii fixed-point theorem.

5. Proving the uniqueness of positive solutions by using Banach contraction mapping principle.
6. Give examples to illustrate the applicability of the result.

Chapter 4

Results and Discussion

4.1 Preliminaries

Definition 4.1.1 *(Royden et al.,2010) Let X be a non-empty set. A map $T : X \to X$ is said to be a self-map with domain of $T = D(T) = X$ and range of $T = R(T) \subset X$.*

Definition 4.1.2 *(Kreyszig et al., 1989) A normed space is a vector space X in which for each vectors x and y, there correspond real numbers, denoted by $\|x\|$ and $\|y\|$, called the norm of x and the norm of y respectively, and have the following properties:*

i) $\|x\| \geq 0$,

ii) $\|x\| = 0 \iff x = 0$,

iii) $\|\alpha x\| = |\alpha|\|x\|, \forall x \in X,$ *and α being a scalar.*

iv) $\|x + y\| \leq \|x\| + \|y\|$.

Definition 4.1.3 *(Royden et al.,2010) A collection $\mathcal{T}$ of real-valued functions on a metric space X is said to be equicontinuous at the point $x \in X$ provided for each $\epsilon > 0$, there is a $\delta > 0$ such that for every $f \in \mathcal{T}$ and $x' \in X$, if $\rho(x', x) < \delta$, then $|f(x') - f(x)| < \epsilon$. The collection $\mathcal{T}$ is said to be equicontinuous on X provided it is equicontinuous at every point in X.*

Definition 4.1.4 *(Kreyszig et al., 1991) A Normed space X is said to be complete if every Cauchy sequence in X converges to a point in X.*

Definition 4.1.5 *(Kreyszig et al., 1991) A Banach space is a complete normed space.*

Definition 4.1.6 *(Kreyszig et al., 1991) A fixed point of a mapping $T : X \to X$ of a set X into itself is an $x \in X$ which is mapped onto itself (is "kept fixed" by T), that is, $Tx = x$, the image Tx coincides with x.*

Definition 4.1.7 *(Kreyszig et al., 1991) Let $X = (X, d)$ be a metric space. A mapping $T : X \to X$ is called a contraction on X if there is a positive real number $\alpha < 1$ such that for all $x, y \in X$:*

$$d(Tx, Ty) \leq \alpha d(x, y)$$

Definition 4.1.8 *(Kreyszig et al., 1991) Consider a metric space $X = (X, d)$, where $X \neq \emptyset$. Suppose that X is complete and let $T : X \to X$ be a contraction on X. Then T has precisely unique fixed point.*

Definition 4.1.9 *(Royden et al.,2010) Let E be a real Banach space. A nonempty closed convex set $P \subset E$ is called a cone, if it satisfies the following two conditions:*

i) $y \in P, \alpha \geq 0 \implies \alpha y \in P$, and

ii) $y \in P$ and $-y \in P \implies y = 0$.

Definition 4.1.10 *Let X and Y be Banach spaces and $T : X \to Y$ an operator. T is said to be completely continuous, if T is continuous and if it maps compact set into pre-compact sets.*

Definition 4.1.11 *(Andrews et al., 1998) The function $\Gamma : (0, \infty) \to \mathbb{R}$, defined by*

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt, \quad x > 0$$

is called Euler's Gamma function (or Euler's integral of the second kind).

Definition 4.1.12 *(Abbas et al.,2012) The Riemann-Lioville fractional integral of order $\alpha > 0$ of a continuous function $h : (0, +\infty) \to \mathbb{R}$ is given by*

$$I_{0+}^{\alpha} h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} h(s)\, ds, \quad t > 0,$$

provided that the right-hand side is point-wise defined on $(0, +\infty)$.

Definition 4.1.13 *(Abbas et al.,2012) The Riemann-Lioville fractional derivative of order $\alpha > 0$ of a continuous function $h : (0, +\infty) \to \mathbb{R}$ is given by*

$$D_{0+}^{\alpha} h(t) = \frac{1}{\Gamma(n - \alpha)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{h(s)}{(t - s)^{\alpha - n + 1}}\, ds,$$

where n is the smallest integer not less than α, provided that the right-hand side is point-wise defined on $(0, +\infty)$.

Lemma 4.1.1 *(Diethelm et al., 2010)*

i) If $u \in L^1[0,1]$, $\rho > \sigma > 0$, and $n \in \mathbb{N}$, then

$$D^{\sigma}_{0+} I^{\rho}_{0+} u(t) = I^{\rho-\sigma}_{0+} u(t), \quad D^{\sigma}_{0+} I^{\sigma}_{0+} u(t) = u(t),$$

$$\frac{d^n}{dt^n}\left(D^{\sigma}_{0+} u(t)\right) = D^{n+\sigma}_{0+} u(t).$$

ii) If $v \geq 0$, $\sigma > 0$, then

$$D^{v}_{0+} t^{\sigma} = \frac{\Gamma(\sigma+1)}{\Gamma(\sigma - v + 1)} t^{\sigma - v}.$$

iii) Let $\alpha > 0$. Then the following equality holds for $u \in L^1[0,1]$ and $D^{\alpha}_{0+} u \in L^1[0,1]$:

$$I^{\alpha}_{0+} D^{\alpha}_{0+} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n},$$

where $c_i \in \mathbb{R}$, $i = 1, 2, 3, \ldots, n$, and $n - 1 < \alpha \leq n$.

Let $-(u''(t) + k^2 u(t)) = y(t)$,t $\in (0,1)$. The boundary value problem

$$\begin{cases} D^{\eta-2}_{0+}(-(u''(t) + k^2 u(t))) + f(t, u(t)) = 0, t \in (0,1) \\ u''(0) = -k^2 u(0), \ldots, u^{(5)}(0) = -k^2 u'''(0), u'''(1) = -k^2 u'(1), \end{cases}$$

becomes

$$\begin{cases} D^{\eta-2}_{0+} y(t) + f(t, u(t)) = 0, & t \in (0,1) \\ y(0) = y'(0) = 0, & y'(1) = 0. \end{cases}$$

Lemma 4.1.2 *(Mustafa et al., 2016) Let $g \in C(0,1)$. Then the boundary value problem*

$$\begin{cases} D^{\eta-2}_{0+} y(t) + g(t) = 0, t \in (0,1) \\ y(0) = y'(0) = 0, y'(1) = 0 \end{cases}$$

has a unique solution

$$y(t) = \int_0^1 H(t,s)g(t)ds, \quad \text{where}$$

$$H(t,s) = \begin{cases} \dfrac{(1-s)^{\eta-4}}{\Gamma(\eta-2)} t^{\eta-3}, & 0 \le t \le s \le 1 \\ \dfrac{(1-s)^{\eta-4} t^{\eta-3} - (t-s)^{\eta-3}}{\Gamma(\eta-2)}, & 0 \le s \le t \le 1. \end{cases}$$

Proof:

$$D_{0+}^{\eta-2} y(t) + g(t) = 0, \quad t \in (0,1),$$

$$\Longrightarrow D_{0+}^{\eta-2} y(t) = -g(t),$$

$$\Rightarrow I_{0+}^{\eta-2} D_{0+}^{\eta-2} y(t) = -I_{0+}^{\eta-2} g(t).$$

From Lemma 4.1, we have:

$$y(t) + c_1 t^{\eta-3} + c_2 t^{\eta-4} + c_3 t^{\eta-5} + c_4 t^{\eta-6} + c_5 t^{\eta-7} = -\frac{1}{\Gamma(\eta-2)} \int_0^t (t-s)^{\eta-3} g(s)ds,$$

$$y(t) = \frac{1}{\Gamma(\eta-2)} \int_0^t (t-s)^{\eta-3} g(s)ds + c_1 t^{\eta-3} + c_2 t^{\eta-4} + c_3 t^{\eta-5} + c_4 t^{\eta-6} + c_5 t^{\eta-7}.$$

From the first BC, we have $c_2 = 0 = c_3 = c_4 = c_5$.
Hence, we have

$$y(t) = \frac{1}{\Gamma(\eta-2)} \int_0^t (t-s)^{\eta-3} g(s)ds + c_1 t^{\eta-3}.$$

$$y'(t) = \frac{\eta-3}{\Gamma(\eta-2)} \int_0^t (t-s)^{\eta-4} g(s)ds + (\eta-3)c_1 t^{\eta-4}.$$

Applying the second BC yields

$$y'(1) = \frac{\eta-3}{\Gamma(\eta-2)} \int_0^1 (1-s)^{\eta-4} g(s)ds + (\eta-3)c_1 = 0,$$

$$c_1 = \frac{-1}{\Gamma(\eta-2)} \int_0^1 (1-s)^{\eta-4} g(s)ds.$$

Thus

$$y(t) = \frac{1}{\Gamma(\eta-2)}\int_0^t (t-s)^{\eta-3}g(s)ds + \frac{-1}{\Gamma(\eta-2)}\int_0^1 (1-s)^{\eta-4}t^{\eta-3}g(s)ds$$

$$= \frac{1}{\Gamma(\eta-2)}\left[\int_0^t (t-s)^{\eta-3}g(s)ds - \int_0^t (1-s)^{\eta-4}t^{\eta-3}g(s)ds + \int_t^1 (1-s)^{\eta-4}t^{\eta-3}g(s)ds\right]$$

$$= \frac{1}{\Gamma(\eta-2)}\left[\int_0^t \left[(1-s)^{\eta-4}t^{\eta-3} - (t-s)^{\eta-3}\right]g(s)ds + \int_t^1 (1-s)^{\eta-4}t^{\eta-3}g(s)ds\right]$$

$$= \int_0^1 H(t,s)g(s)ds.$$

The proof is complete. □

By direct computation, we see that $H(t,s)$ satisfy properties of Green's function.

Lemma 4.1.3 *The function H(t,s) satisfies the following properties:*

i) If $0 \le t \le s \le 1$, *then* $0 \le H(t,s) \le H(s,s) \le \frac{1}{\Gamma(\eta-2)}$, *for all t,s* $\in [0,1]$.

ii) If $0 \le s \le t \le 1$, *then* $0 \le H(t,s) \le H(s,s) \le \frac{1}{\Gamma(\eta-2)}$, *for all* $t,s \in [0,1]$.

Proof:

i) Since $0 \le H(t,s)$ and H is increasing in t, we get $H(t,s) \le H(s,s)$. Furthermore, $H(s,s) \le H(1,s) = \frac{(1-s)^{\eta-4}}{\Gamma(\eta-2)} \le \frac{1}{\Gamma(\eta-2)}$.

ii) Since $0 \le H(t,s)$ and H is decreasing in t, we also have $H(t,s) \le H(s,s)$. $H(s,s) \le H(1,s) = \frac{(1-s)^{\eta-4}-(1-s)^{\eta-3}}{\Gamma(\eta-2)} \le \frac{(1-s)^{\eta-4}}{\Gamma(\eta-2)} \le \frac{1}{\Gamma(\eta-2)}$.

□

Lemma 4.1.4 *For* $0 \le t,s \le 1$, *we have* $\min_{t\in(0,1]} H(t,s) = b^{\eta-3}H(1,s)$, *where* $b \in (0,1)$ *is a constant.*

Proof: case i: Let $t \leq s$. Then $H(t,s)$ is increasing in t, and we have:

$$\min_{t\in(0,b]} H(t,s) \leq \frac{(1-s)^{\eta-4}}{\Gamma(\eta-2)} b^{\eta-3} \qquad = b^{\eta-3} H(1,s).$$

Case ii: Let $t \geq s$. Then $H(t,s)$ is decreasing in t, and we have:

$$\begin{aligned}
\min_{t\in[b,1]} H(t,s) &\geq \frac{(1-s)^{\eta-4}(1-b)^{\eta-3} - (1-b-s)^{\eta-3}}{\Gamma(\eta-2)} \\
&\geq \frac{(1-s)^{\eta-4}(1-b)^{\eta-3} - (1-b-(1-b)s)^{\eta-3}}{\Gamma(\eta-2)} \\
&= (1-b)^{\eta-3}\left[\frac{(1-s)^{\eta-4} - (1-s)^{\eta-3}}{\Gamma(\eta-2)}\right] \\
&= (1-b)^{\eta-3} H(1,s) \\
&\geq b^{\eta-3} H(1,s).
\end{aligned}$$

Therefore, from case (i) and (ii), we have

$$\min_{t\in[0,1]} H(t,s) = b^{\eta-3} H(1,s),$$

where $b \in (0,1)$ is a constant. □

Let $y(t) = -(u''(t) + k^2 u(t))$.
The BVP(1.1) is equivalent to the classical BVP:

$$\begin{aligned}
u''(t) + k^2 u(t) &= -\int_0^1 H(t,s) f(s,u(s))ds, \\
\alpha u(0) - \beta u'(0) &= \int_0^1 h_1(s) u(s) ds, \\
\gamma u(1) + \delta u'(1) &= \int_0^1 h_2(s) u(s) ds.
\end{aligned} \tag{4.1}$$

Now we consider the BVP

$$\begin{aligned}
u''(t) + k^2 u(t)) &= -g(t), \\
\alpha u(0) - \beta u'(0) &= 0, \\
\gamma u(1) + \delta u'(1) &= 0.
\end{aligned} \tag{4.2}$$

The complementary solution is

$$u_c(t) = c_1 \sin kt + c_2 \cos kt, \tag{4.3}$$

where c_1 and c_2 are arbitrary constants.
By applying the boundary conditions, we see that $c_1 = 0 = c_2$ and the associated homogenous BVP has trivial solution $u(t) = 0$. If such a solution exists, there exists a non-trivial solution $G_0(t, s)$ such that (4.2) has a solution of the form:

$$u(t) = \int_0^1 G_0(t, s)g(s)\, ds, \tag{4.4}$$

where $G_0(t, s)$ is called Green's function for the associated homogenous BVP of the BVP (4.2). we will prove this in lemma 4.2.3.

4.2 Construction of Green's Function

(L.T. Wesen et al., 2016)
Consider the associated homogenous BVP for (4.2),

$$u''(t) + k^2u(t) = 0, \tag{4.5}$$

$$\alpha u(0) - \beta u'(0) = 0, \tag{4.6}$$

$$\gamma u(1) + \delta u'(1) = 0. \tag{4.7}$$

Assume that $G(t, s)$ is a solution of (4.5). Then,

$$G''(t, s) + k^2G(t, s) = 0, \tag{4.8}$$

$$\alpha G(0, s) - \beta G'(0, s) = 0, \tag{4.9}$$

$$\gamma G(1, s) + \delta G'(1, s) = 0. \tag{4.10}$$

The solution of Eq.(4.8) is

$$G(t, s) = c_1(s) \sin kt + c_2(s) \cos kt. \tag{4.11}$$

Applying the boundary condition (4.9) yields

$$\alpha c_2(s) - \beta c_1(s) = 0 \implies c_2(s) = \frac{\beta}{\alpha} c_1(s).$$

Substituting the expression for c_2 in Eq.(4.11) yields

$$G_1(t,s) = c_1(s)\sin kt + \frac{\beta}{\alpha} c_1(s)\cos kt = \frac{1}{\alpha} c_1(s)[\alpha \sin kt + \beta k \cos kt].$$

Hence,

$$G_1(t,s) = d_1(s)[\alpha \sin kt + \beta k \cos kt], \tag{4.12}$$

where $d_1(s) = \frac{1}{\alpha} c_1(s)$.

Applying the boundary condition(4.10) yields

$$(\gamma \sin k + \delta k \cos k) c_1(s) + (\gamma \cos k - \delta k \sin k) c_2(s) = 0 \implies c_2(s) = \frac{\gamma \sin k + \delta k \cos k}{\delta k \sin k - \gamma \sin k} c_1(s).$$

Substituting the expression for c_2 in Eq.(4.11) yields

$$G_2(t,s) = \frac{1}{\gamma \cos kk - \delta k \sin k} c_1(s)[\gamma \sin k(1-t) + \delta k \cos k(1-t)].$$

Hence,

$$G_2(t,s) = d_2(s)[\gamma \sin k(1-t) + \delta k \cos k(1-t)], \tag{4.13}$$

where $d_2(s) = \frac{1}{\gamma \cos k - \delta k \sin k} c_1(s)$.
Collecting the results in (4.12) and (4.13), we get

$$G_0(t,s) = \begin{cases} G_1(t,s), & 0 \le t \le s \le 1, \\ G_2(t,s), & 0 \le s \le t \le 1. \end{cases}$$

$$= \begin{cases} d_1(s)[\alpha \sin kt + \beta k \cos kt], & 0 \le t \le s \le 1 \\ d_2(s)[\delta k \cos k(1-t) + \gamma \sin k(1-t)], & 0 \le s \le t \le 1 \end{cases}$$

Let $\phi_1(t) = \alpha \sin kt + \beta k \cos kt$, $\phi_2(t) = \gamma \sin k(1-t) + \delta k \cos k(1-t)$, and

$$w(\phi_1, \phi_2) = (\alpha\delta + \beta\gamma)k^2 \cos k + (\alpha\gamma - \beta\delta k^2)k \sin k.$$

Choose

$$d_1(s) = \frac{\phi_2(s)}{w}, \quad d_2(s) = \frac{\phi_1(s)}{w},$$

where ϕ_1 and ϕ_2 are two linearly independent solutions for the homogenous equation satisfying the first and the second boundary condition respectively, and w is the Wronskian of ϕ_1 and ϕ_2. So the Green's function is given by:

$$G_0(t,s) = \begin{cases} \dfrac{\phi_1(t)\phi_2(s)}{w}, 0 \le t \le s \le 1, \\ \dfrac{\phi_1(s)\phi_2(t)}{w}, 0 \le s \le t \le 1. \end{cases}$$

Substituting the expressions for ϕ_1 and ϕ_2, we get:

$$G_0(t,s) = \begin{cases} \dfrac{(\alpha \sin kt + \beta k \cos kt)(\gamma \sin k(1-s) + \delta k \cos k(1-s))}{(\alpha\delta + \beta\gamma)k^2 \cos k + (\alpha\gamma - \beta\delta k^2)k \sin k}, 0 \le t \le s \le 1, \\ \dfrac{(\gamma \sin k(1-t) + \delta k \cos k(1-t))(\alpha \sin ks + \beta k \cos ks)}{(\alpha\delta + \beta\gamma)k^2 \cos k + (\alpha\gamma - \beta\delta k^2)k \sin k}, 0 \le s \le t \le 1. \end{cases} \tag{4.14}$$

Thus $G_0(t,s)$ is the Green's function of the associated homogenous BVP (4.5)-(4.7).

Lemma 4.2.1 *The functions $\phi_1(t)$ and $\phi_2(t)$ satisfy the following properties:*

(i) $0 < \phi_1(t) \le \phi_1(1)$.

(ii) $0 < \phi_2(t) < \phi_2(0)$.

(iii) $\phi_1(t) \ge \gamma_0$.

(iv) $\phi_2(t) \ge \gamma_0$.

where $\gamma_0 = \min\left\{\frac{\beta k \cos k}{\alpha + \beta k}, \frac{\delta k \cos k}{\gamma + \delta}\right\} \in (0,1)$, *for* $(t,s) \in [0,1] \times [0,1]$.

Proof:

i) $0 < \phi_1(t) = \alpha \sin kt + \beta k \cos kt \le \alpha \sin k + \beta k \cos k = \phi_1(1)$.

ii) $0 < \phi_2(t) = \delta k \cos k(t-1) - \gamma \sin k(t-1) \le \delta k \cos k - \gamma \sin k < \delta k \cos k + \gamma \sin k = \phi_2(0)$.

iii) $\phi_1(t) = \alpha \sin kt + \beta k \cos kt \ge \beta k \cos kt \ge \beta k \cos k \ge \frac{\beta k \cos k}{\alpha + \beta k} \ge \gamma_0$.

iv) $\phi_2(t) = \gamma \sin k(1-t) + \delta k \cos k(1-t) \geq \delta k \cos k(1-t)) \geq \frac{\delta k \cos k}{\alpha + \delta k} \geq \gamma_0$.

□

Lemma 4.2.2 *The Green's function $G_0(t,s)$ satisfies the following inequalities:*

i) $G_0(t,s) > 0$, *for all* $t, s \in (0,1)$, $G_0(s,t) \leq G_0(s,s) \leq \frac{M^2}{w}$, *for all* $(t,s) \in [0,1] \times [0,1]$,

iii) $G_0(t,s) \leq M G_0(s,s)$, *for all* $(t,s) \in [0,1] \times [0,1]$,

iii) $\gamma_0 G_0(s,s) \leq G_0(t,s)$, *for all* $, s) \in [0,1] \times [0,1]$,

where

$$M = \max\left\{ \frac{\alpha + \beta k}{\beta k \cos k}, \frac{\gamma + \delta k}{\delta k \cos k} \right\}, \quad \gamma_0 = \frac{1}{M}$$

Proof:

i) $G_0(t,s), \phi_1(t), \phi_2(t) > 0$ for all $t \in (0,1)$, since $\min\limits_{t \in (0,1)} \phi_1(t) = \beta k > 0$ and $\min\limits_{t \in (0,1)} \phi_2(t) = \delta k \cos k + \gamma \sin k > 0$, we used the proof in (Liu et al., 2010) for remaining condition.

ii) Let $s \in [0,1]$ and $t \leq s$. Then,

$$\frac{G_0(t,s)}{G_0(s,s)} = \frac{\alpha \sin kt + \beta k \cos kt}{\alpha \sin ks + \beta k \cos ks} \leq \frac{\alpha + \beta k}{\beta k \cos k}.$$

Let $s \in [0,1]$ and $t \geq s$. Then,

$$\frac{G_0(t,s)}{G_0(s,s)} = \frac{\gamma \sin k(1-t) + \delta k \cos k(1-t)}{\gamma \sin k(1-s) + \delta k \cos k(1-s)} \leq \frac{\gamma + \delta k}{\delta k \cos k}.$$

iii) This inequality follows from a similar procedure as in (ii).

Thus, we have $G_0(t,s) \leq M G_0(s,s)$ for all $(t,s) \in [0,1] \times [0,1]$. □

Lemma 4.2.3 *The boundary value problem*

$$\begin{aligned} u''(t) + k^2 u(t) &= -g(t), \\ \alpha u(0) - \beta u'(0) &= 0, \\ \gamma u(1) + \delta u'(1) &= 0 \end{aligned} \tag{4.15}$$

has a unique solution

$$u(t) = \int_0^1 G_0(t,s) g(s) ds, \tag{4.16}$$

where $G_0(t,s)$ is as given in Eq.(4.14).

Proof: The general solution of Eq.(4.15) is given by

$$u(t) = c_1\phi_1 + c_2\phi_2 + s(t), \tag{4.17}$$

where $s(t) = \phi_1(t)v_1(t) + \phi_2(t)v_2(t)$ is its particular solution. Using the method of variation of parameters,

$$v_1(t) = -\int_0^t \frac{\phi_2(s)g(s)}{w} ds, \quad v_2(t) = \int_0^t \frac{\phi_1(s)g(s)}{w} ds.$$

Substituting these expressions in (4.13), we get

$$u(t) = c_1\phi_1 + c_2\phi_2 + \phi_2(t)\int_0^t \frac{\phi_1(s)g(s)}{w} ds - \phi_1(t)\int_0^t \frac{\phi_2(s)g(s)}{w} ds.$$

By the triviality of the complementary solution, we have $c_1 = c_2 = 0$. Then, we get

$$\begin{aligned} u(t) &= \int_t^1 \frac{\phi_2(t)\phi_1(s)}{w} g(s)ds - \int_0^t \frac{\phi_1(t)\phi_2(s)}{w} g(s)ds \\ &= \int_t^1 \frac{\phi_2(t)\phi_1(s)}{w} g(s)ds + \int_0^t \frac{\phi_1(t)\phi_2(s)}{w} g(s)ds \\ &= \int_0^t \frac{\phi_1(t)\phi_2(s)}{w} g(s)ds + \int_t^1 \frac{\phi_2(t)\phi_1(s)}{w} g(s)ds \\ &= \int_0^1 \left[\frac{\phi_1(t)\phi_2(s)}{w} + \frac{\phi_2(t)\phi_1(s)}{w}\right] g(s)ds \\ &= \int_0^1 G_0(t,s) g(s)ds, \end{aligned}$$

where $G_0(t,s)$ is as given in (4.14). $\square$

Now we are in the position to construct a Green's function for the non-homogenous IBVP problem (4.18) below:

$$\begin{aligned} u''(t) + k^2 u(t) &= -g(t), \\ \alpha u(0) - \beta u'(0) &= \int_0^1 h_1(s) u(s)\, ds, \\ \gamma u(1) + \delta u'(1) &= \int_0^1 h_2(s) u(s)\, ds. \end{aligned} \tag{4.18}$$

Define

$$a(t) = \frac{k\phi_2(t)}{w}, \qquad\qquad b(t) = \frac{k\phi_1(t)}{w},$$

then $a(t)$ and $b(t)$ are the solutions of the boundary value problems:

$$\begin{aligned} a''(t) + k^2 a(t) &= 0, \quad t \in (0,1) \\ \alpha a(0) - \beta a'(0) &= 1, \\ \gamma a(1) + \delta a'(1) &= 0 \end{aligned}$$

and

$$\begin{aligned} b''(t) + k^2 b(t) &= 0, \quad t \in (0,1) \\ \alpha b(0) - \beta b'(0) &= 0, \\ \gamma b(1) + \delta b'(1) &= 1, \end{aligned}$$

respectively.

As in (Hao et al., 2019), we denote

$$v_1 = 1 - \int_0^1 h_1(t)a(t)\,dt, \qquad v_2 = 1 - \int_0^1 h_2(t)b(t)\,dt,$$

$$v_3 = \int_0^1 h_2(t)a(t)\,dt, \qquad v_4 = \int_0^1 h_1(t)b(t)\,dt,$$

$$\mu_1(s) = \frac{v_2 \int_0^1 G_0(t,s)h_2(t)\,dt + v_4 \int_0^1 G_0(t,s)h_1(t)\,dt}{v_1v_2 - v_3v_4},$$

$$\mu_2(s) = \frac{v_1 \int_0^1 G_0(t,s)h_1(t)\,dt + v_3 \int_0^1 G_0(t,s)h_2(t)\,dt}{v_1v_2 - v_3v_4}.$$

This formula is proved in (Liu et al., 2010).

Assumtions

Throughout this study we use the following assumptions:

M1) $v_1 > 0$, and $v_1v_2 - v_3v_4 > 0$

M2) $h_1, h_2 \in C[0,1]$, bounded in $[0,1]$ and $f : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous, and $h_1 \neq h_2$

M3) There exists $k \neq \pm\sqrt{\frac{\alpha\gamma}{\beta\delta}}$ such that $w \neq 0$

M4) If $\alpha\gamma > \beta\delta$, then there exists $k \leq \sqrt{\frac{\alpha\gamma}{\beta\delta}}$ such that $w > 0$

Lemma 4.2.4 *The functions $a(t), b(t), \mu_1(s)$ and $\mu_2(s)$ satisfy the following properties:*

i) $a(t)$ is strictly decreasing, positive on $[0,1]$ and $\frac{\gamma_0}{\gamma+\delta k} \leq a(t) \leq M$,

ii) $b(t)$ is strictly increasing, positive on $[0,1]$ and $\frac{\gamma_0}{\alpha+\beta k} \leq b(t) \leq M$, for $t \in [0,1]$, where $M = \max\left\{\frac{\alpha+\beta k}{\beta k\cos k}, \frac{\gamma+\delta k}{\delta k\cos k}\right\}$ and $\gamma_0 = \frac{1}{M}$

iii) $\mu_1(s)$ and $\mu_2(s)$ are nonnegative and bounded on $[0,1]$.

Proof: Since k$\neq 0$, both $a(t)$ and $b(t)$ are positive.

i)

$$a(t) = \frac{k(\gamma\sin k(1-t) + \delta k\cos k(1-t))}{(\alpha\delta + \beta\gamma)k^2\cos k + (\alpha\gamma - \beta\delta k^2)k\sin k}$$

$$\geq \frac{\delta k \cos k}{(\alpha\delta+\beta\gamma)k\cos k+(\alpha\gamma-\beta\delta k^2)\sin k} \geq \frac{\delta k \cos k}{\delta k(\alpha-\beta k)+\gamma(\alpha+\beta k)}$$

$$= \frac{\delta k \cos k}{(\gamma+\delta k)(\alpha+\beta k)} \geq \frac{1}{M(\gamma+\delta k)} = \frac{\gamma_0}{\gamma+\delta k}.$$

On the other hand,

$$a(t) = \frac{k(\gamma \sin k(1-t)+\delta k \cos k(1-t))}{(\alpha\delta+\beta\gamma)k^2\cos k+(\alpha\gamma-\beta\delta k^2)k\sin k}$$

$$= \frac{\gamma \sin k(1-t)+\delta k \cos k(1-t)}{(\alpha\delta+\beta\gamma)k\cos k+(\alpha\gamma-\beta\delta k^2)\sin k}$$

$$\leq \frac{\gamma+\delta k}{(\alpha\delta+\beta\gamma)k\cos k+(\alpha\gamma-\beta\delta k^2)\sin k}$$

$$\leq \frac{\gamma+\delta k}{(\alpha\delta+\beta\gamma)k\cos k} \leq \frac{\gamma+\delta k}{\delta k\cos k} \leq M.$$

ii)

$$b(t) = \frac{\alpha \sin kt+\beta k \cos kt}{(\alpha\delta+\beta\gamma)k\cos k+(\alpha\gamma-\beta\delta k^2)\sin k}$$

$$\geq \frac{\beta k \cos kt}{\beta k(\alpha-\gamma k)k\cos k+(\alpha\gamma-\beta\delta k^2)\sin k}$$

$$\geq \frac{\beta k \cos kt}{(\alpha\delta+\beta\gamma)k+\alpha\gamma-\beta\delta k^2}$$

$$= \frac{\beta k \cos kt}{(\gamma-\delta k)\beta k+\alpha(\gamma+\delta k)}$$

$$= \frac{\beta k \cos kt}{(\gamma+\delta k)(\alpha+\beta k)} \geq \frac{1}{M(\alpha+\beta k)} = \frac{\gamma_0}{\alpha+\beta k}$$

On the other hand,

$$b(t) = \frac{\alpha+\beta k}{(\alpha\delta+\beta\gamma)k\cos k+(\alpha\gamma-\beta\delta k^2)\sin k}$$

$$\leq \frac{\alpha+\beta k}{(\alpha\delta+\beta\gamma)k\cos k} \leq \frac{\alpha+\beta k}{\beta\gamma k\cos k} \leq \frac{\alpha+\beta k}{\beta k\cos k} \leq M.$$

iii) positivity of $\mu_1(s)$ and $\mu_2(s)$ follows from assumtion (M1)-(M3)

$\square$

Lemma 4.2.5 *Assume that* ***(M1)-(M3)*** *hold. Then u is a solution of the BVP(4.18) if and only if u can be expressed as*

$$u(t) = \int_0^1 G(t,s)g(s)\,ds, \qquad t \in [0,1], \tag{4.19}$$

where

$$G(t,s) = G_0(t,s) + a(t)\mu_1(s) + b(t)\mu_2(s), \tag{4.20}$$

and

$$a(t) = \frac{k\phi_2(t)}{w},$$
$$b(t) = \frac{k\phi_1(t)}{w},$$

with $G_0(t,s)$ as in Eq.(4.14).

Proof: For ϕ_1 and ϕ_2, we have the following conditions:

(i) $\phi_1(0) = \beta k$, $\phi_1'(0) = \alpha k$, $\phi_1(1) = \alpha \sin k + \beta k \cos k$, $\phi_1'(1) = \alpha k \cos k - \beta k^2 \sin k$.

(ii) $\phi_2(0) = \delta k \cos k + \gamma \sin k$, $\phi_2'(0) = \delta k^2 \sin k - \gamma k \cos k$, $\phi_2(1) = \delta k$, $\phi_2'(1) = -\gamma k$.

Since ϕ_1 and ϕ_2 are linearly independent, the solution of the BVP can be written as:

$$u(t) = c_1\phi_1 + c_2\phi_2 + \int_0^1 G_0(t,s)g(s)\,ds, \tag{4.21}$$

where c_1 and c_2 are arbitrary constants.

Applying the first boundary condition, we get:

$$\alpha[c_1\phi_1(0) + c_2\phi_2(0) + \int_0^1 G_0(0,s)g(s)\,ds]$$
$$- \beta[c_1\phi_1'(0) + c_2\phi_2'(0) + \int_0^1 G_0'(0,s)g(s)\,ds] = \int_0^1 h_1(s)u(s)\,ds$$
$$\Longrightarrow \alpha[\beta k c_1 + (\delta k\cos k + \gamma\sin k)c_2 + \int_0^1 G_0(0,s)g(s)\,ds]$$
$$- \beta[c_1\alpha k + (\delta k^2\sin k - \gamma k\cos k)c_2 + \int_0^1 G_0'(0,s)g(s)\,ds] = \int_0^1 h_1(s)u(s)\,ds$$
$$\Longrightarrow (\alpha\beta k - \alpha\beta k)c_1 + (\alpha\delta k\cos k + \alpha\gamma\sin k - \beta\delta k^2\sin k + \beta\gamma k\cos k)c_2$$
$$+ \int_0^1 [\alpha G_0(0,s) - \beta G_0'(0,s)]g(s)\,ds = \int_0^1 h_1(s)u(s)\,ds$$

Since $\alpha G_0(0,s) - \beta G_0'(0,s) = 0$, we have:

$$\frac{(\alpha\delta k^2\cos k + \alpha\gamma k\sin k - \beta\delta k^3\sin k + \alpha\gamma k^2\cos k)c_2}{k} = \int_0^1 h_1(s)u(s)ds$$
$$c_2 = \frac{k}{w}\int_0^1 h_1(s)u(s)ds.$$

In similar procedure we get

$$c_1 = \frac{k}{w}\int_0^1 h_2(s)u(s)\,ds.$$

Substituting these values in Eq.(4.21),we get

$$u(t) = \int_0^1 G_0(t,s)g(s)\,ds + \frac{k}{w}\phi_1(t)\int_0^1 h_2(s)u(s)\,ds + \frac{k}{w}\phi_2(t)\int_0^1 h_1(s)u(s)\,ds.$$

But, we have defined that: $a(t) = \frac{k}{w}\phi_2(t)$, and $b(t) = \frac{k}{w}\phi_1(t)$ in our previous discussion. Hence,

$$u(t) = \int_0^1 G_0(t,s)g(s)ds + a(t)\int_0^1 h_1(s)u(s)ds + b(t)\int_0^1 h_2(s)u(s)ds.$$

Now, using the proof in (Benchohra et al., 2011; Fokas, 2008) we have

$$\int_0^1 h_1(s)u(s)\,ds = \mu_1(s)g(s), \quad \int_0^1 h_2(s)u(s)\,ds = \mu_2(s)g(s).$$

Thus,

$$u(t) = \int_0^1 \big[G_0(t,s) + a(t)\mu_1(s) + b(t)\mu_2(s)\big]g(s)\,ds = \int_0^1 G(t,s)g(s)ds.$$

□

Lemma 4.2.6 *Assume that* ***(M1) - (M2)*** *holds and let*

$$\phi(s) = G_0(s,s) + \mu_1(s) + \mu_2(s).$$

Then

$$\gamma_0\phi(s) \le G(t,s) \le M\phi(s),\ t,s \in [0,1].$$

Proof: From Lemma 4.2.2, we have

$$G(t,s) = G_0(t,s) + a(t)\mu_1(s) + b(t)\mu_2(s) \le MG_0(s,s) + \frac{\gamma+\delta k}{\delta k\cos k}\mu_1(s) + \frac{\alpha+\beta k}{\beta k\cos k}\mu_2(s)$$

$$\le M\left[G_0(s,s) + \mu_1(s) + \mu_2(s)\right] = M\phi(s).$$

and

$$\begin{aligned} G(t,s) &= G_0(t,s) + a(t)\mu_1(s) + b(t)\mu_2(s) \\ &\ge \gamma_0 G_0(s,s) + \frac{\delta k\cos k}{\gamma+\delta k}\mu_1(s) + \frac{\beta k\cos k}{\alpha+\beta k}\mu_2(s) \\ &\ge \gamma_0\left[G_0(s,s) + \mu_1(s) + \mu_2(s)\right] = \gamma_0\phi(s), t,s \in [0,1]. \end{aligned}$$

□

By using Lemma 4.1.2 and 4.2.5, a solution of the integral equation

$$u(t) = \int_0^1 G(t,s)\int_0^1 H(t,\tau)f(\tau,u(\tau))d\tau ds, \quad t \in [0,1],$$

is a solution of the BVP (1.1).

The integral equation can be written as a convolution of G and H. That is,

$$
\begin{aligned}
u(t) &= \int_0^1 G(t,s) \int_0^1 H(t,\tau) f(\tau, u(\tau)) d\tau ds, \quad t \in [0,1] \\
&= \int_0^1 \mathcal{G}(t,s) f(s, u(s)) ds,
\end{aligned}
$$

where

$$
\mathcal{G}(t,s) = G(t,s) \int_0^1 H(t,\tau)) d\tau.
$$

Lemma 4.2.7 *The function $\mathcal{G}(t,s)$ has the following properties:*

i) If $(t,s) \in (0,1) \times (0,1)$ and $t < s$, then $0 < \mathcal{G}(t,s) < \mathcal{G}(s,s)$.

ii) $\gamma_0 \overline{\mathcal{G}}(s) \leq \mathcal{G}(t,s) \leq M\overline{\mathcal{G}}(s)$, $t, s \in [0,1]$,

where

$$
\overline{\mathcal{G}}(s) = \int_0^1 \phi(\tau) H(\tau, s) d\tau, s \in (0,1)
$$

and $\gamma_0 = \min\left\{\frac{\beta k \cos k}{\alpha + \beta k}, \frac{\delta k \cos k}{\gamma + \delta}\right\} = \frac{1}{M} \in (0,1)$, for $(t,s) \in [0,1] \times [0,1]$.
Proof:

i) By using Lemma 4.1.3, for $t < s$, we have $0 < H(t,s) < H(s,s)$. Taking the integral with respect to τ yields

$$
0 < \int_0^1 H(\tau, s) d\tau < \int_0^1 H(s,s) d\tau,
$$

Now, Multiplying by $G(t,s)$ yields

$$
0 < G(t,s) \int_0^1 H(\tau, s) d\tau < G(t,s) \int_0^1 H(s,s) d\tau = \mathcal{G}(s,s).
$$

ii) From Lemma 4.2.6, we have $G(t,s) \leq M\phi(s)$. Multiplying by $\int_0^1 H(\tau,s)d\tau$ yields

$$
G(t,s) \int_0^1 H(\tau, s) d\tau \leq M \int_0^1 \phi(\tau) H(\tau, s) d\tau = M\overline{\mathcal{G}}(s).
$$

Hence, we get

$$\mathcal{G}(t,s) \leq M\overline{\mathcal{G}}(s).$$

since $\gamma_0 \in (0,1)$ and it is minimum , we have $\mathcal{G}(t,s) \geq \gamma_0 \mathcal{G}(s,s) \geq \gamma_0 \overline{\mathcal{G}}(s)$.

Hence (ii) follows. $\square$

Let $E = C[0,1]$ be a Banach space, the set of continuous functions in $[0,1]$, with the norm $\|u\| = \sup_{t\in[0,1]} |u(t)|$. Then $P = \{u \in E : u(t) \geq 0, \min\limits_{0\leq t\leq 1} u(t) \geq \gamma_0 \|u\|\}$ is a cone.

Now we define the operator $T : E \to E$ as

$$Tu(t) = \int_0^1 \mathcal{G}(t,s) f(s,u(s)) ds, \quad t \in [0,1] \tag{4.22}$$

If $u \in P$ is a fixed point of T, then u is a positive solution of BVP (1.1).

Lemma 4.2.8 *(The Arzela-Ascoli Theorem, Royden et al., 2010) Let X be a compact metric space and $\{f_n\}$ a uniformly bounded, equicontinuous sequence of real-valued functions on X. Then $\{f_n\}$ has a subsequence that converges uniformly on X to a continuous function f on X.*

Lemma 4.2.9 *(Krasnoselskii et al., 1964) Let E be a Banach space and P be a cone in E. Assume that Ω_1 and Ω_2 are bounded open subsets of E with $\theta \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$ and let $A : P \cap (\overline{\Omega}_1 \setminus \Omega_1) \to P$ be a completely continuous operator. If the following conditions are satisfied:*

- *i) $\|Ax\| \leq \|x\|$ for all $x \in P \cap \partial\Omega_1$ and $\|Ax\| \geq \|x\|$ for all $x \in P \cap \partial\Omega_2$, or*
- *ii) $\|Ax\| \geq \|x\|$ for all $x \in P \cap \partial\Omega_1$ and $\|Ax\| \leq \|x\|$ for all $x \in P \cap \partial\Omega_2$,*

then A has at least one fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

4.3 Main Results

4.3.1 Existence and uniqueness of positive solutions

Lemma 4.3.1 *Assume that* ***(M1)-(M3)*** *holds. Then the operator T on u is a self-map.*

Proof: If $u \in P$, then $Tu(t) \geq 0$ on $[0,1]$, using Lemma 4.2.7,

$$\begin{aligned}\min_{0\leq t\leq 1} Tu(t) &= \min_{0\leq t\leq 1}\left[\int_0^1 \mathcal{G}(t,s)f(s,u(s))ds\right]\\ &\geq \gamma_0\left[\int_0^1 \mathcal{G}(t,s)f(s,u(s))ds\right]\\ &= \gamma_0\|Tu\|.\end{aligned}$$

where $\gamma_0 = \min\left\{\frac{\beta k\cos k}{\alpha+\beta k}, \frac{\delta k\cos k}{\gamma+\delta k}\right\} \in (0,1)$, for $(t,s) \in [0,1]\times[0,1]$.

Thus, $Tu \in P$ and hence $T : P \to P$. □

Lemma 4.3.2 *Let $T : P \to E$ be the operator defined by*

$$Tu(t) = \int_0^1 \mathcal{G}(t,s)f(s,u(s))ds, \quad t \in [0,1].$$

Then $T : P \to P$ is completely continuous.

Proof: The operator $T : P \to E$ is continuous in view of the non-negativeness and continuity of the function $\mathcal{G}(t,s)$ and $f(t,u(t))$. Let $\Omega \subset P$ be bounded. Then there exists a positive constant $R_1 > 0$ such that $\|u\| \leq R_1$, $u \in \Omega$.

Define

$$R = \max_{t\in[0,1],u\in\Omega} |f(t,u)|,$$

and

$$L = M\int_0^1 \overline{\mathcal{G}}(s)ds.$$

For $u \in \Omega$, we have

$$\begin{aligned}
|Tu(t)| &= \left| \int_0^1 \mathcal{G}(t,s) f(s,u(s)) ds \right| \\
&\leq \int_0^1 |\mathcal{G}(t,s)||f(s,u(s))| ds \\
&\leq \int_0^1 M\overline{\mathcal{G}}(s) R ds \\
&= RM \int_0^1 \overline{\mathcal{G}}(s) ds \\
&= RL.
\end{aligned}$$

Hence we have

$$|Tu(t)| \leq RL.$$

Thus, the operator $T(\Omega)$ is bounded. Since $\mathcal{G}(t,s)$ is continuous on $[0,1]\times[0,1]$, it is uniformly continuous on $[0,1] \times [0,1]$. Hence, for every $\varepsilon > 0$ there exists $\delta \in (0, \varepsilon/\alpha - 1)$ such that

$|\mathcal{G}(t_1,s) - \mathcal{G}(t_2,s)| < \varepsilon \quad$ for $|t_1 - t_2| < \delta$ and $(t_1,s),(t_2,s) \in [0,1] \times [0,1]$.

For any $u \in P \cap \partial\Omega$ and $t_1, t_2 \in [0,1]$ with $|t_1 - t_2| < \delta$, we have

$$\begin{aligned}
|(Tu)(t_1) - (Tu)(t_2)| &\leq \int_0^1 |\mathcal{G}(t_1,s) - \mathcal{G}(t_2,s)||f(s,u(s))| ds \\
&\leq \varepsilon \int_0^1 |f(s,u(s))| ds \\
&= \varepsilon R.
\end{aligned}$$

Hence, T is equicontinuous. Consequently, by Arzela-Ascoli Theorem, it has a uniformly convergent subsequence on $P \cap \partial\Omega$ and hence compact on $P \cap \partial\Omega$.

Thus, $T : P \cap \partial\Omega \to P$ is completely continuous. $\square$

Denote

$$f_0^s = \lim_{x\to 0^+} \sup_{t\in[0,1]} \frac{f(t,x)}{x}, \qquad f_0^i = \lim_{x\to 0^+} \inf_{t\in[0,1]} \frac{f(t,x)}{x},$$
$$f_\infty^s = \lim_{x\to \infty} \sup_{t\in[0,1]} \frac{f(t,x)}{x}, \qquad f_\infty^i = \lim_{x\to \infty} \inf_{t\in[0,1]} \frac{f(t,x)}{x},$$

and

$$L = M\int_0^1 \overline{\mathcal{G}}(s)\,ds, \qquad K_1 = \frac{1}{\gamma_0^2 L f_\infty^i}, \qquad K_2 = \frac{1}{L f_0^s},$$
$$K_3 = \frac{1}{\gamma_0^2 L f_0^i}, \qquad K_4 = \frac{1}{L f_\infty^s}.$$

From the expression $\phi(\tau)$ and $H(\tau,s)$, we obtain $0 < L < \infty$.

Theorem 4.3.3 *Assume that* (M_2) *holds. If* $f_0^s, f_\infty^i \in (0,\infty)$ *and* $K_1 < K_2$, *then the BVP (1.6) has at least one positive solution.*

Proof: We use the extension of the proof by X.Hao(Hao et al, 2019) for $\lambda = 1$. For any interval of the form (K_1, K_2) containing 1 , there exists $0 < \zeta < f_\infty^i$ such that

$$\frac{1}{\gamma_0^2 L(f_\infty^i - \zeta)} \le 1 \le \frac{1}{L(f_0^s + \zeta)}.$$

By the definition of f_0^s, there exists $R_1 > 0$ such that

$$f(t,x) \le (f_0^s + \zeta)x, \quad t \in [0,1], \quad 0 \le x \le R_1.$$

Set $\Omega_1 = \{u \in E : \|u\| < R_1\}$. For any $u \in P \cap \partial\Omega_1$ and $t \in [0,1]$, we have

$$Tu(t) = \int_0^1 \mathcal{G}(t,s) f(s,u(s))\,ds$$

Applying the norm yields

$$\begin{aligned}
\|Tu(t)\| &= \left\|\int_0^1 \mathcal{G}(t,s)f(s,u(s))\,ds\right\| \\
&\le M\int_0^1 \overline{\mathcal{G}}(s)(f_0^s+\zeta)\|u(s)\|\,ds \\
&= L(f_0^s+\zeta)\|u\| \le \|u\|.
\end{aligned}$$

Thus,

$$\|Tu\| \le \|u\|, \quad u \in P\cap\partial\Omega_1. \tag{4.23}$$

Again, by the definition of f_∞^s, there exists $(\overline{R}_2) > 0$ such thatk8

$$f(t,x) \ge (f_\infty^s - \zeta)x, \quad t\in[0,1], \quad x \le (\overline{R}_2).$$

Choose $R_2 = \max\{2R_1, (\overline{R}_2)/\gamma_0\}$ and define $\Omega_2 = \{u \in E : \|u\| < R_2\}$. Then for any $u \in P\cap\partial\Omega_2$ and $t\in[0,1]$, we obtain $u(t) \ge \gamma_0\|u\| \ge \overline{R}_2$ and

$$Tu(t) \ge \int_0^1 \gamma_0\mathcal{G}(s)(f_\infty^i-\zeta)u(s)\,ds$$

$$\begin{aligned}
\|Tu(t)\| &\ge \left\|\int_0^1 \gamma_0\mathcal{G}(t,s)f(s,u(s))\,ds\right\| \\
&\ge \gamma_0^2\left\|\int_0^1 \mathcal{G}(s)(f_\infty^i-\zeta)u(s)\,ds\right\| \\
&\ge \gamma_0^2\int_0^1 M\overline{\mathcal{G}}(s)(f_\infty^i-\zeta)\|u(s)\|\,ds \\
&= \gamma_0^2 L(f_\infty^i-\zeta)\|u\| \ge \|u\|.
\end{aligned}$$

Thus,

$$\|Tu\| \ge \|u\|, \quad u \in P\cap\partial\Omega_2. \tag{4.24}$$

By (4.23), (4.24), and Lemma 4.2.9, we conclude that T has a fixed point $u \in P\cap(\overline{\Omega}_2\setminus\Omega_1)$. $\square$

Theorem 4.3.4 *Assume that (M1)-(M3) holds. Suppose that* $f_0^i, f_\infty^s \in (0,\infty)$ *and* $K_3 < K_4$*. Then the BVP (1.6) has at least one positive solution.*

Proof: We use the extension of the proof by X.Hao(Hao et al, 2019) for $\lambda = 1$. For any interval of the form (K_3, K_4) containing 1, there exists $0 < \zeta < f_0^i$ such that

$$\frac{1}{\gamma_0^2 L(f_0^i - \zeta)} \leq 1 \leq \frac{1}{L(f_\infty^s + \zeta)}.$$

By the definition of f_0^i, there exists $R_3 > 0$ such that

$$f(t,x) \geq (f_0^i - \zeta)x, \quad t \in [0,1], \quad 0 \leq x \leq R_3.$$

Set $\Omega_3 = \{u \in E : \|u\| < R_3\}$. For any $u \in P \cap \partial\Omega_3$ and $t \in [0,1]$, we have

$$\begin{aligned} Tu(t) &\geq \int_0^1 \gamma_0 \overline{\mathcal{G}}(s)(f_0^i - \zeta)u(s)\,ds \\ &\geq \gamma_0^2 L(f_0^i - \zeta)\|u\| \\ &\geq \|u\|. \end{aligned}$$

Thus,

$$\|Tu\| \geq \|u\|, \quad u \in P \cap \partial\Omega_3. \tag{4.25}$$

We define $f^* : [0,1] \times [0,+\infty) \to [0,+\infty)$ as follows:

$$f^*(t,x) = \max_{u \in [0,x]} f(t,u), \quad t \in [0,1], \quad x \geq 0.$$

Then for any $t \in [0,1]$ and $u \in [0,x]$, we have $f(t,u) \leq f^*(t,x)$, showing that $f^*(t,x)$ is nondecreasing in x. By the proof in (H. Wang, 2003), we have

$$\lim_{x\to\infty} \sup \max_{t\in[0,1]} \frac{f^*(t,x)}{x} \leq f_\infty^s.$$

From this inequality, there exists $(\overline{R_4}) > 0$ such that

$$\frac{f^*(t,x)}{x} \leq \lim_{x\to\infty} \sup \max_{t\in[0,1]} \frac{f^*(t,x)}{x} + \zeta \leq f_\infty^s + \zeta, \quad x \geq (\overline{R_4}), \quad t \in [0,1].$$

Then $f^*(t,x) \le (f^s_\infty + \zeta)x$ for $x \ge (\overline{R_4}), \quad t \in [0,1]$.

We define now $R_4 = \max\{2R_3, (\overline{R_4})/\gamma_0\}$ and $\Omega_4 = \{u \in E : \|u\| < R_4\}$. For any $u \in P \cap \partial\Omega_3$ and $t \in [0,1]$, we have $u(t) \ge \gamma_0\|u\| \ge R_4$.
Thus

$$Tu(t) \le \int_0^1 M\overline{\mathcal{G}}(s) f^*(s, \|u\|)ds \le L(f^s_\infty + \zeta)\|u\| \le \|u\|.$$

Therefore, we obtain

$$\|Tu\| \le \|u\|, \quad u \in P \cap \partial\Omega_4. \tag{4.26}$$

By (4.26), (4.26) and Lemma 4.2.9, we conclude that T has a fixed point $u \in P \cap (\overline{\Omega_4} \setminus \Omega_3)$. □

Theorem 4.3.5 *Assume that (M1)-(M3) holds, $f^s_\infty > 1$, and that f satisfies Lipschitz condition. i.e., there is a constant $K > 0$ such that*

$$|f(t,u) - f(t,v)| \le K|u - v|, \quad \textit{for all } t \in [0,1] \textit{ and } u, v \in [0,+\infty).$$

Then for $(t,u), (t,v) \in [0,1] \times \mathbb{R}$ with $0 < \alpha < 1$, the boundary value problem (1.1)-(1.2) has a unique positive solution on P.

Proof: We will prove that the operator T defined by Eq.(4.22) is a contraction on P. By Lemma 4.3.1, if $u \in P$, then $Tu \in P$. If we define a metric d for any $u, v \in [0,+\infty)$, we get

$$\begin{aligned}
d(Tu(t), Tv(t)) = |(Tu)(t) - (Tv)(t)| &= \left| \int_0^1 \mathcal{G}(t,s) \left[f(s,u(s)) - f(s,v(s))\right] ds \right| \\
&\le \int_0^1 \mathcal{G}(t,s) |f(s,u(s)) - f(s,v(s))| ds \\
&= K|u - v| \int_0^1 \mathcal{G}(s) ds \\
&\le K|u - v| \int_0^1 M\overline{\mathcal{G}}(s) ds \\
&= LK|u - v| \\
&= LKd(u,v),
\end{aligned}$$

where K is a Lipschitz constant.

Now, choose $K = K_4 = \frac{1}{Lf_\infty^s}$. Then LK $= \frac{1}{f_\infty^s}$.

$$d(Tu(t), Tv(t)) \leq \alpha|u - v|$$

$$= \alpha d(u, v),$$

where $\alpha = \frac{1}{f_\infty^s}$. Since , $f_\infty^s > 1$, then $\alpha \in (0, 1)$.
We have

$$\Rightarrow d(Tu(t), Tv(t)) \leq \alpha d(u, v).$$

Hence, T is a contraction mapping on P. Therefore, by the Banach contraction mapping principle, T has a unique fixed point on P, and the boundary value problem (1.1)–(1.2) has a unique positive solution on P. □

Corollary 4.3.6 *Let $T : P \to P$ be a contractive mapping on a cone P of a Banach space E. Then T has a unique fixed point on P*

Proof: Let $u, u' \in P$ be two fixed points of T in E. Then we have $Tu = u$ and $Tu' = u'$.

We claim that $u = u'$. Since T is a contraction on P, we have

$$\|u - u'\| = \|Tu - Tu'\| \leq \alpha\|u - u'\|, \quad \alpha \in [0, 1).$$

Rearranging, we get

$$0 = (1 - \alpha)\|u - u'\| \leq 0.$$

Since $1 - \alpha > 0$, this implies that $\|u - u'\| = 0$, and hence $u = u'$. □

4.3.2 An Example

Let $\alpha = \beta = \delta = 1, \gamma = 2, \eta = \frac{9}{2}, k = 1$, and $h_1(s) = s, h_2(s) = 2s$.

We consider the following fractional integral BVP:

$$\begin{aligned} -D_{0+}^{5/2}(u''(t) + u(t)) + f(t, u(t)) &= 0, \\ u''(0) = -u(0), \quad u'''(0) = -u'(0) \quad u'''(1) &= -u'(1), \\ u(0) - u'(0) &= \int_0^1 su(s)\, ds, \\ 2u(1) + u'(1) &= \int_0^1 2su(s)\, ds. \end{aligned} \tag{4.27}$$

We compute

$$v_1 = 1 - \int_0^1 h_1(t)a(t)\, dt, \qquad v_2 = 1 - \int_0^1 h_2(t)b(t)\, dt,$$
$$v_3 = \int_0^1 h_2(t)a(t)\, dt, \qquad v_4 = \int_0^1 h_1(t)b(t)\, dt,$$

$$\begin{aligned} M = \max\left\{\frac{\alpha + \beta k}{\beta k \cos k}, \frac{\gamma + \delta k}{\delta k \cos k}\right\}, \gamma_0 = \frac{1}{M}, \ L = M \int_0^1 \overline{\mathcal{G}}(s)\, ds, \\ \overline{\mathcal{G}}(s) = \int_0^1 \phi(\tau) H(\tau, s) d\tau, \\ \phi(s) = G_0(s, s) + \mu_1(s) + \mu_2(s), \quad s \in (0, 1). \end{aligned}$$

Direct computation shows that:

$$v_1 = 0.3486, v_2 = 1.6398, v_3 = 1.3028, v_4 = -0.3199, M = 3.00046, L = 113.133, , \gamma_0 = 0.3333.$$

So assumption (M1)-(M3) is satisfied.

1. We choose $f(t, x) = (\cos(\frac{\pi(1-t)}{2}) + 1)\frac{5x+1/3}{2x+90}x$, then
$f_0^s = \frac{1}{135}, \quad f_\infty^i = \frac{5}{2}, \quad K_1 \approx 0.0318, \quad K_2 \approx 1.1938,$

By Theorem 4.3.4, we conclude that the BVP(4.26) has at least one positive solution.

2. We choose $f(t,x) = (t+1)\frac{x+20}{10(35x+1)}x$, then

$$f_0^i = 2, \quad f_\infty^s = \frac{1}{175}, \quad K_3 \approx 0.0399, \quad K_4 \approx 1.5469,$$

By Theorem 4.3.4, we conclude that the BVP has at least one positive solution. Moreover, f satisfies Lipschitz condition with Lipschitz constant $K = \frac{1}{175}$, $\alpha = LK = 0.6465 < 1$. Hence by Theorem 4.3.5, we conclude that this solution is unique.

Chapter 5

Conclusion and Recommendation

5.1 Conclusion

In this study, we considered a fractional differential equation with integral boundary conditions. Comparing with the motivation papers, the inner function under the derivative and the terms under integral boundary conditions are newly introduced. By using the definition of fractional derivative and integral, we constructed the corresponding Green's function for the fractional part. Then, we applied each boundary conditions to construct a Green's function for resulting classical BVP, which is an equivalent integral equation for the given boundary value problem. We used Guo-Krasnoseliskii's fixed-point theorem on a cone to determine the existence of fixed point of the integral equation and established the existence of positive solutions of the FBVP. We also considered the uniqueness of positive solutions by using Banach contraction mapping theorem, which was not focused by the paper we considered. For such class of BVPs, this method better proves the existence and uniqueness of positive solutions.

5.2 Recommendation

This study focused on the existence and uniqueness of positive solutions for fractional integral boundary value problems. To prove existence theorem, we used Guo-Krasnoselskii's fixed point theorem on a cone. Other interested researchers may conduct research on the existence, uniqueness and multiplicity of positive solutions by considering different coefficients, more general parameter dependence and variable order by applying other methods.

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