Some Common Fixed Point Results of Generalized $\alpha - \theta$-Geraghty Type Contraction Mappings in the Setting of b-Metric-Like Spaces via C_G-Simulation Functions

A Thesis Submitted to the Department of Mathematics in Partial Fulfillment for the Requirements of the Degree of Masters of Science in Mathematics

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May, 2024
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Declaration

I, the undersigned declare that, the thesis entitled "Common Fixed Point Results for Generalized $\alpha-\theta$-Geraghty Type Contraction Mappings in the Setting of b-Metric-Like Spaces via C_G-Simulation Functions " is original and it has not been submitted to any institution elsewhere for the award of any academic degree or like, where other sources of information that have been used, they have been acknowledged.

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Acknowledgment

First of all, I am indebted to the almighty God who gave me long life and helped me to pass through different ups and downs to reach this time. Next, my Special heartfelt thanks go to my advisor, Dr. Kidane Koyas and co-adviser Mr. Mustefa Abduletif for their unreserved support, advice, and guidance throughout the preparation of this research work. Also, I would like to thank all my friends who gave me initiation while I was preparing this research work. Last but not least. I would like to thank my family for their encouragement through out my study period.

Abstract

In this thesis work we introduced generalized $\alpha - \theta$-Geraghty type contraction mappings in the setting of b-metric-like spaces via C_G-simulation functions and proved existence and uniqueness of common fixed point for mappings introduced. Our result extends and generalize many related fixed-point results in the existing literature, in particular that of Taqbibt et al., (2023). Also, we provided examples in support of our main results. In this research under taking, we followed analytical study design and used secondary sources of data such as published paper related books.

Contents

Chapter 1

Introduction

1.1 Background of the study

The Banach contraction principle is one of the most fundamental results in fixed point theory. Due to its usefulness and applications in many disciplines, several authors have improved, extended and generalized this basic result of Banach by defining new contractive conditions and replacing the metric space by more general abstract spaces. For example, the concept of a b-metric space was introduced by Czerwik, (1993) as generalization of the metric space and studied new fixed point results, Pant and Panicker, (2016) extended some fixed point theorems from metric spaces to b-metric spaces. Alghamdi et al., (2013) introduced the concept of a b-metric-like space as a generalization of partial metric spaces, metric-like spaces and b-metric spaces. After that, Hussain et al., (2014), Chen et al., (2015) and Aydi et al., (2016) proved some fixed point theorems in the setting of b-metric-like spaces. On the other hand, Geraghty, (1973) generalized the Banach contraction principle by considering an auxiliary function. After that, many scholars extend and generalize Geraghty contraction mappings in several derection [see e.g., Amini-Harandi and Emami, (2010), Caballero et al., (2012), Gordjiet al., (2012), Cho et al., (2013), Karapinar, (2014), Arshad and Hussain, (2017)]. Also the notion of Z-contraction was introduced by Khojasteh et al., (2015) using simulation functions, and they presented a generalized version of the Banach contraction principle. Olgun et al., (2016) studied fixed point results for generalized Z-contraction. Karapinar, (2016) further generalized the results of Samet et al., (2012) and Khojasteh et al., (2015) by introducing the notion of α-admissible Z-contraction. Chandok et al., (2021) combined simulation functions and $\mathscr{C}$-class functions, leading to the existence and uniqueness of the point of coincidence, which generalized the results in Khojasteh et al., (2015) and Olgun et al., (2016). Chandok, (2015) introduced the notion of (α, θ)-admissible mappings and obtained fixed point theorems for such mappings. Alsamir et al., (2019) proved fixed point theorems for an (α, θ)-admissible Z-contraction mapping in complete metric-like spaces. Very recently, Taqbibt et al., (2023) introduced and proved fixed point results for $\alpha - \theta$-Geraghty contraction mapping using C_G-simulation functions in matric-like spaces. Inspired and motivated by the works of above researchers in this work, we restrict ourselves on the merging of one of the most interesting generalizations of metric spaces, metric-like spaces and b-metric spaces and partial metric spaces namely, b-metric-like spaces and prove the existance and Uniqueness of common fixed point results for generalized α, θ-Geraghty contraction mappings using C_G-simulation function Ψ in the setting of b-matric-like spaces.

1.2 Statement of the Problem

Geraghty, (1973) introduced Geraghty contraction mappings by using an auxiliary function as a generalization of Banach contraction principle. The notion of Z-contraction was introduced by Khojasteh et al., (2015) using simulation functions, and they generalize this principle. Samet et al., (2012) introduced the concept of $\alpha-$admissibility and extended the Banach contraction principle. Also, Chandok, (2015) introduced the notion of (α, θ)-admissible mappings and obtained new fixed point theorems for introduced mappings. Alsamir et al., (2019) studied fixed point theorems for an (α, θ)-admissible Z-contraction mappings in complete metric-like spaces. Very recently, Taqbibt et al., (2023) introduced and proved fixed point results for $\alpha - \theta$-Geraghty contraction mappings using C_G-simulation functions in the setting of matric-like spaces.

However, common fixed point results for generalized $\alpha - \theta$-Geraghty type contraction mappings using C_G-simulation functions in the setting of b-matric-like spaces are not yet studied. Inspired and motivated by the above researchers in this research work, we introduce generalized $\alpha-\theta$-Geraghty type contraction mappings using C_G-simulation functions, establishing and proving new common fixed point theorems for the mappings introduced in the setting of b-matric-like spaces.

1.3 Objectives of the study

1.3.1 General objective

The general objective of this study was to study common fixed point theorems for a generalized $\alpha-\theta$-Geraghty type contraction mappings using C_G-simulation functions in the setting of b-metric-like spaces.

1.3.2 Specific objectives

This study has the following specific objectives

- To Establish common fixed point theorems for a generalized $\alpha-\theta$-Geraghty contraction mappings using C_G-simulation functions in the setting of b-metric-like spaces.
- To prove the existence of common fixed points for a generalized $\alpha-\theta$-Geraghty type contraction mappings using C_G-simulation functions in the setting of b-metric-like spaces.
- To verify the uniqueness of common fixed point for a generalized $\alpha-\theta$-Geraghty type contraction mappings using C_G-simulation functions in the setting of b-metric-like spaces.
- To provide examples in the support of our main results.

1.4 Significance of the study

The result of this study may have the following importance

- It may be used as a reference for any researcher who has interest in doing research in the area.
- It may give basic research skill to the researcher.
- It may be applied to solve existence of solution of some integral and differential equations.

1.5 Delimitation of the Study

This study was delimited to proving the existence and uniqueness of common fixed point results for generalized $\alpha - \theta$-Geraghty type contraction mappings using C_G-simulation functions in the setting of b-metric-like spaces.

Chapter 2

Review of Related literatures

Fixed point theory is one of the most important topics in development of nonlinear and mathematical analysis in general. Also, fixed point theory has been used effectively in many other branches of science, such as chemistry, physics, biology, economics, computer science and all engineering fields and so on.

Let X be a nonempty set and $T : X \to X$ be a self mapping. We say that x is a fixed point of T if $T(x) = x$. Also, let X be a nonempty set and $S, T : X \to X$ are mappings, then we say that x is a common fixed point of S and T if $x = Sx = Tx$.

In 1922, Banach, (1922) introduced the following well known fixed point result which is called Banach contraction principle and it is one of the pivotal results in nonlinear analysis.

Theorem 2.1 *(Banach, 1922) Let (X, d) be a complete metric space and T be a contraction on X, i.e., there exists $r \in [0, 1)$ such that $d(Tx, Ty) \leq rd(x, y)$ for all $x, y \in X$. Then T has a unique fixed point.*

Due to its importance and fruitful applications, several authors have obtained many interesting extensions and generalizations of the Banach contraction principle in several direction. The notion of a distance was generalized by Fr´echet under the name of metric that satisfies certain axioms. Inspired by this notion, several researchers improve and generalize the notion of a metric in several directions (symmetric, quasi-metric, ultra metric, b-metric, b-metric like etc.) Among them, we focus on the notion of a b-metric-like, since it has a wide potential in applications of several quantitative sciences, in particular in computer sciences. The notion of b-metric-like was introduced by Alghamdi et al., (2013) as a generalization of partial metric spaces, metric-like spaces and b-metric spaces. One of the interesting results was given by Geraghty, (1973) in the setting of complete metric spaces by considering an auxiliary function, which generalized the Banach Contraction principle. Also, Chandok, (2015) introduced the notion of (α, θ)-admissible mappings and obtained new fixed point theorems for introduced mappings. Alsamir et al., (2019) studied fixed point theorems for an (α, θ)-admissible Z-contraction mappings in complete metric-like spaces. Very recently, Taqbibt et al., (2023) introduced and proved fixed point results for $\alpha - \theta$-Geraghty contraction mappings using C_G-simulation functions in the setting of matric-like spaces. Inspired and motivated by the above researchers in this research work, we introduce generalized $\alpha - \theta$-Geraghty type contraction mappings using C_G-simulation functions, establishing and proving new common fixed point theorems for the mappings introduced in the setting of b-matric-like spaces.

Chapter 3

Methodology

3.1 Study period and site

The study was conducted at Jimma University under the department of mathematics from September, 2023 G.C to June, 2024 G.C.

3.2 Study Design

In this research work we employed analytical design.

3.3 Source of Information

The relevant sources of information for this study were books, published articles and related studies from internet.

3.4 Mathematical Procedure of the Study

In this research under taking, we followed the standard procedures. The procedures are:

1. Introducing a generalized $\alpha - \theta$-Geraghty type contraction mappings using C_G-simulation functions in b-metric-like spaces.

2. Establishing common fixed point theorems for generalized $\alpha-\theta$-Geraghty type contraction mappings using C_G-simulation functions in b-metric-like spaces.

3. Constructing sequences for introduced mappings.

4. Showing that constructed sequences are Cauchy

5. Showing the convergences of the sequences

6. Proving the existence and uniqueness of common fixed points for generalized $\alpha-\theta$-Geraghty type contraction mappings using C_G-simulation functions in b-metric-like spaces.

7. Giving example in support of the main findings

Chapter 4

Preliminaries and Main Results

4.1 Preliminaries

Now, we state some notation, definitions and theorems which are useful for our work as follows.

Notation Through out this work we denotes:

(i) $\mathbb{R}^+ = [0, \infty)$;

(ii) $\mathbb{N}$ is the set of natural numbers;

(iii) $\mathbb{R}$ is the set of all real numbers;

(iv) $\mathscr{F}_G = \{\beta = \mathbb{R}^+ \to [0,1)$ such that $\lim\limits_{n\to\infty} \beta(t_n) = 1$ implies $\lim\limits_{n\to\infty} (t_n) = 0$ $\}$;

(v) $\mathscr{F}_{Gs} = \{\beta = \mathbb{R}^+ \to [0, \frac{1}{s})$ such that $\lim\limits_{n\to\infty} \beta(t_n) = \frac{1}{s}$ implies $\lim\limits_{n\to\infty} (t_n) = 0$ $\}$.

Definition 4.1 *(Czerwik, 1993) Let X be a nonempty set and $s \geq 1$ be a given real number. A function $d : X \times X \to \mathbb{R}^+$ is a b-metric on X if, for all, $x, y, z \in X$, the following conditions hold:*

(i) $d(x,y) = 0$ if and only if $x = y$;

(ii) $d(x,y) = d(y,x)$;

(iii) $d(x,z) \leq s[d(x,y) + d(y,z)]$ (b-triangular inequality).

Then the pair (X,d) is called a b-metric space.

It should be noted that, the class of b-metric spaces is effectively larger than that of metric spaces; every metric is a b-metric with s = 1, while the converse is not true.

Definition 4.2 *(Amini-Harandi, 2012) Let X be nonempty set and $d : X \times X \to \mathbb{R}^+$ be a function satisfying the following conditions for all $x, y, z \in X$:*

a) if $d(x,y) = 0$, then $x = y$;
b) $d(x,y) = d(y,x)$;
c) $d(x,z) \leq d(x,y) + d(y,z)$.
Then the pair (X,d) is called metric-like space.

Definition 4.3 *(Alghamdi, et al., 2013) Let X be a nonempty set, $s \geq 1$ be a real number then a mapping $d : X \times X \to \mathbb{R}^+$ is called b-metric-like if for all $x,y,z \in X$ the following hold:*
a) if $d(x,y) = 0$, then $x = y$;
b) $d(x,y) = d(y,x)$;
c) $d(x,z) \leq s[d(x,y) + d(y,z)]$.
The pair(X,d) is called a b-metric-like space.

Every metric-like is b-metric-like and every b-metric is also b-metric-like, but the converses of these facts need not be true.

Example 4.1 *(Aydi et al., 2017) Let $X = \{1,2,3\}$ and $d : X \times X \to \mathbb{R}^+$ be defined by*
$d(x,y) = 3$ if $x = y$,
$d(x,y) = 1$ otherwise.
Then, (X,d) is a b-metric-like space with coefficient $s = \frac{3}{2}$. Clearly, d is not a b-metric and metric as $d(2,2) = 3 \neq 0$. Also d is not a metric-like as $d(2,2) = 3 > 2 = d(2,3) + d(3,2)$.

Definition 4.4 *(Alghamdi, et al., 2013) Let (X,d) be a b-metric-like space and $\{x_n\}$ be a sequence in X we say that*
a. $\{x_n\}$ is converges to $x \in X$ if and only if $d(x,x) = \lim_{n\to\infty} d(x,x_n)$;
b. $\{x_n\}$ is a Cauchy sequence if and only if $\lim_{n,m\to\infty} d(x_n,x_m)$ exists and is finite;
c. (X,d) is complete if and only if every Cauchy sequence in X is convergent.

Definition 4.5 *(Chandok, 2015) Let X be a nonempty set, $T : X \to X$ and $\alpha, \theta : X \times X \to \mathbb{R}^+$ be a functions. We say that T is an (α,θ)-admissible mapping if for all $x,y \in X$ satisfying:*

$$\alpha(x,y) \geq 1 \text{ and } \theta(x,y) \geq 1 \text{ imply that } \alpha(Tx,Ty) \geq 1 \text{ and } \theta(Tx,Ty) \geq 1.$$

Definition 4.6 *(Geraghty, 1973). An operator $T : X \to X$ is called a Geraghty contraction if there exists a function $\beta \in \mathscr{F}_G$ which satisfies for all $x,y \in X$ the condition $d(Tx,Ty) \leq \beta(d(x,y))d(x,y)$.*

Theorem 4.1 *(Geraghty, 1973). Let (X,d) be a complete metric space. If $T : X \to X$ is a Geraghty contractive mapping, then T has a unique fixed point.*

In 2015, Khojasteh et al., (2015) provides a class Θ of functions $Z : \mathbb{R}^{+2} \to \mathbb{R}$ which satisfies the following assumptions
(Z_1) : $Z(0,0) = 0$;
(Z_2): $Z(x,y) < y - x$ for every $x,y > 0$;
(Z_3) : If $\{x_n\}$, $\{y_n\}$ are sequences in $\mathbb{R}^+$ such that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n > 0$, then $\limsup_{n\to\infty} Z(x_n,y_n) < 0$ is called simulation functions.

Definition 4.7 *(Ansari, 2014) A map $G : \mathbb{R}^{+2} \to \mathbb{R}$ is said to be a C-class function, if it is continuous and verifies the following assumptions*
(i) $G(x,y) \leq x$;
(ii) $G(x,y) = x$ means that either $x = 0$ or $y = 0$, for every $x,y \in \mathbb{R}^+$.

Definition 4.8 *(Liu et al., 2018) A map $G : \mathbb{R}^{+2} \to \mathbb{R}$ is said to satisfy condition C_G, if there exists $C_G \geq 0$ with*
(G_1) $G(x,y) > C_G$ implies that $x > y$;
(G_2) $G(y,y) \leq C_G$, for each $y \in \mathbb{R}^+$.

Definition 4.9 *(Liu et al., 2018) A map $\Psi : \mathbb{R}^{+2} \to \mathbb{R}$ is said to be a C_G-simulation function, if it is verifies the following assumptions:*
(i) $\Psi(x,y) < G(y,x)$ for each $x,y > 0$, with $G : \mathbb{R}^{+2} \to \mathbb{R}$ is a C-class function which satisfies the condition C_G;
(ii) if $\{x_n\}$, $\{y_n\}$ are sequences in $\mathbb{R}^+$ with $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n > 0$ and $x_n < y_n$, then $\limsup_{n\to\infty} \Psi(x_n,y_n) < C_G$.

Definition 4.10 *(Taqbibt et al., 2023) Let (X,d) be a metric-like space, $T : X \to X$ be a maps, and $\alpha, \theta : X^2 \to \mathbb{R}^+$ be a function. We say that T is $\Psi_{C_G} - \alpha - \theta$-Geraghty contraction with respect to a C_G-simulation function Ψ if there exists*

$\beta \in \mathscr{F}_G$ *such that*

$$\Psi(\alpha(x,y)\theta(x,y)d(Tx,Ty),\beta(M(x,y))M(x,y)) \geq C_G$$

for every $x,y \in X$, *where*

$$M(x,y) = \max\{d(x,y),d(x,Tx),d(y,Ty)\}.$$

Theorem 4.2 *(Taqbibt et al., 2023) Let* (X,d) *be a complete metric-like space and* $T : X \to X$ *be a* $\Psi_{C_G} - \alpha - \theta$*-Geraghty contraction satisfying:*
(1) T *is an* (α,θ)*-admissible;*
(2) there is $X_0 \in X$ *such that* $\alpha(x_0,Tx_0) \geq 1$ *and* $\theta(x_0,Tx_0) \geq 1$;
(3) Ethier
(3a) T *is d-continuous, or*
(3b) If $\{x_n\}$ *in* X *with* $\theta(x_n,x_{n+1}) \geq 1$ *and* $\alpha(x_n,x_{n+1}) \geq 1$ *for every* $n \in N$ *and* $\lim_{n\to\infty} d(x_n,u) = d(u,u)$, *then we have* $\theta(x_n,u) \geq 1$ *and* $\alpha(x_n,u) \geq 1$ *for every* $n \in N$.
Then, T *possesses a unique fixed point* $u \in X$ *with* $d(u,u) = 0$.

Inspired and motivated by the work of Taqbibt et al., (2023) the main purpose of this research work is to introduce generalized $\Psi_{C_G} - \alpha - \theta$-Geraghty type contraction mappings with C_G-simulation function Ψ and establish new common fixed point theorems for the mappings introduced in the setting of complete b-metric-like spaces. This current work extend and generalize many comparable fixed point results in the existing literature, in particular that of Taqbibt et al., (2023) in the setting of complete metric-like spaces.

4.2 Main Results

In this section we introduced generalized $\Psi_{C_G} - \alpha - \theta$-Geraghty type contraction mappings with respect to C_G-simulation function Ψ in the setting of b-metric like spaces and proved existence and uniqueness of common fixed point for mappings introduced.

Definition 4.11 *Let X be a nonempty set, $S,T : X \to X$ and $\alpha, \theta : X \times X \to R^+$ be a functions. We say that the pair (S,T) is an (α,θ)-admissible mapping if for all $x, y \in X$ satisfying:*

$$\alpha(x,y) \geq 1 \text{ and } \theta(x,y) \geq 1 \text{ imply that } \alpha(Sx,Ty) \geq 1 \text{ and } \theta(Sx,Ty) \geq 1.$$

Definition 4.12 *Let (X,d) be a b-metric like spaces with $s \geq 1$, $S,T : X \to X$ be two mappings and $\alpha, \theta : X \times X \to \mathbb{R}^+$ be functions. Then (S,T) is called a pair of generalized $\Psi_{C_G} - \alpha - \theta$-Geraghty type contraction mappings with respect to C_G-simulation function Ψ if there exists $\beta \in \mathscr{F}_{Gs}$ such that for all $x, y \in X$.*

$$\Psi(\alpha(x,y)\theta(x,y)s^3 d(Sx,Ty), \beta(M(x,y))M(x,y)) \geq CG, \tag{4.1}$$

where

$$M(x,y) = max\{d(x,y), d(x,Sx), d(y,Ty), \frac{d(x,Ty)+d(y,Sx)}{4s}\}.$$

Theorem 4.3 *Let (X,d) be a complete b-metric like space with $s \geq 1$, $\alpha, \theta : X \times X \to \mathbb{R}^+$ be functions and $S,T : X \to X$ be two mappings. Suppose that the following conditions hold:*
(i) (S,T) be a pair of (α,θ)-admissible mappings;
(ii) there exists $x_0 \in X$ such that $\alpha(x_0,Sx_0) \geq 1$ and $\theta(x_0,Tx_0) \geq 1$;
(iii) (S,T) is a pair of generalized $\Psi_{C_G} - \alpha - \theta$-Geraghty type contraction mappings ;
(iv) Either
(iva) S and T are continuous,
or
(ivb) if $\{x_n\} \subset X$ with $\alpha(x_n,x_{n+1}) \geq 1$ and $\theta(x_n,x_{n+1}) \geq 1$ for each $n \in \mathbb{N} \cup \{0\}$ and $\lim_{n\to\infty} d(x_n,x) = d(x,x)$, then $\alpha(x_n,x) \geq 1$ and $\theta(x_n,x) \geq 1$ for each $n \in \mathbb{N} \cup \{0\}$.
Then, S and T have a common fixed point $x \in X$.

Proof: By (ii) above there exist $x_0 \in X$ such that $\alpha(x_0,Sx_0) \geq 1$ and $\theta(x_0,Tx_0) \geq 1$. We construct a sequence x_n in X by $x_{2n+1} = Sx_{2n}$, $x_{2n+2} = Tx_{2n+1}$ for each $n \in \mathbb{N} \cup \{0\}$. If $x_{2n_0} = x_{2n_0+1} = x_{2n_0+2}$ for some $n_0 \in \mathbb{N} \cup \{0\}$, then x_{2n_0} is a common fixed point of S and T which completes the proof. Now ,we suppose $x_{2n} \neq x_{2n+1} \neq x_{2n+2}$ for all $n \in \mathbb{N} \cup \{0\}$.

Using the (α, θ)-admissiblity of (S,T), we have $\alpha(x_0, Sx_0) = \alpha(x_0, x_1) \geq 1$ implies $\alpha(Sx_0, Tx_1) = \alpha(x_1, x_2) \geq 1$. Continuing by this manner, we obtain $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$. Similarly, $\theta(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$. Hence,

$$\alpha(x_n, x_{n+1}) \geq 1 \text{ and } \theta(x_n, x_{n+1}) \geq 1 \text{ for all } n \in \mathbb{N} \cup \{0\}. \tag{4.2}$$

Since (S,T) is a generalized $\Psi_{C_G} - \alpha - \theta$-Geraghty type contraction mappings with respect to C_G-simulation function Ψ, by taking $x = x_{2n}$ and $y = x_{2n+1}$ in (4.1), we get

$$\Psi(\alpha(x_{2n}, x_{2n+1})\theta(x_{2n}, x_{2n+1})s^3 d(Sx_{2n}, Tx_{2n+1}), \beta(M(x_{2n}, x_{2n+1}))M(x_2n, x_{2n+1}) \geq C_G.$$

And thus

$$\begin{aligned} C_G &\leq \Psi(\alpha(x_{2n}, x_{2n+1})\theta(x_{2n}, x_{2n+1})s^3 d(Sx_{2n}, Tx_{2n+1}), \beta(M(x_{2n}, x_{2n+1}))M(x_2n, x_{2n+1})) \\ &< G(\beta(M(x_{2n}, x_{2n+1}))M(x_{2n}, x_{2n+1}), \alpha(x_{2n}, x_{2n+1})\theta(x_{2n}, x_{2n+1})s^3 d(Sx_{2n}, Tx_{2n+1})). \end{aligned}$$

Using Condition (G_1) of Definition 4.8 and (4.2), we have

$$\begin{aligned} d(x_{2n+1}, x_{2n+2}) &= d(Sx_{2n}, Tx_{2n+1}) \\ &\leq \alpha(x_{2n}, x_{2n+1})\theta(x_{2n}, x_{2n+1})s^3 d(Sx_{2n}, Tx_{2n+1}) \\ &< \beta(M(x_{2n}, x_{2n+1}))M(x_{2n}, x_{2n+1}) \\ &< \frac{1}{s} M(x_{2n}, x_{2n+1}) \\ &\leq M(x_{2n}, x_{2n+1}), \end{aligned} \tag{4.3}$$

where

$$
\begin{aligned}
M(x_{2n},x_{2n+1}) &= \max\Bigg\{d(x_{2n},x_{2n+1}),d(x_{2n},Sx_{2n}),d(x_{2n+1},Tx_{2n+1}),\\
&\qquad \frac{(d(x_{2n},Tx_{2n+1})+d(x_{2n+1},Sx_{2n})}{4s}\Bigg\}\\
&= \max\Bigg\{d(x_{2n},x_{2n+1}),d(x_{2n},x_{2n+1}),d(x_{2n+1},x_{2n+2}),\\
&\qquad \frac{(d(x_{2n},x_{2n+2})+d(x_{2n+1},x_{2n+1})}{4s}\Bigg\}.
\end{aligned}
$$

But,

$$
\begin{aligned}
\frac{(d(x_{2n},x_{2n+2})+d(x_{2n+1},x_{2n+1})}{4s} &\leq \frac{s[d(x_{2n},x_{2n+1})+3d(x_{2n+1},x_{2n+2})]}{4s}\\
&= \frac{d(x_{2n},x_{2n+1})+3d(x_{2n+1},x_{2n+2})}{4}\\
&\leq \max\{d(x_{2n},x_{2n+1}),d(x_{2n+1},x_{2n+2})\}.
\end{aligned}
$$

So, $M(x_{2n},x_{2n+1})=\max\{d(x_{2n},x_{2n+1}),d(x_{2n+1},x_{2n+2})\}$.
If $M(x_{2n},x_{2n+1})=d(x_{2n+1},x_{2n+2})$, then from (4.3), we obtain

$$d(x_{2n+1},x_{2n+2})<d(x_{2n+1},x_{2n+2}),$$

which is a contradiction. Therefore, $M(x_{2n},x_{2n+1})=d(x_{2n},x_{2n+1})$ and consequently from (4.3), we get $d(x_{2n+1},x_{2n+2})<d(x_{2n},x_{2n+1})$, for all $n\in\mathbb{N}\cup\{0\}$.
Thus, $\{d(x_n,x_{n+1})\}$ is strictly deacreasing sequences of nonnegative real numbers, so there exists $r\geq 0$, such that $\lim_{n\to\infty} d(x_n,x_{n+1})=r$. Now, we claim $r=0$. Suppose on the contrary, $r>0$, from (4.3), we have

$$\lim_{n\to\infty}\alpha(x_n,x_{n+1})\theta(x_n,x_{n+1})s^3d(Sx_n,Tx_{n+1})=r$$

and

$$\lim_{n\to\infty}\beta(d(x_n,x_{n+1}))d(x_n,x_{n+1})=r.$$

Let $a_n = \alpha(x_n, x_{n+1})\theta(x_n, x_{n+1})s^3 d(Sx_n, Tx_{n+1})$ and $b_n = \beta(d(x_n, x_{n+1}))d(x_n, x_{n+1})$. Using (4.1) and (ii) of Definition 4.9, we get

$$C_G \leq \limsup_{n\to\infty} \Psi(a_n, b_n) < C_G,$$

which is a contradiction. Hence, $\lim_{n\to\infty} d(x_n, x_{n+1}) = r = 0$.

Now, we show that $\{x_n\}$ is a Cauchy sequence in X. Assume on the contrary, it is not a Cauchy sequence. Hence, there exists $\varepsilon > 0$ for which we can find subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ with $n_k > m_k > k$ for all k such that

$$d(x_{n_k}, x_{m_k}) \geq \varepsilon \tag{4.4}$$

and n_k is the smallest number satisfaying (4.4). So, we have

$$d(x_{n_k-1}, x_{m_k}) < \varepsilon.$$

Using the triangle inequality, we obtain

$$\begin{aligned} \varepsilon &\leq d(x_{n_k}, x_{m_k}) \\ &\leq sd(x_{n_k}, x_{n_k-1}) + sd(x_{n_k-1}, x_{m_k}) \\ &< sd(x_{n_k}, x_{n_k-1}) + s\varepsilon. \end{aligned} \tag{4.5}$$

Letting upper limit as $k \to \infty$ in (4.5), we get

$$\varepsilon \leq \limsup_{k\to\infty} d(x_{n_k}, x_{m_k}) \leq s\varepsilon.$$

Again, applying the triangle inequality, we have

$$\begin{aligned} \varepsilon &\leq d(x_{n_k}, x_{m_k}) \\ &\leq sd(x_{n_k}, x_{n_k+1}) + sd(x_{n_k+1}, x_{m_k}) \\ &\leq sd(x_{n_k}, x_{n_k+1}) + s^2 d(x_{n_k+1}, x_{n_k}) + s^2 d(x_{n_k}, x_{m_k}). \end{aligned}$$

Taking the upper limit as $k \to \infty$ in the above inequality, we get

$$\frac{\varepsilon}{s} \leq \limsup_{k\to\infty} d(x_{n_k+1}, x_{m_k}) \leq s^2\varepsilon. \tag{4.6}$$

Similarly,

$$\begin{aligned}
\varepsilon &\leq d(x_{n_k}, x_{m_k}) \\
&\leq sd(x_{n_k}, x_{n_k+1}) + sd(x_{n_k+1}, x_{m_k}) \\
&\leq sd(x_{n_k}, x_{n_k+1}) + s^2 d(x_{n_k+1}, x_{m_k+1}) + s^2 d(x_{m_k+1}, x_{m_k}) \\
&\leq sd(x_{n_k}, x_{n_k+1}) + s^3 d(x_{n_k+1}, x_{m_k}) + s^3 d(x_{m_k}, x_{m_k+1}) + s^2 d(x_{m_k+1}, x_{m_k}).
\end{aligned}$$

Taking the upper limit as $k \to \infty$ in the above inequality combaning with (4.6), we get

$$\frac{\varepsilon}{s^2} \leq \limsup_{k\to\infty} d(x_{n_k+1}, x_{m_k+1}) \leq s^3\varepsilon. \tag{4.7}$$

Using (4.1), we obtain

$$\begin{aligned}
C_G &\leq \Psi(\alpha(x_{n_k}, x_{m_k})\theta(x_{n_k}, x_{m_k})s^3 d(Sx_{n_k}, Tx_{m_k}), \beta(M(x_{n_k}, x_{m_k}))M(x_{n_k}, x_{m_k})) \\
&< G(\beta(M(x_{n_k}, x_{m_k}))M(x_{n_k}, x_{m_k}), \alpha(x_{n_k}, x_{m_k})\theta(x_{n_k}, x_{m_k})s^3 d(Sx_{n_k}, Tx_{m_k})).
\end{aligned}$$

So,

$$\alpha(x_{n_k}, x_{m_k})\theta(x_{n_k}, x_{m_k})s^3 d(Sx_{n_k}, Tx_{m_k}) < \beta(M(x_{n_k}, x_{m_k}))M(x_{n_k}, x_{m_k}).$$

Using (4.2), the above inequality follows that

$$\begin{aligned}
d(x_{n_k+1}, x_{m_k+1}) &= d(Sx_{n_k}, Tx_{m_k}) \\
&\leq \alpha(x_{n_k}, x_{m_k})\theta(x_{n_k}, x_{m_k})s^3 d(Sx_{n_k}, Tx_{m_k} \\
&< \beta(M(x_{n_k}, x_{m_k}))M(x_{n_k}, x_{m_k}) \\
&< \frac{1}{s} M(x_{n_k}, x_{m_k}) \\
&\leq M(x_{n_k}, x_{m_k}), \qquad (4.8)
\end{aligned}$$

where

$$\begin{aligned}
M(x_{n_k},x_{m_k}) &= \max\left\{d(x_{n_k},x_{m_k}),d(x_{n_k},Sx_{n_k}),d(x_{m_k},Tx_{m_k}),\frac{d(x_{n_k},Tx_{m_k})+d(x_{m_k},Sx_{n_k})}{4s}\right\} \\
&= \max\left\{d(x_{n_k},x_{m_k}),d(x_{n_k},x_{n_k+1}),d(x_{m_k},x_{m_k+1}),\right. \\
&\quad \left.\frac{d(x_{n_k},x_{m_k+1})+d(x_{m_k},x_{n_k+1})}{4s}\right\} \\
&\leq \max\left\{d(x_{n_k},x_{m_k}),d(x_{n_k},x_{n_k+1}),d(x_{m_k},x_{m_k+1}),\right. \\
&\quad \left.\frac{s[d(x_{n_k},x_{m_k})+d(x_{m_k},x_{m_k+1})]+d(x_{m_k},x_{n_k+1})}{4s}\right\}.
\end{aligned}$$

Taking upper limit as $k\to\infty$ in the above inequality, we get

$$\limsup_{k\to\infty} M(x_{n_k},x_{m_k}) \leq \max\{s\varepsilon,\frac{s^2\varepsilon+s^2\varepsilon}{4s}\} = \max\{s\varepsilon,\frac{s\varepsilon}{2}\} = s\varepsilon. \tag{4.9}$$

Using (4.7) and (4.9) from (4.8), we obtain

$$\lim_{k\to\infty} \alpha(x_{n_k},x_{m_k})\theta(x_{n_k},x_{m_k})s^3 d(Sx_{n_k},Tx_{m_k}) = s\varepsilon$$

and

$$\lim_{k\to\infty} \beta(M(x_{n_k},x_{m_k}))M(x_{n_k},x_{m_k}) = s\varepsilon.$$

Let

$$a_n = \alpha(x_{n_k},x_{m_k})\theta(x_{n_k},x_{m_k})s^3 d(Sx_{n_k},Tx_{m_k})$$

and

$$b_n = \beta(M(x_{n_k},x_{m_k}))M(x_{n_k},x_{m_k}).$$

Using (ii) of Definition 4 and (4.1), we have

$$C_G \leq \limsup_{k\to\infty} \Psi(a_n,b_n) < C_G,$$

which is a contradiction. Therefore, $\{x_n\}$ is a Cauchy sequence in X. Since (X,d) is a complete b-metric like space, there is some $x \in X$ such that

$$\lim_{n\to\infty} d(x_n,x) = d(x,x) = \lim_{n,m\to\infty} d(x_n,x_m) = 0, \tag{4.10}$$

which is implies that $d(x,x) = 0$.
Suppose (iv_a) is holds, then we have

$$x = \lim_{k\to\infty} x_{2k+1} = \lim_{k\to\infty} Sx_{2k} = S\lim_{k\to\infty} x_{2k} = Sx,$$

and

$$x = \lim_{k\to\infty} x_{2k+2} = \lim_{k\to\infty} Tx_{2k+1} = T\lim_{k\to\infty} x_{2k+1} = Tx.$$

Hence, $x = Sx = Tx$, that is, the pair (S,T) has a common fixed point $x \in X$.
Next, suppose (iv_b) holds with $d(Sx,x) > 0$ and $d(Tx,x) > 0$. We have $\alpha(x_n,x) \geq 1$ and $\theta(x_n,x) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$. Using (4.1), we get

$$\begin{aligned} C_G &\leq \Psi(\alpha(x_n,x)\theta(x_n,x)s^3 d(Sx_n,Tx), \beta(M(x_n,x))M(x_n,x)) \\ &< G(\beta(M(x_n,x))M(x_n,x), \alpha(x_n,x)\theta(x_n,x)s^3 d(Sx_n,Tx)), \end{aligned}$$

which implies that

$$\begin{aligned} \alpha(x_n,x)\theta(x_n,x)s^3 d(Sx_n,Tx)) &< \beta(M(x_n,x))M(x_n,x)),x_{m_k}) \\ &< \frac{1}{s}M(x_n,x) \\ &\leq M(x_n,x), \end{aligned} \tag{4.11}$$

where

$$\begin{aligned} M(x_n,x) &= \max\left\{ d(x_n,x), d(x_n,Sx_n), d(x,Tx), \frac{d(x_n,Tx)+d(x,Sx_n)}{4s} \right\} \\ &= \max\left\{ d(x_n,x), d(x_n,x_{n+1}), d(x,Tx), \frac{d(x_n,Tx)+d(x,x_{n+1})}{4s} \right\} \\ &\leq \max\left\{ d(x_n,x), d(x_n,x_{n+1}), d(x,Tx), \frac{sd(x_n,x)+sd(x,Tx)+d(x,x_{n+1})}{4s} \right\}. \end{aligned}$$

By taking limit $n \to \infty$ in the above inequality, we have

$$\lim_{n\to\infty} M(x_n,x) = d(x,Tx). \tag{4.12}$$

Then, from (4.11), we obtain

$$\begin{aligned} \frac{1}{s}d(x_{n+1},Tx) &\leq d(x_{n+1},Tx) \\ &= d(Sx_n,Tx) \\ &\leq \alpha(x_n,x)\theta(x_n,x)s^3 d(Sx_n,Tx) \\ &< \beta(M(x_n,x))M(x_n,x)) \\ &< \frac{1}{s}M(x_n,x) \\ &\leq M(x_n,x). \end{aligned}$$

So,

$$\frac{d(x_{n+1},Tx)}{sd(x,Tx)} < \frac{\beta(M(x_n,x))M(x_n,x)}{d(x,Tx)} < \frac{M(x_n,x)}{sd(x,Tx)}.$$

By using (4.12) and taking limit as $n \to \infty$ in the above inequality, we get

$$\lim_{n\to\infty} \beta(M(x_n,x)) = \frac{1}{s}.$$

Since $\beta \in \mathscr{F}_{Gs}$, then

$$\lim_{n\to\infty} M(x_n,x) = 0,$$

which is a contradiction. Therefore, $d(x,Tx) = 0$, that is, $Tx = x$.

Similarly, we get that $Sx = x$. Hence, x is a common fixed point of S and T. $\square$

In order to establish uniqueness of common fixed point, we need the the following additional condition.

Condition U: $\alpha(x,y) \geq 1$ and $\theta(x,y) \geq 1$ for all $x,y \in Fix(S,T)$, where $Fix(S,T)$ denote the set of all common fixed points of S and T.

Theorem 4.4 *Adding condition (U) to the hypotheses of Theorem 4.3, we obtain the uniqueness of the common fixed point of S and T.*

Proof: Suppose that there exists $x,y \in X$ such that $Sx = Tx = x$, $Sy = Ty = y$ and $x \neq y$. Then,

$$d(x,x) = d(y,y) = 0. \tag{4.13}$$

Using (4.1), we get

$$\begin{aligned} C_G &\leq \Psi(\alpha(x,y)\theta(x,y)s^3 d(Sx,Ty), \beta(M(x,y))M(x,y)) \\ &< G(\beta(M(x,y))M(x,y), \alpha(x,y)\theta(x,y)s^3 d(Sx,Ty), \end{aligned}$$

So,

$$\begin{aligned} \alpha(x,y)\theta(x,y)s^3 d(Sx,Ty) &< \beta(M(x,y))M(x,y) \\ &< \frac{1}{s}M(x,y) \\ &\leq M(x,y). \end{aligned} \tag{4.14}$$

By (4.13), we have

$$\begin{aligned} M(x,y) &= \max\left\{ d(x,y), d(x,Sx), d(y,Ty), \frac{d(x,Ty)+d(y,Sx)}{4s} \right\} \\ &= \max\left\{ d(x,y), d(x,x), d(y,y), \frac{d(x,y)+d(y,x)}{4s} \right\} \\ &= d(x,y). \end{aligned}$$

Using Condition(U), from (4.14), we get

$$d(x,y) \leq \alpha(x,y)\theta(x,y)s^3 d(Sx,Ty) < d(x,y),$$

which is a contradiction. Therefore, $x = y$, that is, common fixed point of S and T is unique. □

Corollary 4.5 *Let (X,d) be a complete b-metric like space, $\alpha, \theta : X \times X \to \mathbb{R}^+$ and $S,T : X \to X$ be a mappings with*

$$\Psi(\alpha(x,y)\theta(x,y)s^3 d(Sx,Ty), \beta(N(x,y))N(x,y)) \geq CG,$$

where $N(x,y) = \max\{d(x,y), d(x,Sx), d(y,Ty)\}$. Also assume the following conditions hold:
(i) (S,T) be a pair of (α,θ)-admissible mappings;
(ii)There exists $x_0 \in X$ such that $\alpha(x_0,Sx_0) \geq 1$ and $\theta(x_0,Tx_0) \geq 1$;
(iii) the pair (S,T) is continuous or if $\{x_n\} \subset X$ with $\alpha(x_n,x_{n+1}) \geq 1$ and $\theta(x_n,x_{n+1}) \geq 1$ for each $n \in \mathbb{N} \cup \{0\}$ and $\lim_{n\to\infty} d(x_n,x) = d(x,x)$, then $\alpha(x_n,x) \geq 1$ and $\theta(x_n,x) \geq 1$ for each $n \in \mathbb{N} \cup \{0\}$.
Then, (S,T) has a common fixed point $x \in X$.

Proof: The result follows from Theorem 4.3 by taking for all $x,y \in X$

$$N(x,y) = \max\{d(x,y), d(x,Sx), d(y,Ty)\} \leq M(x,y).$$

□

The following corollary is obtained by taking $S = T$ in Theorem 4.3

Corollary 4.6 *Let (X,d) be a complete b-metric like space with $s \geq 1$, $\alpha, \theta : X \times X \to \mathbb{R}^+$ and $T : X \to X$ be a mappings with*

$$\Psi(\alpha(x,y)\theta(x,y)s^3 d(Tx,Ty), \beta(M(x,y))M(x,y)) \geq CG,$$

where $M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{4s}\}$.
Suppose the following conditions hold:
(i) T *is* (α,θ)*-admissible mappings;*
(ii)There exists $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geq 1$ *and* $\theta(x_0, Tx_0) \geq 1$*;*
(iii) the pair T is continuous or if $\{x_n\} \subset X$ *with* $\alpha(x_n, x_{n+1}) \geq 1$ *and* $\theta(x_n, x_{n+1}) \geq 1$ *for each* $n \in \mathbb{N} \cup \{0\}$ *and* $\lim_{n\to\infty} d(x_n, x) = d(x,x)$*, then* $\alpha(x_n, x) \geq 1$ *and* $\theta(x_n, x) \geq 1$ *for each* $n \in \mathbb{N} \cup \{0\}$.
Then, T *has a fixed point* $x \in X$.

If we consider in Theorem 4.3, $\alpha(x,y) = \theta(x,y) = 1$ for each $x,y \in X$ we get the following corollary

Corollary 4.7 *Let* (X,d) *be a complete b-metric like space with* $s \geq 1$ *and* $S,T : X \to X$ *be a mappings. Asuume that there is is a* C_G*-simulation fanction* Ψ *with for all* $x,y \in X$

$$\Psi(s^3 d(Sx,Ty), \beta(M(x,y))M(x,y)) \geq CG.$$

Then, S and T has a unique common fixed point $x \in X$.

By taking $\alpha(x,y) = \theta(x,y) = 1$ for each $x,y \in X$ in Corollary 4.6, we get the following result.

Corollary 4.8 *Let* (X,d) *be a complete b-metric like space with* $s \geq 1$ *and* $T : X \to X$ *be a mappings. Asuume that there is is a* C_G*-simulation fanction* Ψ *with for all* $x,y \in X$

$$\Psi(s^3 d(Tx,Ty), \beta(M(x,y))M(x,y)) \geq CG.$$

Then, T *has a unique fixed point* $x \in X$.

Now, we give an example in support of our main result.

Example 4.2 *Let* $X = \{0,1,2\}$ *and* $d : X \times X \to \mathbb{R}^+$ *be defined by*
$d(0,0) = 0$
$d(1,1) = 4$
$d(2,2) = 64$
$d(0,1) = d(1,0) = 1$

$d(0,2) = d(2,0) = 16$
$d(1,2) = d(2,1) = 25$. *Then,* (X,d) *is a b-metric-like space with coefficient* $s = 2$. *Clearly,* d *is not not a b-metric and metric since* $d(1,1) = 4 \neq 0$. *Also,* d *is not a metric-like because* $d(1,2) = 25 > 17 = d(1,0) + d(0,2)$. *Let* $\alpha, \theta : X \times X \to \mathbb{R}^+$ *be defined by*

$$\alpha(x,y) = \begin{cases} \frac{3}{2} & \text{if } (x,y) \in \{0,1\}, \\ \frac{1}{8} & \text{if } x \in X \text{ and } y = 2, \\ \frac{1}{2} & \text{if } x = 2 \text{ and } y \in \{0,1\}, \end{cases}$$

and

$$\theta(x,y) = \begin{cases} 2 & \text{if } (x,y) \in \{0,1\}, \\ \frac{1}{5} & \text{if } x \in X \text{ and } y = 2, \\ 1 & \text{if } x = 2 \text{ and } y \in \{0,1\}. \end{cases}$$

Let $S,T : X \to X$ *be defined by*

$$S(x) = \begin{cases} 1 & \text{if } x \in \{0,1\}, \\ 0 & otherwise. \end{cases}$$

and

$$T(x) = \begin{cases} 1 & \text{if } x \in \{0,1\}, \\ 2 & otherwise. \end{cases}$$

Next, we conder the following mappings
$G(x,y) = x - y$;
$\Psi(y,x) = \frac{3}{4}y - x$
$\beta(t) = \frac{1}{t+4}$ *for every* $t,x,y \in \mathbb{R}^+$. *Clearly,* $\beta \in \mathscr{F}_{Gs}$, *and* G *and* Ψ *are respectively C-class function and* C_G*-simulation function. Then, for all* $x,y \in X$ *such that* $\alpha(x,y) \geq 1$ $\theta(x,y) \geq 1$ *implies* $\alpha(Sx,Ty) \geq 1$ *and* $\theta(Sx,Ty) \geq 1$, *that is,* (S,T) *is an* (α,θ)*-admissible. On the other hand, if we take* $x_0 = 1$, *then the condition (ii) of Theorem 4.3 is satisfied.*
Now, we consider the following cases:
Case 1: *If* $(x,y) = (0,0)$, *we have*

$$
\begin{aligned}
M(0,0) &= \max\{d(0,0), d(0,S0), d(0,T0), \frac{d(0,T0)+d(0,S0)}{8}\} \\
&= \max\{d(0,0), d(0,1), d(0,1), \frac{d(0,1)+d(0,1)}{8}\} = 1.
\end{aligned}
$$

Then,

$$
\begin{aligned}
\Psi(\alpha(0,0)\theta(0,0)s^3 d(S0,T0), \beta(M(0,0))M(0,0)) &= \Psi(\frac{3}{2}.2.2^3.4, \beta(1).1 \\
&= \Psi(96, \frac{1}{5}) = 72 - \frac{1}{5}.
\end{aligned}
$$

Case 2: *If* $(x,y) = (0,1)$*, we have*

$$
\begin{aligned}
M(0,1) &= \max\{d(0,1), d(0,S0), d(0,T1), \frac{d(0,T1)+d(1,S0)}{8}\} \\
&= \max\{d(0,1), d(0,1), d(1,1), \frac{d(0,1)+d(1,1)}{8}\} = 4.
\end{aligned}
$$

Then,

$$
\begin{aligned}
\Psi(\alpha(0,1)\theta(0,1)s^3 d(S0,T1), \beta(M(0,1))M(0,1)) &= \Psi(\frac{3}{2}.2.2^3.4, \beta(4).4) \\
&= \Psi(96, \frac{1}{2}) = 72 - \frac{1}{2}.
\end{aligned}
$$

Case 3: *If* $(x,y) = (0,2)$*, we have*

$$
\begin{aligned}
M(0,2) &= \max\{d(0,2), d(0,S0), d(2,T2), \frac{d(0,T2)+d(2,S0)}{8}\} \\
&= \max\{d(0,2), d(0,1), d(2,2), \frac{d(0,2)+d(2,1)}{8}\} = 64.
\end{aligned}
$$

Then,

$$\begin{aligned}\Psi(\alpha(0,2)\theta(0,2)s^3 d(S0,T2),\beta(M(0,2))M(0,2)) &= \Psi(\frac{1}{8}.\frac{1}{5}.2^3.25,\beta(64).64)\\ &= \Psi(5,\frac{64}{68})=\frac{15}{4}-\frac{16}{17}.\end{aligned}$$

Case 4: *If $(x,y)=(2,1)$, we have*

$$\begin{aligned}M(2,0) &= \max\{d(2,0),d(2,S2),d(0,T0),\frac{d(2,T0)+d(0,S2)}{8}\}\\ &= \max\{d(2,0),d(2,0),d(0,1),\frac{d(2,1)+d(0,0)}{8}\}=16.\end{aligned}$$

Then,

$$\begin{aligned}\Psi(\alpha(2,0)\theta(2,0)s^3 d(S2,T0),\beta(M(2,0))M(2,0)) &= \Psi(\frac{1}{2}.1.2^3.1,\beta(16).16)\\ &= \Psi(4,\frac{16}{20})=3-\frac{4}{5}.\end{aligned}$$

Case 5: *If $(x,y)=(1,1)$, we have*

$$\begin{aligned}M(1,1) &= \max\{d(1,1),d(1,S1),d(1,T1),\frac{d(1,T1)+d(1,S1)}{8}\}\\ &= \max\{d(1,1),d(1,1),d(1,1),\frac{d(1,1)+d(1,1)}{8}\}=4.\end{aligned}$$

Then,

$$\begin{aligned}\Psi(\alpha(1,1)\theta(1,1)s^3 d(S1,T1),\beta(M(1,1))M(1,1)) &= \Psi(\frac{3}{2}.2.2^3.4,\beta(4).4)\\ &= \Psi(96,\frac{1}{2})=72-\frac{1}{2}.\end{aligned}$$

Case 6: *If $(x,y)=(1,2)$, we have*

$$
\begin{aligned}
M(1,2) &= \max\{d(1,2), d(1,S1), d(2,T2), \frac{d(1,T2)+d(2,S1)}{8}\} \\
&= \max\{d(1,2), d(1,1), d(2,2), \frac{d(1,2)+d(2,1)}{8}\} = 64.
\end{aligned}
$$

Then,

$$
\begin{aligned}
\Psi(\alpha(1,2)\theta(1,2)s^3 d(S1,T2), \beta(M(1,2))M(1,2)) &= \Psi(\frac{1}{8}.\frac{1}{5}.2^3.25, \beta(64).64) \\
&= \Psi(5, \frac{64}{68}) = \frac{15}{4} - \frac{16}{17}.
\end{aligned}
$$

Case 7: *If* $(x,y) = (2,1)$*, we have*

$$
\begin{aligned}
M(2,1) &= \max\{d(2,1), d(2,S2), d(1,T1), \frac{d(2,T1)+d(1,S2)}{8}\} \\
&= \max\{d(2,1), d(2,0), d(1,1), \frac{d(2,1)+d(1,0)}{8}\} = 25.
\end{aligned}
$$

Then,

$$
\begin{aligned}
\Psi(\alpha(2,1)\theta(2,1)s^3 d(S2,T1), \beta(M(2,1))M(2,1)) &= \Psi(\frac{1}{2}.1.2^3.1, \beta(25).25) \\
&= \Psi(4, \frac{25}{29}) = 3 - \frac{25}{29}.
\end{aligned}
$$

Case 8: *If* $(x,y) = (2,2)$*, we have*

$$
\begin{aligned}
M(2,2) &= \max\{d(2,2), d(2,S2), d(2,T2), \frac{d(2,T2)+d(2,S2)}{8}\} \\
&= \max\{d(2,2), d(2,2), d(2,2), \frac{d(2,2)+d(2,0)}{8}\} = 64.
\end{aligned}
$$

Then,

$$\begin{aligned}\Psi(\alpha(2,2)\theta(2,2)s^3 d(S2,T2),\beta(M(2,2))M(2,2)) &= \Psi(\frac{1}{8}.\frac{1}{5}.2^3.16,\beta(64).64)\\ &= \Psi(\frac{16}{5},\frac{16}{17}) = \frac{12}{5}-\frac{16}{17}.\end{aligned}$$

Finally, for all $x,y \in X$, we have

$$\begin{aligned} 0 &\leq \Psi(\alpha(x,y)\theta(x,y)s^3 d(Sx,Ty),\beta(M(x,y))M(x,y))\\ &< G(\alpha(x,y)\theta(x,y)s^3 d(Sx,Ty),\beta(M(x,y))M(x,y)). \end{aligned} \tag{4.15}$$

Then, by using the inequality (4.15) *and the Definition 4.12, the mappings* (S,T) *is a pair of generalized* $\Psi_{C_G}-\alpha-\theta$*-Geraghty type contraction with* $C_G = 0$*. Thus the assumptions of Theorem 4.3 are verified. So, S and T possesses a common fixed point in X, which is* $x = 1$.

Chapter 5

Conclusion and Future scope

5.1 Conclusion

Taqbibt, et al. (2023) introduced the concept of an $\alpha-\theta$-Geraghty type contraction mapping using CG-simulation in metric-like space and studied fixed point results for the mappings introduced. In this research work, we introduced Generalized $\alpha-\theta$-Geraghty type contraction mappings with respect to C_G-simulation function Ψ in the setting of b-metric like space and proved proved the existence and uniqueness of common fixed points for the introduced mappings. Our results extend and generalize many fixed point results in the existing literature particularly that of Taqbibt, et al. (2023). We have also supported the main result of this research work by applicable example.

5.2 Future scope

There are some published results related to the existence of fixed point theorems of mappings defined on b-metric like space. The researcher believes the search for the existence and uniqueness of common fixed points for Generalized $\alpha-\theta$-Geraghty type contraction mappings with respect to C_G-simulation function Ψ in b-metric like space is an active area of study. So, any interested researchers can use this opportunity and conduct their research work in this area.

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