AN ITERATIVE ALGORITHM FOR SOLVING SPLIT EQUALITY FIXED POINT PROBLEM OF A FINITE FAMILY OF PSEUDO-PSEUDO CONTRACTIVE MAPPINGS IN HILBERT SPACE

A RESEARCH SUBMITTED TO THE DEPARTMENTOF MATHEMATICS INPARTIAL FULFILLMENT FOR THE DEGREE OF MASTERS OF SCIENCE IN MATHEMATICS

By: ABIOT ABITE AKIRSO

Advisor : Dr. Getahun Bekele (PhD)

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Jimma, Ethiopia

Declaration

I undersigned declare that, this thesis paper entitled that an iterative algorithm for solving split equality problem of a finite family of pseudopseudo contractive mappings in Hilbert space is my own original work and it has not been submitted to any institution or university elsewhere for the award of any academic degree or like. Where other sources of information have been used or quoted, they have been indicated and acknowledged. Name: Abiot Abite
Signature: ....
Date:..
The work has been done under the supervision of:
Name: Getahun Bekele (Ph.D.)
Signature:..................................
Date:..

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Abstract

In this thesis we introduced an iterative algorithm for solving split equality fixed point problem of a finite family of pseudo-pseudo contractive mappings in Hilbert space and proved the strong convergence of a sequence generated by proposed algorithm to a solution of the problem in Hilbert spaces provided that the mappings are uniformly continuous. Finally, we applied our main results to solve fixed point equality problem. Our results extended and generalized many results in the literature

Acronym

Throughout this research, we denotes the following.

- H is real Hilbert space.
- C is nonempty close and convex subset of Hilbert space.
- $\|.\|$ is the norm space.
- $\langle .,. \rangle$ is the inner product space.
- SEP is a split equality problem.
- SEPFPP is a split equality problem fixed point problem.

Contents

Chapter 1

Introduction

1.1 Background of the study

The theory of non-linear functional analysis is a rich and fascinating area of research that has applications in engineering, computer science, mathematical sciences, and social sciences. Fixed point techniques have been widely used in various fields such as science, economics, and engineering. Many authors have developed iterative algorithms for approximating fixed points of mappings, including Mann (1953), Berinde (2007), Browder (1968), Khan (2008), and Krasnoselskii (1955).

Authors have also introduced hybrid Mann and hybrid Ishikawa algorithms for obtaining strong convergence of sequences to fixed points of Lipschitz pseudo-contractive mappings, as seen in the works of Liu et al. (2011) and Marino et al. (2009). In 2020, Zegeye and Wega introduced a new class of mappings, pseudo-pseudo contrastive, and established an iterative algorithm that strongly converges to fixed points of pseudo-pseudo contrastive mappings, assuming the mapping T is uniformly continuously sequentially weakly continuous. Researchers have studied various algorithms for approximating solutions of the Split Equality Fixed Point Problem (SEFPP) for non-linear mappings in Hilbert spaces, such as in the works of Moudafi (2014), Zhao (2014), Chang et al. (2015), Boikanyo. O.A (2019), and Wega (2022). The SEFPP was first introduced by Moudafi in 2014 and mathematically formulated as finding x and y with the property.

In 2015, Zhao introduced an iterative process for quasi-non-expansive mappings, proving convergence under certain assumptions without prior knowledge of norms. Similarly, Chang et al. in 2015 introduced an iterative process for quasi-pseudo contrastive mappings, establishing weak convergence to a solution of the SEFPP.

Motivated by previous works, this thesis aims to introduce a new iterative algorithm for approximating solutions of SEFPP for pseudo-pseudo contrastive mappings in

Hilbert spaces and includes an application to convex minimization problems. Our results extend and generalize existing literature.

Inspired and motivated by the above research works the purpose of this thesis is to introduce a new iterative algorithm for approximating a common solution of the sum of two monotone mappings and fixed point problem a pseudo-pseudo contractive mapping in Hilbert spaces. Moreover, we give an application to the convex minimization problem. Our results extend and generalize many results in the literature.

Now, we recall some definitions that the researcher will need in the following sequel.

Definition 1.1.1 *Let $T : C \longrightarrow H$ be mapping,*

i) T is called $L-$ Lipschitz mapping with Lipschtz constant $L > 0$ if

$$||Tx - Ty|| \leq L||x - y||$$

for all $x, y \in C$. If $0 \leq L < 1$, then T is called contraction. If $L = 1$, then T is called nonexpansive.

ii) T is called pseudocontractive mapping if for all $x, y \in C$ we have that

$$\langle\, x - y\,,\ Tx - Ty\rangle \leq ||x - y||^2.$$

iii) T is called to be $\alpha-$strictly pseudocontractive mapping, if there exists a constant $\alpha > 0$ such that for all $x, y \in C$,

$$\langle\, x - y\,,\ Tx - Ty\rangle \leq ||x - y||^2 - \alpha||(x - y) - (Tx - Ty)||^2.$$

iv) T is called $T : C \longrightarrow H$ is said to be pseudo-pseudocontractive mapping provided that for each $x, y \in C$, we have:

$$\langle\, x - Tx, y - x\rangle \geq 0 \ \ implies \ \ \langle\, y - Ty, y - x\rangle.$$

We remark that the class pseudo-pseudocontractive mappings are more general than the classes of mappings mentioned in (i)-(iii) above.

Definition 1.1.2 *The operator T is called sequentially weakly continuous if for each sequence x_n,we have x_n converges weakly to x implies Tx_n converges to Tx.*

Now, we recall some notions and definitions that we will need in our work. Throughout this research, (VIP) denotes for variational inequality problem, (SEP) denotes the split equality problem, (SEVIP) denotes for split equality variational inequality problem, H denotes a Hilbert space and C denotes closed, nonempty and convex subset of a Hilbert space.

Now, we recall some definitions that the researcher will need in the following sequel.

Definition 1.1.3 *Let $T : C \longrightarrow H$ be mapping,*

i) T is called monotone mapping provided that for each $x, y \in C$, we have:

$$\langle\, Tx - Ty, y - x\rangle \geq 0.$$

ii) T is called is said to be pseudomonotone mapping provided that for each $x, y \in C$, we have:

$$\langle\, Tx, y - x\rangle \geq 0 \ \ \textit{implies} \ \ \langle\, Ty, y - x\rangle \geq 0.$$

We remark that the class pseudomonotone mappings are more general than the class of monotone mappings.

1.2 Statements of the Problem

In previous years, various iterative methods have been proposed to address split equality fixed point problems in Hilbert Spaces for different classes of mappings. Recently, Boikanyo (2019) introduced an iterative method for quasi-pseudo-contractive mappings that demonstrates strong convergence to a solution of the split equality fixed point problem. Additionally, Wega (2022) developed an iterative algorithm to approximate a solution of the SEFPP for pseudocontractive mappings and proved the strong convergence of the sequence generated by this method to a solution in Hilbert spaces. However, a scheme that achieves strong convergence to a solution of the split equality fixed point problem for pseudo-pseudo contractive mappings in real Hilbert spaces has not yet been explored. Inspired and motivated by the research works of Boikanyo (2019), Wega (2022), we investigated a new iterative scheme for solving the split equality fixed point problem of a finite family of pseudo-pseudo-contractive mappings in Hilbert spaces.

1.3 Objectives of the Study

1.3.1 General Objective

The general objective of this thesis was to study iterative algorithms for solving the split equality fixed point problem of a finite family of pseudo-pseudo-contractive mappings in Hilbert spaces.

1.3.2 Specific Objectives

The specific objectives of this thesis is to:

- investigate an iterative algorithm for solving the split equality fixed point problem of a finite family of pseudo-pseudo-contractive mappings in Hilbert spaces.
- prove the sequence generated by the proposed algorithm is bounded in Hilbert spaces.
- apply our main result to solve the fixed point equality problems.

1.4 Significance of the Study

The outcome of this study have the following importance:

- It may generalize the study of the class of mapping into more general classes of mapping.
- It generalized the study fixed points of pseudocontractive mappings in Hilbert spaces.
- Approximation of a solution of an iterative algorithm for solving the split equality problem of a finite family of pseudo-pseudo-contractive mappings in Hilbert spaces.
- It may provide some background information for other researchers who want to conduct a research on related topics.

1.5 Delimitation of the Study

This study was delimited to study iterative algorithms for solving the split equality fixed point problem of a finite family of pseudo-pseudo-contractive mappings in Hilbert spaces.

Chapter 2

Review of Related Literatures

The split equality monotone inclusion problem (SEMIP) studied by Moudafi, (2014) is formulated as the problem of finding p and q with the property:

$$p \in C, q \in D, \text{such that} Ap = Bq, \tag{2.1}$$

where C and D are closed and convex subsets of H_1 and H_2, respectively, $S : H_1 \to H_1$ and $T : H_2 \to H_2$ are nonlinear maps and $A : H_1 \to H_3$ and $B : H_2 \to H_3$ are bounded linear operators. This problem has received much attention of researchers due to its applications in many disciplines. This kind of problem allows asymmetric and partial relations between the variables x and y. There are a vast number of applications of the SEFP in many fields such as phase retrieval, medical image reconstruction, game theory, decomposition methods for PDEs, applications in game theory, and intensity-modulated radiation therapy treatment planning. As a result of the broad spectrum of its applications, there has been various research output for approximating solutions of the SEFP. Note that each nonempty, closed and convex subset of a Hilbert space in (2.1) can be regarded as a set of solutions of the variational inequality problem and hence we obtain the split equality variational inequality problem (SEVIP) which consists of finding two points,

$$p \in VI(C,T), q \in VI(D,S) \text{such that} Ap = Bq, \tag{2.2}$$

where $T : C \to H_1$ and $S : D \to H_2$, be nonlinear mappings. The Fixed point theory is attracted many authors' attention due to its extraordinary utility and broad applicability in many areas of applied mathematics (most notably, inverse problems which arise from phase retrievals and in medical image reconstruction.

Fixed point results give conditions under mappings of fixed point theory in which the desired iterative method converges to the solution. In the last one century the theory of fixed point has been reached as a powerful and important tool in the study of nonlinear problems. Banach (1922), introduced an iterative algorithm called Picard iteration for the class of contraction mappings and given by:

$$x_0 \in C, x_{n+1} = Tx_n, \quad n \geq 0. \tag{2.3}$$

The sequence generated by algorithm converges strongly to a unique fixed point of contraction mapping. However, this method in general failed to converge if T is not a contraction mapping. As a result many researchers introduced different types of algorithms for approximating fixed points mappings in Hilbert spaces (see, e.g., Mann (1953), Halpern (1964), Ishikawa (1974)). For approximating a fixed point of Mann introduced an iterative algorithm called Mann iteration Mann (1954), for approximating fixed points of nonexpansive mappings and it is given by

$$x_0 \in C, x_{n+1} = (1-\alpha_n)x_n + \alpha_n Tx_n, \text{ for } n \geq 0, \tag{2.4}$$

where $\{\alpha\}$ is a real sequence in the interval $(0,1)$ satisfying certain conditions. However, it is worth mentioning that the sequence generated by this scheme does not always converge strongly to a fixed point of nonexpansive mapping T. To obtain strong convergence of this method to a fixed point of T one has to impose compactness assumption on C (see, e.g., Chidume (1981), Kirk (1981)). Halpern (1964) introduced an iterative scheme called Halpern-iteration and it is given by

$$u,\ x_0 \in C,\ x_{n+1} = \alpha_n u + (1-\alpha_n)Tx_n,\ n \leq 1. \tag{2.5}$$

He proved that the sequence generated by algorithm (2.5) converges to a fixed point of nonexpansive mapping T. Ishikawa (1974), construct an iterative scheme called Ishikawa-iteration for approximating fixed points of the class of pseudocontractive mappings which is more general than the class of nonexpansive mapping and the scheme is given by

$$\begin{aligned} x_0 \in C,\ y_n &= (1-\beta_n)x_n + (1-\beta_n)Tx_n, \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n Ty_n \text{ for } n \geq 0. \end{aligned} \tag{2.6}$$

Where $\{\beta_n\}$ and $\{\alpha_n\}$ are real sequences in the interval $(0,1)$ satisfying certain conditions. He proved that the sequence generated by algorithm (2.6) converges strongly to a fixed point T provided that T is Lipschitz pseudocontractive mapping and C is a compact convex subset of a Hilbert space H.

We note that several authors have also established different iterative algorithms for approximating a common fixed point of a finite family of nonlinear mappings (see, e.g., Bauschke (1996), Zhou (2008), Daman and Zegeye (2012)).

In 1996, Bauschke (1996) introduced a Halpern type algorithm for approximating a common fixed point for a finite family of nonexpansive mappings and proved that the sequence generated by his method converges strongly to a common fixed point of a finite family of nonexpansive mappings in Hilbert spaces.

In 2008, Zhou (2008) studied an iterative algorithm for approximating a common fixed point of a finite family of pseudocontractive mappings and proved the sequence generated by their method converges weakly to a common fixed point of a finite family of pseudocontractive mappings in Hilbert spaces.

In 2012, Daman and Zegeye (2012) established and proved strong convergence of Halpern-Ishikawa iterative methods to a common fixed points of a finite family of Lipschitz pseudocontractive mappings in Hilbert spaces.

Recently, Zegeye and Wega introduced an iterative algorithm for a finite family of Lipschitz pseudocontractive mappings and proved strong convergence of a sequence generated by their method converges strongly to a common fixed point of a finite family of Lipschitz pseudocontractive mappings in Hilbert spaces.

2.1 Preliminaries

First we recall some known definitions, Lemmas and Theorems which are used in our subsequent analysis and recall some properties of the projection. Let C be a nonempty closed and convex of a Hilbert space H. for $x \in H$, the projection mapping $P_c : H \to C$ is defined by

$$\|P_c x - x\| = inf_{y \in c}\|x - y\|, \tag{2.7}$$

hence, P_c satisfies: $\|P_c x - P_c y\|^2 \leq \langle P_c x - P_c y, x - y\rangle$, for all $x, y \in H$.

Lemma 2.1.1 *For all $x, y \in H$, it is known that the following inequality hold.*

i) $\|x + y\| \leq \|x\| + \|y\|$

ii) $\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2)$

iii) $\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y\rangle$

iv) $\|x + y\|^2 = \|x\|^2 + 2\langle x + y, y\rangle$

Lemma 2.1.2 *(Xu, 2002). Let $\{\alpha_n\}$ be a sequence of nonnegative real numbers that satisfying the following relation:*

$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n \gamma_n$ *for* $n \geq n_0$ *where* $\{\alpha_n\} \subseteq (0, 1)$ *and* $\{\gamma_n\} \subseteq \mathbb{R}$, *satisfies* $lim_{x\to\infty}\alpha_n = 0$,

$$\sum_{n=1}^{\infty} \alpha_n = \infty, \text{ and } lim \quad sup_{n\to\infty}\gamma_n \leq 0. \text{ Then } lim_{n\to\infty}a_n = 0.$$

Lemma 2.1.3 *(Albert,1996). Let C be complex subset of a real Hilbert space H and let $x \in H$, then*

$$x_0 = P_c x \text{ if and only if } \langle z - x_0, x - x_0\rangle \leq 0, \text{ for every } z \in C.$$

Lemma 2.1.4 *(Zegeye and Shahzad). Let H be a real Hilbert space. Then for all $x_i \in H$ and$\alpha_i \in [0, 1]$ for $i = 1, 2, 3, ...n$, such that $\alpha_1 + \alpha_2 + \alpha_3 + ... + \alpha_n = 1$, then the following holds*
$\|\alpha_0 x_0 + \alpha_1 x_1 + ... + \alpha_n x_n\|^2 = \sum_{i=0}^{n} \alpha_i \|x_i\|^2 - \sum_{0 \leq i,j \leq n} \alpha_i \alpha_j \|x_i - x_j\|^2$.

Lemma 2.1.5 *(Mainge, 2008). Let* $\{a_n\}$*, be sequence of real numbers such that there exists a subsequence* $\{n_i\}$ *of* $\{n\}$ *such that* $\alpha_{ni} < \alpha_{ni+1}$*, for* $i \in N$*..Then there exists a non decreasing sequence* $\{m_k\} \subset N$*,, such that* $m_k \to \infty$*,and the following properties are satisfied for numbers* $k \in N$*:*

$$\alpha_{mk} \leq \alpha_{mk+1} \ \ in\, fact \ \ m_k = max\{j \leq k : \alpha_j \leq \alpha_{j+1}\}.$$

Lemma 2.1.6 *(He,2006). Let C, be a nonempty, closed and convex subset of H Let* $r(x)$*, be a real valued function on H and defined* $K := \{x \in C : r(x) \leq 0\}$*. If K, is nonempty and r is L-Lipshitz continuous with* $L > 0$*, then* $||P_K x - x|| \geq \frac{1}{L}\max\{r(x), 0\}$*, for* $x \in C$*.*

Chapter 3

Methodology

In this section, the study area and period, mathematical study design, source of information and mathematical procedures will be presented.

3.1 Study Area and Period

The study was conducted at Jimma University under the Department of Mathematics from January 2024 G.C. to June 2024 G.C and conceptually the study focused on studying study iterative algorithms for solving the split equality fixed point problem of a finite family of pseudo-pseudo-contractive mappings in Hilbert spaces.

3.2 Study Design

In order to achieve the objectives of this study was employ analytical methods of designing.

3.3 Sources of Information

The relevant sources of information for this study will be published articles, books of different mathematics that are related to our research topic.

3.4 Methodology

In this thesis, we were follow the standard mathematical procedures given below:

- Establishing an iterative algorithm and constructing theorems for solving the split equality fixed point problem of a finite family of pseudo-pseudo-contractive mappings in Hilbert spaces.
- Proving strong convergence of the sequence proposed by the method to a solution of split equality fixed point problem of a finite family of pseudo-pseudo-contractive mappings in Hilbert spaces.
- Applying our main result to convex solve fixed point equality problems.

Chapter 4

Main Results

In this section we introduce an iterative algorithm for solving the split equality fixed point problem of a finite family of pseudo-pseudo-contractive mappings in Hilbert spaces.

In this section, we shall make use of the following assumptions:

Assumption 1:

A1: Let $T_1, T_2 : H_1 \to H_1$ and $S_2, S_2 : H_2 \to H_2$ be uniformly continuous pseudomonotone mappings on bounded subset of H_1 and H_2, respectively.

A2: Let $\Omega := \{(p,q) \in H_1 \times H_2 : p \in F(T_1) \cap F(T_2), q \in F(S_1) \cap F(S_2)$ and $Ap = Bq\} \neq \emptyset$, where $A : H_1 \to H_3$ and $B : H_2 \to H_3$ are bounded linear mappings with adjoints A^* and B^*, respectively.

A3: Let $\iota \in (0,1), \mu > 0$ and $\delta \in [\underline{\delta}, \bar{\delta}] \subset (0, \frac{1}{\mu})$

A4: Let $\{\alpha_n\} \subset (0, \varepsilon)$ for some constant real number $\varepsilon > 0$ be a real sequence such that,

$$lim_{n\to\infty}\alpha_n = 0, \text{ and } \sum_{n=1}^{\infty} \alpha_n = \infty.$$

A5: Let $g_1 : H_1 \times H_1 \to H_1 \times H_1$ and $g_2 : H_2 \times H_2 \to H_2 \times H_2$ be contraction mappings with constants $\alpha_1, \alpha_2 \in (0, \frac{1}{\sqrt{2}})$, respectively and we denote $\alpha = \max\{\alpha_1, \alpha_2\}$.

A6 : Let the sequence γ_n satisfies

$$0 < \xi \leq \gamma_n \leq \frac{||Ax_n - Bt_n||^2}{||A^*(Ax_n - \mathscr{B}t_n)||^2 + ||B^*(Bt_n - Ax_n)||^2},\ for\ n \in \Upsilon$$

otherwise, $\gamma_n = \gamma > 0$, such that the indexes $\Upsilon = \{n \in \mathbb{N} : Ax_n - Bt_n \neq 0\}$.

Algorithm 1: For arbitrary $(x_0, t_0) \in H_1 \times H_2$, define an iterative algorithm by

Step 1. Compute

$$\begin{cases} z_{i,n} = (1-\delta)x_n + \delta T_i x_n \text{ and } d_i(x_n) = x_n - z_{i,n}, \text{ for } i = 1,2, \\ u_{i,n} = (1-\delta)x_n + \delta S_i x_n \text{ and } d_i(t_n) = t_n - u_{i,n}, \text{ for } i = 1,2. \end{cases} \tag{4.1}$$

Step 2. Compute

$$\begin{cases} y_{i,n} = x_n - \Upsilon_{i,n} d(x_n), \text{ for } i = 1,2. \\ v_{i,n} = t_n - \Upsilon'_{i,n} d(t_n), \text{ for } i = 1,2., \end{cases} \tag{4.2}$$

where, $\Upsilon_{i,n} = \iota^{j_{i,n}}$ such that $j_{i,n}$ is the smallest nonnegative integer j_i satisfying

$$\langle \iota^{j,i}(d_i(x_n)) + T_i(x_n - \iota^{j,i} d_i(x_n)) - T_i x_n, d`_i(x_n)\rangle \leq \mu \|d_i(x_n)\|^2,$$

and $\Upsilon'_n = \iota^{j'_{i,n}}$ such that $j'_{i,n}$ is the smallest nonnegative integer j'_i satisfying

$$\langle \iota^{j',i} d'_i(t_n) + S_i(t_n - \iota^{j',i} d'_i(t_n)) - S_i t_n, d'_i(t_n)\rangle \leq \mu \|d'_i(t_n)\|^2.$$

Step 3. Compute

$$\begin{cases} a_n = P_{C_{i,n}}(x_n - \gamma_n A^*(Ax_n - Bt_n), \\ b_n = P_{D_{i,n}}(t_n - \gamma_n B^*(Bt_n - Ax_n), \\ w_n = \theta_n a_n + \beta_n p_{1,n} + \eta_n p_{2,n}, \\ r_n = \theta_n b_n + \beta_n q_{1,n} + \eta_n q_{2,n} \end{cases} \tag{4.3}$$

where $C_{i,n} = \{x \in H : h_{i,n} = \langle y_{i,n} - T_i y_{i,n}, x - y_{i,n}\rangle \leq 0\}$, $D_{i,n} = \{t \in H : e_{i,n} = \langle v_{i,n} - S_i v_{i,n}, t - v_{i,n}\rangle \leq 0\}$ and $\{\theta_n\}, \{\beta_n\}, \{\eta_n\} \subset [\rho, 1)$ for $\rho > 0$ such that $\beta_n + \theta_n + \eta_n = 1$ for all $n \geq 0$ and $p_{i,n} = P_{C_{i,n}} x_n$, $q_{i,n} = P_{D_{i,n}} t_n$ for $i = 1,2$.

Step 4. Compute

$$\begin{cases} x_{n+1} = \alpha_n g_1(x_n) + (1-\alpha_n) w_n, \\ t_{n+1} = \alpha_n g_2(t_n) + (1-\alpha_n) r_n. \end{cases} \tag{4.4}$$

Step 5. Set $n := n+1$ and go to **Step 1**.

Remark 4.1 *Note that if we have $d(x_n) = d'(x_n) = 0$ for some $n \in N$, then we get, $x_n \in \Omega = F(T_i) \cap F(S_i)$, since we have $d(x_n) = x_n - z_n = 0$ implies that, $x_n = (1-\delta)x_n + \delta T_i x_n$ which gives us, $\delta(T_i x_n - x_n) = 0$, and hence $T_i x_n = x_n$. Similarly, $d'(t_n) - t_n - u_n = 0$, implies, $S_i t_n = t_n$. Thus, $(x_n, t_n) \in \Omega$. For the rest of the study we consider only the case that this equality does not hold.*

Lemma 4.0.1 *Suppose that the assumption $A_3 - A_2$ hold, and $\{x_n\}, \{y_n\}, \{z_n\}, \{u_n\}, \{v_n\}$ are sequences, generated by Algorithm 1.Then the search rules in step two are well defined.*

Proof: Since $\iota \in (0,1)$, T_i and S_i are uniformly continuous on H, We have

$$\langle \iota^j (d(x_n) + T_i(x_n - \iota^j d(x_n)) - T_i x_n, d(x_n) \rangle \to 0 \text{ as } j \to \infty,$$

and

$$\langle \iota^{j'} (d'(x_n) + S_i(x_n - \iota^{j'} d'(x_n)) - S_i x_n, d'(x_n) \rangle \to 0 \text{ as } j' \to \infty.$$

Moreover, since $\|d(x_n)\| > 0$ and $\|d'(x_n)\| > 0$ there exist a non-negative integers j_n and j'_n , satisfying the inequalities in Step 2. □

Lemma 4.0.2 *Suppose that the assumption $A_1 - A_3$ hold. If $\{x_n\}, \{y_n\}, \{z_n\}, \{u_n\}, \{v_n\}$ are sequences generated by Algorithm 1 then, $\langle x_n - T_i x_n, d(x_n) \rangle = \frac{1}{\delta}\|d(x_n)\|^2$ and*

$$\langle x_n - S_i x_n, d(x_n) \rangle = \frac{1}{\delta}\|d'(x_n)\|^2 \tag{4.5}$$

Proof: From equations (4.1), we have, $z_n = (1-\delta)x_n + \delta T_i x_n$ which gives us, $z_n - x_n = \delta(T_i x_n - x_n)$ and hence

$$\frac{z_n - x_n}{\delta} = T_i x_n - x_n \tag{4.6}$$

Thus, from the fact that $d(x_n) = x_n - z_n$, we get

$$\begin{aligned}
\langle x_n - T_i x_n, d(x_n) \rangle &= \langle \frac{x_n - z_n}{\delta}, x_n - x_n \rangle \\
&= \frac{1}{\delta}\langle x_n - z_n, x_n - z_n \rangle \\
&= \frac{1}{\delta}\|x_n - z_n\|^2
\end{aligned} \tag{4.7}$$

Similarly, we get

$$\langle t_n - S_i t_n, d'(t_n)\rangle = \frac{1}{\delta}\|t_n - u_n\|^2. \tag{4.8}$$

□

Lemma 4.0.3 *Suppose the assumptions $A_1 - A_3$ holds. Let $p \in \Omega$, let $h_n(x_n) = \langle y_n - T_i y_n, x_n - y_n\rangle$, and let $e_{n,i}(t_n) = \langle y = v_n - S_i v_n, t_n - v_n\rangle$. Then, $h_{n,i}(p) \leq 0, \quad e_{n,i}(p) \leq 0, \quad h_{n,i}(x_n) \geq \Upsilon_n(\frac{1}{\delta} - \mu)\|d(x_n)\|^2$, and $e_{n,i}(t_n) \geq \Upsilon_n'(\frac{1}{\delta} - \mu)\|d'(t_n)\|^2$ In particular, if $d(x_n) \neq 0$ and $d'(t_n) \neq 0$, then $h(x_n) > 0$ and $g(t_n) > 0$.*

Proof: For the fact that $(p,q) \in \Omega$, we have

$$\langle p - T_i p, y_n - p\rangle \geq 0. \tag{4.9}$$

This inequality and the fact that T_i is pseudo-pseudocontractive mapping, we obtain

$$h_{n,i}(p) = \langle y_n - T_i y_n, y_n - p\rangle \geq 0,$$

which gives us,

$$h_{n,i}(p) = \langle y_n - T_i y_n, p - y_n\rangle \leq .0$$

In addition, from Step 2, of Algorithm 1, we have,

$$\begin{aligned} h_{n,i}(x_n) &= \langle y_n - T_i y_n, x_n - y_n\rangle \\ &= \langle y_n - T_i y_n, x_n - (x_n - \Upsilon_n d(x_n)\rangle \\ &= \langle y_n - T_i y_n, d(x_n)\rangle \end{aligned}$$

Furthermore, from the inequalities in Step 2, we have,

$$\langle x_n - y_n + T_i y_n - T_i x_n, d(x_n)\rangle \leq \mu\|d(x_n)\|^2,$$

which implies

$$\langle y_n - T_i y_n, d(x_n)\rangle \geq \langle x_n - T_i y_n, d(x_n)\rangle - \mu\|d(x_n)\|^2 \tag{4.10}$$

From Lemma 4.0.2 and inequality above, we obtain

$$\langle y_n - T_i y_n, d(x_n)\rangle \geq (\frac{1}{\delta} - \mu)\|d(x_n)\|^2 \tag{4.11}$$

By combining (4.10) and (4.11), we obtain,

$$h_{n,i}(x_n) \geq \Upsilon_n(\frac{1}{\delta} - \mu)\|d(x_n)\|^2.$$

Similarly, we obtain,

$$e_{n,i}(t_n) \geq \Upsilon'_n(\frac{1}{\delta} - \mu)\|d'(t_n)\|^2.$$

□

Theorem 4.0.4 *Suppose the assumptions $A_1 - A_4$ hold. Then, the sequence $\{(x_n, t_n)\}$, generated by the Algorithm 1 is bounded in Hilbert space, $C \times D$.*

Proof: Now, from Lemma 2.1.4, we get

$$\begin{aligned}
\|w_n - p\|^2 &= \|\theta_n a_n + \beta_n p_{1,n} + \eta_n p_{2,n} - p\|^2 \\
&= \|\theta_n(a_n - p) + \beta_n(P_{C_{1,n}} x_n - p) + \beta_n(P_{C_{2,n}} x_n - p)\|^2 \\
&\leq \theta_n\|a_n - p\|^2 + \beta_n\|P_{C_{1,n}} x_n - p\|^2 + \eta_n\|P_{C_{2,n}} x_n - p\|^2 \\
&\leq \theta_n\|a_n - p\|^2 + \beta_n[\|x_n - p\|^2 - ||P_{C_{1,n}} x_n - x_n||^2] \\
&\quad + \eta_n[\|x_n - p\|^2 - ||P_{C_{2,n}} x_n - x_n||^2].
\end{aligned} \tag{4.12}$$

Similarly, we obtain

$$\begin{aligned}
\|r_n - q\|^2 &\leq \theta_n\|b_n - p\|^2 + \beta_n[\|t_n - q\|^2 - ||P_{D_{1,n}} t_n - t_n||^2] \\
&\quad + \eta_n[\|t_n - q\|^2 - ||P_{D_{2,n}} t_n - t_n||^2].
\end{aligned} \tag{4.13}$$

Thus, by adding inequalities (4.12) and (4.13), we get

$$\begin{aligned}\|w_n - p\|^2 + \|r_n - q\|^2 \leq\ & \theta_n[\|a_n - p\|^2 + \|b_n - p\|^2] \\ & +\beta_n[\|x_n - p\|^2 - ||P_{C_{1,n}}x_n - x_n||^2] \\ & +\eta_n[\|x_n - p\|^2 - ||P_{C_{2,n}}x_n - x_n||^2] \\ & +\beta_n[\|t_n - q\|^2 - ||P_{D_{1,n}}t_n - t_n||^2] \\ & +\eta_n[\|t_n - q\|^2 - ||P_{D_{2,n}}t_n - t_n||^2]. \qquad (4.14)\end{aligned}$$

In addition from (4.3), we obtain

$$\begin{aligned}\|a_n - p\|^2 &= ||P_C(x_n - \gamma_n A^*(Ax_n - Bt_n) - p||^2 \\ &\leq ||x_n - \gamma_n A^*(Ax_n - Bt_n||^2 - ||a_n - (x_n - \gamma_n A^*(Ax_n - Bt_n)||^2 \\ &\leq ||x_n - p||^2 + \gamma_n^2||A^*(Ax_n - Bt_n)||^2 - \gamma_n||Ax_n - Bt_n||^2 \\ & \quad -||x_n - a_n - \gamma_n A^*(Ax_n - Bt_n)||^2. \qquad (4.15)\end{aligned}$$

Similarly, we get

$$\begin{aligned}\|b_n - q\|^2 \leq\ & ||t_n - q||^2 + \gamma_n^2||B^*(Bt_n - Ax_n)||^2 - \gamma_n||Ax_n - Bt_n||^2 \\ & -||t_n - b_n - \gamma_n B^*(Bt_n - Ax_n)||^2. \qquad (4.16)\end{aligned}$$

By adding inequalities (4.15) and (4.16), we get

$$\begin{aligned}\|a_n - p\|^2 + \|b_n - q\|^2 \leq\ & ||x_n - p||^2 + ||t_n - q||^2 \\ & +\gamma_n^2[||A^*(Ax_n - Bt_n)||^2 + ||B^*(Bt_n - Ax_n)||^2] \\ & -2\gamma_n||Ax_n - Bt_n||^2 - ||x_n - a_n - \gamma_n A^*(Ax_n - Bt_n)||^2 \\ & -||t_n - b_n - \gamma_n B^*(Bt_n - Ax_n)||^2. \qquad (4.17)\end{aligned}$$

Moreover, (4.17) and (A6), we obtain

$$\begin{aligned}\|a_n - p\|^2 + \|b_n - q\|^2 \leq\ & ||x_n - p||^2 + ||t_n - q||^2 - \gamma_n||Ax_n - Bt_n||^2 \\ & -||x_n - a_n - \gamma_n A^*(Ax_n - Bt_n)||^2 \\ & -||t_n - b_n - \gamma_n B^*(Bt_n - Ax_n)||^2. \qquad (4.18)\end{aligned}$$

Now, by substituting (4.18) in (4.14), we get

$$\begin{aligned}
\|w_n-p\|^2+\|r_n-q\|^2 \leq\ & \|x_n-p\|^2+\|t_n-p\|^2 \\
& -\beta_n[||P_{C_{1,n}}x_n-x_n||^2+||P_{D_{1,n}}t_n-t_n||^2] \\
& -\eta_n[||P_{C_{2,n}}x_n-x_n||^2+||P_{D_{2,n}}t_n-t_n||^2] \\
& -\theta_n\gamma_n||Ax_n-Bt_n||^2-\theta_n||x_n-a_n-\gamma_nA^*(Ax_n-Bt_n)||^2 \\
& -\theta_n||t_n-b_n-\gamma_nB^*(Bt_n-Ax_n)||^2. \qquad (4.19)
\end{aligned}$$

From (4.19), Lemma 2.1.4 and (A5), we get

$$\begin{aligned}
||x_{n+1}-p||^2+||t_{n+1}-q||^2 =\ & ||\alpha_n g_1(x_n)+(1-\alpha_n)w_n-p||^2 \\
& +||\alpha_n g_2(t_n)+(1-\alpha_n)r_n-q||^2 \\
\leq\ & \alpha_n||g_1(x_n)-p||^2+(1-\alpha_n)||w_n-p||^2 \\
& +\alpha_n||g_2(t_n)-q||^2+(1-\alpha_n)|r_n-p||^2 \\
\leq\ & \alpha_n(||g_1(x_n)-g_1(p)||+||g_1(p)-p||)^2+(1-\alpha_n)||x_n-p||^2 \\
& +\alpha_n(||g_2(t_n)-g_2(q)||+||g_2(q)-q||)^2+(1-\alpha_n)||t_n-q||^2 \\
\leq\ & \alpha_n[\alpha^2||x_n-p||^2+||g_1(x_n)-p||^2]+(1-\alpha_n)||x_n-p||^2 \\
& +\alpha_n[\alpha^2||t_n-q||^2+||g_2(t_n)-q||^2]+(1-\alpha_n)||t_n-q||^2 \\
\leq\ & 2\alpha_n(\alpha^2||x_n-p||^2+||g_1(p)-p||^2)+(1-\alpha_n)||x_n-p||^2 \\
& +2\alpha_n(\alpha^2||t_n-q||^2+||g_2(q)-q||^2)+(1-\alpha_n)||t_n-q||^2 \qquad (4.20)
\end{aligned}$$

By setting $R_n(p,q)=||x_n-p||^2+||t_n-q||^2$, from inequality (4.20), we get

$$\begin{aligned}
R_{n+1}(p,q) \leq\ & (1-\alpha_n(1-2\alpha^2))R_n(p,q) \\
& +2\alpha_n(||g_1(p)-p||^2+||g_2(q)-q||^2) \\
\leq\ & \max\Big\{R_n(p,q),\frac{2}{1-2\alpha^2}(||g_1(p)-p||^2+||g_2(q)-q||^2)\Big\},
\end{aligned}$$

and hence by induction

$$R_n(p,q)\leq \max\Big\{R_0(p,q),\frac{2(||g_1(p)-p||^2+||g_2(q)-q||^2)}{1-2\alpha^2}\Big\},$$

which implies that $\{x_n\}$, $\{t_n\}$ and hence $\{y_{i,n}\}$, $\{v_{i,n}\}$, $\{T_ix_n\}$ and $\{S_iv_n\}$ for $i=1,2$

are bounded. $\square$

Theorem 4.0.5 *Suppose the assumption* $(A1)-(A6)$ *hold. Then, the sequence* $\{(x_n,t_n)\}$*, generated by A Algorithm 1 converges strongly to* $(p,q)=P_{\Omega}(g_1(p),g_2(q))$.

Proof: Now, let $(p,q)=P_{\Omega}(g_1(p),g_2(q))$. Then, from equation 2.7, for $i=1,2$, we have,

$$\|p-p_{i,n}\|^2 \le \|p-x_n\|^2 - \|x_n-p_{i,n}\|^2.$$

Similarly, we get

$$\|q-q_{i,n}\|^2 \le \|q-t_n\|^2 - \|t_n-q_{i,n}\|^2, \tag{4.21}$$

Since for $i=1,2$, $(I-T_i)$ is bounded on bounded subset of H_1 ,Then their exists $L_i>0$, such that $\|(I-T_i)y_n\| \le L_i$, for all $n \ge 0$ and $i=1,2$. Thus,

$$\begin{aligned}
|h_{i,n}(z)-h_{i,n}(w)| &= |\langle (I-T_i)y_{i,n}, z-y_{i,n}\rangle - \langle (I-T_i)y_{i,n}, w-y_{i,n}\rangle| \\
&= |\langle (I-T_i)y_{i,n}, z-w\rangle \\
&\le \|(I-T_i)y_{i,n}\|\|z-w\| \\
&\le L_i\|z-w\|,
\end{aligned}$$

which gives us that $h_{i,n}$ is L_i- Lipschitz continuous on H_1. Thus, from Lemma 2.1.6 and Lemma 4.0.3, we obtain

$$\|x_n-p_{i,n}\|^2 \ge \frac{h_{i,n}x_n}{2L_i^2} \ge \Upsilon_{i,n}^2(\frac{1}{\delta}-\mu)^2\|d_i(x_n)\|^4 \tag{4.22}$$

Thus, from (4.21) and (4.22), for $i=1,2$, we get

$$\|p-p_{i,n}\|^2 \le \|p-x_n\|^2 - \Upsilon_{i,n}^2(\frac{1}{\delta}-\mu)^2\|d_i(x_n)\|^4. \tag{4.23}$$

Similarly, for $i=12$, we get

$$\|q-q_{i,n}\|^2 \le \|q-t_n\|^2 - \Upsilon_{i,n}'^2(\frac{1}{\delta}-\mu)^2\|d_i(t_n)\|^4. \tag{4.24}$$

Now, from Lemma 2.1.4, ((4.23)), (4.19) and (4.15), we get

$$\begin{aligned}
\|w_n - p\|^2 + \|r_n - q\|^2 &= \|\theta_n a_n + \beta_n p_{1,n} + \eta_n p_{2,n} - p\|^2 \\
&\quad + \|\theta_n b_n + \beta_n q_{1,n} + \eta_n q_{2,n} - q\|^2 \\
&\leq \theta_n \|a_n - p\|^2 + \beta_n \|p_{1,n} - p\|^2 + \eta_n \|p_{2,n} - p\|^2 \\
&\quad + \theta_n \|b_n - q\|^2 + \beta_n \|q_{1,n} - q\|^2 + \eta_n \|q_{2,n} - q\|^2 \\
&\leq \theta_n \|x_n - p\|^2 + \beta_n \|x_n - p\|^2 \eta_n \|x_n - p\|^2 + \eta_n \|x_n - p\|^2 \\
&\quad + \theta_n \|t_n - q\|^2 + \beta_n \|t_n - q\|^2 \eta_n \|t_n - q\|^2 + \eta_n \|t_n - q\|^2 \\
&\quad - \Big(\Upsilon_{1,n}^2 (\frac{1}{\delta} - \mu)^2 \|d_1(x_n)\|^4 + \Upsilon_{2,n}^2 (\frac{1}{\delta} - \mu)^2 \|d_2(x_n)\|^4\Big) \\
&\quad - \Big(\Upsilon_{1,n}'^2 (\frac{1}{\delta} - \mu)^2 \|d_1(t_n)\|^4 + \Upsilon_{2,n}'^2 (\frac{1}{\delta} - \mu)^2 \|d_2(t_n)\|^4\Big) \\
&\leq \|x_n - p\|^2 + \|t_n - q\|^2 \\
&\quad - \Big(\Upsilon_{1,n}^2 (\frac{1}{\delta} - \mu)^2 \|d_1(x_n)\|^4 + \Upsilon_{2,n}^2 (\frac{1}{\delta} - \mu)^2 \|d_2(x_n)\|^4\Big) \\
&\quad - \Big(\Upsilon_{1,n}'^2 (\frac{1}{\delta} - \mu)^2 \|d_1(t_n)\|^4 + \Upsilon_{2,n}'^2 (\frac{1}{\delta} - \mu)^2 \|d_2(t_n)\|^4\Big) \qquad (4.25)
\end{aligned}$$

By Lemma 2.1.1, Lemma 2.1.4 and (4.25), we obtain

$$\begin{aligned}
R_{n+1}(p,q) &= \|\alpha_n g_1(x_n) + (1-\alpha_n) w(n) - p\|^2 + \|\alpha_n g_2(t_n) + (1-\alpha_n) r(n) - q\|^2 \\
&\leq \|\alpha_n (g_1(x_n) - g_1(p)) + (1-\alpha_n)(w_n - p) + \alpha_n (g_1(p) - p)\|^2 \\
&\quad + \|\alpha_n (g_2(t_n) - g_2(q)) + (1-\alpha_n)(r_n - q) + \alpha_n (g_2(q) - q)\|^2 \\
&\quad + 2\alpha_n [\langle g_1(p) - p, x_{n+1} - p\rangle + \langle g_2(q) - q, t_{n+1} - q\rangle] \\
&\leq \alpha\alpha_n \|x_n - p\|^2 + (1-\alpha_n)\|w_n - p\|^2 + \alpha\alpha_n \|t_n - q\|^2 + (1-\alpha_n)\|r_n - q\|^2 \\
&\quad + 2\alpha_n \|g_1(x_n) - p\| \|x_{n+1} - x_n\| + 2\alpha_n \|g_2(t_n) - q\| \|t_{n+1} - t_n\| \\
&\quad + 2\alpha_n [\langle g_1(p) - p, x_n - p\rangle + \langle g_2(q) - q, t_n - q\rangle] \\
&\leq \big(1 - (1-\alpha)\alpha_n\big) R_n(p,q) \\
&\quad + 2\alpha_n \|g_1(x_n) - p\| \|x_{n+1} - x_n\| + 2\alpha_n \|g_2(t_n) - q\| \|t_{n+1} - t_n\| \\
&\quad + 2\alpha_n [\langle g_1(p) - p, x_n - p\rangle + \langle g_2(q) - q, t_n - q\rangle] \\
&\quad - (1-\alpha_n)\Big(\Upsilon_{1,n}^2 (\frac{1}{\delta} - \mu)^2 \|d_1(x_n)\|^4 + \Upsilon_{2,n}^2 (\frac{1}{\delta} - \mu)^2 \|d_2(x_n)\|^4\Big) \\
&\quad - (1-\alpha_n)\Big(\Upsilon_{1,n}'^2 (\frac{1}{\delta} - \mu)^2 \|d_1(t_n)\|^4 + \Upsilon_{2,n}'^2 (\frac{1}{\delta} - \mu)^2 \|d_2(t_n)\|^4\Big),
\end{aligned}$$

which gives us

$$
\begin{aligned}
&(1-\alpha_n)\Big(\Upsilon_{1,n}^2(\frac{1}{\delta}-\mu)^2\|d_1(x_n)\|^4+\Upsilon_{2,n}^2(\frac{1}{\delta}-\mu)^2\|d_2(x_n)\|^4\Big)\\
&+(1-\alpha_n)\Big(\Upsilon_{1,n}'^2(\frac{1}{\delta}-\mu)^2\|d_1(t_n)\|^4+\Upsilon_{2,n}'^2(\frac{1}{\delta}-\mu)^2\|d_2(t_n)\|^4\Big)\\
&\leq R_n(p,q)-R_{n+1}(p,q)\\
&+2\alpha_n\|g_1(x_n)-p\|\|x_{n+1}-x_n\|+2\alpha_n\|g_2(t_n)-q\|\|t_{n+1}-t_n\|\\
&+2\alpha_n[\langle g_1(p)-p,x_n-p\rangle+\langle g_2(q)-q,t_n-q\rangle]. \qquad (4.26)
\end{aligned}
$$

In addition, from (4.19), we get

$$
\begin{aligned}
R_{n+1}(p,q) \quad \leq \quad & R_n(p,q)+2\alpha_n[\langle g_1(p)-p,x_{n+1}-p\rangle+\langle g_2(q)-q,t_{n+1}-q\rangle]\\
&-\beta_n[||P_{C_{1,n}}x_n-x_n||^2+||P_{D_{1,n}}t_n-t_n||^2]\\
&-\eta_n[||P_{C_{2,n}}x_n-x_n||^2+||P_{D_{2,n}}t_n-t_n||^2]\\
&-\theta_n\gamma_n||Ax_n-Bt_n||^2-\theta_n||x_n-a_n-\gamma_nA^*(Ax_n-Bt_n)||^2\\
&-\theta_n||t_n-b_n-\gamma_nB^*(Bt_n-Ax_n)||^2. \qquad (4.27)
\end{aligned}
$$

Next, we that the sequence $\{R_n(p,q)\}$ converges strongly to zero. For this we consider two cases as follows:

Case 1: Assume that there exist $n_0 \in N$, such that the sequence of real numbers $\{R_n(p,q)\}$ is decreasing for all $n \geq n_0$. Thus, the sequence $\{R_n(p,q)\}$ convergent and hence from (4.27) and the fact that $\alpha_n \to 0$, we obtain

$$\lim_{n\to\infty}||P_{C_{1,n}}x_n-x_n||^2=\lim_{n\to\infty}||P_{C_{2,n}}x_n-x_n||^2=0,$$

$$\lim_{n\to\infty}||P_{D_{1,n}}t_n-t_n||=\lim_{n\to\infty}||P_{D_{2,n}}t_n-t_n||=0,$$

$$\lim_{n\to\infty}||Ax_n-Bt_n||=0,$$

and

$$\lim_{n\to\infty}||x_n-a_n-\gamma_nA^*(Ax_n-Bt_n)||=\lim_{n\to\infty}||t_n-b_n-\gamma_nB^*(Bt_n-Ax_n)||=0,$$

which implies that

$$\lim_{n\to\infty}||x_n-a_n||\leq||x_n-a_n-\gamma_nA^*(Ax_n-Bt_n)||^2+||\gamma_nA^*(Ax_n-Bt_n)||=0, \quad (4.28)$$

and

$$\lim_{n\to\infty} ||t_n - b_n|| \le ||t_n - b_n - \gamma_n B^*(Bt_n - Ax_n)||^2 + ||\gamma_n B^*(Bt_n - Ax_n)|| = 0, \quad (4.29)$$

In addition, from (4.26), we have

$$\lim_{n\to\infty} \Upsilon_{1,n}^2 \|d_1(x_n)\|^4 = \lim_{n\to\infty} \Upsilon_{i,n}^2 \|d_2(x_n)\|^4 = 0,$$

and

$$\lim_{n\to\infty} \Upsilon_{1,n}'^2 \|d_1(t_n)\|^4 = \lim_{n\to\infty} \Upsilon_{2,n}'^2 \|d_2(t_n)\|^4 = 0.$$

Then, from this we obtain that

$$\lim_{n\to\infty} \Upsilon_{1,n} \|d_1(x_n)\|^2 = \lim_{n\to\infty} \Upsilon_{2,n} \|d_2(x_n)\|^2 = 0, \quad (4.30)$$

and

$$\lim_{n\to\infty} \Upsilon_{1,n}' \|d_1(t_n)\|^2 = \lim_{n\to\infty} \Upsilon_{2,n}' \|d_2(t_n)\|^2 = 0. \quad (4.31)$$

Since the sequence $\{(x_n, t_n)\}$ is bounded, there exists a subsequence $\{(x_{n_k}, (x_{n_k})\}$, of $\{(x_n, t_n)\}$ which converges weakly to $(\overline{p}, \overline{q}) \in H_1 \times H_2$ and

$$\limsup_{n\to\infty} [\langle g_1(p) - p, x_n - p\rangle + \langle g_2(q) - p, t_n - q\rangle] \quad (4.32)$$
$$= \lim_{k\to\infty} [\langle g(p) - p, x_{n_k} - p\rangle + g_2(q) - q, t_{n_k} - q\rangle]. \quad (4.33)$$

Now, we prove that for $i = 1, 2$

$$\lim_{k\to\infty} \|x_{n_k} - z_{i,n_k}\| = 0, \lim_{k\to\infty} \|t_{n_k} - u_{i,n_k}\| = 0 \quad (4.34)$$

First consider the case, when $\liminf_{k\to\infty} \Upsilon_{n_k} > 0$ In this case there is $\Upsilon > 0$ such that $\Upsilon_{n_k} > \Upsilon > 0$, for all $k \in N$. Thus, we have

$$\|x_{n_k} - z_{n_k}\|^2 = \frac{1}{\Upsilon_{n_k}} \Upsilon_{n_k} \|x_{n_k} - z_{n_k}\|^2 \le \frac{1}{\Upsilon} \Upsilon_{n_k} \|x_{n_k} - z_{n_k}\|^2.$$

From this inequality and ((4.31)), we obtain

$$\lim_{k\to\infty} \|x_{n_k} - z_{n_k}\|^2 = 0$$

and hence

$$\lim_{k\to\infty} \|x_{n_k} - z_{n_k}\|$$

Second consider, when $\lim inf_{k\to\infty}\Upsilon_{i,n_k} = 0$. In this case

$$\lim_{k\to\infty} \Upsilon_{i,n_k} = 0 \text{ and } \lim_{k\to\infty} \|x_{n_k} - z_{i,n_k}\|^2 = c > 0 \tag{4.35}$$

Consider, $y'_{i,n_k} = \frac{1}{\iota}\Upsilon_{i,n_k} z_{i,n_k} + (1 - \frac{1}{\iota}\Upsilon_{i,n_k} x_{i,n_k}$
Thus, from (4.35), we have

$$\lim_{k\to\infty} \|y'_{i,n_k} - z_{i,n_k}\| = \lim_{k\to\infty} \frac{1}{\iota}\Upsilon_{i,n_k}\|.x_{n_k} - z_{i,n_k}\| = 0 \tag{4.36}$$

From inequality in Step 2 and definition of y'_{i,n_k}, we obtain

$$\begin{aligned} \mu\|x_{n_k} - z_{i,n_k}\|^2 &< \langle x_{n_k} - y'_{i,n_k} + (I-T_i)y'_{i,n_k} - (I-T_i)x_{n_k}, x_{n_k} - z_{i,n_k}\rangle \\ &\leq \langle x_{n_k} - y'_{i,n_k}, x_{n_k} - z_{i,n_k}\rangle + \langle (I-T_i)y'_{i,n_k} - (I-T_i)x_{n_k}, x_{n_k} - z_{i,n_k}\rangle \\ &\leq \|x_{n_k} - y'_{i,n_k}\|\|x_{n_k} - z_{i,n_k}\| + \| + (I-T_i)y'_{i,n_k} - (I-T_i)x_{n_k}\|\|x_{n_k} - z_{i,n_k}\| \end{aligned} \tag{4.37}$$

From (4.36), (4.37) and the fact that $(I-T_i)$ is uniformly continuous, we get $\lim_{n\to\infty} \|x_{n_k} - z_{i,n_k}\| = 0$, which contradict (4.35). In a similar way we can show that $\lim_{n\to\infty} \|t_{n_k} - u_{i,n_k}\| = 0$ Thus, from this fact the equations (4.34) hold. Moreover, since $\{(x_{n_k}, (x_{n_k})\}$, which converges weakly to $(\overline{p}, \overline{q})$, then $x_{n_k} \rightharpoonup \overline{p}$ and $t_{n_k} \rightharpoonup \overline{q}$. Thus, from (4.35) we get $\overline{p} \in F(T_1) \cap F(T_2)$ and $\overline{q} \in F(S_1) \cap F(S_2)$.
Next we show that $A\overline{p} = B\overline{q}$. But, observe that from Lemma 2.1.1 (ii) we get

$$\begin{aligned} ||A\overline{p} - B\overline{q}||^2 &= ||A\overline{p} - Ax_{n_k} + Bt_{n_k} - B\overline{q} + Ax_{n_k} - Bt_{n_k}||^2 \\ &\leq ||Ax_{n_k} - Bt_{n_k}||^2 + 2\langle A\overline{p} - B\overline{q}, A\overline{p} - Ax_{n_k} + Bt_{n_k} - B\overline{q}\rangle, \\ &\to 0 \text{ as } k \to \infty, \end{aligned} \tag{4.38}$$

and this implies $A\overline{p} = B\overline{q}$. That is $(\overline{p}, \overline{q}) \in \Omega$. From the definition of x_{n+1} and t_{n+1}, we have $\|x_{n+1} - w_n\| = \alpha_n\|g_1(x_n) - w_n\| \to 0$, as $n \to \infty$, and $\|t_{n+1} - r_n\| = \alpha_n\|g_2(t_n) - r_n\| \to 0$, as $n \to \infty$, since $\alpha_n \to \infty$, as $n \to \infty$. From (4.26) and (4.28), we get

$$
\begin{aligned}
\|x_{n+1}-x_n\| &\leq \|x_{n+1}-w_n\|+\|w_n-x_n\| \\
&\leq \|x_{n+1}-w_n\|+\|\theta_n a_n+\beta_n p_{1,n}+\eta_n p_{2,n}-x_n\| \\
&\leq \|x_{n+1}-w_n\|+\theta_n\|a_n-x_n\| \qquad (4.39)\\
&\quad +\beta_n\|p_{1,n}-x_n\|+\eta_n\|p_{2,n}-x_n\| \to 0 \text{ as } n\to 0. \qquad (4.40)
\end{aligned}
$$

Moreover,

$$
\|x_n-w_n\| \leq \theta_n\|a_n-x_n\|+\beta_n\|p_{1,n}-x_n\|+\eta_n\|p_{2,n}-x_n\| \to 0, \text{ as } n\to 0. \qquad (4.41)
$$

Thus, from (4.39) and (4.41), we obtain

$$
\begin{aligned}
\|x_{n+1}-x_n\| &= \|x_{n+1}-w_n+w_n-x_n\| \\
&\leq \|x_{n+1}-w_n\|+\|w_n-x_n\| \to 0, \text{ as } n\to 0. \qquad (4.42)
\end{aligned}
$$

Similarly we can show that

$$
\|t_{n+1}-t_n\| \to 0, \text{ as } n\to 0. \qquad (4.43)
$$

From (4.32) and Lemma 2.1.3, we have

$$
\begin{aligned}
\limsup_{n\to\infty}[\langle g_1(p)-p,x_n-p\rangle+\langle g_2(q)-q,t_n-q\rangle] &\leq \lim_{k\to\infty}[\langle g_1(p)-q,x_{n_k}-p\rangle \\
&\quad +\langle g_2(q)-q,t_{n_k}-q\rangle] \\
&= \langle g_1(p)-p,\overline{p}-p\rangle+\langle g_2(q)-q,\overline{q}-q\rangle \\
&\leq 0. \qquad (4.44)
\end{aligned}
$$

Now, we show that the sequence $\{R_n(p,q)\}$ converges strongly to 0. Indeed, from Lemma 2.1.1 and 4.25, we obtain

$$
\begin{aligned}
R_{n+1}(p,q) &\leq (1-(1-\alpha)2\alpha_n)R_n(p,q) \\
&\quad +\alpha_n(1-\alpha)\frac{2}{1-\alpha}[\langle g_1(p)-p,x_{n+1}-p\rangle+\langle g_2(q)-q,t_{n+1}-q\rangle] \qquad (4.45)
\end{aligned}
$$

Finally, from (4.45), (4.42), (4.44) and Lemma 2.1.2, we get $R_n(p,q)\to 0$, as $n\to\infty$

and hence $x_n \to p$ and $t_n \to q$ as $n \to \infty$.

Case 2: Suppose that there exists a subsequence $\{R_{n_j}(p,q)\}$ of $\{R_n(p,q)\}$ such that

$$R_{n_j}(p,q) < R_{n_j+1}(p,q), \text{ for } j \geq 0. \tag{4.46}$$

Thus by Lemma 2.1.5 there exists a non-decreasing sequence $\{m_k\}$,of the set of positive integer of numbers such that $m_k \to 0$, as $k \to \infty$,
$\|x_{m_k} - p\|^2 \leq \|x_{m_k+1} - p\|^2$ and

$$\max\{R_{m_k}(p,q), R_k(p,q)\} \leq R_{m_k+1}(p,q) \text{ for all } k \geq 1. \tag{4.47}$$

Following the method of Case 1, we obtain

$$\lim_{k\to\infty} ||P_{C_{1,m_k}} x_{m_k} - x_{m_k}||^2 = \lim_{k\to\infty} ||P_{C_{2,m_k}} x_{m_k} - x_{m_k}||^2 = 0,$$

$$\lim_{k\to\infty} ||P_{D_{1,m_k}} t_{m_k} - t_{m_k}|| = \lim_{k\to\infty} ||P_{D_{2,m_k}} x_{m_k} - t_{m_k}|| = 0,$$

$$\lim_{k\to\infty} ||Ax_{m_k} - Bt_{m_k}|| = 0,$$

and

$$\lim_{k\to\infty} ||x_{m_k} - a_{m_k} - \gamma_{m_k} A^*(Ax_{m_k} - Bt_{m_k})|| = \lim_{k\to\infty} ||t_{m_k} - b_{m_k} - \gamma_{m_k} B^*(Bt_{m_k} - Ax_{m_k})|| = 0,$$

In addition, by following the method of Case 1, from the inequality (4.32), for $i = 1,2$, we obtain

$$\lim_{k\to\infty} \|x_{m_k} - p_{i,m_k}\| = \lim_{k\to\infty} \|t_{n_k} - q_{i,m_k}\| = 0.$$

In addition, for $i = 1,2$

$$\lim_{k\to\infty} \|x_{m_k} - z_{i,m_k}\| = 0 = \lim_{k\to\infty} \|t_{m_k} - u_{i,m_k}\| = 0,$$

$$\lim_{k\to\infty} \|x_{m_k} - x_{m_k+1}\| = 0 = \lim_{k\to\infty} \|t_{m_k} - t_{m_k+1}\| = 0,$$

and

$$\limsup_{k\to\infty} [\langle g_1(p) - p, x_{m_k+1} - p\rangle + \langle g_2(q) - q, t_{m_k+1} - q\rangle] \leq 0. \tag{4.48}$$

Now, from (4.45), we get

$$\begin{aligned} R_{m_k+1}(p,q) &\leq (1-(1-\alpha)\alpha_{m_k})R_{n_k}(p,q) \\ &\quad +\alpha_{m_k}(1-\alpha)\frac{2}{1-\alpha}[\langle g_1(p)-p, x_{m_k+1}-p\rangle \\ &\quad +\langle g_2(q)-q, t_{m_k+1}-q\rangle] \\ &\leq (1-(1-\alpha)2\alpha_n)R_{m_k+1}(p,q) \\ &\quad +\alpha_{m_k}(1-\alpha)\frac{2}{1-\alpha}[\langle g_1(p)-p, x_{m_k+1}-p\rangle \\ &\quad +\langle g_2(q)-q, t_{m_k+1}-q\rangle] \end{aligned} \tag{4.49}$$

which implies that

$$\begin{aligned} \alpha_{m_k}(1-\alpha)R_{m_k+1}(p,q) &\leq \alpha_{m_k}(1-\alpha)\frac{2}{1-\alpha}[\langle g_1(p)-p, x_{m_k+1}-p\rangle \\ &\quad +\langle g_2(q)-q, t_{m_k+1}-q\rangle]. \end{aligned} \tag{4.50}$$

Thus, from (4.46) and (4.50), we have

$$\begin{aligned} R_k(p,q) \leq R_{m_k+1}(p,q) &\leq \frac{2}{1-\alpha}[\langle g_1(p)-p, x_{m_k+1}-p\rangle \\ &\quad +\langle g_2(q)-q, t_{m_k+1}-q\rangle]. \end{aligned} \tag{4.51}$$

Hence using (4.48), we get

$$\begin{aligned} \limsup_{k\to\infty} R_k(p,q) &\leq \limsup_{k\to\infty} \frac{2}{1-\alpha}[\langle g_1(p)-p, x_{m_k+1}-p\rangle \\ &\quad +\langle g_2(q)-q, t_{m_k+1}-q\rangle] \leq 0, \end{aligned} \tag{4.52}$$

which implies

$$\limsup_{k\to\infty} R_k(p,q) = 0, \tag{4.53}$$

and hence $x_k \to p$ and $t_k \to q$ as, $k \to \infty$. $\square$

4.1 Application

In this section we present some applications of Theorem 4.0.5 to show that it is applicable.

4.1.1 Split Equality Fixed Point Problem

If in Algorithm 1, we assume $B = I_2$. Thus, split equality fixed point problem reduces to split equality fixed point problem and the method of proof of Theorem 4.0.5 provides the following corollary for approximating a solution of split equality fixed point problem for pseudo-pseudocontractivemappings in Hilbert spaces.

Corollary 4.1.1 *Suppose Assumption $(A3)-(A_6)$ hold. Let $T_i : H_1 \to H_1$ and $S_i : H_2 \to H_2$ be uniformly continuous pseudo-pseudocontractive mappings on bounded subset of H_1 and H_2, respectively for $i = 1,2,...,m$ such that $\Omega := \{(p,q) \in H_1 \times H_2 : p \in \cap_i^m F(T_i), q \in \cap_i^m F(S_i)$ and $Ap = q\} \neq \emptyset$. Then, the sequence $\{(x_n,t_n)\}$ generated by Algorithm 1 converges strongly to an element $(p,q) = P_\Omega(g_1(p), g_2(q))$.*

Chapter 5

Conclusion and Future scope

5.1 Conclusion

In this thesis, we investigated an iterative algorithm for solving the split equality fixed point problem of a finite family of pseudo-pseudo-contractive mappings in Hilbert spaces.
In addition, we also proved a strong convergence of a sequence generated by the proposed algorithm to solving the split equality fixed point problem of a finite family of pseudo-pseudo-contractive mappings in Hilbert spaces. Our result generalizes many results in the literature.

5.2 Future Scope

In this thesis we obtained results on solving the split equality fixed point problem of a finite family of pseudo-pseudo-contractive mappings in Hilbert spaces However, extending this result to a Banach spaces more general Hilbert spaces is an open problem. So, any interested researchers can use this opportunity to conduct their research work in this area.

References

Baillon, J. B., Haddad, G. (1977). Quelques proprits des oprateurs angle-borns et n-cycliquement monotones. Israel Journal of Mathematics, 26, 137-150. 15.

Boikanyo, O. A., Zegeye, H. (2020). The split equality fixed point problem for quasi-pseudo-contractive mappings without prior knowledge of norms. Numerical Functional Analysis and Optimization, 41(7), 759-777.

Boikanyo, O. A., Zegeye, H. (2020). The split equality fixed point problem for quasi-pseudo-contractive mappings without prior knowledge of norms. Numerical Functional Analysis and Optimization, 41(7), 759-777.

Censor, Y.,Bortfeld, T., Martin, B., Trofimov, A. (2006). A unified approach for inversion problems in intensity-modulated radiation therapy. Physics in Medicine Biology, 51(10), 2353.

Censor, Y., Elfving, T. (1994). A multiprojection algorithm using Bregman projections in a product space. Numerical Algorithms, 8, 221-239.

Chang, S. S., Wang, L., Qin, L. J. (2015). Split equality fixed point problem for quasi-pseudo-contractive mappings with applications. Fixed Point Theory and Applications, 2015(1), 1-12.

Chidume, C. E. (2014). Strong convergence and stability of Picard iteration sequences for a general class of contractive-type mappings. Fixed Point Theory and Applications, 2014(1), 1-10.

Chidume, C. E., Nnakwe, M. O. (2019). Iterative algorithms for split variational inequalities and generalized split feasibility problems with applications. J. Nonlinear Var. Anal, 3(2), 127-140.

Chidume, C., Ndambomve, P., Bello, A. (2015). The split equality fixed point problem for demi-contractive mappings. Journal of Nonlinear Analysis and Optimization: Theory Applications, 6(1), 61-69.

Journal of Computational Mathematics and Data Science, 5, 100066. Journal of mathematical analysis and applications, 241(1), 46-55.

Kwelegano, K. M., Zegeye, H., Boikanyo, O. A. (2022). An Iterative method for split equality variational inequality problems for non-Lipschitz pseudo monotone mappings. Rendiconti del Circolo Matematico di Palermo Series 2, 71(1), 325-348.

M. Osilike, D.I. Igbokwe, (2000). Weak and strong convergence theorems for fixed points of pseudo contractions and solutions of monotone type operator equations. Comput. Math. Appl. 40, 559567 (2000).

Moudafi, A. (2014). Alternating CQ-algorithm for convex feasibility and split fixed-point problems. J. Nonlinear Convex Anal, 15(4), 809-818.

Moudafi, A., Al-Shemas, E. (2013). Simultaneous iterative methods for split equality problem. Trans. Math. Program. Appl, 1(2), 1-11.

Rockafellar, R. (1970). On the maximal monotonicity of subdifferential mappings. Pacific Journal of Mathematics, 33(1), 209-216.

Takele, M. H., Reddy, B. K. (2019). Convergence theorems for common fixed point of the family of non self and non expansive mappings in real Banach spaces. Applications and Applied Mathematics: An International Journal (AAM), 14(4), 13.

Tang, Y. C., Peng, J. G., Liu, L. W. (2011). Strong convergence theorem for pseudo-contractive mappings in Hilbert spaces. Nonlinear Analysis: Theory, Methods Applications, 74(2), 380-385.

Wega, G. B. (2022). An iterative algorithm for split equality g fixed point problem. Journal of Computational Mathematics and Data Science, 5, 100066.

Wega, G. B. (2022). An iterative algorithm for split equality g fixed point problem. Wega, G. B., Zegeye, H. (2020). Convergence results of forward backward method for a zero of the sums of maximally monotone mappings in Banach spaces. Computational and Applied Mathematics, 39, 1-16.

Zhao, J. (2015). Solving split equality fixed-point problem of quasi-non expansive mappings without prior knowledge of operator's norms. Optimization, 64(12), 2619- 2630.