Finite Difference Method on Piecewise Mesh for Singularly Perturbed Reaction Diffusion Equation with Robin Type Boundary Conditions



A Research Submitted to the Department of Mathematics, Jimma University in Partial Fulfillment for the Requirements of the Degree of Masters of Science in Mathematics (Numerical Analysis)

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Jimma, Ethiopia

Declaration

I undersigned declare that, this research entitled "**Finite Difference Method on Piecewise Mesh for Singularly Perturbed Reaction Diffusion Equation with Robin Type Boundary Conditions"** is my own original work and it has not been submitted for the award of any academic degree or the like in any other institution or university, and that all the sources I have used or quoted have been indicated and acknowledged as complete references.

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Abstract

In this study, we develop a finite difference approach on piecewise mesh to solve singularly perturbed reaction-diffusion equations with Robin boundary conditions. Two numerical examples are provided to demonstrate the method's applicability. The results confirm that the maximum absolute error and convergence order validate the method's accuracy. Moreover, the proposed approach outperforms several existing methods in terms of solution precision. Numerical experiments show improved convergence with fewer errors, reinforcing the theoretical analysis. Consequently, the method produces accurate solution in approximating exact solutions.

Chapter One
Introduction

1.1. Background of the Study

Mathematical models use sets of equations to represent specific features or behaviors of physical systems. Many physical phenomena, modeled by differential equations with defined conditions, involve singularly perturbed differential equations where the highest derivative is multiplied by a small positive parameter, known as the perturbation parameter [1]. Solutions to these equations are critical in applied mathematics, frequently arising in fields such as fluid mechanics, elasticity, quantum mechanics, aerodynamics, and oceanography, among others [2, 3, 4, 5].

Outstanding to the importance of these problems in real life situations, the requisite to develop numerical methods for approximation its solution is valuable. Numerical methods are techniques that are used to approximate mathematical modeled problems. We need approximations since we either cannot solve analytically or the analytical method is intractable, [2]. Mostly, finite-difference methods are a class of numerical techniques for solving differential equations with approximating derivatives by finite difference approximations. This methods obtained by converting differential equations into a system of equation that can be solved by appropriate iterative techniques.

Singularly perturbed problems broadly categorized into reaction-diffusion and convection-diffusion types, accordingly the order of differential equation reduced by two or one when the perturbation parameter approaching to zero, [6, 7, 8, 9]. Also, the problems may be classified in different types regarding the type's defined conditions like initial or boundary (Dirichlet, Robin or mixed conditions). Particularly, the linear second order singularly perturbed reaction-diffusion problems are categorized into with Dirichlet or robin boundary conditions exhibiting twin (both left and right) boundary layers. Thus, in this work, the study will focus on numerical solution of the linear second order singularly perturbed reaction diffusion ordinary differential equation with robin type boundary conditions.

Numerical solutions for the singularly perturbed problems depend on perturbation parameter that causes the solution varies rapidly in some parts and slowly in some other parts of the domain, [7,

8]. Due to this, if one body tries to solve these types of problems using standard numerical methods, then less accurate solutions are obtained. As a result, it should be important to develop appropriate numerical methods for such problems, whose convergence does not affected by the perturbation parameter. Thus, it is necessary to formulate a uniformly convergent numerical method for robin type singularly perturbed reaction diffusion problems. There are several numerical methods presented for solving the different types of the singularly perturbation reaction-diffusion problems mentioned in the literature, [2, 5, 10, 11, 12, 13, 14]. Most authors in these literatures developed higher order finite difference methods for the singularly perturbation reaction-diffusion problems with Dirichlet boundary conditions.

Mainly, different numerical methods developed for solving the robin type singularly perturbed reaction diffusion problems, [15, 16, 17, 18]. Hence, several numerical approximation techniques were presented to solve singularly perturbed reaction diffusion differential equation with Robin-type boundary conditions. Further, the error estimates indicate that the proposed techniques are parameter-uniform second-order convergent and are numerically stable. Furthermore, numerical experiments performed and presented to validate the theoretical results. Henceforth, different finite difference methods have been adopted for solving the Robin type singularly perturbed reaction-diffusion problems. However, yet, the obtained accuracy of the numerical solution and the rate of convergence are takes attention to be work more. Therefore, the main objective of this study is to offer a finite difference method on piecewise mesh for solving singularly perturbed reaction diffusion equation with Robin type boundary conditions.

1.2 Statement of the Problem

Numerous numerical methods have been proposed for solving linear second-order singularly perturbed reaction-diffusion differential equations with Robin boundary conditions. From the existing literature, it is observed that the highest order of convergence achieved by these methods is typically of order two. However, due to the presence of a small positive parameter in the differential equation and the complexity of Robin boundary conditions, obtaining accurate solutions using traditional numerical methods remains challenging. This highlights the significance of exploring alternative numerical approaches to these problems.

Thus, developing an accurate finite difference method on a piecewise mesh is crucial for addressing these challenges. The study aims to investigate and answer the following key questions:

i. How can an accurate finite difference method on a piecewise mesh be formulated to solve singularly perturbed reaction-diffusion differential equations with Robin boundary conditions?
ii. How can the stability of the proposed scheme be rigorously analyzed?
iii. To what extent does the accuracy and rate of convergence of the proposed scheme offer advantages over existing methods?

1.3 Objectives of the Study

1.3.1 General objective

The general objective of this study is to develop an accurate finite difference method on piecewise mesh for solving the singularly perturbed reaction diffusion differential equation with Robin type boundary conditions.

1.3.1 Specific objectives

The specific objectives of the study are:

1. To formulate an accurate finite difference method on a piecewise mesh for solving singularly perturbed reaction-diffusion differential equation with Robin boundary conditions.
2. To conduct a stability analysis of the proposed numerical scheme.
3. To evaluate the accuracy obtained by the formulated scheme in comparison to some existing method.

1.4 Delimitation of the Study

This study is delimited to study a finite difference approximation on piecewise mesh for solving the singularly perturbed reaction diffusion differential equation of the form:

$$-\varepsilon y''(x) + b(x)y(x) = f(x), \quad 0 < x < 1, \tag{1.1}$$

subject to the Robin type boundary conditions:

$$\begin{cases} a_0 y(0) - a_1\sqrt{\varepsilon}\, y'(0) = \eta_1, \\ \\ d_0 y(1) + d_1\sqrt{\varepsilon}\, y'(1) = \eta_2, \end{cases} \tag{1.2}$$

where $0 < \varepsilon << 1$ is the singular perturbation parameter, and $a_0, a_1, d_0, d_1, \eta_0, \eta_1$ with $a_1, d_1 \neq 0$ are given constants. Assume that the functions $b(x)$ and $f(x)$ are sufficiently smooth such that $b_1 \geq b(x) \geq b_0, \ \forall x \in [0, 1]$. The constant terms in robin boundary conditions satisfy $a_j, d_j \geq 0$ and $a_j + d_j > 0, \ j = 0,1$ with $a_0 \neq a_1\sqrt{b(0)}$. Then, the problem in Eqn. (1.1) and (1.2) has a unique solution $y(x) \in C^2(0, 1) \bigcap C^1[0, 1]$ that typically exhibits exponential boundary layers at $x = 0$ and $x = 1$ when the perturbation parameter ε tends to zero, [15].

1.5 Significance of the Study

The outcomes of this study:

- To provide an accurate numerical solution for the addressed problem.
- To serve as a reference for researchers aiming to develop higher-order numerical methods for singularly perturbed reaction-diffusion equations with Robin boundary conditions.
- To assist graduate students in acquiring essential research skills and scientific methodologies.

Chapter Two

Review of Related Literature

2.1 Singularly Perturbed Problems

Singular perturbation problem was first introduced by Prandtl [19] during his report on fluid motion with small friction in a seven page report presented at the third international congress of mathematics in Heidelberg in 1904 in which he demonstrated that fluid flow past over a body. Prandtl [19] originally introduces the term 'boundary layer' but this term comes in to more general following the work of Wasow [20]. The study of many theoretical and applied problems in science and technology leads to boundary value problems for singularly perturbed differential equations. The second order differential equations in which its second order derivative term is multiplied by a small positive parameter (say ε) is called second-order singularly perturbed differential equations. These equations subject to the defined conditions (initial and or boundary or interval boundary) is known as second order singularly perturbed problems.

Kadalbajoo and Reddy [21] gave survey of various asymptotic and numerical methods developed from 1908−1986 for the determination of approximate solution of singular perturbation problems of various kinds. Kadalbajoo and Patidar extended the work done by Kadalbajoo and Reddy and surveyed the work done by various researchers in the area of singular perturbation from 1984 − 2000. In this work they considered one dimensional problem only and discussed the work done on linear, non-linear, semi-linear and quasi-linear singularly perturbed problems [22].

The area of singular perturbations is a field of increasing interest to applied mathematicians. During the last few years much progress has been made in the theory and in the computer implementation of the numerical treatment of singular perturbation problems. The existence of boundary layer theory in various fields of applied mathematics, for instance, fluid mechanics, fluid dynamics, quantum mechanics, plasticity, chemical-reactor theory, aerodynamics, plasma dynamics, rarefied-gas dynamics, meteorology, diffraction theory, reaction-diffusion processes and radiating flows and other domains of the great world of fluid motion [23].

Generally, second-order singularly perturbed problems broadly can be categorized in terms of linear or nonlinear, convection-diffusion or reaction-diffusion, Dirichlet or robin boundary

condition types, [7, 8]. Particularly, linear second order singularly perturbed reaction-diffusion problems have different types depending on boundary conditions (Dirichlet, robin or mixed). Hence, second order singularly perturbed reaction-diffusion boundary value problems exhibits boundary and/ or interior layers, [7, 8]. In this review section, we investigate the solution methods for the class of second-order singularly perturbed differential equations with robin boundary conditions.

2.2 Methods for singularly perturbed reaction-diffusion with Dirichlet boundary conditions

Also, in this subsection, we give a quick description of the developed numerical methods for the linear second-order singularly perturbed reaction-diffusion problem of the form:

$$\begin{cases} -\varepsilon y''(x) + b(x)y(x) = f(x), & 0 < x < 1, \\ y(0) = \alpha, \\ y(1) = \beta, \end{cases} \tag{2.1}$$

where, $0 < \varepsilon < 1$ is a singular perturbation parameter, α and β are given constant numbers. For the existence and uniqueness of the solution, the functions $b(x)$ and $f(x)$ are assumed to be sufficiently smooth and bounded $b(x) \geq \min_{\forall x \in [0,1]} b(x) = b_0 > 0$ for small positive constant b_0. Most methods like in [5, 11, 12, 13, 14] and others on reaction-diffusion problems presented to solve the problems in Eq. (2.1) were developed on uniform mesh and the methods are finite difference based. Principally, the higher order (fourth, sixth, eighth and tenth) order compact finite difference method is presented for solving linear singularly perturbed reaction–diffusion problems. These methods described by replacing the derivative in the differential equation by finite difference approximations that transformed to linear systems of algebraic equations in the form of a three-term recurrence relation, which can be solved using Thomas algorithm. Both the theoretical error bounds and the numerical rate of convergence have been established for the methods.

Further, most of the solution methodologies presented in [5, 11, 12, 13] to solve Eq. (2.1) were works for the constant coefficient reaction-term. Furthermore, the methods were gives accurate solution when the perturbation and uniform mesh-size parameters are nearly comparable. Hence, standard numerical methods give valuable accurate solution when mesh-size parameter is too smaller than perturbation parameter which is costly to solve algebraic equations. Thus, this

implies to think of alternative method to solve Eq. (2.1) for the variable coefficient of the reaction term and when the perturbation parameter is very small compared to non-uniform mesh sizes of the solution domain.

2.3 Methods for singularly perturbed reaction-diffusion equations with Robin type boundary conditions

In this subsection, we try to see the developed numerical methods for the linear second-order robin type singularly perturbed reaction-diffusion problem of the form:

$$\begin{cases} -\varepsilon y''(x) + b(x)y(x) = f(x), \quad 0 < x < 1, \\ \\ a_0 y(0) - a_1\sqrt{\varepsilon}\, y'(0) = \eta_1, \\ d_0 y(1) + d_1\sqrt{\varepsilon}\, y'(1) = \eta_2, \end{cases} \tag{2.2}$$

where $0 < \varepsilon << 1$ is the singular perturbation parameter, and $a_0, a_1, d_0, d_1, \eta_0, \eta_1$ with $a_1, d_1 \neq 0$ are given constants. Assume that the functions $b(x)$ and $f(x)$ are sufficiently smooth such that $b_1 \geq b(x) \geq b_0, \ \forall x \in [0, 1]$. The constant terms in Robin boundary conditions satisfy $a_j, d_j \geq 0$ and $a_j + d_j > 0, \ j = 0,1$ with $a_0 \neq d_0\sqrt{b_0}$, [15, 16]. Several numerical methods which involve both the cubic spline and classical finite difference schemes are presented to solve the problem in Eq. (2.2) in [15, 16, 17, 18]. The error estimates indicate that the provided methods are parameter-uniform second-order convergent and numerically stable. For each of the formulated methods, numerical experiments performed and presented to validate the theoretical results.

As observed from the more recent existing literatures in [15, 16, 17, 18] regarding the robin type singularly perturbed reaction diffusion problems described by the governing equation in Eq. (2.2) is solved by using various numerical methods like robust and the family of spline methods that were developed based on uniform and non-uniform or adaptive mesh discretization techniques. It is also observable from these literatures that the maximum rate or order of convergence of the developed methods is order two. However, yet, the obtained accuracy of the numerical solution and the rate of convergence are takes attention to be work more. Therefore, this implies takes attention to think of an alternative method to solve Eq. (2.2) which may give more accurately solution and higher order convergence.

Chapter Three
Research Methodology

3.1 Study Site and Period

The study conducted at Jimma University under college of natural science department of mathematics in numerical analysis stream from June - October 2024.

3.2 Study Design

The study employed both documentary review and experimental design on an accurate computational approach on piecewise mesh for solving singularly perturbed reaction-diffusion differential equation with Robin type boundary conditions.

3.3 Sources of Information

The relevant source of information for this study were journals, books, published articles, related articles from internets and the experimental results obtained by writing MATLAB code for the proposed numerical scheme.

3.4 Mathematical procedures

In order to achieve the stated specific objectives, the study followed the following procedures:

i. Defining the singularly perturbed reaction-diffusion equation with Robin type boundary conditions.
ii. Discretizing the solution domain or interval by defining non-uniform mesh.
iii. Formulating a finite difference scheme on the discretized mesh.
iv. Establishing the stability of the formulated scheme.
v. Writing MATLAB code for the formulated scheme.
vi. Provide numerical illustrations to validate the applicability scheme.

CHAPTER FOUR

DESCRIPTION OF THE METHOD, RESULT AND DISCUSSION

4.1 Description of the problem

In this section, we consider the Robin type singularly perturbed reaction diffusion problem of the form:

$$\begin{cases} Ly(x) \equiv -\varepsilon y''(x) + b(x)y(x) = f(x), & 0 < x < 1, \\ B_L y(0) \equiv a_0 y(0) - a_1\sqrt{\varepsilon}\, y'(0) = \eta_1, \\ B_R y(1) \equiv d_0 y(1) + d_1\sqrt{\varepsilon}\, y'(1) = \eta_2, \end{cases} \tag{4.1}$$

where $0 < \varepsilon << 1$ is the singular perturbation parameter, and $a_0, a_1, d_0, d_1, \eta_0, \eta_1$ with $a_1, d_1 \neq 0$ are given constants. The differential operators are denoted by L, B_L, B_R.

The functions $b(x)$ and the source $f(x)$ function are assume to be sufficiently smooth functions such that $b_1 \geq b(x) \geq b_0 > 0, \ \forall x \in [0, 1]$. Also, assume that the given constant values given in robin type boundary conditions satisfy $a_j, d_j \geq 0$ and $a_j + d_j > 0, \ j = 0,1$.Then, the problem in Eq. (4.1) has a unique solution $y(x) \in C^2(0, 1) \bigcap C^1[0, 1]$ that typically exhibits boundary layers both at $x = 0$ and $x = 1$ when the perturbation parameter sufficiently small. The existence and uniqueness of a continuous solution to the defined problem are thoroughly discussed in [15, 16]. In this work, our primary focus is on the accurate finite difference method applied to a piecewise mesh for modeling the singularly perturbed reaction-diffusion equation with Robin-type boundary conditions, as presented in Eq. (4.1).

4.2 Piecewise Mesh Generation of the solution domain

We construct a piecewise mesh that contains more number of nodal points in the layer regions than non-layer region. Based on the nature of the defined problem that involves both left and right boundary layers, the domain $[0,1]^N$, $N \geq 4$ is divided into three subintervals, $[0, \tau], [\tau, 1-\tau], [1-\tau, 1]$ where the chosen transition parameter,

$$\tau = \min\left\{\frac{1}{4},\ 2\sqrt{\varepsilon}\ln(N)\right\}, \tag{4.2}$$

denotes the width of the boundary layers, [7, 8]. The domain $[0,1]^N$ is obtained by putting a non-uniform mesh with $\frac{N}{4}$ mesh elements in both the layer intervals and a uniform mesh with $\frac{N}{2}$ mesh elements in the outer layer region. A general piecewise mesh

$$[0,1]^N = \{0 = x_0, x_1, x_2, ..., x_N = 1\}$$

with step size will be defined:

$$h_i = x_i - x_{i-1} = \begin{cases} \frac{4\tau}{N},\ i = 1, 2, ..., \frac{N}{4} \\ \frac{2(1-2\tau)}{N},\ i = \frac{N}{4}+1, ..., \frac{3N}{4}, \\ \frac{4\tau}{N},\ i = \frac{3N}{4}+1, ..., N. \end{cases} \tag{4.3}$$

4.3 Formulation of the Finite Difference Scheme on Piecewise Mesh

Now, in order to develop the finite difference approximation for the problems in Eq. (4.1), the interval $[0, 1]$ is divided into *N* sub-intervals. For this, we introduce the set of grid points from Eq. (4.3) as:

$$x_i = x_{i-1} + h_i, \quad \begin{cases} i = 1, 2, \ldots, N-1, \\ h_i = x_i - x_{i-1}, \\ 0 = x_0 < x_1 < \ldots < x_N = 1. \end{cases} \tag{4.4}$$

Let $y(x_i)$ be the exact solution to the problem in Eq. (4.1) , and y_i be an approximation to $y(x_i)$ at the nodal point x_i. Then, adopt that $y(x)$ has continuous higher order derivatives on $[0, 1]$, and then to find a description of the finite difference approximation, we use Taylor's series expansion in order to get the formula for y_i'' as:

$$y_{i+1} = y_i + h_{i+1}y_i' + \frac{h_{i+1}^2}{2!}y_i'' + \frac{h_{i+1}^3}{3!}y_i''' + \frac{h_{i+1}^4}{4!}y_i^{(4)} + \frac{h_{i+1}^5}{5!}y_i^{(5)} + \ldots \tag{4.5}$$

$$y_{i-1} = y_i - h_i y_i' + \frac{h_i^2}{2!}y_i'' - \frac{h_i^3}{3!}y_i''' + \frac{h_i^4}{4!}y_i^{(4)} - \frac{h_i^5}{5!}y_i^{(5)} + \ldots \tag{4.6}$$

Multiplying Eq. (4.5) by h_i and Eq. (4.6) by h_{i+1}, and then adding the two equation yields:

$$h_i y_{i+1} + h_{i+1}y_{i-1} = h_i y_i + h_{i+1}y_i + h_i h_{i+1}(\frac{h_{i+1}+h_i}{2})y_i'' + h_i h_{i+1}(\frac{h_{i+1}^2 - h_i^2}{6})y_i''' + h_i h_{i+1}(\frac{h_{i+1}^3 + h_i^3}{24})y_i^{(4)} + h_i h_{i+1}(\frac{h_{i+1}^4 - h_i^4}{120})y_i^{(5)} + \ldots$$

From this equation let solve for y_i'', and then re-arranged in the form:

$$y_i'' = \frac{2}{h_{i+1}+h_i}(\frac{y_{i+1}-y_i}{h_{i+1}} - \frac{y_i - y_{i-1}}{h_i}) + \tau_1, \tag{4.7}$$

with the considered truncation term is

$$\tau_1 = -(\frac{h_{i+1}-h_i}{3})y_i''' - (\frac{h_{i+1}^3 + h_i^3}{12(h_{i+1}+h_i)})y_i^{(4)} + \ldots$$

Now, substituting Eq. (4.7) into the discrete form of Eq. (4.1) gives:

$$\frac{-2\varepsilon}{h_{i+1}+h_i}(\frac{y_{i+1}-y_i}{h_{i+1}} - \frac{y_i - y_{i-1}}{h_i}) + b_i y_i + \tau_2 = f_i, \tag{4.8}$$

where $\tau_2 = -\varepsilon\tau_1$.

This can be rearranged in the three-term recurrence relations scheme of the form:

$$E_i y_{i-1} + F_i y_i + G_i y_{i+1} = H_i, \quad i = 1, 2, \ldots, N-1, \tag{4.9}$$

where, $E_i = \dfrac{-2\varepsilon}{h_i(h_{i+1}+h_i)}$, $F_i = \dfrac{2\varepsilon}{h_i h_{i+1}} + b_i$, $G_i = \dfrac{-2\varepsilon}{h_{i+1}(h_{i+1}+h_i)}$, and $H_i = f_i$.

4.4 Finite Difference Scheme for the Boundary Conditions

For $i = 0$, at $x_0 = 0$, we have the discretized form of the first boundary condition, $B_L y(0)$ as

$$a_0 y_0 - a_1 \sqrt{\varepsilon} y_0' = \eta_1 \tag{4.10}$$

In order to formulate the finite difference approximation to Eq. (4.10), adapting the Taylor series expansion,

$$y_{i+1} = y_i + h_{i+1} y_i' + \frac{h_{i+1}^2}{2} y_i'' + \frac{h_{i+1}^3}{3!} y_i''' + \frac{h_{i+1}^4}{4!} y_i^{(4)} + \ldots \tag{4.11}$$

Considering Eq. (4.11) at the nodal point x_0, to approximation the first derivative in boundary condition, we obtain

$$y_0' = \frac{y_1 - y_0}{h_1} - \frac{h_1}{2} y_0'' + TB_L, \tag{4.12}$$

where the local truncation error for the left boundary condition is

$$TB_L = -\frac{h_1^2}{3!} y_0''' - \frac{h_1^3}{4!} y_0^{(4)} + \ldots \ .$$

From the second order differential equation Eq. (4.1) that obtained at the nodal point x_0 as:

$$y_0'' = \frac{b_0 y_0 - f_0}{\varepsilon}. \tag{4.13}$$

Substituting Eq. (4.13) into Eq. (4.12) and rearranging yields

$$[a_0 + \frac{a_1\sqrt{\varepsilon}}{h_1} + \frac{a_1 h_1 b_0}{2\sqrt{\varepsilon}}]y_0 - \frac{a_1\sqrt{\varepsilon}}{h_1} y_1 = \eta_1 + \frac{a_1 h_1}{2\sqrt{\varepsilon}} f_0. \tag{4.14}$$

Similarly, for $i = N, \text{at } x_N$, we have the discretized form of the second boundary condition, $B_R y(1)$ as

$$d_0 y_N + d_1\sqrt{\varepsilon} y'_N = \eta_2 . \tag{4.15}$$

At any nodal point x_i, the Taylors' series expansion written as

$$y_{i-1} = y_i - h_i y'_i + \frac{h_i^2}{2} y''_i - \frac{h_i^3}{3!} y'''_i + \frac{h_i^4}{4!} y_i^{(4)} + ... \tag{4.16}$$

Then, adapting this expansion for the nodal point x_N, to approximation the first derivative in the second boundary condition, we get:

$$y'_N = \frac{y_N - y_{N-1}}{h_N} + \frac{h_N}{2} y''_N + TB_R, \tag{4.17}$$

where the local truncation error for the second boundary condition is

$$TB_R = -\frac{h_N^2}{3!} y'''_N + \frac{h_N^3}{4!} y_N^{(4)} + ... \ .$$

Also, substituting Eq. (4.17) into Eq. (4.15) and after rearranging, we get:

$$-\frac{d_1\sqrt{\varepsilon}}{h_N} y_{N-1} + [d_0 + \frac{d_1\sqrt{\varepsilon}}{h_N} + \frac{d_1 h_N b_N}{2\sqrt{\varepsilon}}]y_N = \eta_2 + \frac{d_1 h_N}{2\sqrt{\varepsilon}} f_N. \tag{4.18}$$

Thus, the formulated numerical scheme in the three Eqs. (4.9), (4.14) and (4.18) constitute the system of $(N+1)\times(N+1)$ linear algebraic equations, that give the approximations $y_0, y_1, ..., y_N$ of the solution $y(x)$ at the nodal points, $x_0, x_1, ..., x_N$, with piecewise mesh length h_i. This can be re-written in matrix form

$$MY = R, \tag{4.19}$$

where the three matrices are defined

$$M = (m_{ij})_{(N+1)\times(N+1)} = \begin{bmatrix} B_0 & C_0 & 0 & \cdots & 0 \\ A_1 & B_1 & C_1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & A_{N-1} & B_{N-1} & C_{N-1} \\ 0 & \cdots & 0 & A_N & B_N \end{bmatrix}, \quad Y = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{N-1} \\ y_N \end{bmatrix}, \quad R = \begin{bmatrix} H_0 \\ H_1 \\ \vdots \\ H_{N-1} \\ H_N \end{bmatrix},$$

with the specific entries are denoting:

For $i = 0$, $B_0 = a_0 + \dfrac{a_1\sqrt{\varepsilon}}{h_1} + \dfrac{a_1 h_1 b_0}{2\sqrt{\varepsilon}}$, $C_0 = -\dfrac{a_1\sqrt{\varepsilon}}{h_1}$, $H_0 = \eta_1 + \dfrac{a_1 h_1}{2\sqrt{\varepsilon}} f_0$.

For $i = 1, \ldots, N-1$: $A_i = E_i$, $B_i = F_i$, $C_i = G_i$, and $H_i = f_i$.

For $i = N$: $A_N = \dfrac{-d_1\sqrt{\varepsilon}}{h_N}$, $B_N = d_0 + \dfrac{d_1\sqrt{\varepsilon}}{h_N} + \dfrac{d_1 h_N b_N}{2\sqrt{\varepsilon}}$, $H_N = \eta_2 + \dfrac{d_1 h_N}{2\sqrt{\varepsilon}} f_N$.

4.5 Algorithm and its Stability

The obtained finite difference approximation yields an $(N+1)\times(N+1)$ algebraic tri-diagonal system that can be solved by Gaussian elimination method. Hence, in using this method, we have the algorithm:

$$\text{Forward Elimination} \rightarrow \begin{cases} W_0 = \dfrac{C_0}{B_0}, \\ T_0 = \dfrac{R_0}{B_0}, \\ W_i = \dfrac{C_i}{B_i - A_i W_{i-1}}, \quad i = 1, 2, \ldots, N+1, \\ T_i = \dfrac{R_i - A_i T_{i-1}}{B_i - A_i W_{i-1}}, \quad i = 1, 2, \ldots, N+1. \end{cases}$$

$$\text{Backward Elimination} \rightarrow \begin{cases} \text{Setting } C_N = 0, \\ y_{N+1} = T_{N+1}, \\ y_i = T_i - W_i y_{i+1}, \quad i = N,\ N-1,\ \ldots\ 2, 1. \end{cases}$$

Theorem 1: (Stability)

Let M be a coefficient matrix of the tri-diagonal system, in Eq. (4.31). Then, the matrix M is an irreducible and diagonally dominant matrix and hence the scheme is stable. [27]

Proof: To estimate the stability of the formulated method, we show that the coefficient matrix of the method satisfy the diagonal dominance. First, to check this, consider that $b(x) \geq b_0 > 0,\ \forall x \in [0, 1]$ and for $i = 1, \ldots, N-1$, from Eq. (4.16), we have:

$$B_i \geq A_i + C_i,\ \ (F_i \geq E_i + G_i), \quad i = 1, 2,\ \ldots, N-1,$$

Second, we consider the Robin boundary conditions at the two extremities $i = 0$ and $i = N$.

For $i = 0,\ B_0 > C_0$,

For $i = N$: $B_N > A_N$.

Thus, this implies that for each row of M, the sum of the two off-diagonal elements is less than or equal to the modulus of the diagonal element. Therefore, M is diagonally dominant. Hence, M is irreducible matrix. Henceforth, the Gaussian elimination algorithm is used for solving the formulated scheme in Eq. (4.19) is stable for the described scheme.

4.6 Truncation Error for the Formulated Scheme

While formulating the finite difference approximations in Eqs. (4.7), (4.12), and (4.17), we obtain the truncation terms:

$$TE = \begin{cases} \tau_1 = -(\frac{h_{i+1}-h_i}{3})y_i''' - (\frac{h_{i+1}^3+h_i^3}{12(h_{i+1}+h_i)})y_i^{(4)} + \ldots \\ TB_L = -\frac{h_1^2}{3!}y_0''' - \frac{h_1^3}{4!}y_0^{(4)} + \ldots \\ TB_R = -\frac{h_N^2}{3!}y_N''' + \frac{h_N^3}{4!}y_N^{(4)} + \ldots \end{cases} \quad (4.20)$$

If $h_{i+1} = h_i$, then $\tau_1 = -(\frac{h_{i+1}^3+h_i^3}{12(h_{i+1}+h_i)})y_i^{(4)}$. Thus, assume that around the transition point $h_{i+1} \neq h_i$ and let denote the smallest truncation term by:

$$|TE| \leq \begin{cases} \left|\frac{h_{i+1}-h_i}{3}\right| \\ \frac{h_1^2}{3!} \\ \frac{h_N^2}{3!} \end{cases} \leq h^2. \quad (4.21)$$

Therefore, the finite difference scheme formulated in Eq. (4.19) is a second-order convergent scheme.

4.7 Improving Accuracy of the Solution (Richardson Extrapolation)

To enhance the accuracy of the solution, we use the Richardson extrapolation technique. The key idea is that if the leading term in the truncation error of an approximation formula is known, we can combine two approximations, calculated with different mesh sizes h and $\frac{h}{2}$, to achieve a higher-order approximation. This method accelerates convergence by taking a linear combination of two approximations from nested meshes, resulting in a more accurate solution.

From Eq. (4.21), which involves the truncation error of the formulated method is $O(h^2)$. Hence, we have:

$$\left|y(x_i)-y_i\right| \leq Ch^2 \tag{4.22}$$

where $y(x_i)$ and y_i are exact and approximate solutions respectively, C is constant independent of mesh sizes h. Let $[0,1]^{2N}$ be the mesh obtained by bisecting each mesh interval in $[0,1]^N$ and denote the approximation of the solution on $[0,1]^{2N}$ by $y_{\frac{i}{2}}$. Consider Eq. (4.22) works for any $h \neq 0$, which implies:

$$y(x_i)-y_i \cong Ch^2+R^N, \qquad x_i \in [0,1]^N \tag{4.23}$$

So that, it works for any $\frac{h}{2} \neq 0$ yields:

$$y(x_i)-y_{\frac{i}{2}} \cong C\frac{h^2}{4}+R^{2N}, \qquad x_i \in [0,1]^{2N} \tag{4.24}$$

where the remainders, R^N and R^{2N} are $O(h^4)$. Eliminating C from the system of equations formed by both Eqs. (4.15) and (4.16) leaders to:

$$3y(x_i)-\left(4y_{\frac{i}{2}}-y_i\right) \cong O(h^4), \tag{4.25}$$

which suggests that the extrapolated solution as:

$$\left(y_i\right)^{ext}=\frac{1}{3}\left(4y_{\frac{i}{2}}-y_i\right) \tag{4.26}$$

is also an approximation of $y(x_i)$. Using this approximation to evaluate the truncation error, we obtain:

$$\left|y(x_i)-\left(y_i\right)^{ext}\right| \leq Ch^4 \tag{4.27}$$

The third solution, based on the two solutions from Eq. (4.26), is the Richardson extrapolation method. This method is applied to the finite difference approximation formulated in Eq. (4.19) to improve the convergence rate from second order to fourth order.

4.8 Numerical Examples and Results

To validate the proposed method and demonstrate its computational convergence, we tested it using various model examples. The maximum absolute error is calculated as follows:

$$E_\varepsilon^N = \max_{x_i \in [0,1]^N} | y(x_i) - y_i |,$$

where $y(x_i)$ and y_i respectively, denotes the exact and approximate solutions. Further, the order of convergence calculated by the formula:

$$R = \frac{\log(E_\varepsilon^N) - \log(E_\varepsilon^{2N})}{\log(2)}.$$

Example 1: Consider the singularly perturbed problem, [15]:

$$\begin{cases} -\varepsilon y''(x) + (1 + x(1-x))y(x) = f(x), \quad 0 < x < 1, \\ y(0) - \sqrt{\varepsilon} y'(0) = -1 - \sqrt{\varepsilon} + e^{\frac{-1}{\sqrt{\varepsilon}}}, \\ y(1) + \sqrt{\varepsilon} y'(1) = -1 - \sqrt{\varepsilon} + e^{\frac{-1}{\sqrt{\varepsilon}}}. \end{cases}$$

.

The source function is given by

$$f(x) = 1 + x(1-x) + [2\sqrt{\varepsilon} - x(1-x)^2]e^{\frac{-x}{\sqrt{\varepsilon}}} + [2\sqrt{\varepsilon} - x^2(1-x)]e^{\frac{-(1-x)}{\sqrt{\varepsilon}}}.$$

The exact solution to the problem that satisfies the above boundary conditions is given by

$$y(x) = 1 + (x-1)e^{\frac{-x}{\sqrt{\varepsilon}}} - xe^{\frac{-(1-x)}{\sqrt{\varepsilon}}}.$$

Example 2: Consider the singularly perturbed problem, [15]:

$$\begin{cases} -\varepsilon y''(x) + y(x) = -\cos^2(\pi x) = 2\varepsilon\pi^2\cos(2\pi x), \quad 0 < x < 1, \\ y(0) - 2\sqrt{\varepsilon}y'(0) = 2 - \dfrac{4e^{\frac{-1}{\sqrt{\varepsilon}}}}{1+e^{\frac{-1}{\sqrt{\varepsilon}}}}, \\ y(1) + 3\sqrt{\varepsilon}y'(1) = \dfrac{3(1-e^{\frac{-1}{\sqrt{\varepsilon}}})}{1+e^{\frac{-1}{\sqrt{\varepsilon}}}}. \end{cases}$$

.

The exact solution that satisfies the given boundary conditions is given by

$$y(x) = \frac{e^{\frac{-(1-x)}{\sqrt{\varepsilon}}} + e^{\frac{-x}{\sqrt{\varepsilon}}}}{1+e^{\frac{-1}{\sqrt{\varepsilon}}}} - \cos^2(\pi x)$$

Table 1: Comparison of computed maximum absolute errors for Example 1 by the proposed method on uniform and piecewise mesh

$\varepsilon \downarrow N \rightarrow$	64	128	256	512	1024	2048
On Piecewise Mesh						
10^{-3}	1.2907e-04	8.3584e-06	5.2721e-07	3.3029e-08	2.0656e-09	1.2910e-10
10^{-6}	1.7631e-04	3.2237e-05	4.3353e-06	3.4296e-07	1.6870e-08	1.5445e-09
10^{-12}	2.4406e-04	6.0997e-05	1.5240e-05	3.8055e-06	9.4910e-07	2.3614e-07
On Uniform Mesh						
10^{-3}	1.2907e-04	8.3584e-06	5.2721e-07	3.3029e-08	2.0656e-09	1.2910e-10
10^{-6}	5.4645e-01	2.9843e-01	9.6954e-02	1.5994e-02	1.5021e-03	1.0660e-04
10^{-12}	9.9940e-01	9.9881e-01	9.9762e-01	9.9523e-01	9.9049e-01	9.8105e-01

Table 2: Computed maximum absolute errors and rate of convergence for Example 1 by the proposed methods on piecewise mesh

$\varepsilon \downarrow N \rightarrow$	64	128	256	512	1024	2048
Fourth Order						
10^{-3}	1.2907e-04	8.3584e-06	5.2721e-07	3.3029e-08	2.0656e-09	1.2910e-10
	3.9488	3.9868	3.9966	3.9991	4.0000	
10^{-4}	1.4232e-04	1.7046e-05	1.8395e-06	1.8490e-07	1.7636e-08	1.6144e-09
	3.0616	3.2120	3.3145	3.3901	3.4495	
10^{-5}	1.3765e-04	1.6484e-05	1.7787e-06	1.7879e-07	1.7053e-08	1.5611e-09
	3.0619	3.2122	3.3145	3.3902	3.4494	
Second Order						
10^{-3}	1.6113e-02	4.1194e-03	1.0358e-03	2.5931e-04	6.4851e-05	1.6214e-05
	1.9677	1.9917	1.9980	1.9995	1.9999	
10^{-4}	1.6798e-02	5.8409e-03	1.9219e-03	6.0967e-04	1.8832e-04	5.6982e-05
	1.5240	1.6037	1.6564	1.6948	1.7246	
10^{-5}	1.6487e-02	5.7326e-03	1.8863e-03	5.9835e-04	1.8483e-04	5.5925e-05
	1.5241	1.6036	1.6565	1.6948	1.7246	

Table 3: Comparison of maximum absolute errors for Example 1

$\varepsilon \downarrow N \rightarrow$	64	128	256	512	1024	2048
Present Method						
10^{-3}	1.2907e-04	8.3584e-06	5.2721e-07	3.3029e-08	2.0656e-09	1.2910e-10
10^{-4}	1.4232e-04	1.7046e-05	1.8395e-06	1.8490e-07	1.7636e-08	1.6144e-09
10^{-5}	1.1765e-04	1.6484e-05	1.7787e-06	1.7879e-07	1.7053e-08	1.5611e-09
Result in [15]						
10^{-3}	2.34e-04	5.82e-05	1.45e-05	3.63e-06	9.07e-07	2.27e-07
10^{-4}	4.99e-04	1.59e-04	4.61e-05	1.16e-05	2.91e-06	7.26e-07

10^{-5}	1.20e-04	4.20e-05	1.32e-05	3.80e-06	1.12e-06	3.30e-07

Table 4: Comparison of maximum absolute errors for Example 2

$\varepsilon \downarrow N \rightarrow$	64	128	256	512	1024	2048
Present Method						
10^{-1}	2.9550e-08	1.8466e-09	1.1542e-10	7.2056e-12	6.2839e-13	1.0038e-13
10^{-2}	2.1567e-06	1.3535e-07	8.4682e-09	5.2942e-10	3.3070e-11	2.0753e-12
10^{-3}	2.0676e-04	1.3455e-05	8.4963e-07	5.3240e-08	3.3298e-09	2.0924e-10
10^{-4}	2.5196e-04	3.0337e-05	3.2798e-06	3.2987e-07	3.1471e-08	2.8809e-09
Result in [15]						
10^{-1}	3.208e-04	7.957e-05	2.005e-05	5.007e-06	1.252e-06	3.131e-07
10^{-2}	4.066e-04	1.017e-04	2.544e-05	6.359e-06	1.590e-06	3.975e-07
10^{-3}	3.998e-04	1.003e-04	2.509e-05	6.274e-06	1.569e-06	3.922e-07
10^{-4}	3.859e-04	9.938e-05	2.504e-05	6.271e-06	1.568e-06	3.922e-07

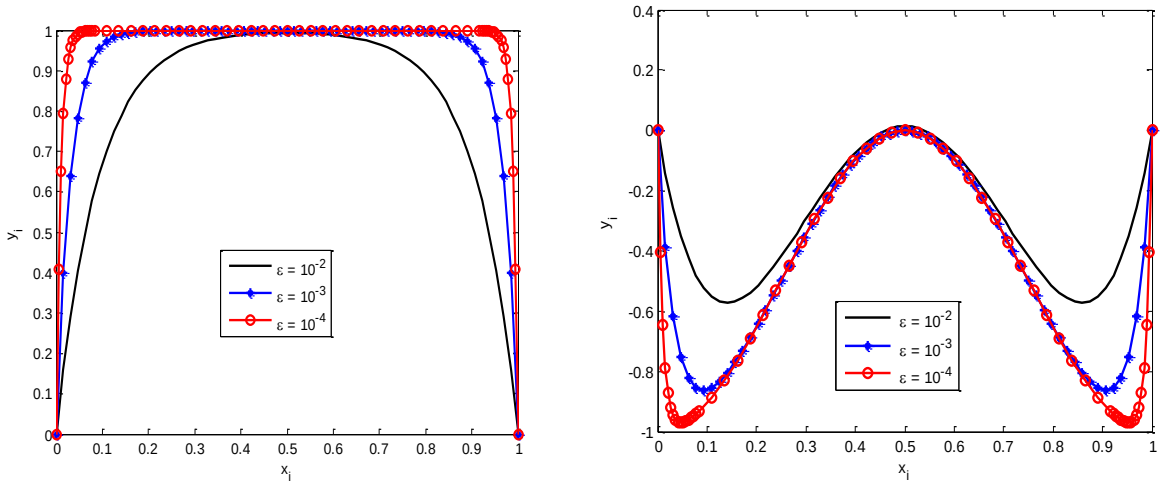


Figure 1: Effects of perturbation parameter on the solution for Examples 1 and 2 when, $N = 64$.

4.9 Discussion

This work presents a finite difference approximation on piecewise mesh to solve the singularly perturbed reaction-diffusion equation with Robin boundary conditions. First, we discretize the solution domain based on the lengths of the piecewise mesh length. Next, we replace the second-order derivative with finite difference approximations. The two Robin boundary conditions are also approximated using second-order finite differences that derived from Taylor series expansions.

As shown in Eq. (4.19), this method results in a system of linear algebraic equations. Since the sum of the two off-diagonal elements is less than or equal to the diagonal element, the coefficient matrix is diagonally dominant. Therefore, we can solve the system using the Gaussian elimination method, and the scheme in Eq. (4.19) demonstrates stability for the described method. To validate the proposed method, we provide two examples, with numerical results displayed in Tables 1– 4 and Figures 1. Specifically, Figures illustrate the effects of the perturbation parameter and boundary layer behaviors. Notably, as seen in Table 1, the proposed method fails to produce accurate solutions on a uniform mesh when a small positive perturbation

parameter is present. To address this issue, we employ a piecewise mesh that features a finer mesh in the layer region and coarser mesh in the outer layer.

The maximum absolute errors and convergence rates presented in Tables 2 - 4 confirm the superiority of the proposed method, which yields more accurate solutions than some existing methods in the literature. Furthermore, the results indicate that as the number of mesh point's increases, the maximum absolute error decreases in each row, demonstrating the convergence of the proposed method.

Chapter Five

Conclusion and Future Work

5.1. Conclusion

A finite difference method on a piecewise mesh is developed for solving the singularly perturbed reaction-diffusion equation with Robin boundary conditions. Two numerical examples are modeled to illustrate the method. The results demonstrate that the maximum absolute error and order of convergence validate the method's competence. Furthermore, the proposed method yields better accurate solutions than some existing approaches in the literature. Numerical experiments confirm that it achieves better convergence with fewer errors, supporting the theoretical findings. As a result, the method well approximates the exact solution.

5.2. Future Work

In this study, a finite difference method on a piecewise mesh is presented for solving the singularly perturbed reaction-diffusion equation with Robin boundary conditions. The proposed approach can also be extended to higher-order convergence methods to achieve well precise solutions for the problem under consideration.

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